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The Kato Square Root Problem on Vector Bundles with Generalised Bounded Geometry

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Abstract We consider smooth, complete Riemannian manifolds which are exponentially locally doubling. Under a uniform Ricci curvature bound and a uniform lower bound on injectivity radius, we prove a Kato square root estimate for certain coercive operators over the bundle of finite rank tensors. These results are obtained as a special case of similar estimates on smooth vector bundles satisfying a criterion which we call *generalised bounded geometry*. We prove this by establishing quadratic estimates for perturbations of Dirac type operators on such bundles under an appropriate set of assumptions.

Keywords Kato square root problem · Square roots of elliptic operators · Quadratic estimates · Holomorphic functional calculi · Dirac type operators · Generalised bounded geometry

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1 Introduction

In this paper, we consider the *Kato square root problem* for uniformly elliptic operators on smooth vector bundles $\mathcal{V}$ over smooth, complete Riemannian mani-

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folds $\mathcal{M}$ which are at most exponentially locally doubling. Let ∇ be a connection on the bundle and h its metric. Define the uniformly elliptic operator $\mathrm{L}_A u = -a \operatorname{div}(A_{11}\nabla u) - a \operatorname{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u$ where $\operatorname{div} = -\nabla^*$ and where the a, A_{ij} are L^∞ coefficients. Under an appropriate bounded geometry assumption, we show that $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$ and that $\|\sqrt{\mathrm{L}_A}u\| \simeq \|u\|_{\mathrm{W}^{1,2}}$ for all $u \in \mathrm{W}^{1,2}(\mathcal{V})$.

The case of trivial, flat bundles was considered by Morris in [16] (and his thesis [14]). In particular, he obtains solutions to the Kato problem on Euclidean submanifolds under *extrinsic* curvature bounds. The novelty of our work is that we dispense with the requirement of an embedding and prove much more general results under *intrinsic* assumptions. We take the perspective of preserving the thrust of the harmonic analytic argument from the Euclidean context and as a consequence, we are forced to perform a detailed and intricate analysis of the geometry.

Our work utilises the foundation laid by Axelsson et al. in [6], and the perspective developed in [5] by the same authors. The ideas in both these papers have their roots in the solution of the Kato conjecture by Auscher et al. in [4]. See also the surveys of Hofmann [11], McIntosh [13], the book by Auscher and Tchamitchian [2], and the recent survey [12] by Hofmann and McIntosh.

The idea of the authors of [6] is to consider closed, densely defined, nilpotent, operators Γ, along with perturbations B_1, B_2 and then to establish quadratic estimates of the form

$$\int_0^\infty \left\| t\Pi_B(I + t^2\Pi_B)^{-1}u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

where $\Pi_B = \Gamma + B_1\Gamma^* B_2$. In [5], the authors illustrate that for *inhomogeneous* operators, it is enough to establish a certain *local* quadratic estimate, for which we need bounds on the integral from 0 to 1, since the integral from 1 to ∞ is straightforward in this case. The proof of this quadratic estimate proceeds by reduction to a Carleson measure estimate.

The techniques developed in [4,6] and [16] rely upon being able to take averages of functions over subsets, and defining constant vectors in key aspects of the proof. This is a primary obstruction to generalising these techniques to non-trivial bundles. To circumvent this obstacle, we formulate a condition which we call *generalised bounded geometry*. This condition captures a uniform locally Euclidean structure in the bundle. The existence of a dyadic decomposition (below a fixed scale) provides a decomposition of the manifold in a way that allows us to work on a fixed set of coordinates in the bundle. We can picture this decomposition of the bundle as a sort of abstract polygon—Euclidean regions separated by a boundary of null measure. Under the condition of generalised bounded geometry, and using this decomposition, we are then able to adapt the arguments of [16] and [6] in order to obtain a Kato square root estimate.

The intuition behind generalised bounded geometry is the existence of *harmonic coordinates* under Ricci bounds, along with a uniform lower bound on injectivity radius. We use this to show that the bundle of (p, q) tensors satisfies generalised bounded geometry. As a consequence, we obtain a Kato square root estimate for tensors under a coercivity condition which is automatically satisfied for scalar-valued functions. We highlight the scalar-valued version as a central theorem of this paper.

Theorem 1.1 (Kato square root problem for functions) *Let* $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose that the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and}$$

$$\mathrm{Re}(\langle A_{11}\nabla v, \nabla v\rangle + \langle A_{10}v, \nabla v\rangle + \langle A_{01}\nabla v, v\rangle + \langle A_{00}v, v\rangle) \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$*. Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$.

We also prove Lipschitz estimates for small perturbations of our operators, similar to those in §6 of [6], which are a direct consequence of considering operators with *complex* bounded measurable coefficients.

In [7], the first author proves quadratic estimates for operators Π_B on doubling measure metric spaces under an appropriate set of assumptions. We conclude this paper by demonstrating how to extend the quadratic estimates which we obtain on a manifold, to the more general setting of a complete metric space equipped with a Borel-regular measure that is exponentially locally doubling.

2 Preliminaries

2.1 Notation

Throughout this paper, we use the Einstein summation convention. That is, whenever there is a repeated lowered index and a raised index (or conversely), we assume summation over that index. By $\mathbb{N}$ we denote natural numbers not including 0 and we let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We denote the integers by $\mathbb{Z}$. We take the liberty to sometimes write $a \lesssim b$ for two real quantities a and b. By this, we mean that there is a constant $C > 0$ such that $a \leq Cb$. By $a \simeq b$ we mean that $a \lesssim b$ and $b \lesssim a$. For a function (and indeed, a section) f, we denote its support by $\mathrm{spt}\, f$. We denote an open ball centred at x of radius r by $B(x, r)$. The radius of a ball B (open or closed) is denoted by $\mathrm{rad}(B)$. For $\Omega \subset \mathcal{M}$, we denote the diameter of Ω by $\mathrm{diam}\ \Omega = \sup\{d(x, y) : x, y \in \Omega\}$.

2.2 Manifolds and Vector Bundles

In this section, we introduce terminology that allows us to describe the class of manifolds in which we obtain our results. Furthermore, we introduce the function spaces that we shall use. We also prove a key result that allows us to construct Sobolev spaces on vector bundles.

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g and Levi-Civita connection ∇. Let $d\mu$ denote the volume measure induced by g. We follow the notation of [16] in the following definition.

Definition 2.1 (*Exponential volume growth*) We say that $\mathcal{M}$ has *exponentially locally doubling volume growth* if there exists $c \geq 1,\ \kappa, \lambda \geq 0$ such that

$$0 < \mu(B(x, tr)) \leq ct^{\kappa} \mathrm{e}^{\lambda tr} \mu(B(x, r)) < \infty \tag{E$_{\text{loc}}$}$$

for all $t \geq 1,\ r > 0$ and $x \in \mathcal{M}$.

Let $\mathcal{V}$ be a smooth vector bundle of rank N with metric h and connection ∇. Let $\pi_{\mathcal{V}} : \mathcal{V} \to \mathcal{M}$ denote the canonical projection. Since we are interested in considering "sections" which may have low regularity, we deviate from the usual definition of $\mathbf{\Gamma}(\mathcal{V})$ by dropping the requirement that they be differentiable. More precisely, we write $\mathbf{\Gamma}(\mathcal{V})$ to be the space of measurable functions $\omega : \mathcal{M} \to \mathcal{V}$ such that $\omega(x) \in \pi_{\mathcal{V}}^{-1}(x)$. We impose no regularity assumptions (other than measurability) and we call these *sections* of $\mathcal{V}$. We write $\mathrm{L}^1_{\text{loc}}(\mathcal{V}) = \mathrm{L}^1_{\text{loc}}(\mathcal{M}, \mathcal{V})$ to denote the sections $\gamma \in \mathbf{\Gamma}(\mathcal{V})$ such that

$$\int_K |\gamma(x)|_{\mathrm{h}(x)}\ d\mu(x) < \infty$$

for each compact $K \subset \mathcal{M}$. Similarly, we define $\mathrm{L}^2(\mathcal{V})$. The spaces $\mathrm{C}^k(\mathcal{V})$ are then the k-differentiable sections and $\mathrm{C}^k_{\mathrm{c}}(\mathcal{V})$ are k-differentiable sections with compact support. The following proposition is necessary to define Sobolev spaces on $\mathcal{V}$.

Proposition 2.2 *The connection* $\nabla : \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is a densely defined, closable operator.*

Proof That $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V})$ is dense in $\mathrm{L}^2(\mathcal{V})$ is shown by a mollification argument after covering the manifold by countably many compact sets corresponding to coordinate charts. Since $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V}) \subset \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$, we have that ∇ is a densely defined operator.

We show that ∇ is a closable operator by reduction to the well known fact that $\nabla = \mathrm{d}$ is closable on scalar-valued functions. Fix $u_n \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$ such that $u_n \to 0$ and $\nabla u_n \to v$. For each $x \in \mathcal{M}$, we can find an open set K_x such that $\overline{K_x}$ is compact and with a local trivialization $\psi_x : \overline{K_x} \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(\overline{K_x})$. Let $\{K_x\}$ be a collection of such sets and let $\{K_j\}$ denote a countable subcover. Now, fix K_j and let $\{e^i\}$ denote an orthonormal frame (by a Gram–Schmidt procedure) in $\overline{K_j}$. Write $u_n = (u_n)_i e^i$ and note that

$$\nabla u_n = \nabla(u_n)_i \otimes e^i + (u_n)_i \nabla e^i$$

and therefore,

$$\sum_i |\nabla(u_n)_i - v_i|^2 = \left|\nabla(u_n)_i \otimes e^i - v\right|^2 \leq |\nabla u_n - v|^2 + \sup_l \left|\nabla e^l\right|^2 |u_n|^2 .$$

Since by assumption $\overline{K_i}$ is compact and the basis e^i is smooth, we have that $\sup_l \left|\nabla e^l\right|^2 \leq C_j$ for some C_j dependent on K_j. Thus,

$$\|\nabla(u_n)_i - v_i\|^2_{\mathrm{L}^2(K_j)} \to 0.$$

Thus, we have that $(u_n)_i \to 0$ and $\nabla(u_n)_i \to v_i$ in $\mathrm{L}^2(K_j)$, and by the closability of ∇ on functions, we have that $v_i = 0$ almost everywhere in K_j. Thus, $v = 0$ almost everywhere in K_j and consequently $v = 0$ almost everywhere in $\mathcal{M}$. This proves that ∇ is a closable operator. □

As a consequence of this proposition, we can define the Sobolev space $\mathrm{W}^{1,2}(\mathcal{V})$ as the completion of functions $\mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$ in the graph norm of ∇. The closure of ∇ is then denoted by the same symbol, namely $\nabla : \mathrm{W}^{1,2}(\mathcal{V}) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$.

The higher order Sobolev spaces $\mathrm{W}^{k,2}(\mathcal{V})$ are defined as subsets of $\mathrm{W}^{1,2}(\mathcal{V})$ in the usual manner. To keep some accord with tradition, when we consider the situation $\mathcal{V} = \mathcal{M} \times \mathbb{C}$, we write $\mathrm{L}^2(\mathcal{M})$ in place of $\mathrm{L}^2(\mathcal{V})$ and similarly for the function spaces C^k, C^k_c and $\mathrm{W}^{k,2}$.

We also highlight that we have the following important density result.

Proposition 2.3 $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ *is dense in* $\mathrm{W}^{1,2}(\mathcal{V})$.

Proof Since we assume that $\mathcal{M}$ is complete, the Hopf–Rinow theorem guarantees that closed balls are compact. Fix some centre $x \in \mathcal{M}$ and consider closed balls $\overline{B(x,r)}$. For such balls, we can find $\eta_r \in \mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ such that spt η_r is compact, $0 \le \eta \le 1$, $\eta_r \equiv 1$ on $B(x,r)$ and $\sup |\nabla \eta_r| \to 0$ as $r \to \infty$. Now, for any $u \in \mathrm{W}^{1,2}(\mathcal{V})$, $\eta_r u$ is compactly supported. Also,

$$\nabla(\eta_r u) = \nabla \eta_r \otimes u + \eta_r \nabla u$$

and

$$\begin{aligned} \|\nabla(\eta_r u) - \nabla u\| &\le \|\nabla \eta_r \otimes u\| + \|\eta_r \nabla u - \nabla u\| \\ &\le \sup_{x \in \mathcal{M}} |\nabla \eta_r| \, \|u\| + \|\eta_r \nabla u - \nabla u\| \to 0 \end{aligned}$$

as $r \to \infty$. Thus, it suffices to consider $u \in \mathrm{W}^{1,2}(\mathcal{V})$ with spt u compact. But for each such u, an easy mollification argument in each local trivialization will yield a sequence $u_n \in \mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ such that $\|u_n - u\|_{\mathrm{W}^{1,2}} \to 0$. □

Remark 2.4 We remark that this proof does not generalise to higher order Sobolev spaces. In fact, even for $k = 2$, the best known result for the density of $\mathrm{C}^\infty_\mathrm{c}(\mathcal{M})$ in $\mathrm{W}^{2,2}(\mathcal{M})$ is under uniform lower bounds on injectivity radius along with uniform lower bounds on Ricci curvature. See §3 of [10] and in particular, Proposition 3.3 in [10].

