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Stability of Monge–Ampère Energy Classes

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Abstract We show that the non-pluripolar product of positive currents is a bimeromorphic invariant. Under some natural assumptions, we show that the (weighted) energy associated with big cohomology classes are also bimeromorphic invariants. We compare the weighted energy functionals of currents with respect to different cohomology classes and establish quantitative estimates between big capacities.

Keywords Kähler classes · Big classes · Monge–Ampère equations · Monge–Ampère energy classes · Monge–Ampère capacity · Alexander–Taylor capacity

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1 Introduction

Let X be a compact n-dimensional Kähler manifold, $T_1 = \theta_1 + dd^c\varphi_1, \ldots, T_p = \theta_p + dd^c\varphi_p$ be closed positive (1, 1)-currents where θ_j are smooth representatives of the cohomology classes $\{T_j\}$. Denote by $\theta_1 + dd^c V_{\theta_1}, \ldots, \theta_p + dd^c V_{\theta_p}$ the canonical currents with minimal singularities. Following the construction of Bedford–

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Taylor [2] in the local setting, it has been shown in [5] that

$$\mathbf{1}_{\bigcap_j \{\varphi_j > V_{\theta_j} - k\}} (\theta_1 + dd^c \max(\varphi_1, V_{\theta_1} - k)) \wedge \cdots \wedge (\theta_p + dd^c \max(\varphi_p, V_{\theta_p} - k))$$

is non-decreasing in k and converges to the so-called *non-pluripolar product*

$$\langle T_1 \wedge \cdots \wedge T_p \rangle .$$

The resulting positive (p, p)-current does not charge pluripolar sets, and it is always *well defined* and *closed*.

Given α a big cohomology class, a positive closed $(1, 1)$-current $T \in \alpha$ is said to have *full Monge–Ampère mass* if

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha),$$

and we then write $T \in \mathcal{E}(X, \alpha)$. In [5] the authors also define *weighted energy functionals* E_χ (for any weight χ) in the general context of a big class extending the case of a Kähler class ([13]). The space of currents with finite weighted energy is denoted by $\mathcal{E}_\chi(X, \alpha)$.

The aim of the present paper is to show the invariance of the non-pluripolar product and establish stability properties of energy classes.

Theorem A *The non-pluripolar product is a bimeromorphic invariant.*
More precisely, fix $\alpha \in H^{1,1}(X, \mathbb{R})$ a big class and $f : X - - > Y$ a bimeromorphic map. Then

1) $f_\star \langle T^n \rangle = \langle (f_\star T)^n \rangle$ *for any positive closed* $T \in \alpha$.

Furthermore, if $f_\star\big(\mathcal{T}_\alpha(X)\big) = \mathcal{T}_{f_\star \alpha}(Y)$, *then*

2) $f_\star(\mathcal{E}(X, \alpha)) = \mathcal{E}(Y, f_\star \alpha)$;
3) $f_\star(\mathcal{E}_\chi(X, \alpha)) = \mathcal{E}_\chi(Y, f_\star \alpha)$ *for any weight* $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$.

Here $\mathcal{T}_\alpha(X)$ denotes the set of all positive and closed currents in the big class α and $\mathcal{T}_{f_\star \alpha}(Y)$ is the set of all positive closed currents in the image class. The condition on the image of positive currents ensures that the push-forward of a current with minimal singularities is still with minimal singularities; this easily implies that the volumes are preserved, i.e., $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star \alpha)$. We show conversely in Proposition 3.5 that the condition $f_\star\big(\mathcal{T}_\alpha(X)\big) = \mathcal{T}_{f_\star \alpha}(Y)$ is equivalent to $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star \alpha)$ in complex dimension 2, by using the existence of Zariski decompositions.

