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Convergence of Harmonic Maps

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Abstract In this paper we prove a compactness theorem for sequences of harmonic maps which are defined on converging sequences of Riemannian manifolds.

Keywords Harmonic maps · Gromov-Hausdorff convergence · Convergence of maps

Mathematics Subject Classification 53C23 · 53C43

1 Introduction

Harmonic maps are critical points of the energy functional defined on the space of maps between Riemannian manifolds. This theory was developed by Eells and Sampson [9] in the 1960s. The notion of harmonic maps on smooth metric measure spaces was introduced by Lichnerowicz in [21]. Harmonic maps between singular spaces have been studied since the early 1990s in the works of Gromov and Schoen in [16], Korevaar and Schoen in [20] and Jost in [19]. Eells and Fuglede describe the application of the methods of [20] to the study of maps between polyhedra [8].

A smooth metric measure space is a triple $(M, g, \Phi\, \mathrm{dvol}_M)$, where (M, g) is an n-dimensional Riemannian manifold, dvol_M denotes the corresponding Riemannian volume element on M, and Φ is a smooth positive function on M. These spaces have been used extensively in geometric analysis and they arise as smooth collapsed measured Gromov–Hausdorff limits in the works of Cheeger and Colding [3–5], Fukaya [11] and Gromov [15]. They have been studied recently by Morgan [24]. See also

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works of Lott [23], Qian [28], Fang et al. [10], Wei and Wylie [34], Wu [33], Su and Zhang [31] and Munteanu and Wang [26].

In this paper, we are going to study the behavior of harmonic maps under convergence. Let $\mathcal{M}(n, D)$ denote the set of all compact Riemannian manifolds (M, g) such that $\dim(M) = n, \operatorname{diam}(M) < D$, and the sectional curvature $\sec_g$ satisfies $|\sec_g| \leq 1$, equipped with the measured Gromov–Hausdorff topology. Let $(M_i, g_i, \mathrm{dvol}_{M_i})$ in $\mathcal{M}(n, D)$ be a sequence of manifolds which converges to a smooth metric measure space $(M, g, \Phi\, \mathrm{dvol}_M)$. Suppose $f_i : (M_i, g_i) \to (N, h)$ is a sequence of harmonic maps. We are interested in knowing under what circumstances the f_i converge to a harmonic map f on the smooth metric measure space $(M, g, \Phi\, \mathrm{dvol}_M)$.

When a sequence of manifolds (M_i, g_i) in $\mathcal{M}(n, D)$ converges to a metric space X, according to Fukaya [12], X is a quotient space $Y/O(n)$, where Y is a smooth manifold. Indeed Y is the limit point of the sequence of frame bundles, $F(M_i)$, over the manifolds M_i and X has the structure of a Riemannian polyhedron $(X, g_X, \Phi_X \mu_g)$ where μ_g is the Riemannian volume element related to the metric g_X on X.

We state the main result of this paper which is a compactness theorem for sequences of harmonic maps.

Theorem 1.1 *Let (M_i, g_i) be a sequence of smooth Riemannian manifolds in $\mathcal{M}(n, D)$ which converges to a metric measure space $(X, g, \Phi\mu_g)$ in the measured Gromov–Hausdorff topology. Suppose (N, h) is a compact Riemannian manifold. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of harmonic maps such that $\|e_{g_i}(f_i)\|_{L^\infty} < C$, where $\|e_{g_i}(f_i)\|_{L^\infty}$ is the L^∞-norm of the energy density of the map f_i and C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (X, g, \Phi\mu_g) \to (N, h)$, and this map is a harmonic map in $\mathcal{H}^1((X, \Phi\mu_g), N)$.*

By $\mathcal{H}^1(X, N)$ we mean

$$\{f \in \mathcal{H}^1(X, \mathbb{R}^q) \;\Big|\; f(x) \in N \text{ for almost all } x \in M\},$$

where $\mathcal{H}^1(X, \mathbb{R}^q)$ is the standard Sobolev space and N is isometrically embedded in $\mathbb{R}^q$. In this work we use the notations $\mathcal{H}^1$ and $W^{1,2}$ interchangeably. For the notion of convergence of maps we refer the reader to the Definition 2.11.

The rest of this paper is organized as follows. In the first section we introduce our main notations and preliminary results needed for the rest of this paper. In the second section, we prove Theorem 1.1. We divide the proof into three cases. In Sect. 3.1 we consider the non-collapsing case, Proposition 3.1. Moreover using the regularity results for harmonic maps in the work of Schoen and Lin [22,30] we study Theorem 1.1 under less restrictive assumption of uniform boundedness of the energy of the maps f_i (see Propositions 3.3, 3.4). In Sect. 3.2 we consider the case of collapsing to a Riemannian manifold, Proposition 3.5. As a preliminary step we prove the result under some regularity assumption on the metrics g_i, see Proposition 3.6. The general case is considered in Sect. 3.3. The Appendix is devoted to the study of convergence of the tension fields of the maps f_i under the assumptions of Proposition 3.6.

2 Background

2.1 Harmonic Maps

In this subsection, we first recall the definition of weakly harmonic maps on smooth metric measure spaces. We then briefly review this concept on Riemannian polyhedra. At the end we present some theorems and lemmas that we need in this paper. Let (N, h) be a compact Riemannian manifold and I an isometric embedding $I : N \to \mathbb{R}^q$. Since $I(N)$ is a smooth, compact submanifold of $\mathbb{R}^q$, there exists a number $\kappa > 0$ such that the neighborhood

$$U_\kappa(N) = \left\{ y \in \mathbb{R}^q : \operatorname{dist}(y, N) < \kappa \right\}$$

has the following property: for every y in $U_\kappa(N)$ there exists a unique point $\pi_N(y) \in N$ such that

$$|y - \pi_N(y)| = \operatorname{dist}(y, N)$$

The map $\pi_N : U_\kappa(N) \to N$ defined as above is called the *nearest point projection* onto N.

The Hess π_N defines an element in $\Gamma(TN^* \otimes TN^* \otimes TN^\perp)$ which coincides with the second fundamental form of $I : N \to \mathbb{R}^q$ up to a negative sign

$$\langle \operatorname{Hess} \pi_N(y)(X, Y), \eta \rangle = -\langle \nabla_Y \eta, X \rangle$$

where X and Y are in TN, y in N and η in $TN^\perp$ (see §3 in Moser [25]).

A map $f : (M, g, \Phi \, \mathrm{dvol}_M) \to (N, h)$, belonging to $\mathcal{H}^1_{\mathrm{loc}}((M, \Phi \, \mathrm{dvol}_M), N)$ is called *weakly harmonic* if and only if

$$\Delta I \circ f - \Pi(f)(df, df) + dI \circ f(\nabla \ln(\Phi)) = 0 \tag{1}$$

in the weak sense. Here

$$\Pi(f)(df, df) = \operatorname{trace} \ \operatorname{Hess}(\pi_N)(I \circ f)(dI \circ f, dI \circ f), \tag{2}$$

or in coordinates

$$\Pi(f)(df, df) = \sum g^{ij} \frac{\partial^2 \pi_N^A}{\partial z^B \partial z^C} \frac{\partial f^B}{\partial x^i} \frac{\partial f^C}{\partial x^j}.$$

For $f : (M^n, g) \to (N^m, h)$ and $\eta : M \to \mathbb{R}^q$, we define

$$\Xi_g(f, \eta) = \langle dI \circ f, d\eta \rangle - \langle \Pi(f)(df, df), \eta \rangle. \tag{3}$$

We explain now what we mean by harmonic maps on Riemannian polyhedra. Following Eells and Fuglede [8] on an admissible Riemannian polyhedron X, a continuous weakly harmonic map $u : (X, g, \mu_g) \to (N, h)$ is of class $\mathcal{H}^1_{\rm loc}(X, N)$ and satisfies: for any chart $\eta : V \to \mathbb{R}^n$ on N and any open set $U \subset u^{-1}(V)$ of compact closure in X, the equality

$$\int_U g(\nabla\lambda, \nabla u^k)\, d\mu_g = \int_U \lambda(\Gamma^k_{\alpha\beta} \circ u) g(\nabla u^\alpha, \nabla u^\beta)\, d\mu_g \tag{4}$$

holds for every $k = 1, \dots, n$ and every bounded function $\lambda \in \mathcal{H}^1_0(U)$. Here $\Gamma^k_{\alpha\beta}$ denote the Christoffel symbols on N. Similarly on a polyhedron X with a measure $\Phi\mu_g$, a continuous weakly harmonic map is a map in $\mathcal{H}^1_{\rm loc}((X, \Phi\mu_g), N)$ which satisfies equation (4) with $\Phi d\mu_g$ in place of $d\mu_g$. When the target is compact a continuous map f on an admissible Riemannian polyhedron is harmonic if and only if it satisfies (1) weakly.

Theorem 2.1 (Moser [25], Theorem 3.1) *Let $f \in \mathcal{H}^1(U, N) \cap C^0(U, N)$ be a weakly harmonic map, where U is an open domain in $\mathbb{R}^n$. Then f is smooth.*

The energy functional is lower semi continuous, and we have

Lemma 2.2 (Xin [35]) *Let $S \subset \mathcal{H}^1(M, N)$ be such that the energy functional is bounded on S and S is closed under weak limits. Then S is sequentially compact.*

Now we recall some regularity results for harmonic maps from [30] and [22]. Let M and N be compact Riemannian manifolds. Define

$$\mathcal{F}_\Lambda = \{u \in C^\infty(M, N) :\ u \text{ is harmonic and } E(u) \le \Lambda\}.$$

We have the following results.