For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{M})$ (i.e., a function), we define the *average* of f on some measurable subset $A \subset \mathcal{M}$ with $0 < \mu(A) < \infty$ by $f_A = \fint_A f \, d\mu = \frac{1}{\mu(A)} \int_A f \, d\mu$.

In what follows, we assume that $\mathcal{M}$ satisfies ($\mathrm{E}_{\mathrm{loc}}$) unless stated otherwise.

2.3 Generalised Bounded Geometry

The harmonic analytic techniques we employ in the main proof, along with some of the assumptions we make on the operators under consideration, require us to capture a uniform locally Euclidean structure in the underlying vector bundle. The concept which we describe here is motivated by the fact that injectivity radius bounds on a manifold, coupled with appropriate curvature bounds, give *bounded geometry* on the (p, q) tensor bundle. It provides us with the framework for applying a dyadic decomposition in order to construct a system of global coordinates (no longer smooth), thus allowing us to define key quantities and to construct proofs.

Recalling that $\mathcal{V}$ is equipped with a metric h, we make the following definition.

Definition 2.5 (*Generalised Bounded Geometry*) Suppose there exists $\rho > 0$, $C \geq 1$, such that for each $x \in \mathcal{M}$, there exists a local trivialization $\psi_x : B(x, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x, \rho))$ satisfying

$$C^{-1} I \leq \mathrm{h} \leq C I$$

in the basis $\left\{e^i = \psi_x(y, \tilde{e}^i)\right\}$, where $\left\{\tilde{e}^i\right\}$ is the standard orthonormal basis for $\mathbb{C}^N$. Then, we say that $\mathcal{V}$ has *generalised bounded geometry* or *GBG*. We call ρ the *GBG radius*, and local trivializations ψ_x the *GBG charts*.

Since we can always take ρ to be as small as we like, we assume that $\rho \leq 5$.

For the convenience of the reader we quote Proposition 4.2 from [16].

Theorem 2.6 (Existence of a truncated dyadic structure) *There exists a countable collection of open subsets* $\left\{Q_\alpha^k \subset \mathcal{M} : \alpha \in I_k,\ k \in \mathbb{Z}_+\right\}$, $z_\alpha^k \in Q_\alpha^k$ *(called the* centre *of* Q_α^k*), index sets* I_k *(possibly finite), and constants* $\delta \in (0, 1)$, $a_0 > 0$, $\eta > 0$ *and* $C_1, C_2 < \infty$ *satisfying:*

(i) *for all* $k \in \mathbb{Z}$, $\mu(\mathcal{M} \setminus \cup_\alpha Q_\alpha^k) = 0$,
(ii) *if* $l \geq k$, *then either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$,
(iii) *for each* (k, α) *and each* $l < k$ *there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$,
(iv) $\operatorname{diam} Q_\alpha^k < C_1 \delta^k$,
(v) $B(z_\alpha^k, a_0 \delta^k) \subset Q_\alpha^k$,
(vi) *for all* k, α *and for all* $t > 0$, $\mu\left\{x \in Q_\alpha^k : d(x, \mathcal{M} \setminus Q_\alpha^k) \leq t\delta^k\right\} \leq C_2 t^\eta \mu(Q_\alpha^k)$.

Remark 2.7 This theorem was first proved by Christ in [9] for $k \in \mathbb{Z}$, for doubling measure metric spaces.

Throughout this paper we fix $\mathrm{J} \in \mathbb{N}$ such that $C_1 \delta^{\mathrm{J}} \leq \frac{\rho}{5}$. For $j \geq \mathrm{J}$, we write $\mathscr{Q}^j$ to denote the collection of all cubes Q_α^j, and set $\mathscr{Q} = \cup_{j \geq \mathrm{J}} \mathscr{Q}^j$.

We call $\mathrm{t_S} = \delta^{\mathrm{J}}$ the *scale*. Furthermore, when $t \leq \mathrm{t_S}$, set $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$.

The existence of a truncated dyadic structure allows us to formulate the following system of coordinates on $\mathcal{V}$.

Definition 2.8 (*GBG coordinates of* $\mathcal{V}$) Let x_Q denote the *centre* of each $Q \in \mathscr{Q}^{\mathrm{J}}$. Then, we call the system of coordinates

$$\mathscr{C} = \left\{\psi : B(x_Q, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_Q, \rho)) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}}\right\}$$

the *GBG coordinates*. We call the set

$$\mathscr{C}_{\mathrm{J}} = \left\{ \psi_Q = \psi|_Q : Q \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(Q) \text{ s.t. } Q \in \mathscr{Q}^{\mathrm{J}} \right\}$$

the *dyadic GBG coordinates*. For any cube $Q \in \mathscr{Q}^j$, there is a unique cube $\widehat{Q} \in \mathscr{Q}^{\mathrm{J}}$ satisfying $Q \subset \widehat{Q}$ and we call this cube the *GBG cube of* Q. The *GBG coordinate system* of Q is then $\psi : B(x_{\widehat{Q}}, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_{\widehat{Q}}, \rho)$.

Remark 2.9 Since $\cup \mathscr{Q}^J$ is an almost-everywhere covering of $\mathcal{M}$, the GBG coordinate system of $\mathcal{V}$ defines an almost-everywhere smooth global trivialization of the vector bundle.

We emphasise that throughout this paper, for any cube $Q \in \mathscr{Q}^j$, we denote its GBG cube by $\widehat{Q}$.

A first and important consequence of the GBG coordinates is that it allows us to define the notion of a *constant* section (locally, in the eyes of our GBG coordinates and dyadic decomposition). More precisely, for $x \in Q$, given $w = w_i\ e^i(x) \in \mathcal{V}_x$, where the $\{e^i\}$ is the GBG coordinates of Q, we define $\omega(y) = w_i\ e^i(y)$ whenever $y \in B(x_{\widehat{Q}}, \rho)$ and $\omega(y) = 0$ otherwise. This is an extension of $w \in \mathcal{V}_x$ to the entire GBG coordinate ball $B(x_{\widehat{Q}}, \rho)$. Thus, we call ω the GBG constant section associated to w. Note that $\omega \in \mathrm{L}^\infty(\mathcal{V})$. We remark that this notion is crucial in later parts of the paper. Next, we can define a notion of *cube integration* in the following way. Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$. Then, for any $Q \in \mathscr{Q}$, we can write

$$\int_Q u\ d\mu = \left(\int_Q u_i\ d\mu \right) e^i$$

where $u = u_i e^i$ in the GBG coordinates of Q. Note that $\int_Q u\ d\mu$ is a function from Q to $\mathcal{V}$.

Pursuing a similar vein of thought, we define a *cube average*. Given a cube $Q \in \mathscr{Q}$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$, define $u_Q \in \mathrm{L}^\infty(\mathcal{V})$ by

$$u_Q(y) = \begin{cases} \frac{1}{\mu(Q)} \int_Q u\ d\mu & \text{when } y \in B(x_{\widehat{Q}}, \rho), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.10 We remark that in the average of a function over a general measurable set of positive measure, we do not perform a cutoff as we do here. However, whenever we write u_Q with Q being a cube (even for a function), we shall always assume this definition.

Lastly, we define the *dyadic averaging operator*. For each $t > 0$, define the operator $\mathcal{A}_t : \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V}) \to \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ by

$$\mathcal{A}_t u(x) = \frac{1}{\mu(Q)} \int_Q u(y)\ d\mu(y)$$

whenever $x \in Q \in \mathscr{Q}_t$. We remark that $\mathcal{A}_t : \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$ is a bounded operator with norm bounded uniformly for all $t \leq \mathrm{t_S}$ by a bound depending on the constant C arising in the GBG criterion.

2.4 Functional Calculi of Sectorial Operators

Of fundamental importance to the setup and proof that we present in this paper, is the functional calculus of certain operators. In this section, we introduce the key type of operators that we shall concern ourselves with, and, for the convenience of the reader, recall some facts about functional calculi of these operators. A fuller treatment of this material can be found in [1]. A local version of this theory can be found in [15].

Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces. We say that a linear map $T : \mathcal{D}(T) \subset \mathscr{B}_1 \to \mathscr{B}_2$ is an *operator* with domain $\mathcal{D}(T)$. The range of T is denoted by $\mathcal{R}(T)$ and the null space by $\mathcal{N}(T)$. We say that such an operator is *closed* if the graph $\mathscr{G}(T) = \{(u, Tu) : u \in \mathcal{D}(T)\}$ is closed in the product topology of $\mathscr{B}_1 \times \mathscr{B}_2$. The operator T is *bounded* if $\|Tu\|_{\mathscr{B}_2} \leq \|u\|_{\mathscr{B}_1}$ for all $u \in \mathcal{D}(T) = \mathscr{B}_1$. If $\mathscr{B}_1 = \mathscr{B}_2$ then the *resolvent* set $\rho(T)$ consists of all $\zeta \in \mathbb{C}$ such that $\zeta I - T$ is one-one, onto and has a bounded inverse. The spectrum is then $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

For $0 \leq \omega < \frac{\pi}{2}$, define the bisector $S_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \text{ or } |\arg(-\zeta)| \leq \omega \text{ or } \zeta = 0\}$. The open bisector is then defined as $S_\omega^o = \{\zeta \in \mathbb{C} : |\arg \zeta| < \omega \text{ or } |\arg(-\zeta)| < \omega,\ \zeta \neq 0\}$. An operator T in a Banach space $\mathscr{B}$ is *ω-bisectorial* if it is closed, $\sigma(T) \subset S_\omega$, and the following *resolvent bounds* hold: for each $\omega < \mu < \frac{\pi}{2}$, there exists C_μ such that for all $\zeta \in \mathbb{C} \setminus S_\mu$, we have $|\zeta| \left\|(\zeta I - T)^{-1}\right\| \leq C_\mu$. Depending on context, we refer to such operators simply as bisectorial and ω is then the *angle of bisectoriality*.

Now fix $0 \leq \omega < \mu < \frac{\pi}{2}$ and assume that T is an ω-bisectorial operator in a Hilbert space $\mathscr{H}$. We let $\Psi(S_\mu^o)$ denote the space of all holomorphic functions $\psi : S_\mu^o \to \mathbb{C}$ for which

$$|\psi(\zeta)| \lesssim \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}$$

for some $\alpha > 0$. For such functions, we define a functional calculus similar to the Riesz–Dunford functional calculus by

$$\psi(T) = \frac{1}{2\pi\imath} \oint_\gamma \psi(\zeta)(\zeta I - T)^{-1}\, d\zeta$$

where γ is a contour in S_μ^o enveloping S_ω parameterised anti-clockwise, and the integral is defined via Riemann sums. This integral converges absolutely as a consequence of the decay of ψ, coupled with the resolvent bounds of T. The operator $\psi(T)$ is bounded. We say that T has a *bounded holomorphic functional calculus* if there exists $C > 0$ such that $\|\psi(T)\| \leq C \|\psi\|_\infty$ for all $\psi \in \Psi(S_\mu^o)$.

Now suppose that $\mathscr{H}$ is a Hilbert space, and that $T : \mathcal{D}(T) \subset \mathscr{H} \to \mathscr{H}$ is ω-bisectorial. In this case $\mathscr{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$ where the direct sum is typically non-orthogonal. Let $\mathbf{P}_{\mathcal{N}(T)}$ denote the projection of $\mathscr{H}$ onto $\mathcal{N}(T)$ which is 0 on $\mathcal{R}(T)$.

Let $\mathrm{Hol}^\infty(S^o_\mu)$ denote the space of all bounded functions $f : S^o_\mu \cup \{0\} \to \mathbb{C}$ which are holomorphic on S^o_μ. For such a function f, there exists a uniformly bounded sequence (ψ_n) of functions in $\Psi(S^o_\mu)$ which converges to $f|_{S^o_\mu}$ in the compact-open topology over S^o_μ. If T has a bounded holomorphic functional calculus, and $u \in \mathscr{H}$, then the limit $\lim_{n\to\infty} \psi_n(T)u$ exists in $\mathscr{H}$, and we define

$$f(T)u = f(0)\,\mathbf{P}_{\mathcal{N}(T)}\,u + \lim_{n\to\infty} \psi_n(T)u.$$

This defines a bounded operator independent of the sequence (ψ_n), and we have the bound $\|f(T)\| \le C\,\|f\|_\infty$ for some finite C.