A related problem is to understand what happens to the energy classes if we change cohomology classes on a fixed compact Kähler manifold. Let α, β be big cohomology classes. Given $T \in \mathcal{T}_\alpha(X)$ and $S \in \mathcal{T}_\beta(X)$ so that $T + S \in \mathcal{T}_{\alpha+\beta}(X)$; we wonder whether

$$T \in \mathcal{E}_\chi(X, \alpha) \quad \text{and} \quad S \in \mathcal{E}_\chi(X, \beta) \quad \overset{\Longrightarrow}{\Longleftarrow} \quad T + S \in \mathcal{E}_\chi(X, \alpha + \beta).$$

It turns out that $T+S \in \mathcal{E}_{\chi}(X, \alpha+\beta)$ implies $T \in \mathcal{E}_{\chi}(X, \alpha)$ and $S \in \mathcal{E}_{\chi}(X, \beta)$ in a very general context (Proposition 4.1), but the reverse implication is false in general (see Counterexamples 4.5 and 4.7). We obtain a positive answer under restrictive conditions on the cohomology classes (see Propositions 4.3 and 5.8).

Theorem B *Let α, β be big classes, $T \in \mathcal{T}_{\alpha}(X)$, $S \in \mathcal{T}_{\beta}(X)$ and $\chi \in \mathcal{W}^{-} \cup \mathcal{W}_{M}^{+}$. Then*

1) $T+S \in \mathcal{E}(X, \alpha+\beta)$ implies $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$,
2) $T+S \in \mathcal{E}_{\chi}(X, \alpha+\beta)$ implies $T \in \mathcal{E}_{\chi}(X, \alpha)$ and $S \in \mathcal{E}_{\chi}(X, \beta)$.

If α, β are Kähler, conversely

3) $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$ implies $T+S \in \mathcal{E}(X, \alpha+\beta)$,
4) $T \in \mathcal{E}_{\chi}(X, \alpha)$ and $S \in \mathcal{E}_{\chi}(X, \beta)$ implies $T+S \in \mathcal{E}_{\chi}(X, \alpha+\beta)$.

Proposition C *Assume that $S \in \beta$ has bounded local potentials and that the sum of currents with minimal singularities in α and in β is still with minimal singularities. If $p > n^2-1$, then*

$$T \in \mathcal{E}^{p}(X, \alpha) \Longrightarrow T+S \in \mathcal{E}^{q}(X, \alpha+\beta),$$

where $0<q<p-n^2+1$.

We stress that the condition on the sum of currents having minimal singularities is not always satisfied as noticed in Remark 4.8, but it is a necessary condition if we want the positive intersection class $\langle\alpha \cdot \beta\rangle$ to be multi-linear (see [5]).

In our proof of Proposition **C** we establish a comparison result of capacities which is of independent interest:

Theorem D *Let α be a big class and β be a semipositive class. We assume that the sum of currents with minimal singularities in α and β is still with minimal singularities. Then, for any Borel set $K \subset X$, there exist $C>0$ such that*

$$\frac{1}{C} \operatorname{Cap}_{\theta_{\alpha, \min}}(K) \leq \operatorname{Cap}_{\theta_{\alpha+\beta, \min}}(K) \leq C\left(\operatorname{Cap}_{\theta_{\alpha, \min}}(K)\right)^{\frac{1}{n}},$$

where $\theta_{\alpha, \min}:=\theta_{\alpha}+d d^{c} V_{\theta_{\alpha}}$.

Let us now describe the contents of the article. We first introduce some basic notions such as currents with minimal singularities and finite energy classes, and we recall more or less known facts, e.g., that currents with full Monge–Ampère mass have zero Lelong number on a Zariski open set (Proposition 2.9).

In Sect. 3, we show that the non-pluripolar product is a bimeromorphic invariant (Theorem 3.1). Furthermore, under a natural condition on the set of positive (1, 1)-currents, we are able to prove that weighted energy classes are preserved under bimeromorphic maps (Proposition 3.3).

In the third part of the paper we study the stability of the energy classes (see, e.g., Theorem 4.1 and Proposition 4.3) and we give some counterexamples.

