

# The Kähler Rank of Compact Complex Manifolds

Ionuț Chiose

Received: 15 May 2014 / Published online: 3 February 2015  
© Mathematica Josephina, Inc. 2015

**Abstract** The Kähler rank was introduced by Harvey and Lawson in their 1983 paper as a measure of the *kählerianity* of a compact complex surface. In this work we generalize this notion to the case of compact complex manifolds and we prove several results related to this notion. We show that on class VII surfaces, there is a correspondence between the closed positive forms on a surface and those on a blow-up in a point. We also show that a manifold of maximal Kähler rank which satisfies an additional condition is in fact Kähler.

**Keywords** Compact complex manifolds · Kähler manifolds · Class VII surfaces

**Mathematics Subject Classification** 32J27 primary · 32J15 secondary

## 1 Introduction

In [8], Harvey and Lawson introduced the Kähler rank of a compact complex surface, a quantity intended to measure how far a surface is from being Kähler. A surface has Kähler rank 2 iff it is Kähler. It has Kähler rank 1 iff it is not Kähler but still admits a closed (semi-) positive  $(1, 1)$ -form whose zero-locus is contained in a curve. In the remaining cases, it has Kähler rank 0.

In this paper we generalize the notion of Kähler rank to compact complex manifolds of arbitrary dimension and study its properties.

First, we discuss the problem of the bimeromorphic invariance of the Kähler rank. There are examples that show that it is not a bimeromorphic invariant. However, two bimeromorphic surfaces have the same Kähler rank [2]. This was shown by classifying

---

I. Chiose (✉)

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania  
e-mail: Ionut.Chiose@imar.ro

the surfaces of rank 1. In this paper we take a different approach, local in nature, which was alluded to in [2]. Namely, we study the problem of when a plurisubharmonic function on the blow-up is the pull-back of a smooth function. However, this method leads to an involved system of differential equations, and we were able to solve this system only up to order 3. Thus we obtain:

**Theorem 1.1** *Let  $X$  be a compact, complex, non-Kähler surface with  $b_1(X) = 1$ , and let  $p : X' \rightarrow X$  be the blow-up of  $X$  at a point. Suppose that  $\omega'$  is a closed, positive  $(1, 1)$  form on  $X'$ . Then there exists  $\omega$  a closed positive  $(1, 1)$  form on  $X$  of class  $C^1$  such that  $p^*\omega = \omega'$ .*

Second, we study the manifolds of maximal Kähler rank, i.e., those manifolds that admit a positive  $d$ -closed  $(1, 1)$ -form of strictly positive volume. It is conjectured that such manifolds are in the Fujiki class  $\mathcal{C}$ . Under an additional condition, we prove that they are in fact Kähler:

**Theorem 1.2** *Let  $X$  be a compact complex manifold of dimension  $n$  such that there exists  $\{\alpha\} \in H_{BC}^{1,1}(X, \mathbb{R})$  a nef class such that*

$$\int_X \alpha^n > 0$$

*Suppose moreover that there exists  $h$  a Hermitian metric on  $X$  such that*

$$i\partial\bar{\partial}h = 0, \partial h \wedge \bar{\partial}h = 0$$

*Then  $X$  is Kähler.*

The same method yields a simpler proof of a key theorem of Demailly and Păun in [5].

## 2 Definition and Examples

The Kähler rank of a manifold is the maximal rank a closed positive  $(1, 1)$ -form can reach on the manifold:

**Definition 2.1** Let  $X$  be a compact complex manifold of dimension  $n$ . The Kähler rank of  $X$ , denoted  $Kr(X)$ , is

$$Kr(X) = \max \left\{ k \mid \exists \omega \in C_{1,1}^\infty(X, \mathbb{R}), \omega \geq 0, d\omega = 0, \omega^k \neq 0 \right\} \tag{2.1}$$

The original definition in [8] for surfaces required that the form  $\omega$  appearing in the definition have zeroes in an analytic subset of  $X$ . Corollary 4.3 in [2] shows that the definition above coincides with the one in [8] for surfaces.

*Remark 2.2* Note that if  $Kr(X) = \dim X$  then for every  $p \in \overline{0, n}$  the operator  $\partial : H^{p,0}(X) \rightarrow H^{p+1,0}(X)$  is zero, while, if  $Kr(X) = 0$  then  $\partial : H^{1,0}(X) \rightarrow H^{2,0}(X)$  is onto. Indeed, if  $\sigma \in H^{1,0}(X) \setminus \{0\}$  satisfies  $\partial\sigma = 0$ , then  $i\sigma \wedge \bar{\sigma}$  is a closed, non-zero positive  $(1, 1)$ -form.