In particular the functions χ^+, χ^- and $\mathrm{sgn} = \chi^+ - \chi^-$ belong to $\mathrm{Hol}^\infty(S^o_\mu)$, where $\chi^+(\zeta) = 1$ when $\mathrm{Re}(\zeta) > 0$ and 0 otherwise, and $\chi^-(\zeta) = 1$ when $\mathrm{Re}(\zeta) < 0$ and 0 otherwise. Therefore, if T has a bounded functional calculus in $\mathscr{H}$, then $\chi^\pm(T)$ are bounded projections, and $\mathscr{H} = \mathcal{N}(T) \oplus \mathcal{R}(\chi^+(T)) \oplus \mathcal{R}(\chi^-(T))$. The bounded operator $\mathrm{sgn}(T)$ equals 0 on $\mathcal{N}(T)$, equals 1 on $\mathcal{R}(\chi^+(T))$, and equals -1 on $\mathcal{R}(\chi^-(T))$.

3 Hypotheses

Though our primary goal in this paper is to provide, under an appropriate set of assumptions, a positive answer to the Kato square root problem on vector bundles, we shall pursue a slightly more general setup in the footsteps of [6]. The purpose of this section is to describe this setup as a list of hypotheses, and in later sections, demonstrate how to apply tools from harmonic analysis to prove quadratic estimates under these hypotheses. The Kato square root estimate on vector bundles (and (p, q) tensors) will then be obtained by showing that these hypotheses are satisfied under our geometric assumptions.

Let $\mathscr{H}$ be a Hilbert space and let $\langle\cdot,\cdot\rangle$ denote its inner product. Following [6] and [16], we make the following operator theoretic assumptions.

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is closed, densely defined and *nilpotent*.

(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\mathrm{Re}\,\langle B_1 u, u\rangle \ge \kappa_1\,\|u\|^2 \qquad \text{whenever } u \in \mathcal{R}(\Gamma^*),$$
$$\mathrm{Re}\,\langle B_2 u, u\rangle \ge \kappa_2\,\|u\|^2 \qquad \text{whenever } u \in \mathcal{R}(\Gamma),$$

where $\kappa_1, \kappa_2 > 0$ are constants.

(H3) The operators B_1, B_2 satisfy $B_1B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.

Furthermore, define $\Gamma^*_B = B_1\Gamma^*B_2$, $\Pi_B = \Gamma + \Gamma^*_B$ and $\Pi = \Gamma + \Gamma^*$. The operator Π is self-adjoint, and Π_B is bisectorial, and thus we define the following associated bounded operators:

$$R^B_t = (1 + it\Pi_B)^{-1},\ P^B_t = (1 + t^2\Pi^2_B)^{-1},$$
$$Q^B_t = t\Pi_B(1 + t^2\Pi^2_B)^{-1},\ \Theta^B_t = t\Gamma^*_B(1 + t^2\Pi^2_B)^{-1}.$$

We write R_t, P_t, Q_t, Θ_t on setting $B_1 = B_2 = 1$. The full implications of these assumptions are listed in §4 of [6].

The following additional assumptions are mild, particularly since we wish to apply the theory to differential operators. They are essentially the same as the assumptions made in [6] and [16], but modified for vector bundles. We remark that in (H5) below, $\mathcal{L}(\mathcal{V})$ denotes the vector bundle of all bounded linear maps $T_x : \mathcal{V}_x \to \mathcal{V}_x$ for each $x \in \mathcal{M}$ (where $\mathcal{V}_x$ is the fiber over x). The boundedness is with respect to the metric h_x on the fiber $\mathcal{V}_x$. Note that the local trivializations for this bundle are canonically induced by the local trivializations of $\mathcal{V}$, and in each local trivialization the T_x can be represented as usual by an $N \times N$ matrix.

(H4) The Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$, where $\mathcal{V}$ is a smooth vector bundle with smooth metric h over a smooth, complete Riemannian manifold $\mathcal{M}$ with smooth metric g. Furthermore, $\mathcal{V}$ satisfies the GBG criterion and $\mathcal{M}$ satisfies $(\mathrm{E}_{\mathrm{loc}})$.

(H5) The operators B_1, B_2 are multiplication operators, i.e., there exist $B_i \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$.

(H6) The operator Γ is a first order differential operator. That is, there exists $C_\Gamma > 0$ such that whenever $\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathcal{M})$, we have that $\eta\mathcal{D}(\Gamma) \subset \mathcal{D}(\Gamma)$ and $\mathrm{M}_\eta u(x) = [\Gamma, \eta(x) I]\, u(x)$ is a multiplication operator satisfying

$$\left|\mathrm{M}_\eta u(x)\right| \leq C_\Gamma\, |\nabla\eta|_{\mathrm{T}^*M}\, |u(x)|$$

for all $u \in \mathcal{D}(\Gamma)$ and almost all $x \in \mathcal{M}$.

Remark 3.1 We note as in [16] that (H6) implies the same hypothesis with Γ replaced by either Γ^* or Π.

It is in the following two hypotheses that we make a more substantial departure from [6] and [16]. A significant difference is that we have used the dyadic structure, rather than balls, in their formulation. This cannot be avoided since we are forced to employ quantities which are defined through GBG coordinates.

Recalling the definition of a cube integral in Sect. 2.3, we formulate the following *cancellation* hypothesis.

(H7) There exists $c > 0$ such that for all $Q \in \mathscr{Q}$,

$$\left|\int_Q \Gamma u \, d\mu\right| \leq c\mu(Q)^{\frac{1}{2}}\, \|u\| \quad \text{and} \quad \left|\int_Q \Gamma^* v \, d\mu\right| \leq c\mu(Q)^{\frac{1}{2}}\, \|v\|$$

for all $u \in \mathcal{D}(\Gamma),\ v \in \mathcal{D}(\Gamma^*)$ satisfying $\operatorname{spt} u,\ \operatorname{spt} v \subset Q$.

Lastly, we make the following abstract Poincaré and coercivity hypotheses on the bundle (recalling that $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$).

(H8) There exist $C_P, C_C, c, \tilde{c} > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathscr{N})$, where $\mathscr{N}$ is a normed bundle over $\mathcal{M}$ with norm $|\cdot|_{\mathscr{N}}$ and $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$, satisfying:

-1 (Dyadic Poincaré)

$$\int_B \left|u - u_Q\right|^2 \, d\mu \leq C_P(1 + r^\kappa e^{\lambda c r t})(rt)^2 \int_{\tilde{c}B} \left(|\Xi u|^2_{\mathcal{N}} + |u|^2\right) \, d\mu$$

for all balls $B = B(x_Q, rt)$ with $r \geq C_1/\delta$ where $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$, and

-2 (Coercivity)

$$\|\Xi u\|^2_{\mathrm{L}^2(\mathcal{N})} + \|u\|^2_{\mathrm{L}^2(\mathcal{V})} \leq C_C \, \|\Pi u\|^2_{\mathrm{L}^2(\mathcal{V})} \, .$$

for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

We justify calling this an abstract Poincaré inequality for two reasons. First, the inequality is for sections on a vector bundle, not just for scalar-valued functions. Second, in the usual Poincaré inequality, the operator Ξ is simply ∇. We allow ourselves other possibilities in choosing the operator Ξ here, because it can be useful in the situation of a vector bundle that is in general non-flat and non-trivial.

4 Main Results

4.1 Bounded Holomorphic Functional Calculi and Kato Square Root Type Estimates

In this section we first illustrate how to reduce the main quadratic estimate to a simpler, local quadratic estimate. Then, we present the main theorem of this paper and illustrate its main corollary; a Kato square root type estimate.

We begin with the following adaptation of Proposition 5.2 in [16].

Proposition 4.1 *Suppose that* (Γ, B_1, B_2) *satisfies the hypotheses (H1)–(H3) along with* $\|u\| \lesssim \|\Pi u\|$ *for* $u \in \mathcal{R}(\Pi)$*, and that there exists* $c > 0$ *and* $t_0 > 0$ *such that*

$$\int_0^{t_0} \left\| \Theta_t^B P_t u \right\|^2 \frac{dt}{t} \leq c \, \|u\|^2 \tag{Q1}$$

for all $u \in \mathcal{R}(\Gamma)$*, together with three similar estimates obtained by replacing* (Γ, B_1, B_2) *by* (Γ^*, B_2, B_1)*,* $(\Gamma^*, {B_2}^*, {B_1}^*)$ *and* $(\Gamma, {B_1}^*, {B_2}^*)$*. Then,* Π_B *satisfies*

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$*. Thus,* Π_B *has a bounded holomorphic functional calculus.*

Proof The proof is similar to the proof of Proposition 5.2 in [16]. The assumption that $\|u\| \lesssim \|\Pi u\|$ allows us to handle the integral from t_0 to ∞. □

We use the entire list of hypotheses (H1)–(H8) in Sect. 3 to show that the assumptions of Proposition 4.1 are satisfied. Thus, this yields the main theorem of this paper.

Theorem 4.2 *Suppose that $\mathcal{M}$, $\mathcal{V}$ and (Γ, B_1, B_2) satisfy (H1)–(H8). Then, Π_B satisfies the quadratic estimate*

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{V})$ and hence has a bounded holomorphic functional calculus.

We defer the proof to Sect. 8.

In particular, the projections $\chi^{\pm}(\Pi_B)$ are bounded, as is the operator $\mathrm{sgn}(\Pi_B) = \chi^+(\Pi_B) - \chi^-(\Pi_B)$. Thus we have the following corollary.

Corollary 4.3 (Kato square root type estimate) *Under the hypotheses of Theorem 4.2,*

(i) *there is a spectral decomposition*

$$\mathrm{L}^2(\mathcal{V}) = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$$

where $E_B^{\pm} = \mathcal{R}(\chi^{\pm}(\Pi_B))$ (the direct sum is in general non-orthogonal), and

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B{}^2})$ *with*

$$\|\Gamma u\| + \left\|\Gamma_B^* u\right\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B{}^2} u\|$$

for all $u \in \mathcal{D}(\Pi_B)$.

To prove part (ii), we use the identities $\Pi_B u = \mathrm{sgn}(\Pi_B)\sqrt{\Pi_B{}^2} u$ and $\sqrt{\Pi_B{}^2} u = \mathrm{sgn}(\Pi_B)\Pi_B u$ for all $u \in \mathcal{D}(\Pi_B)$, together with the bound on $\mathrm{sgn}(\Pi_B)$.

4.2 Stability of Perturbations

It is a consequence of the fact that the estimates in Theorem 4.2 hold for a class of operators with complex measurable coefficients B_i, that operators such as $\mathrm{sgn}(\Pi_B)$ are also stable under small perturbations in B.

We provide the following adaption of Theorem 6.4 in [6] with a minor modification. In the following theorem, $\mathscr{H}$ denotes an abstract Hilbert space. We say that a family $\{T(\zeta)\}_{\zeta \in U}$ of ω-bisectorial operators has a *uniformly bounded holomorphic functional calculus* if each operator $T(\zeta)$ has a bounded holomorphic functional calculus on the same sector $S_\mu^o \cup \{0\}$ with a bound which is uniform in $\zeta \in U$.

Theorem 4.4 *(*Holomorphic dependence*) Let $U \subset \mathbb{C}$ be an open set and $B_1, B_2 : U \to \mathcal{L}(\mathscr{H})$ be holomorphic and suppose that $(\Gamma, B_1(\zeta), B_2(\zeta))$ satisfy (H1)–(H3) uniformly for all $\zeta \in U$. Suppose further that $\Pi_{B(\zeta)}$ has a uniformly bounded holomorphic functional calculus on $S_\mu^o \cup \{0\}$ for some $\omega < \mu < \frac{\pi}{2}$ (where ω is the angle of sectoriality). Let $f \in \mathrm{Hol}^\infty(S_\mu^o)$. Then the map $\zeta \mapsto f(\Pi_{B(\zeta)})$ is holomorphic on U.*

Proof This claim is proved in a similar way to the first part of Theorem 6.4 in [6], with the exception that instead of invoking Theorem 2.10 in [6], we note that the uniformly bounded holomorphic functional calculus assumption is sufficient. □

Next, consider the situation of $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$. Adapting the construction of [6], we define the following Hilbert space

$$\mathcal{K} = \mathrm{L}^2\left(\mathcal{M} \times (0, \infty), \mathcal{V}; \frac{d\mu dt}{t}\right).$$

Then, for $\psi \in \Psi(S^o_\mu)$, $t > 0$ and almost all $x \in \mathcal{M}$, we define the operator $\mathcal{S}_{B(\zeta)}(\psi) : \mathscr{H} \to \mathcal{K}$ by $(\mathcal{S}_{B(\zeta)}(\psi)u)(x, t) = \psi(t\Pi_{B(\zeta)})u(x)$.