Theorem 2.3 (Schoen [30]) *Let M and N be compact Riemannian manifolds. Any map u in the weak closure of $\mathcal{F}_\Lambda$ is smooth and harmonic outside a relatively closed singular set of locally finite Hausdorff $(n-2)$-dimensional measure.*

Remark 1 (Schoen [30], Lin [22]) Let u_i be a sequence in $\mathcal{F}_\Lambda$. Then there exists a subsequence which converges weakly to some u in $\mathcal{H}^1(M, N)$. Define

$$\Sigma = \bigcap_{r>0} \left\{x \in M,\ \liminf_{i\to\infty} r^{2-n} \int_{B_r(x)} e(u_i) \ge \epsilon_0\right\}$$

where $\epsilon_0 = \epsilon_0(n, N) > 0$ is a constant independent of u_i as in Theorem 2.2 in [30]. If we consider a sequence of Radon measures $\mu_i = |du_i|^2 dx$, without loss of generality we may assume $\mu_i \rightharpoonup \mu$ weakly as Radon measures. By Fatou's lemma, we may write

$$\mu = |du|^2 dx + \nu$$

for some non-negative Radon measure ν. We can show that $\Sigma = \operatorname{spt} \nu \cup \operatorname{sing} u$ and ν is absolutely continuous with respect to $H^{n-2}|_\Sigma$. Therefore u_i converges strongly in $\mathcal{H}^1(M, N)$ to u if and only if $|du_i|^2 dx \rightharpoonup |du|^2 dx$ weakly, if and only if $\nu = 0$, if and only if $H^{n-2}(\Sigma) = 0$, if and only if there is no smooth non-constant harmonic map from 2-sphere $\mathbb{S}^2$ into N, e.g., negatively curved manifolds. See Lemma 3.1 in [22] for a complete discussion.

The following reduction theorem shows the relation between the tension fields of equivariant harmonic maps under Riemannian submersions.

Theorem 2.4 (Xin [35], Theorem 6.4) *Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be Riemannian submersions, H_1 the mean curvature vector of the submanifold F_1 in E_1 and B_2 the second fundamental form of the fiber submanifold F_2 in E_2. Let $f : E_1 \to E_2$ be a horizontal equivariant map and $\bar{f}$ its induced map from M_1 to M_2 with tension field $\tau(\bar{f})$. Let $f^\perp$ be the restriction of f to the fiber F_1. Then we have the following formula*

$$\tau(f) = \tau^*(\bar{f}) + B_2(f_*(e_t), f_*(e_t)) - f_*(H_1) + \tau(f^\perp)$$

where $\{e_t\}$, $t = n_1 + 1, \ldots, m_1$ is a local orthonormal frame field on the fiber F_1 and $\tau^*(\bar{f})$ denotes the horizontal lift of $\tau(\bar{f})$.

2.2 Hölder Spaces on Manifolds

Let (M, g) be a Riemannian manifold and let ∇ be the Levi–Civita connection on M. Let V be a vector bundle on M equipped with the Euclidean metric on its fibers. Let $\hat{\nabla}$ be a connection on V preserving these metrics. Let $C^k(M)$ be the space of all continuous, bounded functions f that have k continuous, bounded derivatives and define the norm $\|\cdot\|_{C^k}$ on $C^k(M)$ by $\|f\|_{C^k} = \sum_{j=0}^k \sup_M |\nabla^j f|$.

Now we define the Hölder space $C^{0,\alpha}(M)$ for $\alpha \in (0, 1)$. The function f on M is said to be Hölder continuous with exponent α, if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

is finite. The vector space $C^{0,\alpha}(M)$ is the set of continuous, bounded functions on M which are Hölder continuous with exponent α and the norm $C^{0,\alpha}(M)$ is $\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_\alpha$.

In the same way, we shall define Hölder norms on spaces of sections v of a vector bundle V over M equipped with Euclidean metrics in the fibers as above. Let $\delta(g) = \mathrm{injrad}(M, g)$ be the injectivity radius of the metric g on M which we suppose to be positive and set

$$[v]_\alpha = \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \frac{|v(x) - v(y)|}{d(x, y)^\alpha} \tag{5}$$

We now interpret $|v(x) - v(y)|$. When $x \neq y \in M$, and $d(x, y) \leq \delta(g)$, there is unique geodesic γ of length $d(x, y)$ joining x and y in M. Parallel translation along

γ using $\hat{\nabla}$ identifies the fibers of V over x and y and the metrics on the fibers. With this understanding the expression $|v(x) - v(y)|$ is well defined.

Define $C^{k,\alpha}(M)$ to be the set of f in $C^k(M)$ for which $[\nabla^k f]_\alpha$ defined by (5) exists as a section in the vector bundle $\bigotimes^k T^*M$ with its natural metric and connection. The Hölder norm on $C^{k,\alpha}(M)$ is $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha$.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $F : \Omega \to \mathbb{R}^q$ is bounded and Hölder continuous. Let $Q : \mathbb{R}^q \to \mathbb{R}^p$ be a quadratic function. Then $Q \circ F : \Omega \to \mathbb{R}^p$ is also Hölder continuous and*

$$[Q \circ F]_\alpha \leq A \sup_\Omega \|F\|_{\mathbb{R}^q} [\|F\|_{\mathbb{R}^q}]_\alpha,$$

where A is a constant.

In the above lemma by a quadratic function we mean

$$Q(y) = \sum_{i,j=1}^{q} Q_{ij} y_i y_j, \qquad Q_{ij} \in C^1(\overline{\Omega}).$$

We have

Corollary 2.6 *Let $f \in C^{1,\alpha}(M, N)$, then*

$$[\Pi(f)(df, df)]_{C^\alpha} \leq A \cdot \|df\|_{L^\infty} \cdot [df]_{C^\alpha}.$$

Proof Let $\{\Omega_j\}$ be an atlas of M, such that $\operatorname{diam}(\Omega_j) \leq \operatorname{injrad}(M)$ and set $F_j = df|_{\Omega_j}$ and $Q = \operatorname{Hess} \pi_N(X, X)$, for an smooth vector field X. Then using the previous lemma and an appropriate partition of unity we will have the result. □

Schauder Estimates

In this part, we give a quick review on the Schauder estimate of solutions to linear elliptic partial differential equations. Suppose (M, g) is compact and L is an elliptic operator, $L = a^{ij}\nabla_i\nabla_j + b_i\nabla_i + c$, where a is a symmetric and positive definite tensor, b is a $C^{0,\alpha}$ vector field on M and c is in $C^{0,\alpha}(M)$ such that L satisfies the conditions

$$\|a\|_{C^{0,\alpha}} + \|b\|_{C^{0,\alpha}} + \|c\|_{C^{0,\alpha}} \leq \Lambda,$$
$$\lambda\|\xi\|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda\|\xi\|^2, \quad \text{for all } x \in M, \text{ and } \xi \in \mathbb{R}^n.$$

Consider the following problem,

$$Lu = f \qquad \text{in } M,$$

if $\partial M = \emptyset$ and

$$\begin{cases} Lu = f & \text{in } M \\ u = g & \text{on } \partial M. \end{cases}$$

if $\partial M \neq \emptyset$. Then we have (cf. Gilbarg and Trudinger [17])

Theorem 2.7 (Schauder Estimate) *If $f \in C^{0,\alpha}(M)$ and $u \in C^2(M)$, then $u \in C^{2,\alpha}(M)$ and we have*

$$\|u\|_{C^{1,\alpha}} \leq C(\|f\|_{L^\infty} + \|u\|_{L^\infty}),$$
$$\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}} + \|u\|_{L^\infty}),$$

where C depends on M, λ, Λ.

Hereafter we present an introduction to the convergence and collapsing theory. Most of the materials in this part was gathered from the work of Rong [29].

2.3 Convergence

Gromov introduced the notion of the Gromov–Hausdorff distance between metric spaces in [15]), based on the notion of Hausdorff distance between subsets A, B in a metric space Z:

$$d_H^Z(A, B) = \inf\{\epsilon > 0 : \ B \subset T_\epsilon(A) \text{ and } A \subset T_\epsilon(B)\}$$

where $T_\epsilon(A) = \{x \in Z : \ d_Z(x, A) < \epsilon\}$ is a tubular neighborhood of a set A.

Definition 2.8 (*Gromov* [15]) Let X and Y be two compact metric spaces. The Gromov–Hausdorff distance between X and Y is defined as

$$d_{GH}(X, Y) = \inf\left\{d_H^Z(\phi(X), \psi(Y)) : \begin{array}{c}\text{for all metric spaces } Z \text{ and isometric embeddings}\\ \phi : X \hookrightarrow Z, \ \psi : Y \hookrightarrow Z\end{array}\right\}$$

Let $\mathcal{MET}$ denote the set of all isometry classes of nonempty compact metric spaces. Then $(\mathcal{MET}, d_{GH})$ is a complete metric space. There is an alternative definition for Gromov–Hausdorff distance given in [15]:

Definition 2.9 (*Gromov* [15]) Let X and Y be two elements of $\mathcal{MET}$. A map $\phi : X \to Y$ is said to be an ϵ-Hausdorff approximation from X to Y, if the following two conditions are satisfied

i. ϵ-onto: $B_\epsilon(\phi(X)) = Y$.
ii. ϵ-isometry: $|d(\phi(x), \phi(y)) - d(x, y)| < \epsilon$ for all $x, y \in X$.

The Gromov–Hausdorff distance $\hat{d}_{GH}(X, Y)$, between X and Y is defined to be the infimum of the positive number ϵ such that there exists ϵ-Hausdorff approximation from X to Y and form Y to X.

The distance $\hat{d}_{GH}$ does not satisfy triangle inequality and $\hat{d}_{GH} \neq d_{GH}$ but onecan show that

$$\tfrac{2}{3} d_{GH} \leq \hat{d}_{GH} \leq 2 d_{GH}$$

Because a sequence in $\mathcal{MET}$ converges with respect to d_{GH} if and only if it converges with respect to $\hat{d}_{GH}$, we will not distinguish $\hat{d}_{GH}$ from d_{GH}.