*Remark 2.3* As in the surface case considered in [8], on a compact complex manifold  $X$  of Kähler rank  $Kr(X) = k$ , there exists a complex analytic canonical foliation  $\mathcal{F}$  of codimension  $k$ . It is defined on the open set

$$\mathcal{B} = \left\{ x \in X \mid \exists \omega \in C_{1,1}^\infty(X, \mathbb{R}), d\omega = 0, \omega \geq 0, \omega^k(x) \neq 0 \right\} \tag{2.2}$$

and is characterized by  $\omega^k|_{\mathcal{F}} = 0, \forall \omega \geq 0, d\omega = 0$ .

*Example 2.4* A compact complex surface  $X$  has Kähler rank 2 if and only if it is Kähler (see remark 4.3 below) and this is equivalent to  $b_1(X)$  even (see [10]). When  $b_1(X)$  is odd but at least 3, then  $H^{1,0}(X) \neq 0$  and if  $\sigma$  is a non-zero holomorphic 1-form on  $X$  then it is  $d$ -closed, hence  $i\sigma \wedge \bar{\sigma}$  is a  $d$ -closed positive  $(1, 1)$ -form on  $X$ . If  $b_1(X) = 1$ , then the main results of [2] and [1] show that the only surfaces of Kähler rank equal to 1 are the Inoue surfaces and some Hopf surfaces. The other known surfaces (the other Hopf surfaces and the Kato surfaces) have Kähler rank 0.

*Example 2.5* In [9] the author constructed an example of a threefold  $X$  which is a proper modification of a Kähler manifold but which is not Kähler. In fact, it is a proper modification  $p : X \rightarrow \mathbb{P}^3$  of the projective space. One can take  $p^*\omega_{FS}$ , where  $\omega_{FS}$  is the Fubini–Study metric, to obtain a closed positive  $(1, 1)$ -form, not everywhere degenerate, on a manifold that is not Kähler. Therefore, unlike the surface case, in higher dimensions there are manifolds of maximal Kähler rank and which are not Kähler.

*Example 2.6* The well-known Iwasawa threefold is the quotient  $H/\Gamma$  where  $H$  is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with complex entries, and  $\Gamma$  is the subgroup of the matrices whose entries have integer real and imaginary entries. Then the holomorphic 1-forms on  $dx, dy$  and  $dz - xdy$  on  $H$  induce three holomorphic 1-forms on  $H/\Gamma$  denoted by  $\sigma_1, \sigma_2$  and  $\sigma_3$  respectively. Then  $d\sigma_3 = -\sigma_1 \wedge \sigma_2$ , hence  $\sigma_3$  is not  $d$ -closed, therefore  $Kr(H/\Gamma) \leq 2$ . But  $\sigma_1$  and  $\sigma_2$  are  $d$ -closed, therefore the form  $\omega = i\sigma_1 \wedge \bar{\sigma}_1 + i\sigma_2 \wedge \bar{\sigma}_2$  is closed and positive, and  $\omega^2 \neq 0$ , therefore the Kähler rank is 2.

*Example 2.7* In [11] the author constructed a Moishezon threefold  $Y$  that contains an algebraic 1-cycle  $\ell$  homologous to zero and which moves and covers the whole  $Y$ . Such a manifold cannot have maximal Kähler rank. Indeed, if  $\omega$  is a closed positive  $(1, 1)$ -form on  $Y$ , and if  $y \in Y$  is arbitrary, let  $\ell'$  be a 1-cycle passing through  $y$  and which is homologous to zero. Then

$$\int_{\ell'} \omega = 0$$

and therefore at  $y$ ,  $\omega$  cannot have rank 3. Therefore  $\omega^3 = 0$ . This example shows that for dimension at least 3 the Kähler rank is not a bimeromorphic invariant. However, it is expected that, if  $Y \rightarrow X$  is the blow-up of a compact complex manifold  $X$  in a point, then  $Kr(X) = Kr(Y)$ .

*Example 2.8* In [6] the authors constructed a complex structure on the connected sum  $\#_k S^3 \times S^3$  of  $k \geq 2$  copies of  $S^3 \times S^3$  and a balanced metric  $g^2$  which is  $i\partial\bar{\partial}$ -exact. Such a manifold has Kähler rank equal to 0. Indeed, if  $\omega$  is a closed positive  $(1, 1)$ -form, then its trace with respect to  $g^2$  is zero, hence the form  $\omega$  has to be 0.

*Remark 2.9* Starting with the above examples, and taking products, one can obtain compact complex manifolds of any dimension  $n \geq 2$  and any Kähler rank  $0 \leq Kr \leq n$ .

### 3 The Bimeromorphic Invariance of the Kähler Rank for Class VII Surfaces

In this section we discuss the bimeromorphic invariance of the Kähler rank on class VII surfaces, the only non-trivial case. We show that the problem can be reduced to a system of differential equations, and then we solve the system up to order 3, thus proving Theorem 1.1.