Theorem 4.5 *Under the hypothesis of Theorem 4.4, and the additional assumption that $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$, whenever $\omega < \mu < \frac{\pi}{2}$ (where ω is the angle of sectoriality), the map $\zeta \mapsto \mathcal{S}_{B(\zeta)}(\psi)$ is holomorphic on U for all $\psi \in \Psi(S^o_\mu)$.*

Proof We note that our choice of $\mathcal{K}$ is an adequate replacement to $\mathcal{K}$ in the proof of Theorem 6.4 in [6]. Also, for $t > 0$, the function $\psi_t(\zeta) = \psi(t\zeta) \in \Psi(S^o_\mu)$ and $\|\psi_t\|_\infty = \|\psi\|_\infty$. Therefore, the uniformly bounded holomorphic functional calculus assumption holds uniformly in $t > 0$ and is again an adequate substitution to run the rest of the argument of the proof of Theorem 6.4 in [6]. □

Corollary 4.6 *Let $\mathscr{H}$, Γ, B_1, B_2, κ_1, κ_2 satisfy (H1)–(H8) and take $\eta_i < \kappa_i$. Set $0 < \hat{\omega}_i < \frac{\pi}{2}$ by $\cos \hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_\infty + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Let $A_i \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$ satisfy*

(i) $\|A_i\|_\infty \leq \eta_i$,
(ii) $A_1 A_2 \mathcal{R}(\Gamma)$, $B_1 A_2 \mathcal{R}(\Gamma)$, $A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$, *and*
(iii) $A_2 A_1 \mathcal{R}(\Gamma^*)$, $B_2 A_1 \mathcal{R}(\Gamma^*)$, $A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Letting $\hat{\omega} < \mu < \frac{\pi}{2}$, we have:

(i) *for all* $f \in \mathrm{Hol}^\infty(S^o_\mu)$,

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\, \|f\|_\infty\,, \qquad \textit{and}$$

(ii) *for all* $\psi \in \Psi(S^o_\mu)$,

$$\int_0^\infty \|\psi(t\Pi_B)u - \psi(t\Pi_{B+A})u\|^2 \frac{dt}{t} \lesssim (\|A_1\|^2_\infty + \|A_2\|^2_\infty)\, \|u\|\,,$$

whenever $u \in \mathscr{H}$.

The implicit constants depend on (H1)–(H8) and η_i.

Proof The argument proceeds in a similar way to the proof of Theorem 6.5 in [6]. The conditions on A_i guarantee that $\tilde{B}_i(\zeta) = (B_i + \zeta A_i)$ satisfies (H3). This is, in fact, a necessary amendment to the original proof of Theorem 6.5 in [6]. □

5 Poincaré Inequalities

In this short section we show that, under appropriate geometric conditions, we can bootstrap the Poincaré inequality on functions to the dyadic Poincaré inequality on the bundle. As in [16], we make the following definition.

Definition 5.1 (*Local Poincaré inequality*) We say that $\mathcal{M}$ satisfies a *local Poincaré inequality* if there exists $c \geq 1$ such that for all $f \in \mathrm{W}^{1,2}(\mathcal{M})$,

$$\|f - f_B\|_{\mathrm{L}^2(B)} \leq c \ \mathrm{rad}(B) \, \|f\|_{\mathrm{W}^{1,2}(B)} \tag{$\mathrm{P_{loc}}$}$$

for all balls B in $\mathcal{M}$ such that $\mathrm{rad}(B) \leq 1$.

Remark 5.2 Note that we allow the Sobolev norm $\|\cdot\|_{\mathrm{W}^{1,2}(B)}$ over the ball on the right of the inequality, rather than simply $\|\nabla \cdot\|$.

The following proposition then illustrates that under appropriate gradient bounds on the GBG coordinate basis, we can obtain a dyadic Poincaré inequality in the bundle.

Proposition 5.3 *Suppose that $\mathcal{M}$ satisfies both (E_{loc}) and (P_{loc}). Furthermore, suppose that that there exists $C_G > 0$ such that in each GBG chart with basis denoted by $\{e^i\}$ we have $|\nabla e^i| \leq C_G$ for each i. Then, for all $u \in \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$, $t \leq \mathrm{t_S}$, $Q \in \mathscr{Q}_t$, and $r \geq \frac{C_1}{\delta}$,*

$$\int_B |u - u_Q|^2 \, d\mu \lesssim (1 + e^{\lambda r t})(rt)^2 \int_B (|\nabla u|^2 + |u|^2) \, d\mu$$

where $B = B(x_Q, rt)$.

Proof First, consider the case $(rt) \geq \frac{\rho}{5}$. Then

$$\|u - u_Q\|^2_{\mathrm{L}^2(B)} \lesssim \|u\|^2_{\mathrm{L}^2(B)} + \|u_Q\|^2_{\mathrm{L}^2(B)} .$$

Recalling that $\widehat{Q}$ is the GBG cube of Q,

$$\begin{aligned}
\int_B |u_Q|^2 \, d\mu &= \int_B \left| \left(\fint_Q u_i \, d\mu \right) \chi_{B(x_{\widehat{Q}},\rho)} e^i \right|^2 \, d\mu \\
&\simeq \sum_i \int_B \left| \fint_Q u_i \, d\mu \right|^2 \chi_{B(x_{\widehat{Q}},\rho)} d\mu \\
&\leq \sum_i \int_B \left(\fint_Q |u_i|^2 \, d\mu \right) d\mu \leq \frac{\mu(B)}{\mu(Q)} \int_B |u|^2 \, d\mu \\
&\lesssim (1 + r^\kappa e^{\lambda r t}) \|u\|^2_{\mathrm{L}^2(B)} .
\end{aligned}$$

Thus, $\|u - u_Q\|^2_{\mathrm{L}^2(B)} \lesssim (1 + r^\kappa e^{\lambda r t})(rt)^2 \|u\|^2_{\mathrm{L}^2(B)}$ since $rt \geq \frac{\rho}{5}$.

Next, suppose that $(rt) < \frac{\rho}{5}$. It is easy to see that we have $B(x_Q, rt) \subset B(x_{\widehat{Q}}, \rho)$ and so,

$$\int_B |u - u_Q|^2 \, d\mu \simeq \sum_i \int_B \left| u_i - \left(\fint_Q u_i \, d\mu \right) \chi_{B(x_{\widehat{Q}},\rho)} \right|^2 d\mu$$
$$\leq \sum_i \int_B |u_i - (u_i)_B|^2 \, d\mu + \sum_i \int_B \left| (u_i)_B - \left(\fint_Q u_i \, d\mu \right) \right|^2 d\mu$$

For the first term, we invoke (P_{loc}) so that

$$\sum_i \int_B |u_i - (u_i)_B|^2 \, d\mu \lesssim (rt)^2 \int_B \left(\sum_i |\nabla u_i|^2 + |u|^2 \right) d\mu.$$

Now, for the second term,

$$\sum_i \int_B \left| (u_i)_B - \left(\fint_Q u_i \, d\mu \right) \right|^2 d\mu \leq \sum_i \frac{\mu(B)}{\mu(Q)} \int_Q |u_i - (u_i)_B|^2 \, d\mu$$
$$\lesssim (1 + r^\kappa e^{\lambda r t})(rt)^2 \int_B \left(\sum_i |\nabla u_i|^2 + |u|^2 \right) d\mu$$

Next, we note that $\nabla u = \nabla(u_i) \otimes e^i + u_i \otimes \nabla e^i$ and therefore, by the hypothesis $|\nabla e^i| \leq C_G$,

$$\sum_i |\nabla u_i|^2 \simeq \left| \nabla u_i \otimes e^i \right|^2 \leq |\nabla u|^2 + |u_i|^2 \left| \nabla e^i \right|^2 \lesssim |\nabla u|^2 + |u| \, .$$

The proof is complete by combining these estimates. □

6 Kato Square Root Estimates for Elliptic Operators

6.1 The Kato Square Root Problem on Vector Bundles

Here, we present the main applications of Theorem 4.2 to uniformly elliptic operators which arise naturally from a connection over a vector bundle. First, we describe a setup of operators which is a generalisation of §1 of [16] (and before that from [5]), making the necessary changes to facilitate the fact that we are working, in general, on a non-trivial bundle.

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. As discussed in Sect. 2, $\nabla : \mathrm{W}^{1,2}(\mathcal{V}) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ is a closed densely defined operator, and so has a well defined adjoint ∇^*, which we denote by $-\operatorname{div} : \mathcal{D}(\operatorname{div}) \subset \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$.

The reason for this notation is because when the connection ∇ and the metric h are compatible, then ∇^* has the form of a divergence in the weak sense of Proposition 6.1 below. First some notation.

For $v \in C^\infty(T^*\mathcal{M} \otimes \mathcal{V})$, define tr ∇v by contracting the first two indices of $\nabla v \in C^\infty(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \mathcal{V})$ over g, to yield a section tr $\nabla v \in C^\infty(\mathcal{V})$.

The connection ∇ and the metric h are *compatible* if the product rule

$$X(\mathrm{h}(Y, Z)) = \mathrm{h}(\nabla_X Y, Z) + \mathrm{h}(Y, \nabla_X Z)$$

is satisfied for every $X \in C^\infty(T\mathcal{M})$ and $Y, Z \in C^\infty(\mathcal{V})$. For such connection and metric pairs, we have div $= -\nabla^* = \mathrm{tr}\,\nabla$ in the following weak sense.

Proposition 6.1 *Suppose that the connection* ∇ *and the metric* h *are compatible. Then, for all* $T \in C_c^\infty(\mathcal{V})$ *and* $P \in C^\infty(T^*\mathcal{M} \otimes \mathcal{V})$,

$$\int_{\mathcal{M}} \langle \nabla T, P \rangle \, d\mu = \int_{\mathcal{M}} \langle T, -\mathrm{tr}\,\nabla P \rangle \, d\mu.$$

Proof Fix $x \in \mathcal{M}$ so that we can locally write $T = T_i e^i$ and $P = P_{kl} dx^k \otimes e^l$. Next, define the $\mathcal{V}$ "inner product" yielding a 1-form by $\langle T, P \rangle_{\mathcal{V}} = T_i P_{kl} \mathrm{h}(e^i, e^l) dx^k$.

Now, to make calculations easier, further assume that $\{x^i\}$ are normal coordinates at x. Then, for any $X = X_k dx^k$, we have that the divergence at x is div $X = \partial_k X_k$. Thus, at x,

$$\begin{aligned} \mathrm{div}\, \langle T, P \rangle_{\mathcal{V}} = \sum_k (\partial_k T_i) P_{kl} \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(\nabla_{\partial_k} e^i, e^l) \\ + \sum_k T_i (\partial_k P_{kl}) \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(e^i, \nabla_{\partial_k} e^l) \end{aligned}$$

by the compatibility of ∇ and h.

A calculation at x then shows that $\nabla T = \nabla(T_i e^i) = \partial_k T_i dx^k \otimes e^i + T_i dx^k \otimes \nabla_{\partial_k} e^i$. Also, tr $\nabla P = \sum_k \partial_k P_{kl} e^l + \sum_k P_{kl} \nabla e^l$, since we assumed normal coordinates at x, making $\nabla dx^k = 0$. Then, a direct calculation shows that

$$\begin{aligned} \langle \nabla T, P \rangle &= \sum_k (\partial_k T_i) P_{kl} \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(\nabla_{\partial_k} e^i, e^l), \quad \text{and} \\ \langle T, \mathrm{tr}\,\nabla P \rangle &= \sum_k T_i (\partial_k P_{kl}) \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(e^i, \nabla_{\partial_k} e^l). \end{aligned}$$

Thus, at x, div $\langle T, P \rangle_{\mathcal{V}} = \langle \nabla T, P \rangle + \langle T, \mathrm{tr}\,\nabla P \rangle$.

By the compactness of spt T, it is easy to see that spt $\langle T, P \rangle_{\mathcal{V}}$, spt $\langle T, \mathrm{tr}\,\nabla P \rangle$ and spt $\langle \nabla T, P \rangle$ are all compact. Thus, we integrate this equation over $\mathcal{M}$ and apply the divergence theorem to obtain the conclusion. □

We pause to introduce some notation. When $\mathcal{W}$, $\tilde{\mathcal{W}}$ are two vector bundles, define the new vector bundle $\mathcal{L}(\mathcal{W}, \tilde{\mathcal{W}})$ to mean the space of all maps $C : \mathcal{W} \to \tilde{\mathcal{W}}$ such that for each $x \in \mathcal{M}$, $C(x) \in \mathcal{L}(\mathcal{W}_x, \tilde{\mathcal{W}}_x)$. This is consistent with the previous notation since $\mathcal{L}(\mathcal{W}) = \mathcal{L}(\mathcal{W}, \mathcal{W})$.

With this notation in mind, let $A_{00} \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$, $A_{01} \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}, \mathcal{V}))$, $A_{10} \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}, \mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$, $A_{11} \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$. Then, define $A \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V} \oplus (\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})))$ by

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}.$$

Furthermore, let $a \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$. Set $B_1, B_2 : \mathscr{H} \to \mathscr{H}$ by

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Moreover, set

$$S = \begin{pmatrix} I \\ \nabla \end{pmatrix}, \; S^* = \begin{pmatrix} I & -\mathrm{div} \end{pmatrix}, \; \Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \text{ and } \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}$$

and define the following divergence form operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$ by

$$\mathrm{L}_A u = aS^*ASu = -a \;\mathrm{div}\,(A_{11}\nabla u) - a \;\mathrm{div}\,(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

We apply Theorem 4.2 to prove the Kato square root problem on vector bundles.