Finally, we compare the Monge–Ampère capacities with respect to different big classes (Theorem 5.6) and we use this result to give a partial positive answer to the stability property of weighted homogeneous classes $\mathcal{E}^{p}$ (Proposition 5.8).

2 Preliminaries

2.1 Big Classes

Let X be a compact Kähler manifold and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a real $(1,1)$-cohomology class.

Recall that α is said to be *pseudo-effective* (*psef* for short) if it can be represented by a closed positive $(1,1)$-current T. Given a smooth representative θ of the class α, it follows from the $\partial\bar{\partial}$-lemma that any positive $(1,1)$-current can be written as $T=\theta+dd^c\varphi$, where the global potential φ is a θ-psh function, i.e., $\theta+dd^c\varphi\geq 0$. Here, d and d^c are real differential operators, defined as

$$d:=\partial+\bar{\partial}, \qquad d^c:=\frac{i}{2\pi}\left(\bar{\partial}-\partial\right).$$

The set of all psef classes forms a closed convex cone, and its interior is by definition the set of all *big* cohomology classes:

Definition 2.1 We say that α is *big* if it can be represented by a *Kähler current*, i.e., there exists a positive closed $(1,1)$-current $T\in\alpha$ that dominates a Kähler form.

2.1.1 Analytic and Minimal Singularities

A positive current $T=\theta+dd^c\varphi$ is said to have *analytic singularities* if there exists $c>0$ such that (locally on X),

$$\varphi=\frac{c}{2}\log\sum_{j=1}^{N}|f_j|^2+u,$$

where u is smooth and $f_1,\dots,f_N$ are local holomorphic functions.

Definition 2.2 If α is a big class, we define its *ample locus* $\mathrm{Amp}\,(\alpha)$ as the set of points $x\in X$ such that there exists a strictly positive current $T\in\alpha$ with analytic singularities and smooth around x.

The ample locus $\mathrm{Amp}\,(\alpha)$ is a Zariski open subset by definition, and it is nonempty thanks to Demailly's regularization result (see [8]).

If T and T' are two closed positive currents on X, then T is said to be *more singular* than T' if their local potentials satisfy $\varphi\leq\varphi'+O(1)$.

Definition 2.3 A positive current T is said to have *minimal singularities* (inside its cohomology class α) if it is less singular than any other positive current in α. Its θ-psh potentials φ will correspondingly be said to have minimal singularities.

Such θ-psh functions with minimal singularities always exist; one can consider, for example,

$$V_\theta:=\sup\{\varphi\ \theta\text{-psh},\ \varphi\leq 0 \text{ on } X\}.$$

Remark 2.4 Let us stress that the sum of currents with minimal singularities does not necessarily have minimal singularities. For example, consider $\pi : X \to \mathbb{P}^2$ the blow-up at one point p and set $E := \pi^{-1}(p)$. Take $\alpha = \pi^\star\{\omega_{FS}\} + \{E\}$ and $\beta = 2\pi^\star\{\omega_{FS}\} - \{E\}$, where ω_{FS} denotes the *Fubini–Study* form on $\mathbb{P}^2$. As we will see in Remark 3.4, currents with minimal singularities in α are of the form $S_{\min} = \pi^\star T_{\min} + [E]$, where $T_{\min}$ is a current with minimal singularities in $\{\omega_{FS}\}$ (i.e., its potential is bounded) and so they have singularities along E. On the other hand, currents with minimal singularities in the Kähler class β have bounded potentials, hence the sum of currents with minimal singularities in α and in β is a current with unbounded potentials. But $\alpha+\beta = 3\pi^\star\{\omega_{FS}\}$ is semipositive, hence currents with minimal singularities have bounded potentials.

2.1.2 Images of Big Classes

It is classical that big cohomology classes are invariant under pull-back and push-forward (see, e.g., [7, Proposition 4.13]).