For the notion of equivariant Gromov–Hausdorff convergence and equivariant measured Gromov–Hausdorff convergence, we refer the reader to Definition 1.5.2 in [29] and Definition 3.11 in [11]. Also for the notion of Lipschitz distance see Definition 3.1 in [15]. Let $\mathcal{MM}$ denotes the class of all pairs (X, μ) of compact metric spaces X equipped with a Borel measure μ on it such that $\mu(X) = 1$. Fukaya in [11] presented a notion of measured Gromov–Hausdorff convergence for the metric measure spaces:

Definition 2.10 (*Fukaya* [11]) Let (X_i, μ_i) be a sequence in $\mathcal{MM}$. We say that (X_i, μ_i) converges to an element (X, μ) in $\mathcal{MM}$ with respect to measured Gromov–Hausdorff topology if there exist Borel measurable ϵ-Hausdorff approximations $f_i : (X_i, \mu_i) \to (X, \mu)$ such that $f_{i*}(\mu_i)$ converges to μ in the weak* topology.

When M is a Riemannian manifold with finite volume, we let $\mu_M = \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}$, where dvol_M denotes the volume element of M and regard (M, μ_M) as an element in $\mathcal{MM}$.

In [14], Grove and Petersen introduced the notion of convergence of maps.

Definition 2.11 *(Grove–Petersen* [14]) Let $(X_i, p_i), (X, p), (Y_i, q_i)$ and (Y, q) be pointed metric spaces such that (X_i, p_i) converges to (X, p) in the pointed Gromov–Hausdorff topology (resp. (Y_i, q_i) converges to (Y, q)). We say that a sequence of maps $f_i : (X_i, p_i) \to (Y_i, q_i)$ converges to a map $f : (X, p) \to (Y, q)$ if there exists a subsequence X_{i_k} such that if $x_{i_k} \in X_{i_k}$ and x_{i_k} converges to x (in $\coprod X_{i_k} \coprod X$ with the admissible metric), then $f_{i_k}(x_{i_k})$ converges to $f(x)$.

A family of maps $f_i : (X_i, d_{X_i}, p_i) \to (Y_i, d_{Y_i}, q_i)$ is called equicontinuous if for any $\epsilon > 0$ there is $\delta > 0$ such that $d_{X_i}(x_i, y_i) < \delta$ implies $d_{Y_i}(f_i(x_i), f_i(y_i)) < \epsilon$ for all x_i, y_i in X_i and for all i. We have

Lemma 2.12 *(Grove–Petersen [14]) Let $(X_i, p_i), (X, p), (Y_i, q_i)$ and (Y, q) be pointed metric spaces such that (X_i, p_i) converges to (X, p) in the pointed Gromov–Hausdorff topology (resp. (Y_i, q_i) converges to (Y, q)). Let $f_i : (X_i, p_i) \to (Y_i, q_i)$ be a sequence of maps. Then*

i. *If f_is are equicontinuous, then there is a uniformly continuous map f and a convergent subsequence X_{i_k} such that f_i converges to f.*
ii. *If f_is are isometries then the limit map $f : (X, p) \to (Y, q)$ is also an isometry.*

2.4 Convergence Theorems, Non-Collapsing

This subsection is devoted to the theory of convergence of manifolds in the non-collapsing case. A sequence of n-manifolds M_i converging to a metric space X is called non-collapsing if $\mathrm{vol}(M_i) \geq v > 0$, and collapsing otherwise. For a non-collapsing sequence of manifolds with bounded sectional curvature there is a uniform lower bound on the injectivity radius of M_i, and thus M_is are diffeomorphic for large i. This result is due to Cheeger–Gromov (Cheeger [7], Peters [27], Greene and Wu [18]) and is formulated as follows.

Theorem 2.13 *Let* (M_i, g_i) *be a sequence of closed Riemannian* n*-manifolds such that* $|\sec_{g_i}| \leq 1$ *and* $\mathrm{vol}(M_i) \geq v > 0$*, and* M_i *converges to a metric space* X*. Then* X *is homeomorphic to a manifold* M *such that for large* i*, and there are diffeomorphisms* $\phi_i : M \to M_i$ *such that the pullback metric converges to a* $C^{1,\alpha}$*-metric* g *on* M *in the* $C^{1,\alpha}$*-topology.*

The following smoothing result concerns the uniform approximation of Riemannian manifolds by smooth ones.

Theorem 2.14 (Bemelmans et al. [2]) *Let* (M, g) *be a compact Riemannian* n*-manifold with* $|\sec_g| < 1$*. For any* $\epsilon > 0$*, there is a smooth metric* g_ϵ *on* M *such that*

$$|g_\epsilon - g|_{C^1} < \epsilon, \quad |\sec_{g_\epsilon}| \leq 1, \quad |\nabla^k \mathrm{R}_{g_\epsilon}| \leq C(n, k) \cdot \epsilon^k.$$

In particular

$$\begin{aligned} \mathrm{e}^{-\epsilon} \,\mathrm{injrad}(M, g) \leq \mathrm{injrad}(M, g_\epsilon) \leq \mathrm{e}^{\epsilon} \,\mathrm{injrad}(M, g),\\ \mathrm{e}^{-\epsilon} \,\mathrm{diam}(M, g) \leq \mathrm{diam}(M, g_\epsilon) \leq \mathrm{e}^{\epsilon} \,\mathrm{diam}(M, g),\\ \mathrm{e}^{-\epsilon} \,\mathrm{vol}(M, g) \leq \mathrm{vol}(M, g_\epsilon) \leq \mathrm{e}^{\epsilon} \,\mathrm{vol}(M, g). \end{aligned}$$

2.5 Convergence Theorems, Collapsing

This subsection is devoted to the theory of convergence of manifolds in the collapsing case. We state some of the main results in this context.

Theorem 2.15 (Fibration theorem, Fukaya [13], Cheeger et al. [6]) *Let* M^n *and* N^m *be compact Riemannian manifolds satisfying*

$$\sec_{M^n} \geq -1, \quad |\sec_{N^m}| \leq 1 \ (m \geq 2), \quad \mathrm{injrad}(N^m) \geq i_0 > 0.$$

Assume M^n *and* N^m *admit isometric compact Lie group* G*-actions. There exists a constant* $\epsilon(n, i_0) > 0$ *such that if* $d_{eqGH}((M^n, G), (N^m, G)) < \epsilon \leq \epsilon(n, i_0)$*, then there is a* C^1*-fibration* G*-invariant map,* $f : (M^n, G) \to (N^m, G)$ *with connected fibers such that*

i. *The diameter of any* f*-fibers is at most* $c_1 \cdot \epsilon$*, where* $c_1 = c_1(n, \epsilon)$ *is such that* $c_1 \to 1$ *as* $\epsilon \to 0$*.*
ii. f *is an almost Riemannian submersion, that is for any vector* $\xi \in TM$ *orthogonal to a fiber,*

$$\mathrm{e}^{-\tau(\epsilon)} \leq \frac{|df(\xi)|}{|\xi|} \leq \mathrm{e}^{\tau(\epsilon)},$$

where $\tau(\epsilon) \to 0$ *as* $\epsilon \to 0$*.*
iii. *If in addition,* $\sec_{M^n} \leq 1$ *then* f *is smooth and the second fundamental form of any fiber satisfies* $|II_{f^{-1}(\bar{x})}| \leq c_2(n)$*, for* $\bar{x}$ *in* N^m*.*

iv. *The fibers are diffeomorphic to an infranilmanifold* $\Gamma\backslash\mathring{N}$*, where* $\mathring{N}$ *is a simply connected nilpotent group,* $\Gamma \subset \mathring{N} \ltimes \mathrm{Aut}(\mathring{N})$*, such that* $[\Gamma, \mathring{N} \cap \Gamma] \leq \omega(n)$.

An easily accessible proof of this theorem can be founded in [29] Theorems 2.1.1 and 5.7.1.

A *pure nilpotent Killing structure* on M^n is a G-equivariant fibration $N_0 \to M^n \to N^m$, with fiber N_0 a nilpotent manifold (equipped with a flat connection) on which parallel fields are Killing fields and the G-action preserves affine fibrations. The underlying G-invariant affine bundle structure is called a pure N_0-structure and a metric for which the N_0-structure becomes a nilpotent Killing structure is called invariant.

Let M^n and N^m be as in Theorem 2.15. Suppose M^n and N^m satisfy the following: for some sequence $A = \{A_k\}$ of real non-negative numbers, for the Riemannian curvature tensor on M and N we have

$$|\nabla^k \,\mathrm{R}\,| \leq A_k. \tag{6}$$

We can construct an invariant metric (invariant under the left action of N_0) such that

$$|\nabla^k(\langle\ ,\ \rangle - (\ ,\ ))| \leq c(n, A) \cdot \epsilon \cdot \mathrm{injrad}(N)^{-(k+1)}, \tag{7}$$

where $\langle\ ,\ \rangle$ denotes the original metric, $(\ ,\ )$ the invariant one, and $c(n, A)$ is a generic constant depending on finitely many A_k and n. For the construction of invariant metric which satisfies inequality (7) see Proposition 4.9 in [6] and the explanation therein. Given such a metric we have a pure nilpotent killing structure.