Theorem 6.2 (Kato square root problem for vector bundles) *Suppose that $\mathcal{M}$ satisfies (E_{loc}) and both $\mathcal{V}$ and $\mathrm{T}^*\mathcal{M}$ have generalised bounded geometry (so that they are equipped with GBG coordinate systems), and that*

(i) *$\mathcal{M}$ satisfies (P_{loc}),*
(ii) *the GBG charts for $\mathrm{T}^*\mathcal{M}$ are induced from coordinate systems on $\mathcal{M}$,*
(iii) *the connection ∇ and the metric h are compatible,*
(iv) *there exists $C > 0$ such that in each GBG chart $\{e^j\}$ for $\mathcal{V}$ and $\{dx^i\}$ for $\mathrm{T}^*\mathcal{M}$, we have that $|\nabla e^j|, |\partial_k \mathrm{h}^{ij}|, |\partial_k \mathrm{g}^{ij}| \leq C$ a.e.,*
(v) *there exist $\kappa_1, \kappa_2 > 0$ such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \textit{and} \qquad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{V})$ and $v \in \mathrm{W}^{1,2}(\mathcal{V})$, and
(vi) *we have that $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{V})$, and there exist $C' > 0$ such that*

$$\left\|\nabla^2 u\right\| \leq C' \,\|(I + \Delta)u\|$$

whenever $u \in \mathcal{D}(\Delta)$.

Then,

(i) Π_B *has a bounded holomorphic functional calculus, and*
(ii) $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$ *with* $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{V})$.

Proof of Theorem 6.2 (assuming Theorem 4.2) We show that (Γ, B_1, B_2) satisfy (H1)–(H8) of Sect. 3 in order to invoke Theorem 4.2.

Nilpotency of Γ is immediate. That Γ is densely defined and closed follows easily from Proposition 2.2. This settles (H1). Also, (H3) is an easy calculation and (H4)–(H5) are immediate. The fact that (H6) is satisfied is an immediate consequence of the Leibniz property of the connection. The conditions (i) and (ii) allow us to invoke Proposition 5.3, thus proving (H8)-1. It remains to demonstrate (H2), (H7), and (H8)-2 hold with $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \oplus \mathrm{L}^2(\mathcal{T}^{(0,2)}\mathcal{M} \otimes \mathcal{V})$ defined by $\Xi(u_1, u_2, u_3) = (\nabla u_1, \nabla u_2, \nabla u_3)$. The domain of Ξ is $\mathcal{D}(\Xi) = \mathrm{W}^{1,2}(\mathcal{V}) \oplus \mathrm{W}^{1,2}(\mathcal{V}) \oplus \mathrm{W}^{1,2}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$.

Fix $u \in \mathcal{R}(\Gamma^*)$. That is, $u = (S^*v, 0)$ for $v \in \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*M \otimes \mathcal{V})$. Thus,

$$\mathrm{Re}\, \langle B_1 u, u \rangle = \mathrm{Re} \left\langle a S^* v, S^* v \right\rangle \geq \kappa_1 \left\| S^* v \right\|^2 = \kappa_1 \|u\|^2 .$$

Next, let $u \in \mathcal{R}(\Gamma)$. Thus, $u = (0, Sv)$ for $v \in \mathrm{L}^2(\mathcal{V})$. Therefore,

$$\mathrm{Re}\, \langle B_2 u, u \rangle = \mathrm{Re}\, \langle ASv, Sv \rangle \geq \kappa_2 \|u\|^2$$

which settles (H2).

We verify (H7). First, let $u = (u_1, u_2, u_3) \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\, u \subset Q$ and $v = (v_1, v_2, v_3) \in \mathcal{D}(\Gamma^*)$ with $\mathrm{spt}\, v \subset Q$. Then, $\Gamma u = (0, Su_1) = (0, u_1, \nabla u_1)$, $\Gamma^* v = (S^*(v_2, v_3), 0) = (v_2 - \mathrm{div}\ v_3, 0, 0)$ and we have that

$$\left| \int_Q \Gamma u \, d\mu \right| = \left| \int_Q u_1 \, d\mu \right| + \left| \int_Q \nabla u_1 \, d\mu \right|$$

and

$$\left| \int_Q \Gamma^* v \, d\mu \right| = \left| \int_Q v_2 - \mathrm{div}\ v_3 \, d\mu \right| \leq \left| \int_Q v_2 \, d\mu \right| + \left| \int_Q \mathrm{div}\ v_3 \, d\mu \right| .$$

By Cauchy–Schwarz,

$$\left| \int_Q u_1 \, d\mu \right| \lesssim \mu(Q)^{\frac{1}{2}} \|u_1\| \leq \mu(Q)^{\frac{1}{2}} \|u\| ,$$

and by a similar computation,

$$\left| \int_Q v_2 \, d\mu \right| \lesssim \mu(Q)^{\frac{1}{2}} \|v\| .$$

To conveniently deal with the two remaining estimates, we omit the indices in u_1 and v_3 and note that it remains to prove

$$\text{(a)} \quad \left|\int_Q \nabla u\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \text{(b)} \quad \left|\int_Q \mathrm{div}\ v\right| \lesssim \mu(Q)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\nabla)$, $v \in \mathcal{D}(\mathrm{div}\,)$ with $\mathrm{spt}\, u,\ \mathrm{spt}\, v \subset Q$.

Before continuing, we remark that every function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ can be written as $f = f_i e^i = \mathrm{h}(f, \mathrm{h}_{ki} e^k) e^i$ in $B(x_{\widehat{Q}}, \rho)$. Here $\mathrm{h}_{ij} = \mathrm{h}(e_i, e_j)$, where $\{e_i\}$ is the dual basis of $\{e^i\}$, and we use the same notation h to denote the induced inner product on $\mathcal{V}^*$ by requiring that $\mathrm{h}_{ij}\mathrm{h}^{jk} = \delta_i^k$. We also remark that every function $F \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ can be written as $F = F_{ij}\, dx^i \otimes e^j = \mathrm{g} \otimes \mathrm{h}(F, \mathrm{g}_{ai} h_{bj}\, dx^a \otimes e^b)\, dx^i \otimes e^j$ in $B(x_{\widehat{Q}}, \rho)$ where $\mathrm{g}_{ij} = \mathrm{g}(\partial_i, \partial_j)$ on $\mathrm{T}\mathcal{M}$.

Turning to the proof of (a), let $u \in \mathrm{W}^{1,2}(\mathcal{V})$ with $\mathrm{spt}\, u \subset Q$. Choose $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$ such that $\mathrm{spt}\, \psi \subset B(x_{\widehat{Q}}, \rho)$ and $\psi = 1$ on Q, and extend $\psi \mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b$ to be zero outside of $B(x_{\widehat{Q}}, \rho)$. Then, by the above remark with $F = \nabla u$, we have the following identity on Q.

$$\begin{aligned}
\int_Q \nabla u &= \int_Q \mathrm{g} \otimes \mathrm{h}(\nabla u, \psi \mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b)\, d\mu\, dx^i \otimes e^j \\
&= \int_{\mathcal{M}} \mathrm{g} \otimes \mathrm{h}(\nabla u, \psi \mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b)\, d\mu\, dx^i \otimes e^j \\
&= \int_{\mathcal{M}} \mathrm{h}\left(u, -\mathrm{tr}\, \nabla(\psi \mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b)\right)\, d\mu\, dx^i \otimes e^j \\
&= \int_Q \mathrm{h}\left(u, -\mathrm{tr}\, \nabla(\mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b)\right)\, d\mu\, dx^i \otimes e^j \\
&= \int_Q -\mathrm{h}(u, \mathrm{g}_{ai} \mathrm{h}_{bj}\, \mathrm{tr}\, \nabla(dx^a \otimes e^b) + \mathrm{tr}\left((\nabla \mathrm{g}_{ai} \mathrm{h}_{bj}) \otimes dx^a \otimes e^b)\right)\, d\mu\, dx^i \otimes e^j.
\end{aligned}$$

We have used Proposition 6.1 (since $\mathrm{g}_{ai} \mathrm{h}_{bj}\, dx^a \otimes e^b$ are smooth), the product rule for ∇, and the linearity of tr. We note that by an easy calculation, we have $|\mathrm{tr}(X)| \lesssim |X|$ for all $x \in \mathcal{M}$ whenever $X \in \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Furthermore, the bound on the metric in each GBG chart implies bounds on $|\mathrm{h}_{ai}|$ and $|\mathrm{g}_{ai}|$, and the bounds in (iv) imply bounds on $|\partial_k \mathrm{h}_{ai}|$ and $\left|\partial_k \mathrm{g}_{bj}\right|$. Since we assumed the connection to be Levi-Civita, we can write ∇dx^a purely in terms of the Christoffel symbols, which in turn can be written in terms of g_{ij}, g^{ij} and $\partial_k \mathrm{g}_{ij}$. Also, $\left|e^b\right|$ and $|dx^a|$ are bounded by the GBG hypothesis, $\left|\nabla e^b\right|$ by (iv), and so we conclude that

$$\left|\mathrm{g}_{ai} \mathrm{h}_{bj}\, \mathrm{tr}\, \nabla(dx^a \otimes e^b) + \mathrm{tr}\left((\nabla \mathrm{g}_{ai} \mathrm{h}_{bj}) \otimes dx^a \otimes e^b\right)\right| \lesssim 1.$$

On combining these estimates, and applying the Cauchy–Schwarz inequality, we conclude that

$$\left|\int_Q \nabla u\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\|$$

as required.

To verify (b), let $v \in \mathcal{D}(\mathrm{div})$ with $\mathrm{spt}\, v \subset Q$, and apply the above remark with $f = \mathrm{div}\ v$ to obtain by a similar argument that

$$\int_Q \mathrm{div}\ v = \int_Q \mathrm{h}(\mathrm{div}\ v, \mathrm{h}_{ki}\ e^k)\ d\mu\ e^i = \int_Q \mathrm{g} \otimes \mathrm{h}(v, \nabla(\mathrm{h}_{ki}\ e^k))\ d\mu\ e^i.$$

Reasoning as before, we have that $\left|\nabla(\mathrm{h}_{ki}\ e^k)\right|$ is bounded and by the Cauchy–Schwarz inequality, we conclude that

$$\left|\int_Q \mathrm{div}\ v\right| \leq \mu(Q)^{\frac{1}{2}} \|v\|$$

as required. This completes the proof of (H7).

To show (H8) let $\Xi(u_1, u_2, u_3) = (\nabla u_1, \nabla u_2, \nabla u_3)$. Upon noting that

$$\left|\nabla(dx^i \otimes e^j)\right| \leq \left|\nabla dx^i\right| \left|e^j\right| + \left|\nabla e^j\right| \left|dx^i\right| \lesssim 1,$$

we apply Proposition 5.3 which proves (H8)-1.

It remains to show (H8)-2. Fix $v = (v_1, v_2, v_3) \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$. Thus, there is $u = (u_1, u_2, u_3) \in \mathcal{D}(\Pi)$ such that $v = \Pi u = (u_2 - \mathrm{div}\ u_3, u_1, \nabla u_1)$. A calculation the shows that

$$\|v\|^2 + \|\Xi v\|^2 = \|u_1\|^2 + 2\|\nabla u_1\|^2 + \left\|\nabla^2 u_1\right\| + \left(\|v_1\|^2 + \|\nabla v_1\|^2\right).$$

Also,

$$\|\Pi v\|^2 = \|(I + \Delta)u_1\|^2 + \|v_1\|^2 + \|\nabla v_1\|^2.$$

But note that

$$\|(I + \Delta)u_1\|^2 = \langle (I + \Delta)u_1, (I + \Delta)u_1 \rangle = \|u_1\|^2 + 2\|\nabla u_1\|^2 + \|\Delta u_1\|^2.$$

Combining these estimates with (iv), we have that $\|v\|^2 + \|\Xi v\|^2 \lesssim \|\Pi v\|$ which proves (H8)-2.

Now, by invoking Theorem 4.2, we conclude that Π_B has a bounded holomorphic functional calculus. By its Corollary 4.3, we obtain that $\left\|\sqrt{{\Pi_B}^2} v\right\| \simeq \|\Pi_B v\|$ for $v \in \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{{\Pi_B}^2})$. Fix $u \in \mathrm{W}^{1,2}(\mathcal{V})$. Then $v = (u, 0, 0) \in \mathcal{D}(\Pi_B)$ and

$$\left\|\sqrt{\mathrm{L}_A} u\right\| = \left\|\sqrt{{\Pi_B}^2} v\right\| \simeq \|\Pi_B v\| = \|u\|_{\mathrm{W}^{1,2}}$$

which finishes the proof. □

Remark 6.3 Instead of taking ∇ to be a connection, we could have considered a *sub-connection*, by which we mean a map $\nabla : \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ that is function linear and satisfies the Leibniz property on a sub-bundle $\mathcal{E}$ of $\mathrm{T}^*\mathcal{M}$, and vanishes outside of $\mathcal{E}$. This is related to the study of square roots of elliptic operators associated with sub-Laplacians on Lie groups [8]. However, it is not clear to us whether a sub-connection is automatically densely defined and closable.