#### 3.1 Preliminaries

Suppose  $X$  is a surface with  $b_1 = 1$  and let  $\pi : X' \rightarrow X$  be the blow-up of  $X$  in a point  $p$ . Let  $\gamma^{0,1}$  be a  $\bar{\partial}$  closed  $(0, 1)$  form on  $X$  which generates  $H^{0,1}(X)$ . Then  $\gamma'^{0,1} = \pi^*\gamma^{0,1}$  generates  $H^{0,1}(X')$ .

Let  $\omega'$  be a closed, positive  $(1, 1)$  form on  $X'$ ; then it is  $d$  exact [8], Proposition 37. We want to show that there exists  $\omega$  on  $X$  such that  $\pi^*\omega = \omega'$ . Then on  $X'$ ,  $\omega'$  can be written as

$$\omega' = \mu \overline{\partial\gamma'^{0,1}} + \bar{\mu} \partial\gamma'^{0,1} + i\partial\bar{\partial}\phi' \tag{3.1}$$

where  $\mu \in \mathbb{C}$  and  $\phi' \in C^\infty(X', \mathbb{R})$ . We need to show that  $\phi'$  is the pull-back of a  $C^\infty$  function  $\phi$  on  $X$ .

Locally on a disk  $\Delta^2 = \{|z| < 1\}$  around  $p$  on  $X$ ,  $\gamma^{0,1}$  is  $\bar{\partial}$  exact, so it can be written as  $\gamma^{0,1}|_{\Delta^2} = \bar{\partial}f$ , where  $f \in C^\infty(\Delta^2)$ . Then on  $\pi^{-1}(\Delta^2)$ ,

$$\omega' = i\partial\bar{\partial} (2\text{Im}(\bar{\mu}f') + \phi') \tag{3.2}$$

where  $f' = \pi^*f$ . Set  $\varphi' = 2\text{Im}(\bar{\mu}f) + \phi'$ . We need to show that  $\varphi'$  is the pull-back of a smooth function on  $\Delta^2$ .

So let  $\pi : \hat{\Delta}^2 \rightarrow \Delta^2$  be the blow-up of the unit disk in  $\mathbb{C}^2$ , let  $E$  be the exceptional divisor, and suppose that locally  $\pi$  is given by  $(z, w) \rightarrow (z, zw) = (z_1, z_2)$ . The exceptional divisor is given by  $\{z = 0\}$ . Let  $\phi'$  be a  $C^\infty$  function on  $\hat{\Delta}^2$ . Then we have

**Proposition 3.1** *There exists  $\varphi$  a  $C^\infty$  function on  $\Delta^2$  such that  $\varphi' = \pi^*\varphi$  if and only if there exist  $A_{\alpha,\beta}^{p,q} \in \mathbb{C}$  such that*

$$\frac{\partial^{\alpha+\beta} \varphi'}{\partial z^\alpha \partial \bar{z}^\beta} \Big|_{z=0} = \sum_{p=0}^\alpha \sum_{q=0}^\beta \binom{\alpha}{p} \binom{\beta}{q} A_{\alpha,\beta}^{p,q} w^p \bar{w}^q \tag{3.3}$$

*Proof* If  $\varphi' = \pi^*\varphi$ , with  $\varphi \in C^\infty(\Delta^2)$ , then, from  $\varphi'(z, w) = \varphi(z, zw)$  and the chain rule, we obtain the above equation with

$$A_{\alpha,\beta}^{p,q} = \frac{\partial^{\alpha+\beta} \varphi}{\partial z_1^p \partial z_2^{\alpha-p} \partial \bar{z}_1^q \partial \bar{z}_2^{\beta-q}}(0) \tag{3.4}$$

Conversely, if  $\varphi'$  satisfies the above conditions on its partial derivatives, then  $\varphi'|_E$  is constant, and it induces a continuous function  $\varphi$  on  $\Delta^2$ . It is actually  $C^\infty$ , with the partial derivatives at 0 equal to  $A_{\alpha,\beta}^{p,q}$  as above.

*Remark 3.1* If the above Eq. 3.3 holds only for  $\alpha + \beta \leq k$ , it follows that  $\varphi'$  is the pull-back of a  $C^k$  function  $\varphi$ .

So in order to prove that  $\varphi'$  is the pull-back of a  $C^\infty$  function  $\varphi$  on  $\Delta^2$ , it is enough to prove that

$$\frac{\partial^{\alpha+\beta} \varphi'}{\partial z^\alpha \partial \bar{z}^\beta} \Big|_{z=0} \tag{3.5}$$

are polynomials in  $w$  and  $\bar{w}$  of degrees  $\alpha$  and  $\beta$  respectively.

### 3.2 The System of Differential Equations

Now we set up the system of differential equations which needs to be solved in order to prove that  $\omega'$  is the pull-back of a smooth  $\omega$ .