Lemma 2.5 *Let $f : X \dashrightarrow Y$ be a bimeromorphic map and $\alpha_X \in H^{1,1}(X, \mathbb{R})$, $\alpha_Y \in H^{1,1}(Y, \mathbb{R})$ be big cohomology classes. Then $f_\star\alpha_X$ and $f^\star\alpha_Y$ are still big classes.*

Note that this is not true in the case of Kähler classes.

2.1.3 Volume of Big Classes

Fix $\alpha \in H^{1,1}_{\text{big}}(X, \mathbb{R})$. We introduce

Definition 2.6 Let $T_{\min}$ be a current with minimal singularities in α and let Ω be a Zariski open set on which the potentials of $T_{\min}$ are locally bounded. Then

$$\text{vol}(\alpha) := \int_\Omega T^n_{\min} > 0 \tag{2.1}$$

is called the volume of α.

Note that the Monge–Ampère measure of $T_{\min}$ is well defined in Ω by [1] and that the volume is independent of the choice of $T_{\min}$ and Ω ([5, Theorem 1.16]).

Let $f : X \to Y$ be a modification between compact Kähler manifolds and let $\alpha_Y \in H^{1,1}(Y, \mathbb{R})$ be a big class. The volume is preserved by pull-backs,

$$\text{vol}(f^\star\alpha_Y) = \text{vol}(\alpha_Y)$$

(see [7]). On the other hand, it is in general not preserved by push-forwards:

Example 2.7 Let $\pi : X \to \mathbb{P}^2$ be the blow-up along $\mathbb{P}^2$ at point p. The class $\alpha_X := \{\pi^\star\omega_{FS}\} - \varepsilon\{E\}$ is Kähler whenever $0 < \varepsilon < 1$ and $\pi_\star\alpha_X = \{\omega_{FS}\}$. Now, $\text{vol}(\alpha_X) = 1 - \varepsilon^2$, while $\text{vol}(\pi_\star\alpha_X) = 1$.

2.2 Finite Energy Classes

Fix X an n-dimensional compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class and $\theta \in \alpha$ a smooth representative.

2.2.1 The Non-Pluripolar Product

Let us stress that since the non-pluripolar product does not charge pluripolar sets,

$$\mathrm{vol}(\alpha) = \int_X \langle T_{\min}^n \rangle.$$

Definition 2.8 A closed positive (1, 1)-current T on X with cohomology class α is said to have *full Monge–Ampère mass* if

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha).$$

We denote by $\mathcal{E}(X, \alpha)$ the set of such currents. If φ is a θ-psh function such that $T = \theta + dd^c\varphi$, the *non-pluripolar Monge–Ampère measure* of φ is

$$\mathrm{MA}\,(\varphi) := \langle (\theta + dd^c\varphi)^n \rangle = \langle T^n \rangle.$$

We will say that φ has *full Monge–Ampère mass* if $\theta + dd^c\varphi$ has full Monge–Ampère mass. We denote by $\mathcal{E}(X, \theta)$ the set of corresponding functions.

Currents with full Monge–Ampère mass have mild singularities.

Proposition 2.9 *A closed positive* (1, 1)*-current* $T \in \mathcal{E}(X, \alpha)$ *has zero Lelong number at every point* $x \in \mathrm{Amp}\,(\alpha)$.

Proof This is an adaptation of [13, Corollary 1.8]. Let us denote $\Omega = \mathrm{Amp}\,(\alpha)$. We claim that for any compact $K \subset\subset \Omega$ there exists a positive closed (1, 1)-current $T_K \in \alpha$ with minimal singularities and such that it is a smooth Kähler form near K. Fix θ a smooth form in α and $T_{\min} = \theta + dd^c\varphi_{\min}$ a current with minimal singularities. By Demailly's regularization theorem [10], in the big class α we can find a strictly positive current with analytic singularities $T_0 = \theta + dd^c\varphi_0$ that is smooth on Ω. Then we define