6.2 Manifolds with Injectivity and Ricci Bounds

In this section, we apply Theorem 6.2 to establish the Kato square root estimates for manifolds which have injectivity radius bounds and Ricci bounds. Our approach is to show that under these conditions, there exist *harmonic* coordinates which gives us bounds on the metric and its derivatives. We then use these coordinates to show that the bundle of (p, q) tensors satisfy the GBG criterion.

We first present the following theorem which is really contained in the observation following Theorem 1.2 in [10].

Theorem 6.4 (Existence of global harmonic coordinates) *Suppose there exist* $\kappa, \eta > 0$ *such that* $\mathrm{inj}(M) \geq \kappa$ *and* $|\mathrm{Ric}| \leq \eta$*. Then, for any* $A > 1$ *and* $\alpha \in (0, 1)$*, there exists* $r_H = r_H(n, A, \alpha, \kappa, \eta) > 0$ *such that* $B(x, r_H)$ *corresponds to a coordinate system satisfying:*

(i) $A^{-1}\delta_{ij} \leq \mathrm{g}_{ij} \leq A\delta_{ij}$ *as bilinear forms and,*

(ii) $\sum_l r_H \sup_{y \in B(x,r_H)} \left|\partial_l \mathrm{g}_{ij}(y)\right| + \sum_l r_H^{1+\alpha} \sup_{y \neq z} \frac{\left|\partial_l \mathrm{g}_{ij}(z) - \partial_l \mathrm{g}_{ij}(y)\right|}{d(y,z)^\alpha} \leq A - 1.$

This immediately gives us the existence of GBG coordinates for tensor fields.

Corollary 6.5 (Existence of GBG coordinates for finite rank tensors) *Under the assumptions that* $\mathrm{inj}(M) \geq \kappa > 0$ *and* $|\mathrm{Ric}| \leq \eta$*, there exist GBG coordinates for* $\mathcal{T}^{(p,q)}\mathcal{M}$*. Furthermore, in each basis* $\{e^i\}$ *in each GBG coordinate system for* $\mathcal{T}^{(p,q)}\mathcal{M}$*, we have that* $\left|\nabla e^i\right| \leq C_{p,q}$*.*

Proof First, we note that the Ricci bounds imply that there exists $\eta \in \mathbb{R}$ such that that $\mathrm{Ric} \geq \eta \mathrm{g}$. Thus, as in the proof of Theorem 1.1 in [16], we conclude that $\mathcal{M}$ satisfies $(\mathrm{E}_{\mathrm{loc}})$.

Fix $A = 2$ and $\alpha = \frac{1}{2}$. The previous theorem guarantees the existence of harmonic coordinates for these choices. Thus, this yields GBG coordinates for $\mathrm{T}\mathcal{M}$ with $2^{-1} \leq \mathrm{g} \leq 2$.

It is an easy calculation to show that $G \simeq I$ implies $G^{-1} \simeq I$ with the same constants for a positive definite matrix G. Thus, we obtain GBG coordinates for $\mathrm{T}^*\mathcal{M}$ with $2^{-1} \leq \mathrm{g} \leq 2$ where we denote the metric on $\mathrm{T}^*\mathcal{M}$ also by g.

Next, if two inner products u, v on vector spaces U, V satisfy $C_1^{-1} \leq u \leq C_1$ and $C_2^{-1} \leq v \leq C_2$, then $(C_1C_2)^{-1} \leq u \otimes v \leq C_1C_2$ on $U \otimes V$. Thus, by induction, we have that $2^{-pq} \leq \mathrm{g} \leq 2^{pq}$ for the metric g on $\mathcal{T}^{(p,q)}\mathcal{M}$.

For the gradient bounds, first consider $\mathrm{T}\mathcal{M} = \mathcal{T}^{(1,0)}\mathcal{M}$. Since we assume our connection is Levi-Civita, we can write the Christoffel symbols Γ_{ij}^k purely in terms of $\partial_a g_{bc}$ and g^{ab}. The Cauchy–Schwarz inequality allows us to bound the g^{ab} and the bounds on $\partial_a g_{bc}$ comes from the theorem. Thus, $\left|\Gamma^k_{ij}\right| \leq C$ and so $|\nabla e_i| \leq C$. An inductive argument then yields the result for $\mathcal{T}^{(p,q)}\mathcal{M}$ with the constant dependent on p and q. □

With this result, we can apply the general Theorem 6.2 to obtain the following solution to the Kato square root problem on finite rank tensors.

Theorem 6.6 (Kato square root problem on finite rank tensors) *Suppose that* $|\mathrm{Ric}| \leq C$, $\mathrm{inj}(M) \geq \kappa > 0$, *and the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and} \qquad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{T}^{(p,q)}\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$. *Suppose further that* $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{V})$ *and that there exists* $C' > 0$ *such that*

$$\left\|\nabla^2 u\right\| \leq C' \,\|(I + \Delta)u\| \tag{R}$$

whenever $u \in \mathcal{D}(\Delta)$. *Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$.

Proof We apply Theorem 6.2. The Ricci bounds imply that there exists $\eta \in \mathbb{R}$ such that that $\mathrm{Ric} \geq \eta \mathrm{g}$. This shows condition (i) as in the proof of Theorem 1.1 in [16]. Conditions (ii) and (iv) are a consequence of Corollary 6.5, and (iii) holds because the connection is Levi-Civita. □

The Riesz transform condition (R) is satisfied automatically for (0, 0) tensors, or in other words, for scalar-valued functions, as we now show.

Proof of Theorem 1.1 Recall the Weitzenböck–Bochner identity

$$\langle \Delta(\nabla f), \nabla f\rangle = \frac{1}{2}\Delta(|\nabla f|^2) + \left|\nabla^2 f\right|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

for all $f \in \mathrm{C}^\infty(\mathcal{M})$. When $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$, we get $\left\|\nabla^2 f\right\| \lesssim \|(I + \Delta) f\|$ by integrating this identity.

The fact that $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{M})$ follows from Proposition 3.3 in [10] as does the density of $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$ in $\mathcal{D}(\Delta)$ Thus, $\left\|\nabla^2 u\right\| \lesssim \|(I + \Delta)u\|$ for $u \in \mathcal{D}(\Delta)$. Hence the hypotheses of Theorem 6.6 hold, so Theorem 1.1 follows as a consequence. □

7 Lipschitz Estimates and Stability

In this short section, we demonstrate Lipschitz estimates for the functional calculus and the stability of the square root.

Let $\tilde{a}$ and $\tilde{A}$ satisfy the same conditions as specified for a and A prior to Theorem 6.2, and set

$$\tilde{B}_1 = \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A} \end{pmatrix}.$$

On noting that $\tilde{B}_i$ satisfy conditions (i)–(iii) of Corollary 4.6, we have the following Lipschitz estimate.

Theorem 7.1 (Lipschitz estimate) *Assume the hypotheses of Theorem 6.2 and fix $\eta_i < \kappa_i$. Suppose that $\tilde{B}_i$ satisfy $\|\tilde{B}_i\|_\infty \leq \eta_i$ for $i = 1, 2$ and set $0 < \hat{\omega}_i < \frac{\pi}{2}$ by $\cos \hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_\infty + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Then, for all $\hat{\omega} < \mu < \frac{\pi}{2}$,*

$$\left\| f(\Pi_B) - f(\Pi_{B+\tilde{B}}) \right\| \lesssim (\|\tilde{B}_1\|_\infty + \|\tilde{B}_2\|_\infty) \|f\|_\infty$$

for all $f \in \mathrm{Hol}^\infty(S^o_\mu)$, and

$$\int_0^\infty \left\| \psi(t\Pi_B)v - \psi(t\Pi_{B+\tilde{B}})v \right\|^2 \frac{dt}{t} \lesssim (\|\tilde{B}_1\|^2_\infty + \|\tilde{B}_2\|^2_\infty) \|v\|,$$

for all $\psi \in \Psi(S^o_\mu)$ and all $v \in \mathscr{H}$. The implicit constants depend in particular on on B_i and η_i. □

We use the coefficients $\tilde{a}$ and $\tilde{A}$ to perturb the coefficients a and A. Then, we construct the following perturbed operator $\mathrm{L}_{A+\tilde{A}}$ defined similar to L_A given by $\mathrm{L}_{A+\tilde{A}} u = (a + \tilde{a}) S^*(A + \tilde{A}) S u$ for $u \in \mathcal{D}(\mathrm{L}_{A+\tilde{A}})$.

Theorem 7.2 (Stability of the square root) *Assume the hypotheses of Theorem 6.2 and fix $\eta_i < \kappa_i$. If $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, then*

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty) \|u\|_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{V})$. The implicit constant depends in particular on A, a and η_i.

Proof Let $\tilde{B}_i$ be given as in the hypothesis of Theorem 7.1, so that $\|\tilde{B}_1\|_\infty = \|\tilde{a}\|_\infty \leq \eta_1$ and $\|\tilde{B}_2\|_\infty = \|\tilde{A}\|_\infty \leq \eta_2$. By Theorem 7.1,

$$\left\| \mathrm{sgn}(\Pi_B)v - \mathrm{sgn}(\Pi_{B+\tilde{B}})v \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty) \|v\|$$

for all $v \in \mathscr{H}$, in particular for all $v = \begin{pmatrix} 0 \\ Su \end{pmatrix}$ with $u \in \mathrm{W}^{1,2}(\mathcal{V})$. Note that $v = \Pi_B \begin{pmatrix} u \\ 0 \end{pmatrix} = \Pi_{B+\tilde{B}} \begin{pmatrix} u \\ 0 \end{pmatrix}$ so $\mathrm{sgn}(\Pi_B)v = \begin{pmatrix} \sqrt{\mathrm{L}_A}\, u \\ 0 \end{pmatrix}$ and $\mathrm{sgn}(\Pi_{B+\tilde{B}})v = \begin{pmatrix} \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \\ 0 \end{pmatrix}$. Thus, on substitution, we obtain the desired result. □

We point out that the conclusions of both these theorems hold if, instead of assuming the hypotheses of Theorem 6.2, we assume the hypotheses of Theorem 6.6. This yields Lipschitz estimates and the stability result for (p, q) tensors. We conclude this section by highlighting the stability of the square root for scalar-valued functions as a corollary.

Corollary 7.3 (Stability of the square root for functions) *Suppose that* $|\mathrm{Ric}| \leq C$*,* $\mathrm{inj}(M) \geq \kappa > 0$ *and the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \textit{and} \qquad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$*. Fix* $\eta_i < \kappa_i$*. If* $\|\tilde{a}\|_\infty \leq \eta_1$*,* $\|\tilde{A}\|_\infty \leq \eta_2$*, then*

$$\left\| \sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u \right\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\, \|u\|_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$*. The implicit constant depends on* $C, \kappa, \kappa_i, A, a$ *and* η_i*.*

8 Harmonic Analysis

8.1 Carleson Measure Reduction

It remains for us to prove Theorem 4.2. The main point is to show that the main local quadratic estimate (Q1) in Proposition 4.1 is a consequence of hypotheses (H1)–(H8).

The proof proceeds by reducing the main quadratic estimate to a *Carleson measure* estimate. Thus, we first recall the notion of a (local) Carleson measure. Set $\mathcal{M}_+ = \mathcal{M} \times (0, t_0]$, for some $t_0 < \infty$. We emphasise that we restrict our considerations to $t \leq t_0$. The *Carleson box* over $Q \in \mathscr{Q}_t$ is then defined as $\mathrm{R}_Q = \overline{Q} \times (0, \ell(Q)]$. A positive Borel measure ν on $\mathcal{M}_+$ is called a *Carleson measure* if there exists $C > 0$ such that $\nu(\mathrm{R}_Q) \leq C\mu(Q)$ for all dyadic cubes $Q \in \mathscr{Q}_t$ for $t \leq t_0$. The Carleson norm $\|\nu\|_{\mathcal{C}}$ is defined by

$$\|\nu\|_{\mathcal{C}} = \sup_{Q \in \mathscr{Q}_t,\ t \leq t_0} \frac{\nu(\mathrm{R}_Q)}{\mu(Q)}$$

Let $\mathcal{C}$ denote the set of all Carleson measures.

The reader will find a more elaborate description of Carleson measures in the classical setting in §II.2 of [17] by Stein. The local construction described here is a fraction of a larger theory explored by Morris in [14] and [16].

With a description of a Carleson measure in hand, we now illustrate how to reduce the *main local quadratic estimate* (Q1) to a Carleson measure estimate. The key is the following consequence of Carleson's theorem, which is a special case of Theorem 4.3 in [16]. Recall the dyadic averaging operator $\mathcal{A}_t$ from Sect. 2.3.