When a sequence of Riemannian n-manifolds with bounded curvature collapses, the limit space can be a singular space. We have

Theorem 2.16 (Singular fibration theorem, Fukaya [12]) *Let* (M_i, g_i) *be a sequence of closed Riemannian n-manifolds with* $|\sec_{g_i}| \leq 1$ *and* $\mathrm{diam}(M_i) \leq D$ *which converges to the closed metric space* (X, d) *in* $\mathcal{MET}$*. Then*

i. *The frame bundles equipped with canonical metrics converge,* $(F(M_i), O(n)) \to (Y, O(n))$*, where* Y *is a manifold.*
ii. *There is an* $O(n)$*-invariant fibration* $\tilde{f}_i : F(M_i) \to Y$ *satisfying the conditions in Theorem 2.15 which becomes for* $\epsilon > 0$*, a nilpotent Killing structure with respect to an* ϵ C^1*-closed metric (with respect to* C^1*-topology). Moreover each fiber on* M_i *has positive dimension.*
iii. *For any* $\bar{x} \in X$*, a fiber* $f_i^{-1}(\bar{x})$ *is singular if and only if* $p^{-1}(\bar{x})$ *is a singular* $O(n)$*-orbit in* Y*.*

For the proof see Theorem 4.1.3 in [29]. In the above theorem, the fibration map $\tilde{f}_i$ descends to a (singular) fibration map $f_i : M_i \to X = Y/O(n)$ such that the following diagram commutes

$$\begin{array}{ccc} F(M_i) & \xrightarrow{\tilde{f}_i} & Y \\ \Big\downarrow{\scriptstyle p_i} & & \Big\downarrow{\scriptstyle p} \\ M_i & \xrightarrow{f_i} & X \end{array}$$

In the following remark we collect the main points that we need from the above theorems and explain the classification in the proof of Theorem 1.1.

Remark 2 When a sequence of Riemannian manifolds M_i converges in $\mathcal{M}(n, D)$ to a metric space X, the frame bundles over M_i equipped with the canonical metrics $\tilde{g}_i$ converge to a manifold Y and $\tilde{f}_i : (F(M_i), \tilde{g}_i, O(n)) \to (Y, O(n))$ is an $O(n)$-invariant fibration map.

To see this let $\tilde{g}_{i\epsilon}$ be the smooth metric on $F(M_i)$ as in Theorem 2.14. Then $(F(M_i), \tilde{g}_{i\epsilon})$ converges to a smooth Riemannian manifold (Y_ϵ, g_ϵ). For a small fixed ϵ_0 and $\epsilon < \epsilon_0$, the sectional curvature on $(F(M_i), \tilde{g}_{i\epsilon})$ is uniformly bounded and we can apply Theorem 2.15 to conclude that there exists an $O(n)$-invariant smooth fibration map $\tilde{f}_{i\epsilon}$. By continuity $(F(M_i), \tilde{g}_{i\epsilon})$ is conjugate to $(F(M_i), \tilde{g}_{i\epsilon_0})$ (by being conjugate we mean there exists $C^{1,\alpha}$-diffeomorphism as in Theorem 2.13). This implies that the convergence of Y_ϵ to Y is the same as the convergence of a sequence of metrics on Y_{ϵ_0}, and therefore $(Y, O(n))$ is conjugate to $(Y_{\epsilon_0}, O(n))$

$$(F(M_i), O(n)) \simeq (F(M_i), \tilde{g}_{i\epsilon_0}, O(n)) \overset{\tilde{f}_{i\epsilon_0}}{\to} (Y_{\epsilon_0}, O(n)) \simeq (Y, O(n)),$$

and it induces a fibration map $(F(M_i), \tilde{g}_i, O(n)) \overset{\tilde{f}_i}{\to} (Y, O(n))$. For more explanations see the proof of Theorem 4.1.3 in [29].

Furthermore, there exists a C^1-close invariant Riemannian metric $\mathring{g}_{i_\epsilon}$ such that $(F(M_i), \mathring{g}_{i_\epsilon}, O(n))$ is a pure nilpotent Killing structure and the fibration map $\tilde{f}_{i\epsilon}$ is a Riemannian submersion considering the induced Riemannian metric on Y_ϵ by this map.

2.6 Density Function

Let $\mathcal{DM}(n, D)$ denote the closure of $\mathcal{M}(n, D)$ in $\mathcal{MM}$ with respect to the measured Gromov–Hausdorff topology. Then $\mathcal{DM}(n, D)$ is compact with respect to the measured Gromov–Hausdorff topology. Let $(M_i, g_i, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}) \in \mathcal{M}(n, D)$ be a sequence of manifolds which converges to a manifold (M, g, μ). Suppose $\psi_i : M_i \to M$ is the fibration map as in Theorem 2.13. For $x \in M$ we define

$$\Phi_i = \frac{\mathrm{vol}(\psi_i^{-1}(x))}{\mathrm{vol}(M_i)},$$

then there exists Φ such that $\Phi = \lim_{i\to\infty} \Phi_i$ and μ is absolutely continuous with respect to dvol_M, $\mu = \Phi\, \mathrm{dvol}_M$ (see §3 in [11]). For the general case when $(X, \mu) \in \mathcal{DM}(n, D)$, we first recall a remark on quotient spaces. Below $S(B)$ denotes the singular part of B.

Remark 3 (Besse [1]) Let (M, g) be a Riemannian manifold and G a closed subgroup of isometries of M. Assume that the projection $p : M \to M/G$ is a smooth submersion. Then there exists a unique Riemannian metric $\check{g}$ on $B = M/G$ such that p is a Riemannian submersion (see Subsection 9.12 in [1]).

We recall that using the general theory of slices for the action of a group of isometries on a Riemannian manifold, one can show that there always exists an open dense submanifold U of M (the union of the principle orbits), such that the restriction $p|_U : U \to U/G$ is a smooth submersion.

Considering now M/G as a Riemannian polyhedron and μ_g as its Riemannian volume element, the restriction of μ_g on U/G is equal to $\mathrm{dvol}_{U/G} = \mathrm{dvol}_{B-S(B)}$.

Now suppose M_i in $\mathcal{M}(n, D)$ converges to a metric space X. We may assume that FM_i with the induced $O(n)$-invariant metric $\tilde{g}_i$ converges to $(Y, g, \Phi_Y \cdot \mathrm{dvol}_Y)$ with respect to the $O(n)$-measured Gromov–Hausdorff topology and g, Φ_Y are $C^{1,\alpha}$-regular. Moreover, since $p_i : F(M_i) \to M_i$ is a Riemannian submersion with totally geodesic fibers, and since the fibers are isometric to each other, it follows that $(FM_i, \mathrm{dvol}_{FM_i})/O(n) = (M_i, \mathrm{dvol}_{M_i})$. Hence by equivariant Gromov–Hausdorff convergence M_i converges to $(X, \nu) = (Y, \Phi_Y \, \mathrm{dvol}_Y)/O(n)$ (see Theorem 0.6 in [11]), and by Remark 3

$$\nu(S(X)) = 0$$

For all x in X we let

$$\Phi_X(x) = \int_{y \in p^{-1}(x)} \Phi_Y(y) \ \mathrm{dvol}_{p^{-1}(x)},$$

where $p : Y \to X$ is the natural projection. For each open set U

$$\nu(U) = \int_U \Phi_X(x) \ \mathrm{dvol}_{X-S(X)} \,.$$

3 Proof of the Convergence Theorem

In this section we are going to prove Theorem 1.1. In the following $\mathcal{M}(n, D)$ denotes the set of all compact Riemannian manifolds (M, g) such that $\dim(M) = n$, $\mathrm{diam}(M) < D$ and the sectional curvature satisfies $|\sec_g| \leq 1$, and $\mathcal{M}(n, D, v)$ the set of Riemannian manifolds in $\mathcal{M}(n, D)$ with volume $\geq v$.

We split the proof in three cases:

Case I: Non-collapsing (M_i, g_i) converge to (M, g) in $\mathcal{M}(n, D, v)$. We first consider the situation where $M_i = M$ and g_i converges to a metric g in $\mathcal{M}(n, D, v)$. Then we study the problem in the general case using Theorem 2.13.

Case II: Collapsing to a manifold (M_i, g_i) converge to (M, g) in $\mathcal{M}(n, D)$ with g a $C^{1,\alpha}$-metric. We first consider the situation when (M_i, g_i) satisfies an additional regularity assumption (see Assumption 1 below). Then we discuss the general case using the fact that there is always a sequence of metrics $g_i(\epsilon)$ on M_i, C^1-close to the the metric g_i which satisfies Assumption 1 as explained in Remark 2.

Case III: Collapsing to a singular space (M_i, g_i) converge to a metric space (X, d) in $\mathcal{M}(n, D)$. When a sequence of manifolds (M_i, g_i) converges in $\mathcal{M}(n, D)$ to a metric space X, the frame bundles over M_i converge to a Riemannian manifold Y, with a $C^{1,\alpha}$-metric and we have $X = Y/O(n)$. The harmonic maps over M_i, induce harmonic maps over $F(M_i)$ and this case reduces to the study of harmonic maps on quotient spaces.

Hereafter we fix an isometric embedding $I : N \to R^q$ and we often denote the composition $I \circ f$ simply by f, unless we need to explicitly distinguish these two maps.

3.1 Case I: Non-collapsing

In this subsection we prove

Proposition 3.1 *Let (M_i, g_i) be a sequence of Riemannian manifolds in $\mathcal{M}(n, D, v)$ which converges to a Riemannian manifold (M, g) in the Gromov–Hausdorff topology. Suppose (N, h) is a compact Riemannian manifold. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of smooth harmonic maps such that $\|e_{g_i}(f_i)\|_{L^\infty} < C$, where C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (M, g) \to (N, h)$ and this map is a smooth harmonic map.*

To go through the proof in this case, we first consider the situation when a sequence of metrics g_i on a manifold M converges to a Riemannian metric g.

Lemma 3.2 *Let g_i be a sequence of Riemannian metrics on a smooth manifold M and suppose (M, g_i) converge to (M, g) in $\mathcal{M}(n, D, v)$. Suppose $f_i : (M, g_i) \to N$ is a sequence of smooth harmonic maps such that*

$$\|e_{g_i}(f_i)\|_{L^\infty} < C,$$

where C is a constant independent of i. Then there exists a subsequence of f_i which converges to some f in the C^k-topology for any $k \geq 0$ and f is also harmonic.