We will use the fact that  $\omega'$  is of rank 1 ([8], Proposition 37), i.e., that

$$\omega' \wedge \omega' = 0 \tag{3.6}$$

and we will show that  $\varphi$  is of class  $C^3$ , i.e., that  $\omega'$  is the pull-back of a  $C^1$  form.

First,  $\omega' = i\partial\bar{\partial}\varphi'$  and it is positive, hence  $\varphi'$  is plurisubharmonic. Restricted to the exceptional divisor  $E$ , it follows that  $\varphi'|_E$  is constant. Hence  $\varphi'$  is the pull-back of a continuous function  $\varphi$  on  $\Delta^2$ .

Next, denote by

$$P_{\alpha,\beta} = \frac{\partial^{\alpha+\beta} \varphi'}{\partial z^\alpha \partial \bar{z}^\beta} \Big|_{z=0} \tag{3.7}$$

which are  $C^\infty$  functions on  $\mathbb{C}$ . Since  $\varphi'$  is defined on the whole  $\hat{\Delta}^2$ , the functions  $P_{\alpha,\beta}$  satisfy the following *growth conditions*:

$$w^\alpha \bar{w}^\beta P_{\alpha,\beta} \left( \frac{1}{w} \right) \tag{3.8}$$

can be extended to  $C^\infty$  functions at 0.

Consider the equation  $\omega' \wedge \omega' = 0$  written in local coordinates  $(z, w)$ :

$$\frac{\partial^2 \varphi'}{\partial z \partial \bar{z}} \cdot \frac{\partial^2 \varphi'}{\partial w \partial \bar{w}} = \frac{\partial^2 \varphi'}{\partial z \partial \bar{w}} \cdot \frac{\partial^2 \varphi'}{\partial w \partial \bar{z}} \tag{3.9}$$

Take

$$\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} \tag{3.10}$$

and restrict it to  $z = 0$ ; we obtain

$$\begin{aligned} & \sum_{p=0}^\alpha \sum_{q=0}^\beta \binom{\alpha}{p} \binom{\beta}{q} P_{p+1,q+1} \frac{\partial^2 P_{\alpha-p,\beta-q}}{\partial w \partial \bar{w}} \\ &= \sum_{p=0}^\alpha \sum_{q=0}^\beta \binom{\alpha}{p} \binom{\beta}{q} \frac{\partial P_{p+1,q}}{\partial \bar{w}} \frac{\partial P_{\alpha-p,\beta-q+1}}{\partial w} \end{aligned} \tag{3.11}$$

which gives a system of partial differential equations in the unknowns  $P_{\alpha,\beta}$  which satisfy the conditions 3.8 and moreover  $\bar{P}_{\alpha,\beta} = P_{\beta,\alpha}$ .

We know that  $P_{0,0}$  is constant, and from

$$P_{1,1} \cdot \frac{\partial^2 P_{0,0}}{\partial w \partial \bar{w}} = \frac{\partial P_{1,0}}{\partial \bar{w}} \cdot \frac{\partial P_{0,1}}{\partial w} \tag{3.12}$$

we obtain that  $P_{1,0}$  is holomorphic, and from the growth condition 3.8 it follows that  $P_{1,0}$  has the desired form, i.e., it is a polynomial in  $w$  of degree 1. This shows that  $\varphi$  is a function of class  $C^1$ .

### 3.3 The Proof of Theorem 1.1

We complete the proof of Theorem 1.1. We show that  $\varphi$  is in fact of class  $C^3$ , hence  $\omega$  is of class  $C^1$ .

For  $\alpha = 2$  and  $\beta = 0$  in 3.11 we obtain

$$P_{1,1} \cdot \frac{\partial^2 P_{2,0}}{\partial w \partial \bar{w}} = 2 \frac{\partial P_{2,0}}{\partial \bar{w}} \cdot \frac{\partial P_{1,1}}{\partial w} \tag{3.13}$$

and for  $\alpha = 1, \beta = 1$  we obtain

$$P_{1,1} \cdot \frac{\partial^2 P_{1,1}}{\partial w \partial \bar{w}} = \frac{\partial P_{2,0}}{\partial \bar{w}} \cdot \frac{\partial P_{0,2}}{\partial w} + \frac{\partial P_{1,1}}{\partial \bar{w}} \cdot \frac{\partial P_{1,1}}{\partial w} \tag{3.14}$$

Set

$$f = \frac{\partial P_{2,0}}{\partial \bar{w}}$$

and  $g = P_{1,1}$ . Then  $f$  and  $g$  satisfy the following properties: they are  $C^\infty$  functions on  $\mathbb{C}$ ;  $g$  has real values; the functions

$$w \bar{w} g \left( \frac{1}{w} \right) \tag{3.15}$$

and

$$\frac{w^2}{\bar{w}^2} \cdot f \left( \frac{1}{w} \right) \tag{3.16}$$

are  $C^\infty$  at 0, and moreover  $f$  and  $g$  satisfy the following equations:

$$\frac{\partial f}{\partial w} \cdot g = 2f \cdot \frac{\partial g}{\partial w} \tag{3.17}$$

$$g \cdot \frac{\partial^2 g}{\partial w \partial \bar{w}} = |f|^2 + \left| \frac{\partial g}{\partial w} \right|^2 \tag{3.18}$$