Proposition 8.1 *For all $u \in \mathscr{H}$ and for all $\nu \in \mathcal{C}$,*

$$\iint_{\mathcal{M}\times(0,t_0]} |\mathcal{A}_t u|^2 \, d\nu(x,t) \lesssim \|u\|^2 \, \|\nu\|_{\mathcal{C}}.$$

Further, recall that whenever $w = w_i \, e^i(x) \in \mathcal{V}_x$ for $x \in Q$ in $\{e^i\}$, the associated GBG coordinates of Q, we define the associated GBG constant section $\omega(y) = w_i \, e^i(y)$ when $y \in B(x_{\widehat{Q}}, \rho)$ and $\omega(y) = 0$ otherwise. Then, we define the *principal part* as $\gamma_t(x)w = (\Theta_t^B \omega)(x)$. With this notation, and for $0 < t_0 < \infty$ to be chosen later, we split the main quadratic estimate in the following way:

$$\begin{aligned}\int_0^{t_0} \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim& \int_0^{t_0} \left\|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\right\|^2 \frac{dt}{t} \\ &+ \int_0^{t_0} \|\gamma_t \mathcal{A}_t (P_t - I)u\|^2 \frac{dt}{t} \\ &+ \int_0^{t_0} \int_M |\mathcal{A}_t u|^2 \, |\gamma_t|^2 \frac{d\mu(x)\, dt}{t}. \end{aligned} \tag{Q2}$$

We call the first two terms on the right of (Q2) the *principal terms*. Proposition 8.1 then allows us to reduce estimating the last term to proving that

$$A \mapsto \int_A |\gamma_t(x)|^2 \frac{d\mu(x)\, dt}{t}$$

is a Carleson measure. We call this term the *Carleson term*.

8.2 Estimation of Principal Terms

In this section, as the title suggests, we illustrate how to estimate the two principal terms of (Q2). We proceed to do so by coupling the existence of exponential off-diagonal bounds with our dyadic Poincaré inequality and cancellation hypothesis. The estimates here are straightforward and are more or less adapted from [6,16] and [3].

First, we quote the following theorem of [16], which is essentially contained in [6].

Proposition 8.2 (Off-diagonal bounds) *Let U_t be either R_t^B, P_t^B, Q_t^B or Θ_t^B for $t \in \mathbb{R}^+$. There exists a $C_\Theta > 0$, which only depends on (H1)–(H6), for every $M > 0$ there exists $c > 0$ with*

$$\|\chi_E U_t u\| \leq c \left\langle \frac{|t|}{d(E,F)} \right\rangle^M \exp\left(-C_\Theta \frac{d(E,F)}{t}\right) \|\chi_F u\|$$

whenever E, F are Borel, spt $u \subset F$.

In [16], Morris points out that Θ_t^B extends to an operator $\Theta_t^B : \mathrm{L}^\infty(\mathcal{V}) \to \mathrm{L}^2_{\mathrm{loc}}(\mathcal{V})$ when $t \in (0, \langle C_\Theta/(2\lambda \langle C_1 \delta^{-1}\rangle)\rangle]$. Since we require this in the harmonic analysis, and recalling the scale t_S which we chose in Sect. 2.3 in interfacing GBG with the dyadic decomposition, we will fix an even smaller scale $t_H \leq t_S$ such that $t_H \leq \langle C_\Theta/(2\lambda \langle C_1\delta^{-1}\rangle)\rangle$. Next, we have the following technical lemma.

Lemma 8.3 *Let $r > 0$ and suppose that $\{B_j = B(x_j, r)\}$ is a disjoint collection of balls. Then, whenever $\eta \geq 1$,*

$$\sum_j \chi_{\eta B_j} \lesssim \eta^\kappa \mathrm{e}^{4\lambda\eta r}.$$

Proof Fix $x \in \mathcal{M}$ and let $\mathscr{C}_x = \{x_j \in \mathcal{M} : x \in B(x_j, \eta r)\}$. It is easy to see that $\sum_j \chi_{\eta B_j}(x) = \operatorname{card}\mathscr{C}_x$. That $x_j \in \mathscr{C}_x$ is equivalent to saying that $d(x, x_j) < \eta r$, and therefore for any $y \in \eta B_j$, $d(x, y) \leq d(x, x_j) + d(x_j, y) < (\eta + 1)r$. That is, $B(x, (\eta + 1)r) \supset B(x_j, r)$ and by the disjointness of $\{B_j\}$, $\mu(B(x, (\eta + 1)r)) \geq \sum_{x_j \in \mathscr{C}_x} \mu(B(x_j, r))$.

Next, note that ($\mathrm{E}_{\mathrm{loc}}$) implies that $\mu(\eta B_j) \lesssim \eta^\kappa \mathrm{e}^{\lambda\eta r} \mu(B_j)$ and therefore,

$$\sum_{x_j \in \mathscr{C}_x} \mu(\eta B_j) \lesssim \eta^\kappa \mathrm{e}^{\lambda\eta r} \mu(B(x, (\eta + 1)r)).$$

Thus, it is enough to compare $\mu(\eta B_j)$ to $\mu(B(x, (\eta+1)r))$. So, take any $y \in B(x, (\eta+1)r)$, and note that $d(x, y) \leq d(x, x_j) + d(x_j, y) < (2\eta + 1)r < 3\eta r$. Thus,

$$\mu(B(x, (\eta + 1)r)) \leq \mu(B(x_j, 3\eta r) \lesssim 3^\kappa \mathrm{e}^{3\lambda\eta r} \mu(B(x_j, \eta r))$$

and the estimate

$$\operatorname{card}\mathscr{C}_x\, \mathrm{e}^{-3\lambda\eta r} \lesssim \sum_{x_j \in \mathscr{C}_x} \frac{\mu(\eta B_j)}{\mu(B(x, (\eta + 1)r))} \lesssim \eta^\kappa e^{\lambda\eta r}$$

completes the proof. □

Proposition 8.4 (First principal term estimate) *For all $u \in \mathcal{R}(\Pi)$,*

$$\int_0^{t_2} \left\| \Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

where $t_2 \leq \min\left\{t_H, \frac{C_\Theta}{4\lambda(c+4\tilde{c})}\right\}$.

Proof Let $v = P_t u$.

(i) First, we note that

$$\left\| \Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u \right\|^2 = \sum_{Q \in \mathscr{Q}_t} \left\| \Theta_t^B (v - v_Q) \right\|^2_{\mathrm{L}^2(Q)}.$$

For each $Q \in \mathscr{Q}_t$, write $B_Q = B(x_Q, \frac{C_1}{\delta} t) \supset Q$ and $C_j(Q) = 2^{j+1} B_Q \backslash 2^j B_Q$. Then, for each such cube Q,

$$\begin{aligned}\int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \, d\mu &= \int_Q \left|\Theta_t^B \left(\sum_{j=0}^{\infty} \chi_{C_j(Q)} (v - v_Q)\right)\right|^2 \, d\mu \\ &\leq \sum_{j=0}^{\infty} \int_Q \left|\Theta_t^B (\chi_{C_j(Q)} (v - v_Q))\right|^2 \, d\mu \\ &\lesssim \sum_{j=0}^{\infty} \left\langle \frac{t}{d(Q, C_j(Q))} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \\ &\quad \int_{C_j(Q)} \left|v - v_Q\right|^2 \, d\mu\end{aligned}$$

(ii) Next, note that by (4.1) in [16]

$$2^j \frac{C_1}{\delta} t \leq d(x_Q, C_j(Q)) \leq d(Q, C_j(Q)) + \operatorname{diam} \, Q$$

which implies that

$$\left\langle \frac{t}{d(Q, C_j(Q))} \right\rangle^M \lesssim 2^{-M(j+1)}.$$

Next, for $j \geq 1$,

$$d(Q, C_j(Q)) \geq 2^j \frac{C_1}{2\delta} t$$

and therefore,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \leq \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^{j+1}\right).$$

For $j = 0$,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) = 1 = \exp\left(\frac{C_\Theta C_1}{4\delta}\right) \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^0\right).$$

Fix $t' > 0$ to be chosen later. Then, for all $t \leq t'$,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \lesssim \exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right)$$

for all $j \geq 0$.

(iii) Combining the estimates in (i) and (ii),

$$\int_Q \left|\Theta_t^B(v - v_Q)\right|^2 \, d\mu \lesssim \sum_{j=0}^{\infty} 2^{-M(j+1)} \exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right) \int_{C_j(Q)} \left|v - v_Q\right|^2 \, d\mu.$$

Since $v = P_t u \in \mathcal{D}(\Pi^2) = \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$, we conclude from (H8)-1 that

$$\int_{C_j(Q)} \left|v - v_Q\right|^2 \, d\mu \lesssim \left(1 + \left(\frac{C_1}{\delta}\right)^{\kappa} 2^{\kappa(j+1)} \exp\left(\frac{\lambda c C_1}{\delta} 2^{j+1} t\right)\right) \left(\frac{C_1}{\delta}\right)^2 2^{2(j+1)} t^2 \int_{\tilde{c} 2^{j+1} B_Q} (|\Xi v|^2 + |v|^2) \, d\mu.$$

Therefore,

$$\begin{aligned}
&\sum_{Q \in \mathcal{Q}_t} \int_Q \left|\Theta_t^B(v - v_Q)\right|^2 \, d\mu \\
&\lesssim \sum_{Q \in \mathcal{Q}_t} \sum_{j=0}^{\infty} 2^{-M(j+1)} \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^{j+1}\right) \\
&\quad \left(1 + \left(\frac{C_1}{\delta}\right)^{\kappa} 2^{\kappa(j+1)} \exp\left(\frac{\lambda c C_1}{\delta} 2^{j+1} t\right)\right) \left(\frac{C_1}{\delta}\right)^2 2^{2(j+1)} t^2 \\
&\quad \int_{\tilde{c} 2^{j+1} B_Q} (|\Xi v|^2 + |v|^2) \, d\mu \\
&\lesssim \sum_{j=0}^{\infty} 2^{-(M-\kappa-2)(j+1)} \left[\exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right)\right. \\
&\quad \left. + \exp\left(-\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'} - \lambda c\right) 2^{j+1} t\right)\right] \\
&\quad t^2 \int_{\mathcal{M}} \sum_{Q \in \mathcal{Q}_t} \chi_{\tilde{c} 2^{j+1} B_Q} (|\Xi v|^2 + |v|^2) \, d\mu.
\end{aligned}$$

(iv) Set $\eta = \tilde{c} 2^{j+1} C_1 / (\delta a_0)$ and $r = a_0 t$ so that $\left\{B(x_Q, a_0 t) \subset Q\right\}$ is disjoint, and invoke Lemma 8.3 to conclude that

$$\chi_{\tilde{c} 2^{j+1} B_Q} \lesssim 2^{\kappa(j+1)} \exp\left(\frac{4\lambda \tilde{c} C_1}{\delta} 2^{j+1} t\right).$$

Combining this with (iii), we have

$$\sum_{Q\in\mathcal{Q}_t}\int_Q \left|\Theta_t^B(v-v_Q)\right|^2\ d\mu$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-(M-2\kappa-2)(j+1)}$$
$$\left[\exp\left(-\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'}-4\lambda\tilde{c}\right)2^{j+1}t\right)\right.$$
$$\left.+\exp\left(-\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'}-\lambda c-4\lambda\tilde{c}\right)2^{j+1}t\right)\right]$$
$$t^2\left(\|\Xi v\|^2+\|v\|^2\right)$$

(v) Now, we choose $t' \leq \mathrm{t_H}$ so that

$$-\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'}-4\lambda\tilde{c}\right)\leq 0 \quad \text{and} \quad -\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'}-\lambda c-4\lambda\tilde{c}\right)\leq 0.$$

That is,

$$\frac{C_\Theta}{4\lambda(c+4\tilde{c})}\leq t'.$$

We can, therefore, set $t' = t_2$ and then,

$$\sum_{Q\in\mathcal{Q}_t}\int_Q \left|\Theta_t^B(v-v_Q)\right|^2\ d\mu \lesssim t^2\sum_{j=0}^{\infty}2^{-(M-2\kappa-2)(j+1)}\ \|\Pi v\|^2$$

by invoking (H8)-2. By choosing $M > 2\kappa+2$, and substituting $v = P_t u$, we have

$$\int_0^{t_2}\left\|\Theta_t^B P_t u-\gamma_t\mathcal{A}_t P_t u\right\|^2\ \frac{dt}{t}\lesssim\int_0^{t_2}\|t\Pi P_t u\|^2\ \frac{dt}{t}\leq\int_0^{\infty}\|t\Pi P_t u\|^2\ \frac{dt}{t}$$
$$\lesssim \|u\|.$$

□

Next, as in [6] and [16], we note the following consequence of (H7).

Lemma 8.5 *Let* $\Upsilon = \Gamma,\ \Gamma^*$ *or* Π. *Then,*

$$\left|\fint_Q \Upsilon u\ d\mu\right|\lesssim\frac{1}{\ell(Q)^\eta}\left(\fint_Q |u|^2\ d\mu\right)^{\frac{\eta}{2}}\left(\fint_Q|\Upsilon u|^2\ d\mu\right)^{1-\frac{\eta}{2}}+\fint_Q |u|^2$$

for all $u\in\mathcal{D}(\Upsilon)$ *and* $Q\in\mathcal{Q}_t$ *where* $t\leq \mathrm{t_S}$.