Proof By Theorem 2.13, the metric g_i converges to g in $\mathcal{M}(n, D, v)$ in the $C^{1,\alpha}$-topology. Using Schauder estimates, f_is have bounded norm in $C^k(M)$ for every $k \geq 0$ and hence converge to a map $f \in C^k(M)$. We have

$$\lim_{i\to\infty} \Delta_{g_i} f_i = \Delta_g f$$

and

$$\lim_{i\to\infty} \Pi(f_i)(df_i, df_i) = \Pi(f)(df, df)$$

The above limits lead to harmonicity of f. □

Using the above lemma we can prove Proposition 3.1.

Proof of Proposition 3.1 Since M_i converges to M in $\mathcal{M}(n, D, v)$, by Theorem 2.13 there is a diffeomorphism $\phi_i : M_i \to M$, such that the pushforward $\bar{g}_i = \phi_{i*}(g_i)$ of the metrics g_i on M_i converges to a $C^{1,\beta}$-metric g. Since the map $\phi_i : (M_i, g_i) \to (M, \bar{g}_i)$ is an isometry

$$e_{g_i}(f_i) = e_{\bar{g}_i}(\bar{f}_i) \tag{8}$$

where $\bar{f}_i$ is the map $f_i \circ \phi_i^{-1}$. f_i is harmonic and so $\bar{f}_i$. Therefore all the assumptions of Lemma 3.2 are satisfied here and the proof of Theorem 1.1 in this case is complete. □

In Lemma 3.2 if we replace the assumption of uniform boundedness of the energy density $\|e_{g_i}(f_i)\|_{L^\infty} < C$ with the assumption uniform bound on the energy $E_{g_i}(f_i) < C$, then the limiting map is not necessarily harmonic (see Theorem 2.3 and Remark 1).

Proposition 3.3 *Let (M_i, g_i) be a sequence of manifolds in $\mathcal{M}(n, D, v)$ which converges to a Riemannian manifold (M, g) in the measured Gromov–Hausdorff topology. Suppose (N, h) is a compact Riemannian manifold which does not carry any harmonic 2-sphere S^2. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of harmonic maps such that $E_{g_i}(f_i) < C$ where C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (M, g) \to (N, h)$, and this map is a weakly harmonic map.*

Proof With the same argument as in the proof of Proposition 3.1 we consider f_i and g_i to be on the manifold M. When we have a sequence of Riemannian manifolds (M, g_i) which converges in $\mathcal{M}(n, d, v)$, the injectivity radius is bounded from below and dvol_{g_i} converges to dvol_g weakly. Therefore if $E_{g_i}(f_i) < C$, C independent of i, then $E_g(f_i)$ is uniformly bounded. Adapting the proof of Remark 1 for our case, f_i converges strongly in $\mathcal{H}^1$ to a map f. Also $\mathrm{Hess}(\pi_N)$ restricted to a neighborhood of N is Lipschitz and $\mathrm{Hess}(\pi_N) \circ f_i$ converges to $\mathrm{Hess}(\pi_N) \circ f$ in $\mathcal{H}^1$-norm (see Lemma 6.4 in Taylor's book [32]) and so therefore $\Pi(f_i)(df_i, df_i)$ converges weakly to $\Pi(f)(df, df)$. We have the same for Δf_i and so f is a weakly harmonic map. □

Under the assumptions of the above theorem one can show more and prove f is stationary harmonic. Under stronger assumptions on N or on the image of f, we can show that the limit map f is strongly harmonic. These results are direct consequences of some of the theorems in [30].

Proposition 3.4 *Let (M_i, g_i) and f_i be as in Proposition 3.3. Then the map f is smooth harmonic, provided that N is a compact Riemannian manifold and we have one of the following conditions:*

i. *(N, h) is a non-positively curved Riemannian manifold.*
ii. *There is no strictly convex bounded function on $f(M)$.*

Proof i. See Proposition 2.1 in [30].
ii. See Corollary 2.4 in [30].

□

3.2 Case II: Collapsing to a Manifold

In this subsection we prove

Proposition 3.5 *Let* (M_i, g_i) *be a sequence of Riemannian manifolds in* $\mathcal{M}(n, D)$ *which converges to a Riemannian manifold* $(M, g, \Phi\, \mathrm{dvol}_M)$ *in the measured Gromov–Hausdorff topology with* $C^{1,\alpha}$*-pair* (g, Φ). *Suppose* (N, h) *is a compact Riemannian manifold. Let* $f_i : (M_i, g_i) \to (N, h)$ *be a sequence of smooth harmonic maps such that* $\|e_{g_i}(f_i)\|_{L^\infty} < C$, *where* C *is a constant independent of* i. *Then* f_i *has a subsequence which converges to a map* $f : (M, g, \Phi\, \mathrm{dvol}_g) \to (N, h)$, *and this map is a weakly harmonic map.*

Before we prove the proposition in general, we will prove the following proposition which has an additional regularity assumption. Then at the end of this subsection, we will apply this proposition to prove case II. Consider the following assumption,

Assumption 1 Let the Riemannian metric g_i be regular on M_i, i.e., there exists a sequence $C = \{C_k\}$ of positive number C_k independent of i, such that

$$|\nabla^k_{g_i} \mathrm{R}_{g_i}| < C_k. \tag{9}$$

Suppose also that the Riemannian metric g_i is an invariant metric with respect to the nil-structure.

We have

Proposition 3.6 *Let* (M_i, g_i) *be a convergent sequence of Riemannian manifolds in* $\mathcal{M}(n, D)$ *(with respect to the measured Gromov–Hausdorff topology) such that* g_i *satisfies the Assumption 1. Let* (M, g, Φ) *be the limit manifold. Suppose* (N, h) *is a compact Riemannian manifold. Let* $f_i : (M_i, g_i) \to (N, h)$ *be a sequence of smooth harmonic maps such that* $\|e_{g_i}(f_i)\|_{L^\infty} < C$, *where* C *is a constant independent of* i. *Then* f_i *has a subsequence which converges to a map* $f : (M, g, \Phi\, \mathrm{dvol}_M) \to (N, h)$ *and this map is a smooth harmonic map.*

Before we prove the Proposition 3.6, we first recall a few remarks from [12,13]. Then we prove Lemma 3.7 which is the main element in the proof of Proposition 3.6.

Remark 4 In [13] Fukaya proves that with the extra regularity assumption (9) on g_i, $(M_i, g_i, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)})$ converges to a smooth Riemannian manifold, with the smooth pair (g, Φ). See Lemma 2.1 in [13]. By Theorem 2.15, we know that for i large enough, there is a fibration map $\psi_i : M_i \to M$. Since g_i is an invariant metric, there exist metrics g_i^M on M such that the maps $\psi_i : (M_i, g_i) \to (M, g_i^M)$ are Riemannian submersions and g_i^M converges to g as in Theorem 2.13.

Remark 5 (Fukaya [12,13]) Take an arbitrary point p_0 in M and choose $p_i \in \psi_i^{-1}(p_0)$. By $|\sec_{g_i}| \leq 1$, at point p_i on M_i the conjugate radius[1] is greater than

[1] The conjugate domain at a point p in a Riemannian manifold M is the largest star shaped domain in which $d\exp_p$ is non-singular and the conjugate radius is the radius of the largest ball in the conjugate domain at p.

some constant name it ρ. We name the pullback of the Riemannian metric g_i by the exponential map, $\exp_{p_i}$ at p_i, $\tilde{g}_i$. Therefore the injectivity radius at 0 is at least the conjugate radius at p_i (see Corollary 2.2.3 in [29]).

Consider the ball $B = B(0, \rho)$ in $T_{p_i}M_i$ with the metric $\tilde{g}_i$. By virtue of the regularity assumption on g_i, $\tilde{g}_i$ will converge to some g_0 in the C^∞-topology. There are local groups G_i converging to a Lie group germ G such that

1. G_i act by isometries on the pointed metric spaces $((B, \tilde{g}_i), 0)$.
2. $((B, \tilde{g}_i), 0)/G_i$ is isometric to a neighborhood of p_i in M_i.
3. G acts by isometries on the pointed metric space $((B, g_0), 0)$.
4. $((B, g_0), 0)/G$ is isometric to a neighborhood of p_0 in M and the action of G is free.

It follows that there is a neighborhood U of p_0 in M and a C^∞ map $s : U \to B$ such that

i. $s(p_0) = 0$.
ii. $P \circ s = Id$, where P denotes the composition of the projection map and the above mentioned isometry in 4.
iii. $d_{(B,g_0)}(s(q), 0) = d_M(q, p_0)$ holds for $q \in M$.

Therefore there is some constant, which we again name ρ, independent of i such that, $M = \bigcup_{j=1}^m B_{\frac{\rho}{2}}(x_j, M)$ and $B_{\frac{\rho}{2}}(x_j, M)$ satisfies the preceding conditions and we can construct a smooth section $s_{i,j} : B_{\frac{\rho}{2}}(x_j, M) \to M_i$ of ψ_i, such that

$$\frac{|(s_{i,j})_*(v)|}{|v|} < C \tag{10}$$

for each $v \in TB_{\frac{\rho}{2}}(x_j, M)$. Here C is a constant independent of i. Hereafter we let $p_{i,j} = \psi_i^{-1}(x_j)$ and by $B(p_{i,j})$ we mean a ball centered at $p_{i,j}$ with radius ρ in $T_{p_{i,j}}M_i$. See section 3 in [12] and section 2 in [13].

Now we show that f_is are almost constant on the fibers of M_i. The following lemma is similar to Lemma 4.3 in [11]. In the following lemma (M_i, g_i) is a convergent sequence in $\mathcal{M}(n, D)$ such that g_i satisfies only (9) and N is a compact Riemannian manifold.