We will show the following

**Proposition 3.2**  $f = 0$  and  $g$  is a quadratic form of rank 1, i.e.,  $g(w) = |a + bw|^2$ .

*Proof* Let  $D_g$  be the non-zero set of  $g$ , i.e.,  $D_g = \{w \in \mathbb{C} | g(w) \neq 0\}$ . If  $D_g = \emptyset$ , then  $g = 0$  and from 3.18 it follows that  $f = 0$ .

If  $D_g = \mathbb{C}$ , then  $g$  is never 0, and from 3.17 it follows that there exists  $h$  holomorphic on  $\mathbb{C}$  such that  $f = \bar{h}g^2$ . We can assume that  $g > 0$  on  $\mathbb{C}$ . Then from 3.18 it follows that  $\ln g$  is subharmonic, hence  $\ln |f|$  is subharmonic on  $D_f = \{w \in \mathbb{C} | f(w) \neq 0\}$ . It follows that  $|f|^2$  is subharmonic on  $\mathbb{C}$  and since  $f$  is bounded (from 3.16), it follows that  $|f|$  is constant. If  $|f| \neq 0$ , then from  $f = \bar{h}g^2$  we obtain that  $i\partial\bar{\partial} \ln g = 0$  and from 3.18 we get that  $|f| = 0$ , contradiction. Hence  $f = 0$  and Eq. 3.18 implies that  $\ln g$  is harmonic, i.e.,  $g = \exp(\text{Re } j)$ , where  $j$  is a holomorphic function on  $\mathbb{C}$ . From condition 3.15 on  $g$  it follows that  $j$  is constant, hence also  $g$  is constant.

Now assume that  $D_g \neq \emptyset, \mathbb{C}$  and denote by  $D'_g$  a connected component of  $D_g$ . Assume that  $g > 0$  on  $D'_g$ . From 3.17 it follows that  $f = \bar{h}g^2$  where  $h$  is a holomorphic function on  $D'_g$ . Again 3.18 implies that  $\ln g$  is subharmonic on  $D'_g$  and so  $\ln |f|$  is subharmonic on  $D'_g \cap D_f$ . Let  $w_0 \in \partial D'_g$  (the boundary of  $D'_g$ ) and set

$$f'(w) = \frac{f(w)}{\sqrt{|w - w_0|}} \tag{3.19}$$

as a function on  $D'_g$ . Since  $\ln |f|$  is subharmonic, it follows that  $\ln |f'|$  is also subharmonic on  $D'_g$ , so  $|f'|^2$  is subharmonic on  $D'_g$ . Moreover,  $f = 0$  on the boundary  $\partial D'_g$  (this follows again from 3.18) except possibly at  $w_0$ , and  $\lim_{w \rightarrow \infty} |f'(w)| = 0$  because  $f$  is bounded at infinity (from 3.16). Since  $f(w_0) = 0$  it follows that  $f'$  can be extended to a continuous function at  $w_0$ , with  $f'(w_0) = 0$ . Hence  $|f'|$  is a subharmonic function on  $D'_g$ ,  $f' = 0$  on  $\partial D'_g \cup \{\infty\}$ , hence from the maximum principle, it follows that  $f' = 0$  on  $D'_g$ , hence also  $f = 0$  on  $D'_g$ . Since  $f = 0$  on  $\{w \in \mathbb{C} | g(w) = 0\}$ , we get that  $f = 0$  on the whole  $\mathbb{C}$ .

So  $g$  satisfies the equation

$$g \cdot \frac{\partial^2 g}{\partial w \partial \bar{w}} = \frac{\partial g}{\partial w} \cdot \frac{\partial g}{\partial \bar{w}} \tag{3.20}$$

and

$$w\bar{w} \cdot g \left( \frac{1}{w} \right) \tag{3.21}$$

is  $C^\infty$  at 0. If  $g$  has two zeroes,  $w_0$  and  $w_1$ ,  $w_0 \neq w_1$ , we consider as above  $D'_g$  a connected component of  $D_g$ . Assume that  $g > 0$  on  $D'_g$ . Then  $\ln g$  is harmonic on  $D'_g$ . Let

$$g'(w) = \frac{g(w)}{\sqrt{|w - w_0|^3} \sqrt{|w - w_1|^3}}$$

Then  $\ln g'$  is harmonic on  $D'_g$ , so  $g'$  is subharmonic. Moreover, it is 0 on the boundary  $\partial D'_g$  of  $D'_g$ , except possibly at  $w_0$  and  $w_1$ . But at  $w_0$ ,  $g(w_0) = 0$  and