The proof of this lemma is the same as that of the proof of Lemma 5.9 in [16]. Thus, we deduce the following.

Proposition 8.6 (Second principal term estimate) *For all* $u \in \mathrm{L}^2(\mathcal{V})$*, we have*

$$\int_0^{\mathrm{t_H}} \|\gamma_t \mathcal{A}_t (P_t - I) u\|^2 \, \frac{dt}{t} \lesssim \|u\|^2 .$$

The proof of this proposition is similar to the proof of Proposition 5.10 in [16] with the principle difference being that we only consider $t \leq \mathrm{t_H}$.

We have demonstrated how to estimate the two principal terms of (Q2).

8.3 Carleson Measure Estimate

We are left with the task of estimating the final term of (Q2). We follow in the footsteps of [4,6] remarking that, in a sense, it is to preserve the main thrust of this Carleson argument that we have constructed the various technologies in this paper. We show in this section that the argument runs as before with some changes that are possible as a consequence of the geometric assumptions which we have made.

In §5 of [16], Morris illustrates how to prove

$$A \mapsto \int_A |\gamma_t(x)|^2 \; d\mu(x) \frac{dt}{t}$$

is a local Carleson measure. At the heart of this proof lies a certain test function. Here, we illustrate how to use the GBG condition to set up a suitable substitute test function $\mathrm{f}^w_{Q,\varepsilon}$. The main complication is to choose a cutoff in a way that we stay inside GBG coordinates of each cube.

Let $\tau > 1$ be chosen later. We note that we can find a smooth function $\eta_Q : \mathcal{M} \to [0, 1]$ such that $\eta = 1$ on $B(x_Q, \tau C_1 \, \ell(Q))$ and $\eta = 0$ on $\mathcal{M} \setminus B(x_Q, 2\tau C_1 \, \ell(Q))$ and satisfying the gradient bound $|\nabla \eta| \lesssim \frac{1}{\ell(Q)}$. The constant here depends on τ.

Let $\widehat{Q}$ be the GBG cube with centre $x_{\widehat{Q}}$. We want to use η to perform a cutoff inside the ball $B(x_Q, \rho)$. That is, we want the ball $B(x_Q, 2\tau C_1 \, \ell(Q)) \subset B(x_Q, \rho)$. A simple calculation then yields that we need to choose $\tau < 3$. Thus, for the sake of the argument, let us fix $\tau = 2$.

Next, let $v \in \mathcal{L}(\mathbb{C}^N)$ and $|v| = 1$. Let $\hat{w}, w \in \mathbb{C}^N$ such that $|\hat{w}| = |w| = 1$ and $v^*(\hat{w}) = w$. Let $\tilde{w} = \eta \, w$. Certainly, we have that $\tilde{w} \in \mathrm{L}^2(\mathcal{V})$ since $\tilde{w} = 0$ outside of $B(x_{\widehat{Q}}, \rho)$. Now, fix $\varepsilon > 0$ and define

$$\mathrm{f}^w_{Q,\varepsilon} = \tilde{w} - \imath \varepsilon \, \ell(Q) \Gamma R^B_{\varepsilon \, \ell(Q)} \tilde{w} = (1 + \imath \varepsilon \, \ell(Q) \Gamma^*_B R^B_{\varepsilon \, \ell(Q)}) \tilde{w}.$$

This allows us to prove a lemma similar to that of Lemma 5.12 in [16].

Lemma 8.7 *There exists $c > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$,*

$$\left\|\mathrm{f}^w_{Q,\varepsilon}\right\| \leq c\mu(Q)^{\frac{1}{2}}, \quad \iint_{\mathrm{R}_Q} \left|\Theta^B_t \mathrm{f}^w_{Q,\varepsilon}\right|^2 \, d\mu \frac{dt}{t} \leq c\frac{\mu(Q)}{\varepsilon^2} \quad \textit{and}$$

$$\left|\fint_Q \mathrm{f}^w_{Q,\varepsilon} - w \, d\mu\right| \leq c\varepsilon^{\frac{\eta}{2}}.$$

The proof of this lemma proceeds exactly as the proof of Lemma 5.12 in [16]. The last estimate relies upon Lemma 8.5.

Setting $\varepsilon = \left(\frac{1}{2c}\right)^{\frac{\eta}{2}}$ we obtain the same conclusion as the author of [16] that

$$\mathrm{Re}\left\langle w, \fint_Q f^w_Q \right\rangle \geq \frac{1}{2}$$

where we have set $f^w_Q = f^w_{Q,\varepsilon}$. Furthermore, a stopping time argument as in Lemma 5.11 in [6] yields the following.

Lemma 8.8 *Let $t_3 = \min\left\{\mathrm{t_H}, \frac{C_\Theta}{4a^3\lambda}\right\}$. There exists $\alpha, \beta > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq t_3$, there exists a collection of subcubes $\{Q_k\} \subset \cup_{t \leq t_3} \mathscr{Q}_t$ of Q such that $E_Q = Q \setminus \cup_k Q_k$ satisfies $\mu(E_Q) \geq \beta\mu(Q)$ and $E^*_Q = \mathrm{R}_Q \setminus \cup_k \mathrm{R}_{Q_k}$ satisfies*

$$\mathrm{Re}\left\langle w, \fint_{Q'} f^w_Q \right\rangle \geq \alpha \quad \textit{and} \quad \fint_{Q'} \left|f^w_Q\right| \leq \frac{1}{\alpha}$$

whenever $Q' \in \cup_{t \leq t_3} \mathscr{Q}_t$ with $Q' \subset Q$ and $\mathrm{R}_{Q'} \cap E^*_Q = \varnothing$.

Let $\sigma > 0$ to be chosen later and let $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$ and define

$$K_{\nu,\sigma} = \left\{\nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left|\frac{\nu'}{|\nu'|} - \nu\right| \leq \sigma\right\}.$$

Proposition 8.9 *Let $t_3 = \min\left\{\mathrm{t_H}, \frac{C_\Theta}{4a^3\lambda}\right\}$. There exists $\sigma, \beta, c > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq t_3$, and $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$, there exists a collection of subcubes $\{Q_k\} \subset \cup_{t \leq t_3} \mathscr{Q}_t$ of Q such that $E_Q = Q \setminus \cup_k Q_k$ satisfies $\mu(E_Q) \geq \beta\mu(Q)$ and $E^*_Q = \mathrm{R}_Q \setminus \cup_k \mathrm{R}_{Q_k}$ satisfies*

$$\iint_{(x,t) \in E^*_Q, \ \gamma_t(x) \in K_{\nu,\sigma}} |\gamma_t(x)|^2 \, d\mu(x) \frac{dt}{t} \leq c\mu(Q).$$

Then, the argument immediately following Proposition 5.11 in [16] illustrates that

$$\iint_{\mathrm{R}_Q} |\gamma_t(x)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \mu(Q)$$

and this is the required Carleson-measure estimate.

On choosing $t_0 = \min\{t_2, t_3\}$ and applying Proposition 8.1, we find that we have bounded all three terms on the right of (Q2), and hence deduced the estimate (Q1) of Proposition 4.1:

$$\int_0^{t_0} \left\| \Theta_t^B P_t u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 .$$

Proof of Theorem 4.2 The invariance of (H1)–(H8) upon replacing (Γ, B_1, B_2) by (Γ^*, B_2, B_1), (Γ^*, B_2^*, B_1), and (Γ^*, B_1^*, B_2^*) proves Theorem 4.2 via Proposition 4.1. □

By establishing Theorem 4.2, we are now in a position to enjoy the full thrust of its consequences. The first is the Kato square root type estimate for perturbations of Dirac type operators on vector bundles as listed in Corollary 4.3. A consequence of these corollaries is the Kato square root problem for vector bundles, as described in Theorem 6.2, the Kato square root problem for finite rank tensors as described in Theorem 6.6 and lastly, the highlighted theorem of this paper, Theorem 1.1, the Kato square root problem for functions. Furthermore, we also can enjoy the holomorphic dependency results of Sect. 4.2, in particular, Corollary 4.6, which illustrates the stability of the functional calculus under small perturbations.

9 Extension to Measure Metric Spaces

In this section, we extend the quadratic estimates to a setting where $\mathcal{M}$ is replaced by an exponentially locally doubling measure metric space $\mathcal{X}$. As a consequence, we also drop the smoothness assumption on the vector bundle $\mathcal{V}$. Similar quadratic estimates on doubling measure metric spaces for trivial bundles are obtained by the first author in [7].

To be precise, let $\mathcal{X}$ be a complete metric space with metric d and let $d\mu$ be a Borel-regular exponentially locally doubling measure. That is, we assume that $d\mu$ satisfies ($\mathrm{E_{loc}}$) with $\mathcal{X}$ in place of $\mathcal{M}$. The underlying space now lacks a differentiable structure and it no longer makes sense to ask the local trivializations and the metric h to be smooth. Instead, we simply require them to be continuous. However, we remark that in applications, the local trivializations would normally be Lipschitz. The fact that $d\mu$ is Borel implies that the local trivializations are measurable. Furthermore, we assume that $\mathcal{V}$ satisfies the GBG condition.

With the exception of (H6), no changes need be made to the hypotheses to (H1)–(H8) in this new setting. To define a suitable (H6), we first define the following quantity as in [7].

Definition 9.1 (*Pointwise Lipschitz constant*) For $\xi : \mathcal{X} \to \mathbb{C}^N$ Lipschitz, define $\mathrm{Lip}\ \xi : \mathcal{X} \to \mathbb{R}$ by

$$\mathrm{Lip}\ \xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x, y)}.$$

We take the convention that $\mathrm{Lip}\ \xi(x) = 0$ when x is an isolated point.

We then define (H6) as in [7].

(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by ξ preserves $\mathcal{D}(\Gamma)$ and $\mathrm{M}_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $\left|\mathrm{M}_\xi(x)\right| \leq m \left|\mathrm{Lip}\ \xi(x)\right|$ for almost all $x \in \mathcal{X}$.

Thus, we have the following theorem.

Theorem 9.2 *Let $\mathcal{X}$ be a complete metric space equipped with a Borel-regular measure $d\mu$ satisfying (E_{loc}). Suppose that that (Γ, B_1, B_2) satisfy (H1)–(H8). Then, Π_B satisfies the quadratic estimate*

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{V})$ and hence has a bounded holomorphic functional calculus.

Proof First, we note that much of the local Carleson theory was originally proved by Morris in §4.3 of [14] in the setting of an exponentially doubling measure metric space. Next, the off-diagonal estimates can be obtained by using the Lipschitz separation Lemma 5.1 in [7]. Also, the construction of the test function and the proof of Lemma 8.5 proceeds similar to the argument in [7]. Thus the arguments of Sect. 8 hold and the theorem is proved by Proposition 4.1. □

As before, we have the following corollaries. The $E_B^\pm$ are the spectral subspaces defined in Sect. 4.

Corollary 9.3 (Kato square root type estimate)

(i) *There is a spectral decomposition*

$$\mathrm{L}^2(\mathcal{V}) = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$$

(where the sum is in general non-orthogonal), and

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{{\Pi_B}^2})$ *with*

$$\|\Gamma u\| + \left\|\Gamma_B^* u\right\| \simeq \|\Pi_B u\| \simeq \|\sqrt{{\Pi_B}^2} u\|$$

for all $u \in \mathcal{D}(\Pi_B)$.

Corollary 9.4 *Let $\mathscr{H}, \Gamma, B_1, B_2, \kappa_1, \kappa_2$ satisfy (H1)–(H8) and take $\eta_i < \kappa_i$. Set $0 < \hat{\omega}_i < \frac{\pi}{2}$ by $\cos \hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_\infty + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Let $A_i \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$ satisfy*

(i) $\|A_i\|_\infty \leq \eta_i$,
(ii) $A_1 A_2 \mathcal{R}(\Gamma)$, $B_1 A_2 \mathcal{R}(\Gamma)$, $A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$, *and*
(iii) $A_2 A_1 \mathcal{R}(\Gamma^*)$, $B_2 A_1 \mathcal{R}(\Gamma^*)$, $A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Letting $\hat{\omega} < \mu < \frac{\pi}{2}$, we have:

(i) *for all $f \in \mathrm{Hol}^\infty(S_\mu^o)$,*

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty) \|f\|_\infty, \text{ and}$$

(ii) *for all* $\psi \in \Psi(S^o_\mu)$,

$$\int_0^\infty \|\psi(t\Pi_B)u - \psi(t\Pi_{B+A})u\|^2 \frac{dt}{t} \lesssim (\|A_1\|_\infty^2 + \|A_2\|_\infty^2)\,\|u\|\,,$$

whenever $u \in \mathscr{H}$.

The implicit constants depend on (H1)–(H8) and η_i.

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