Lemma 3.7 *Let $h_i : M_i \to I(N) \subset \mathbb{R}^q$ be smooth maps which satisfy the Euler–Lagrange equation (1). Suppose $v_i \in T_p(M_i)$ satisfies $(\psi_i)_*(v_i) = 0$, where ψ_i is the fibration map and $v_i', v_i'' \in T_p(M_i)$ ($p \in B_{2\rho/3}(p_{i,j}, M_i)$). Then we have*

$$|v_i \cdot h_i| \le C_1 \cdot \epsilon_i' \cdot |v_i| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}), \tag{11}$$

$$|v_i' \cdot v_i'' \cdot h_i| \le C_2 \cdot |v_i'| \cdot |v_i''| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}), \tag{12}$$

where C_1 and C_2 are some constants independent of i and ϵ_i' is a sequence converging to zero. Also $v_i \cdot h_i = dh_i(v_i)$ denotes the derivative of h_i in the direction of v_i.

Proof We put $\Phi_{i,j} = \exp_{p_{i,j}} : B(p_{i,j}) \to M_i$, $\tilde{g}_{i,j} = \Phi_{i,j*}(g_i)$ and $a = \Phi_{i,j}^{-1}(p)$. We also denote $h_i \circ \Phi_{i,j}$ by $h_{i,j}$.

From the Schauder estimates for elliptic equations (see Theorem 2.7) we have

$$\|h_{i,j}\|_{C^{1,\alpha}} \leq C' \cdot (\|\Delta h_{i,j}\|_{L^\infty} + \|h_{i,j}\|_{L^\infty}), \tag{13}$$

and hence

$$\|v_i' \cdot h_{i,j}\|_{C^\alpha} \leq C' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}), \tag{14}$$

where C' depends on the metric $\tilde{g}_{i,j}$. Since $\Phi_{i,j}$ is an isometry, by the composition formula (see formula 1.4.1 in [35]), we have $\Delta h_{i,j}(x) = \Delta h_i(\Phi_{i,j}(x))$. Also from (13), and the fact that $\tilde{g}_{i,j}$ converges in C^∞

$$\|\Pi(h_{i,j})(dh_{i,j}, dh_{i,j})\|_{C^\alpha} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

where C'' is a constant independent of i. By Eq. (1), we have

$$\|\Delta h_{i,j}\|_{C^\alpha} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

Using Schauder estimates for second derivative, we have

$$\|h_{i,j}\|_{C^{2,\alpha}} \leq C \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}), \tag{15}$$

for some C independent of i and (12) follows.

Now we prove (11) by contradiction. Assume $|v_i| = 1$. Let $\sigma^i(t) = \exp_p^{F_i}(tv_i)$ be a geodesic in the fiber containing p, $F_i \subset M_i$ such that $\frac{d}{dt}|_{t=0}\sigma^i(t) = v_i$. For $0 \leq t \leq \frac{\rho}{5}$ this curve has a lift $l^i(t) \subset B(p_{i,j})$ such that $\Phi_{i,j}(l^i(t)) = \sigma^i(t)$. We have

$$d(\sigma_i(t), p) \leq \operatorname{diam}(F_i) \leq \epsilon_i.$$

By contradiction we assume that there is subsequence of h_i and a positive number A such that

$$|v_i \cdot h_{i,j}| > A \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

We know that

$$v_i \cdot h_i = v_i \cdot h_{i,j} = \left.\frac{d}{dt}\right|_{t=0} h_{i,j} \circ l^i(t).$$

There exist $\beta > 0$ and $\delta > 0$ independent of i such that for any $t < \delta$, we have

$$|h_{i,j} \circ l^i(t) - h_{i,j}(a)| > \beta \cdot t \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}). \tag{16}$$

To explain this, let $h_{i,j} \circ l^i(t) = q_{i,j}(t)$. We know from (15) that

$$| \frac{d}{dt}\bigg|_{t=0} q'_{i,j}(t)| \leq C(\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

so for some fixed δ and $0 < t < \delta$ we have

$$|q'_{i,j}(t) - q'_{i,j}(0)| \leq C' \cdot t \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

On the other hand we have

$$|q'_{i,j}(0)| > A \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

so for δ small enough and $t < \delta$ we have

$$|q'_{i,j}(t)| > \beta \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

Therefore

$$|q_{i,j}(t) - q_{i,j}(0)| = |q'_{i,j}(\theta_i) \cdot t| > \beta \cdot t \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

from which (16) follows.

There exists $b \in B(p_{i,j})$, such that $d(a, b) < \epsilon_i$ and $\Phi_{i,j}(l_i(\delta')) = b$. For a fixed $\delta' < \delta$ we have

$$|h_{i,j}(b) - h_{i,j}(a)| > \beta \cdot \delta' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

If we fix $\{\xi_k\}_{k=0}^{k=n}$ as a coordinate system at the point $a \in B(p_{i,j})$, for some $b' \in B(p_{i,j})$ we have

$$\sum_{k=0}^{k=n} \frac{\partial h_{i,j}}{\partial \xi^k} > C \cdot \beta \cdot \tfrac{\delta'}{\epsilon_i} \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

and this contradicts (14). □

Now we prove Proposition 3.6.

Proof of Proposition 3.6 As we assumed $\|e(f_i)\|_{L^\infty} < c$ and by the Euler–Lagrange equation and Corollary 2.6, we have that $\|\Delta I \circ f_i\|_{L^\infty}$ is uniformly bounded. Moreover, $\|I \circ f_i\|_{L^\infty}$ is uniformly bounded. Using (11), the maps f_is are equicontinuous. By Lemma 2.12, there is a limit map $f : M \to N$ which is continuous.

We consider the following maps on M,

$$\tilde{f}_i = \sum \beta_j \cdot (I \circ f_i) \circ s_{i,j}, \tag{17}$$

where β_j is an arbitrary C^∞ partition of unity associated to $B_{\frac{\varrho}{2}}(x_j, M)$, $s_{i,j}$ is the section associated to ψ_i as mentioned in Remark 5. Along a subsequence, which we again denote by f_i, we have

$$\lim_{i\to\infty} f_i(s_{i,j}(x)) = f(x) \qquad \text{for } x \in B_{\frac{\rho}{2}}(x_j, M),$$

and also

$$\lim_{i\to\infty} \tilde{f}_i(x) = I \circ f(x) \qquad \text{for } x \in B_{\frac{\rho}{2}}(x_j, M).$$

Since the energy density of f_i is bounded and also $s_{i,j}$ satisfies (10), we have $\|e(\tilde{f}_i)\|_{L^\infty}$ is uniformly bounded. By the same argument as above, $\|\tilde{f}_i\|_{C^1}$ is bounded and $\tilde{f}_i$ converge uniformly to $I \circ f$. Moreover ψ_i has bounded second fundamental form (see Theorem 2.6 in [6]) and the same is true for $s_{i,j}$. So $\tilde{f}_i$ has bounded C^2-norm and there is a subsequence of $\tilde{f}_i$ which converges to $I \circ f$ in the C^1-topology.

Choose a local orthonormal frame $\{\bar{e}_k\}_{k=1}^m$ on (M, g_i^M). Denote its horizontal lift on (M_i, g_i) by $\{e_k\}_{k=1}^m$. Suppose $\{e_t\}_{t=m+1}^n$ is a local orthonormal frame field of the fiber F_i in M_i such that $\{e_k, e_t\}$ form a local orthonormal frame field in M_i (note that we omit the index i for the orthonormal frame fields on (M_i, g_i) and (M, g_i^M)). Our aim is to show that f is also weakly harmonic. □

Lemma 3.8 *We have*

$$\lim_{i\to\infty} |\langle dI \circ f_i, d\eta_i\rangle(p) - \langle d\tilde{f}_i, d\eta\rangle(\psi_i(p))| = 0,$$

where $\eta : M \to \mathbb{R}^q$, *is a* C^∞-*map* $\eta_i = \eta \circ \psi_i$, *and* p *in* M_i.

Proof By inequality (11),

$$|\langle dI \circ f_i, d\eta_i\rangle(p) - \sum_{k=1}^{m} \langle di \circ f_i(e_k), d\eta_i(e_k)\rangle(p)| \le C_1 \cdot \epsilon_i'$$

for i large enough where C_1 is a constant independent of i. Let F_i denote the fiber containing p and choose a point q in F_i. By (12), and since $\mathrm{diam}(F_i) \le \epsilon_i$

$$|dI \circ f_i(e_k)(p) - dI \circ f_i(e_k)(q)| \le C_2 \cdot \epsilon_i,$$

and so

$$|dI \circ f_i(e_k)(p) - dI \circ f_i(e_k)(s_{i,j} \circ \psi_i(p))| \le C_2 \cdot \epsilon_i.$$

Because $\psi_i \circ s_{i,j} = \mathrm{Id}$, for $x \in M$ we have

$$\psi_{i*}\left(e_k(s_{i,j}(x)) - s_{i,j*}(\bar{e}_k(x))\right) = 0.$$

By inequality (10), we have

$$|e_k(s_{i,j}(x)) - s_{i,j*}(\bar{e}_k(x))| \le C_3,$$

for some constant C_3 and therefore by (11),

$$|dI \circ f_i(e_k)(p) - d(I \circ f_i) \circ s_{i,j_*}(\bar{e}_k)(\psi_i(p))| \leq C_4 \cdot \epsilon_i.$$

From the convergence of $f_i \circ s_{i,j}$ to f, we have

$$\lim_{i\to\infty} |\sum d\beta_j \cdot (I \circ f_i) \circ s_{i,j} - \sum d\beta_j \cdot (I \circ f)| = 0,$$

So

$$\lim_{i\to\infty} |d\tilde{f}_i - \sum \beta_j \cdot d((I \circ f_i) \circ s_{i,j})| = 0.$$