$$\frac{\partial g}{\partial w}(w_0) = \frac{\partial g}{\partial \bar{w}}(w_0) = 0 \tag{3.22}$$

and the same at  $w_1$ , which implies that  $g'$  is continuous on the whole boundary  $\partial D'_g$ . At infinity,  $g$  approaches 0, and again by the maximum principle we obtain that  $g = 0$  on  $D'_g$ , contradiction. This shows that  $g$  has exactly one zero. Assume that  $g(w_0) = 0$ . Then consider the function

$$g''(w) = \frac{g(w)}{|w - w_0|^2}$$

on  $\mathbb{C} \setminus \{w_0\}$ . Then  $\ln g''$  is harmonic on  $\mathbb{C} \setminus \{w_0\}$ , and it is bounded at infinity. Moreover, since  $g(w_0) = 0$  and  $dg(w_0) = 0$ , it follows that  $g''$  is bounded near  $w_0$ . Hence  $g''$  is a bounded, subharmonic function on  $\mathbb{C} \setminus \{w_0\}$ , so it is constant. Therefore  $g(w) = C|w - w_0|^2$ .

Returning to our previous notations, we showed that  $P_{2,0}$  is holomorphic, hence it is a polynomial of degree 2 in  $w$ , and that  $P_{1,1}$  is a polynomial of degree  $\leq 1$  in  $w$  and  $\bar{w}$ . Hence  $\varphi$  is a function of class  $C^2$  and  $\omega$  is continuous.



Next, we show that if  $P_{1,1} \neq 0$ , then  $\varphi$  is actually  $\mathcal{C}^3$ . First, we can assume, without loss of generality, that  $P_{1,1}$  is constant. Indeed, if  $P_{1,1}(w) = C|w - w_0|^2$ , then we replace the functions  $P_{\alpha,\beta}$  by

$$\frac{1}{(w - w_0)^\alpha (\bar{w} - \bar{w}_0)^\beta} P_{\alpha,\beta}(w) \tag{3.23}$$

and we end up with the same system of differential equations and the same *growth conditions*.

When  $\alpha = 3$  and  $\beta = 0$  in 3.11 we obtain

$$P_{1,1} \cdot \frac{\partial^2 P_{3,0}}{\partial w \partial \bar{w}} = 3 \cdot \frac{\partial P_{3,0}}{\partial \bar{w}} \cdot \frac{\partial P_{1,1}}{\partial w} \tag{3.24}$$

and when  $\alpha = 2$  and  $\beta = 1$  we obtain

$$P_{1,1} \cdot \frac{\partial^2 P_{2,1}}{\partial w \partial \bar{w}} + 2 \cdot P_{2,1} \cdot \frac{\partial P_{1,1}}{\partial w \partial \bar{w}} = \frac{\partial P_{1,1}}{\partial \bar{w}} \cdot \frac{\partial P_{2,1}}{\partial w} + 2 \cdot \frac{\partial P_{2,1}}{\partial \bar{w}} \cdot \frac{\partial P_{1,1}}{\partial w} \tag{3.25}$$

$P_{1,1}$  is a non-zero constant, so the equations imply that both  $P_{3,0}$  and  $P_{2,1}$  are harmonic. By using the *growth conditions* we obtain that  $P_{3,0}$  is holomorphic and that  $P_{2,1}$  has the desired form.

If  $P_{1,1} = 0$ , things get more complicated, but we can still show that  $\varphi$  is of class  $\mathcal{C}^3$ . If  $\omega(0) = 0$ , then for  $\alpha + \beta = 4$  the system 3.11 implies the following equations:

$$3f \cdot \frac{\partial g}{\partial w} = 2 \frac{\partial f}{\partial w} \cdot g \tag{3.26}$$

$$\bar{g} \cdot \frac{\partial f}{\partial w} + 3g \cdot \frac{\partial^2 g}{\partial w \partial \bar{w}} = 3 \frac{\partial g}{\partial w} \cdot \frac{\partial g}{\partial \bar{w}} + 3f \frac{\partial \bar{g}}{\partial w} \tag{3.27}$$

$$2g \cdot \frac{\partial^2 \bar{g}}{\partial w \partial \bar{w}} + 2\bar{g} \frac{\partial^2 g}{\partial w \partial \bar{w}} = \frac{\partial g}{\partial w} \cdot \frac{\partial \bar{g}}{\partial \bar{w}} + 4 \frac{\partial g}{\partial \bar{w}} \cdot \frac{\partial \bar{g}}{\partial w} + f \cdot \bar{f} \tag{3.28}$$

where

$$f = \frac{\partial P_{3,0}}{\partial \bar{w}}$$

and  $g = P_{2,1}$  and we have the corresponding *growth conditions* for  $f$  and  $g$ . This system can be solved by using similar methods as in Proposition 3.2, so we omit it.