Since $\sum_j \beta_j = 1$ we finally have

$$\lim_{i\to} |\langle dI \circ f_i, d\eta_i\rangle(p) - \langle d\tilde{f}_i, d\eta\rangle(\psi_i(p))| = 0.$$

□

Lemma 3.9 *We have*

$$\lim_{1\to\infty} \left|\Pi(f_i)(p)(dI \circ f_i, dI \circ f_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i)\right| = 0.$$

Proof By the proof of the above lemma, we have

$$\lim_{i\to\infty} |df_i(p) - d\tilde{f}_i(\psi_i(p))| = 0.$$

By the same argument as in Lemma 3.8 we can conclude

$$\left|\Pi(f_i)(p)(dI \circ f_i, dI \circ f_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i)\right| \leq C \cdot \left|df_i(p) - d\tilde{f}_i(\psi_i(p))\right|.$$

□

The map $\tilde{f}_i : (M, g_i^M, \mathrm{dvol}_{g_i^M}) \to \mathbb{R}^q$ converges in C^1 to the map $I \circ f$, and Φ_i converges to Φ in the C^∞-topology. Also (M, g_i^M) converges to (M, g) in $\mathcal{M}(n, D, v)$. Therefore we have

$$\left|\int_M \Xi_{g_i^M}(\eta, \tilde{f}_i)\ \Phi_i\ \mathrm{dvol}_{g_i^M} - \int_M \Xi(\eta, f)\ \Phi\ \mathrm{dvol}_g\right| \leq C \cdot \epsilon_i,$$

where $\Xi(\cdot, \cdot)$ is defined by (3). By Lemma 3.8 and 3.9, we have

$$\lim_{i\to\infty} \left|\int_{M_i} \Xi_{g_i}(\eta_i, f_i) \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} - \int_M \Xi_{g_i^M}(\eta, \tilde{f}_i)\psi_{i*}\left(\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}\right)\right| = 0.$$

It follows that

$$\lim_{i\to\infty}\int_{M_i}\Xi_{g_i}(\eta_i, f_i)\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} = \int_M \Xi_g(\eta, f)\ \Phi\,\mathrm{dvol}_M\,. \tag{18}$$

Therefore f is weakly harmonic and since it is continuous, it is also a smooth harmonic map. □

Now we prove Case II without considering Assumption 1.

Proof of Proposition 3.5 By Remark 2 we can obtain a C^1-close metric $g_i(\epsilon)$ to g_i which satisfies (9) and such that the map $\psi_i : (M_i, g_i(\epsilon)) \to (M, \psi_{i*}(g_i(\epsilon)))$ is a Riemannian submersion.

For small ϵ, let $M(\epsilon)$ be the Gromov–Hausdorff limit of a subsequence of $(M_i, g_i(\epsilon))$. By Lemma 2.3 in [12], $(M_i, g_i(\epsilon))$ and $(M(\epsilon), g(\epsilon))$ converge to (M_i, g_i) and (M, g) in $\mathcal{M}(n, D, v)$ respectively.

The map $f_i : (M_i, g_i) \to (N, h)$ is harmonic and since $g_i(\epsilon)$ is C^1-close to g, we have

$$|\Xi_{g_i}(f_i, \eta_i) - \Xi_{g_i(\epsilon)}(f_i, \eta_i)| \le C\cdot\epsilon.$$

By (18), we have

$$\lim_{i\to\infty}\left|\int_{M_i}\Xi_{g_i(\epsilon)}(f_i, \eta_i)\frac{\mathrm{dvol}_{(M_i,g_i(\epsilon))}}{\mathrm{vol}((M_i,g_i(\epsilon)))} - \int_{M(\epsilon)}\Xi_{g(\epsilon)}(f, \eta)\cdot\Phi(\epsilon)\,\mathrm{dvol}_{M(\epsilon)}\right| = 0,$$

and finally since $g(\epsilon)$ converges to g in the $C^{1,\alpha}$-topology, we have the desired result. □

3.3 Case III: Collapsing to a Singular Space

Now we are going to investigate the general case when the sequence converges to a singular space. This means that (M_i, g_i) in $\mathcal{M}(n, D)$ converges to some metric space (X, d). First we recall the following remark from [11].

Remark 6 (Fukaya [11], §7) Let Y be a Riemannian manifold on which $O(n)$ acts by isometry, and let $\theta : Y \to [0,\infty)$ be an $O(n)$-invariant smooth function. Put $X = Y/O(n)$. Let $p : Y \to X$ be the natural projection, $\bar\theta : X \to [0,\infty)$ the function induced from θ, and $S(X)$ the set of all singular points of X. The set $S(X) \subset X$ has a well defined normal bundle on the codimension 2 strata ($X = Y/O(n)$ is a Riemannian polyhedron and $S(X)$ is a subset of the $(n-2)$-skeleton of X). Set

$$\mathrm{Lip}(X, S(X)) = \{u \in \mathrm{Lip}(X) \mid v\cdot u = 0 \text{ if } v \text{ is perpendicular to } S(X)\}.$$

Define $Q_1 : \mathrm{Lip}(Y) \times \mathrm{Lip}(Y) \to [0, \infty)$ and $Q_2 : \mathrm{Lip}(X, S(X)) \times \mathrm{Lip}(X, S(X)) \to [0, 1)$ by

$$Q_1(\tilde{k}, \tilde{h}) = \int_Y \theta \cdot \langle \nabla \tilde{k}, \nabla \tilde{h} \rangle \ \mathrm{dvol}_Y,$$
$$Q_2(k, h) = \int_X \bar{\theta} \cdot \langle \nabla k, \nabla h \rangle \ d\mu_g.$$

It is easy to see that $f \circ p \in \mathrm{Lip}(Y)$ for each f contained in $\mathrm{Lip}(X, S(X))$. Define $p^* : \mathrm{Lip}(X, S(X)) \to \mathrm{Lip}(Y)$ by $p^*(f) = f \circ p$. Let $\mathrm{Lip}_{O(n)}(Y)$ be the set of all $O(n)$-invariant elements of $\mathrm{Lip}(Y)$. Then, we can easily prove the following

Lemma 3.10 *p^* is a bijection between* $\mathrm{Lip}(X, S(X))$ *and* $\mathrm{Lip}_{O(n)}(Y)$*. For elements f and k of* $\mathrm{Lip}(X, S(X))$*, we have*

$$Q_1(f, k) = Q_2(p^*(f), p^*(k)), \tag{19}$$

and

$$\int_Y \theta \cdot p^*(f) p^*(k) \ \mathrm{dvol}_Y = \int_X \bar{\theta} \cdot f k \ d\mu_g. \tag{20}$$

Now we prove the main theorem of this paper.

Proof of Theorem 1.1 We denote by $(Y, g, \Phi_Y \,\mathrm{dvol}_Y)$ the limit space of the frame bundles over M_i, and by (X, d, ν) the limit space of M_i with respect to the measured Gromov–Hausdorff topology. We know $(X, \nu) = (Y, \Phi_Y \,\mathrm{dvol}_Y)/O(n)$ (see Section 2.6). The projection $p_i : (F(M_i), \tilde{g}_i) \to (M_i, g_i)$ is a Riemannian submersion with totally geodesic fibers. So using the reduction formula the map $\bar{f_i} = f_i \circ p_i$ is harmonic on $F(M_i)$ and it is invariant under the action of $O(n)$. Furthermore $\|e_{\tilde{g}_i}(\bar{f_i})\|_\infty$ is bounded (p_i is a Riemannian submersion). Using Case II, $\bar{f_i}$ converge to some map $\bar{f}$ on $(Y, g, \Phi_Y \,\mathrm{dvol}_Y)$. The map $\bar{f}$ satisfies

$$\int_Y \Xi_g(\bar{f}, \eta) \ \Phi_Y \,\mathrm{dvol}_Y = 0,$$

where η is a test function. The map $\bar{f}$ is also $O(n)$ invariant and continuous. Consider a quotient map f such that $\bar{f} = p^*(f)$. First we show that f is in $\mathcal{H}^1((X, \nu), N)$. By the argument in Case II, $\bar{f}$ is in $\mathcal{H}^1((Y, \Phi_Y \,\mathrm{dvol}_Y), N)$ and so by Eq. (19), f has finite energy. Now we show that f is weakly harmonic on (X, ν). By Eq. (19), for η in $\mathrm{Lip}(X, S(X))$

$$\int_Y \langle \nabla I \circ \bar{f}, \nabla p^*(\eta) \rangle \ \Phi_Y \,\mathrm{dvol}_Y = \int_X \langle \nabla I \circ f, \nabla \eta \rangle \ \Phi_X d\mu_g.$$

Furthermore

$$\int_Y \langle \Pi(\bar{f})(\nabla^g(I \circ \bar{f}), \nabla^g(I \circ \bar{f})), p^*(\eta)\rangle \, \Phi_Y \, \mathrm{dvol}_Y$$
$$= \int_X \langle \Pi(f)(\nabla(I \circ f), \nabla(I \circ f)), \eta\rangle \, \Phi_X d\mu_g,$$

and since $\Phi_Y = p^*(\Phi_X)$

$$\int_Y \Xi_g(\bar{f}, p^*(\eta)) \, \Phi_Y \, \mathrm{dvol}_Y = \int_X \Xi(f, \eta) \, \Phi_X d\mu_g,$$

which shows that $f : X \to N$ is a weakly harmonic map. □

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Appendix: Convergence of Tension Field

In this section we study convergence of the tension fields of the maps f_i, $\tau(f_i)$, under the assumptions of Proposition 3.6.

Assume (M_i, g_i), f_i, N to be as in Proposition 3.6. Moreover consider the following assumption

Assumption 2 The section $s_{i,j}$ is almost harmonic,

$$|\tau(s_{i,j})| \leq C \cdot \epsilon_i'', \tag{21}$$

and also

$$|\nabla_{\bar{X}} ds_{i,j}(X)| \leq C \cdot \epsilon_i'', \tag{22}$$

where X is a smooth vector field on M and $\bar{X}$ is its horizontal lift and ϵ_i'' is a sequence which converges to zero.