### 4 Manifolds of Maximal Kähler Rank

In this section we show that a compact complex manifold  $X$  of dimension  $n$  such that  $Kr(X) = n$  and which moreover admits a special Hermitian metric is in fact Kähler:

**Theorem 4.1** *Let  $X$  be a compact complex manifold such that there exists a nef class  $\{\alpha\} \in H_{BC}^{1,1}(X, \mathbb{R})$  such that*

$$\int_X \alpha^n > 0$$

*Suppose moreover that  $X$  supports a Hermitian metric  $h$  such that*

$$i \partial \bar{\partial} h = \partial h \wedge \bar{\partial} h = 0 \tag{4.1}$$

*Then  $\{\alpha\}$  is big and  $h$  is  $\partial + \bar{\partial}$  cohomologous to a Kähler metric. In particular  $X$  is Kähler.*

*Remark 4.1* Here big means that the class  $\{\alpha\}$  contains a Kähler current, i.e., a closed positive current that dominates some Hermitian metric.

*Remark 4.2* Condition 4.1 is needed in order to bound some integrals (see 4.9 below) and it is equivalent to

$$i \partial \bar{\partial} h^k = 0, \quad \forall k = \overline{1, n-1} \tag{4.2}$$

The condition 4.1 appeared in the work [7], where the authors discussed the Monge–Ampère equation on Hermitian manifolds.

*Remark 4.3* When  $n = 2$ , the existence of a Hermitian form satisfying 4.1 is well-known, and we obtain another proof of the fact that a surface of Kähler rank equal to 2 is Kähler. When  $n = 3$  just the equation  $i \partial \bar{\partial} h = 0$  is needed.

*Remark 4.4* The above theorem is a particular case of a conjecture of Demailly and Păun (see [5], Conjecture 0.8) which states that if a manifold admits a nef class of strictly positive self-intersection, the manifold is in Fujiki class  $\mathcal{C}$ , i.e., it is bimeromorphic to a Kähler manifold.

*Proof* First, we show that  $\{\alpha\}$  is big. We need to show that there exists  $\varepsilon_0 > 0$  and a distribution  $\chi$  such that  $\alpha + i \partial \bar{\partial} \chi \geq \varepsilon_0 h$ . According to Lamari’s result [10], Lemma 3.3, this is equivalent to showing that

$$\int_X \alpha \wedge g^{n-1} \geq \varepsilon_0 \int_X h \wedge g^{n-1} \tag{4.3}$$

for any Gauduchon metric  $g^{n-1}$  on  $X$ . So suppose that  $\forall m \in \mathbb{N}, \exists g_m^{n-1}$  a Gauduchon metric such that

$$\int_X \alpha \wedge g_m^{n-1} \leq \frac{1}{m} \int_X h \wedge g_m^{n-1} \tag{4.4}$$

We can assume that

$$\int_X h \wedge g_m^{n-1} = 1 \tag{4.5}$$

and therefore

$$\int_X \alpha \wedge g_m^{n-1} \leq \frac{1}{m} \tag{4.6}$$

Since  $\{\alpha\}$  is nef, for every  $m$  we can find  $\psi_m \in C^\infty(X, \mathbb{R})$  such that  $\alpha + i\partial\bar{\partial}\psi_m \geq -\frac{1}{2m}h$ . The main result of [12] implies that we can solve the equation

$$\left(\alpha + \frac{1}{m}h + i\partial\bar{\partial}\varphi_m\right)^n = C_m g_m^{n-1} \wedge h \tag{4.7}$$

for a function  $\varphi_m \in C^\infty(X, \mathbb{R})$  such that if we set  $\alpha_m = \alpha + \frac{1}{m}h + i\partial\bar{\partial}\varphi_m$ , then  $\alpha_m > 0$ . The constant  $C_m$  is given by

$$C_m = \int_X \left(\alpha + \frac{1}{m}h\right)^n \geq \int_X \alpha^n = C > 0 \tag{4.8}$$

Now

$$\int_X \alpha_m^{n-1} \wedge h = \int_X h \wedge \left(\alpha + \frac{1}{m}h\right)^{n-1} \leq \int_X h \wedge (\alpha + h)^{n-1} = M \tag{4.9}$$

and if we set

$$E = \left\{ \frac{\alpha_m^{n-1} \wedge h}{g_m^{n-1} \wedge h} > 2M \right\} \tag{4.10}$$

then

$$\int_E g_m^{n-1} \wedge h \leq \frac{1}{2} \tag{4.11}$$

Therefore on  $X \setminus E$  we have  $\alpha_m^{n-1} \wedge h \leq 2M g_m^{n-1} \wedge h$ . By looking at the eigenvalues of  $\alpha_m$  with respect to  $h$ , from 4.11 and 4.7, it follows that on  $X \setminus E$  we have

$$\alpha_m \geq \frac{C_m}{2nM}h$$

Therefore

$$\begin{aligned} \int_X \alpha_m \wedge g_m^{n-1} &\geq \int_{X \setminus E} \alpha_m \wedge g_m^{n-1} \geq \frac{C_m}{2nM} \int_{X \setminus E} h \wedge g_m^{n-1} \\ &= \frac{C_m}{2nM} \left( \int_X h \wedge g_m^{n-1} - \int_E h \wedge g_m^{n-1} \right) \geq \frac{C}{4nM} \end{aligned} \tag{4.12}$$

On the other hand

$$\int_X \alpha_m \wedge g_m^{n-1} = \int_X \alpha \wedge g_m^{n-1} + \frac{1}{m} \int_X h \wedge g_m^{n-1} \leq \frac{2}{m} \tag{4.13}$$

contradiction with 4.12.

Therefore  $\{\alpha\}$  is big, and from [5] it follows that  $X$  is in the Fujiki class  $\mathcal{C}$ . Theorem 2.2 in [3] implies that a manifold in the Fujiki class  $\mathcal{C}$  and which is *SKT* (strong Kähler with torsion, i.e., it supports a  $i\partial\bar{\partial}$ -closed Hermitian metric), is in fact Kähler.

*Remark 4.5* A very similar method gives a much simpler proof of a key result in [5] Theorem 0.5 that a nef class on a compact Kähler manifold of strictly positive self-intersection contains a Kähler current. Indeed, suppose  $\{\alpha\}$  is not big, then by Lamari [10] there exists a sequence of Gauduchon metrics such that

$$\int_X \alpha \wedge g_m^{n-1} \leq \frac{1}{m}$$

and

$$\int_X h \wedge g_m^{n-1} = 1$$

If  $h$  is assumed to be Kähler, the proof proceeds as above to obtain a contradiction. This proof is not independent of the proof of Demailly and Păun. In a few words, we replaced the explicit and involved construction of the metrics  $\omega_\varepsilon$  in [5] by the abstract sequence of Gauduchon metrics given by the Hahn–Banach theorem, via Lamari [10]

*Remark 4.6* An adaptation of the proof of Theorem 0.5 in [5] can not work in our case. One of the obstructions is that, if a complex manifold  $X$  admits a Hermitian metric with property 4.1, then it is not clear that  $X \times X$  admits a Hermitian metric with the same property.

*Remark 4.7* We should also point out that a simplified proof of another part of the proof of the Demailly and Păun theorem on the Kähler cone was given recently by Collins and Tosatti [4]. Together with the above proof, one obtains a more compact proof of the main result in [5]

**Acknowledgments** We would like to thank Radu Alexandru Todor for his help with the proof of Proposition 3.2. Ionuț Chiose: Supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme and CNCS Grant PN-II-ID-PCE-2011-3-0269.

## References

1. Brunella, M.A.: A characterization of Inoue surfaces. *Commentarii Mathematici Helvetici* **88**(4), 589–874 (2013)
2. Chiose, I., Toma, M.: On compact complex surfaces of Kähler rank one. *Am. J. Math.* **135**(3), 851–860 (2013)
3. Chiose, I.: Obstructions to the existence of Kähler structures on compact complex manifolds. *Proc. Am. Math. Soc.* **142**(10), 3561–3568 (2014)
4. Collins, T.C., Tosatti, V.: Kähler currents and null loci (2013). [arXiv:1304.5216](https://arxiv.org/abs/1304.5216)
5. Demailly, J.-P., Păun, M.: Numerical characterization of the Kähler cone of a compact Kähler manifold. *Ann. Math.* **159**, 1247–1274 (2004)
6. Fu, J., Li, J., Yau, S.-T.: Balanced metrics on non-Kähler Calabi–Yau threefolds. *J. Diff. Geom.* **90**, 81–130 (2012)
7. Guan, B., Li, Q.: Complex Monge–Ampère equations and totally real submanifolds. *Adv. Math.* **225**(3), 1185–1223 (2010)
8. Harvey, R., Lawson Jr, H.B.: An intrinsic characterization of Kähler manifolds. *Invent. Math.* **74**(2), 169–198 (1983)
9. Hironaka, H.: An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures. *Ann. Math. (2)* **75**, 190–208 (1962)

10. Lamari, A.: Courants kählériens et surfaces compactes. *Ann. Inst. Fourier (Grenoble)* **49**(1), 263–285 (1999)
11. Oguiso, K.: Two remarks on Calabi-Yau Moishezon treefolds. *J. Reine Angew. Math.* **452**, 153–161 (1994)
12. Tosatti, V., Weinkove, B.: The complex Monge–Ampère equation on compact Hermitian manifolds. *J. Am. Math. Soc.* **23**(4), 1187–1195 (2010)