Using Assumption 1 and by Theorem 2.4 we have

$$\begin{aligned}\tau(f_i) &= (\nabla_{e_k} df_i)e_k + (\nabla_{e_t} df_i)e_t \\ &= (\nabla_{e_k} df_i)e_k + \nabla_{f_{i*}(e_t)} f_{i*}(e_t) \\ &\quad - f_{i*}(\nabla_{e_t} e_t)^H - f_{i*}(\nabla_{e_t} e_t)^V \\ &= (\nabla_{e_k} df_i)e_k - f_{i*}(\mathrm{H}_i) + \tau(f_i{}^{\perp})\end{aligned} \tag{23}$$

where $\{e_k, e_t\}$ and $\bar{e}_k$ are as in the proof of Proposition 3.6, $f_i{}^{\perp}$ denotes the restriction of f_i to the fibers F_i, and H_i is the mean curvature vector of the submanifold F_i.

We investigate how each term of the equation above behaves as f_i converges to f.

Lemma 3.11 *We have*

$$\lim_{i\to\infty}\left|dI(\nabla_{e_k}df_i)e_k(p)-\left(\Delta_{g_i^M}\tilde{f}_i-\Pi(\tilde{f}_i)(d\tilde{f}_i,d\tilde{f}_i)\right)(\psi_i(p))\right|=0. \tag{24}$$

Proof By the discussion in the proof of Proposition 3.6, we know that $\tilde{f}_i$ converges to f in the C^1-topology. Using the composition formula we have

$$dI(Bf_i(X_1,X_2))=B(I\circ f_i)(X_1,X_2)-B(\pi_N)(d(I\circ f_i)(X_1),d(I\circ f_i)(X_2)),$$

and so for $k=1,\dots,n$,

$$dI((\nabla_{e_k}df_i)e_k)=(\nabla_{e_k}d(I\circ f_i))e_k-B(\pi_N)(d(I\circ f_i)(e_k),d(I\circ f_i)(e_k)).$$

First we show that

$$\lim_{i\to\infty}|\nabla_{e_k}d(I\circ f_i)e_k(p)-\Delta_{g_i^M}\tilde{f}_i(\psi_i(p))|=0.$$

By definition of $\tilde{f}_i$,

$$\begin{aligned}(\nabla_{\bar{e}_k}d\tilde{f}_i)\bar{e}_k=\sum\big(&d\beta_j(\bar{e}_k)\cdot df_i(s_{i,j*}(\bar{e}_k))\\&+\beta_j\cdot(\nabla_{\bar{e}_k}d(f_i\circ s_{i,j}))\bar{e}_k+\Delta\beta_j\cdot f_i\circ s_{i,j}\big).\end{aligned}$$

and again by the composition formula

$$\tau(f_i\circ s_{i,j})=B_{s_{i,j*}(\bar{e}_k),s_{i,j*}(\bar{e}_k)}f_i+df_i(\tau(s_{i,j})). \tag{25}$$

Since $f_i\circ s_{i,j}$ converges in C^1 to f

$$\lim_{i\to\infty}|\sum d\beta_j(\bar{e}_k)\cdot df_i(s_{i,j*}(\bar{e}_k))|=0,$$
$$\lim_{i\to\infty}\sum\Delta\beta_j\cdot f_i\circ s_{i,j}(x)=\sum\Delta\beta_j\cdot f(x)=0.$$

Also, $\psi_{i*}(e_k-s_{i,j*}(\bar{e}_k))=0$ and so $e_k-s_{i,j*}(\bar{e}_k)$ is vertical. On the other hand

$$|e_k-s_{i,j*}(\bar{e}_k)|\le\epsilon_i.$$

By inequality (11) and almost harmonicity of $s_{i,j}$ (21), the second term on the right hand side of (25) converges to zero. Again by inequality (12) and (22), we have

$$\lim_{i\to\infty}|(\nabla_{e_k}df_i)(e_k-s_{i,j*}(\bar{e}_k))|=0,$$
$$\lim_{i\to\infty}|(\nabla_{(e_k-s_{i,j*}(\bar{e}_k))}df_i)e_k|=0.$$

Finally

$$\lim_{i\to\infty}|(\nabla_{e_k}d(I\circ f_i))e_k(p)-(\nabla_{\bar{e}_k}d\tilde{f}_i)\bar{e}_k(\psi_(p))|=0.$$

We have the same for the second term

$$\lim_{i\to\infty} |\Pi(f_i)(p)(df_i, df_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i)| = 0.$$

□

By the above lemma and $\psi_{i*}(\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}) = \Phi_i \,\mathrm{dvol}_M$, we have

$$\lim_{i\to\infty} \left| \int_{M_i} \langle dI((\nabla_{e_k} df_i)e_k), \eta_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} - \int_M \langle \Delta^{g_i^M} \tilde{f}_i - \Pi(\tilde{f}_i)(d\tilde{f}_i, d\tilde{f}_i), \eta\rangle \,\Phi_i \,\mathrm{dvol}_{g_i^M} \right| = 0,$$

and we conclude

$$\lim_{i\to\infty} \int_{M_i} \langle dI((\nabla_{e_k} df_i)e_k), \eta_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} = \int_M [\langle df, d\eta\rangle + \langle df(\nabla \ln \Phi) - \Pi(f)(df, df), \eta\rangle] \;\Phi \,\mathrm{dvol}_M . \qquad (26)$$

Here η is a test map on M and $\eta_i = \eta \circ \psi_i$. Now we will consider the second and third terms in the decomposition of $\tau(f_i)$.

Lemma 3.12 *With the same assumptions as above*

i. $\lim_{i\to\infty} \int_{M_i} \langle df_i(\mathrm{H}_i), \eta_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} = -\int_M \langle df(\nabla \ln \Phi), \eta\rangle \;\Phi \,\mathrm{dvol}_M.$
ii. $\lim_{i\to\infty} \|\tau(f_i{}^{\perp})\| = 0.$

Here H_i *denotes the mean curvature vector of the fibers* $F_i^x = \psi_i^{-1}(x)$.

Before we prove Lemma 3.12, we prove the following lemma which we need for the proof of part i.

Lemma 3.13 *We have*

$$\int_M \eta d \ln \Phi(X) \;\Phi \,\mathrm{dvol}_M = -\lim_{i\to\infty} \int_{M_i} \eta \langle X, \mathrm{H}_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}. \qquad (27)$$

Proof Suppose X is a smooth vector field on M and X_i its horizontal lift on M_i. The flow θ_t^i of X_i sends fibers to fibers diffeomorphically. By the first variation formula

$$\left.\frac{d}{dt}\right|_{t=0} \theta_t^{i\,*}(\mathrm{dvol}_{F_i^x}) = -\int_{F_i^x} \langle X_i, \mathrm{H}_i^x\rangle \;\mathrm{dvol}_{F_i^x} . \qquad (28)$$

Also

$$\Phi_i(x) = \frac{\mathrm{vol}(\psi_i^{-1}(x))}{\mathrm{vol}(M_i)}.$$

and by (28),

$$d\Phi_i(X)(x) = -\int_{F_i^x} \langle X_i, \mathrm{H}_i^x \rangle \, \frac{\mathrm{dvol}_{F_i^x}}{\mathrm{vol}(M_i)},$$

For an arbitrary η in $C^\infty(M)$, we prove

$$\int_M \eta d\Phi_i(X) \ \mathrm{dvol}_{g_i^M} = -\int_{M_i} \eta_i \langle X_i, \mathrm{H}_i \rangle \, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}. \tag{29}$$

If we consider (U_γ, h_γ) as a local trivialization of the fibration ψ_i, then

$$\int_M \chi_{U_\gamma} d\Phi_i(X) \ \mathrm{dvol}^{g_i^M} = -\int_{U_\gamma} \int_{F_i^x} \chi_{U_\gamma} \langle X_i, \mathrm{H}_i^x \rangle \, \frac{\mathrm{dvol}_{F_i^x}}{\mathrm{vol}(M_i)} \, \mathrm{dvol}_{g_i^M},$$

and so

$$\int_M \chi_{U_\gamma} d\Phi_i(X) \ \mathrm{dvol}_M^{g_i^M} = -\int_{\psi_i^{-1}(U_\gamma)} \langle X_i, \mathrm{H}_i \rangle \, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)},$$

where χ_{U_γ} denotes the characteristic function on U_γ and so we have (29). The functions Φ_i goes to Φ in C^∞ and also $\mathrm{dvol}^{g_i^M}$ goes to dvol_M as i goes to infinity. Letting i go to ∞ on the both sides of (29) and by the definition of weak derivatives

$$\int_M \eta d \ln \Phi(X) \ \Phi \, \mathrm{dvol}_M = -\lim_{i\to\infty} \int_{M_i} \eta \langle X, \mathrm{H}_i \rangle \, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}.$$

□

Proof of Lemma 3.12 Part i follows directly from Lemma 3.13.

To prove part ii consider

$$\tau(f_i{}^\perp) = \nabla_{f_{i*}(e_t)} f_{i*}(e_t) - f_{i*}(\nabla_{e_t} e_t)^V.$$

From (11) and (12)

$$|\nabla_{f_{i*}(e_t)} f_{i*}(e_t)| < C \cdot \epsilon_i',$$
$$\| f_{i*}(\nabla_{e_t} e_t)^V \|_{L^\infty} < C \cdot \epsilon_i' |(\nabla_{e_t} e_t)^V|,$$

where C is a constant independent of i. It follows that

$$\lim_{i\to\infty} \|\tau(f_i{}^\perp)\| = 0.$$

□

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