$$\varphi_C := \max(\varphi_0, \varphi_{\min} - C),$$

where $C >> 1$. Clearly, $T_C = \theta + dd^c\varphi_C$ is the current we were looking for. For any point $x \in \Omega$, let $K = \overline{B(x, r)}$. Let χ be a smooth cut-off function on X such that $\chi \equiv 1$ on $B(x, r) \subset K$ and $\chi \equiv 0$ on $X \setminus B(x, 2r)$, where $r > 0$ is small. Consider a local coordinate system in a neighborhood of x and define the θ-psh function $\psi_\varepsilon = \varepsilon\chi \log\|\cdot\| + \varphi_C$ for ε small enough. Now, if $T = \theta + dd^c\varphi$ has positive Lelong number at point x, then $\varphi \leq \psi_\varepsilon$. On the other hand $T_\varepsilon = \theta + dd^c\psi_\varepsilon$ does not have full Monge–Ampère mass since

$$\int_{\{\psi_\varepsilon \leq \varphi_C - k\}\cap B(x,r)} \mathrm{MA}\,(\psi_\varepsilon^{(k)})$$

does not converge to 0 as k goes to $+\infty$, where $\psi_\varepsilon^{(k)} := \max(\psi_\varepsilon, \varphi_C - k)$ are the "canonical" approximants of ψ_ε ([5, p. 229]). Therefore, by [5, Proposition 2.14], it follows that $T \notin \mathcal{E}(X, \alpha)$. □

We say that a positive closed (1, 1)-current $T \in \alpha$ is pluripolar if it is supported by some closed pluripolar set: if $T = \theta + dd^c\varphi$, T is pluripolar implies that $\operatorname{supp} T \subset \{\varphi = -\infty\}$.

Lemma 2.10 *For $j = 1, \ldots, p$, let $\alpha_j \in H^{1,1}(X, \mathbb{R})$ be a big class and $T_j \in \alpha_j$. If T_1 is pluripolar, then*

$$\langle T_1 \wedge \cdots \wedge T_p\rangle = 0.$$

Proof First note that, since the non-pluripolar product does not put mass on pluripolar sets, we have

$$\mathbf{1}_{X\setminus A}\,\langle T_1 \wedge \cdots \wedge T_n\rangle = \langle T_1 \wedge \cdots \wedge T_n\rangle,$$

with A the closed pluripolar set supporting T_1. Now, let ω be a Kähler form on X. In view of [5, Proposition 1.14], upon adding a large multiple of ω to the T_j's we may assume that their cohomology classes are Kähler classes. We can thus find Kähler forms ω_j such that $T_j = \omega_j + dd^c\varphi_j$. Let U be a small open subset of $X \setminus A$ on which $\omega_j = dd^c\psi_j$, where $\psi_j \leq 0$ is a smooth psh function on U, so that $T_j = dd^c u_j$ on U. By definition, on the plurifine open subset

$$O_k := \bigcap_j \{u_j > -k\}$$

we must have $\mathbf{1}_{O_k}\langle dd^c u_1 \wedge \cdots \wedge dd^c u_p\rangle = \mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k)$. Since u_1 is a smooth potential on U, $u_1 > -k$ for k big enough, and furthermore, since T_1 is supported by A, we have that $dd^c u_1 = 0$. So, clearly

$$\mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k) = 0$$

and hence the conclusion. □

2.2.2 Weighted Energy Classes

By a *weight function*, we mean a smooth increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. We let

$$\mathcal{W}^- := \left\{\chi : \mathbb{R}^- \to \mathbb{R}^- \mid \chi \text{ convex increasing}, \chi(0) = 0, \chi(-\infty) = -\infty\right\}$$

and

$$\mathcal{W}^+ := \left\{\chi : \mathbb{R}^- \to \mathbb{R}^- \mid \chi \text{ concave increasing}, \chi(0) = 0, \chi(-\infty) = -\infty\right\}$$

denote the sets of convex/concave weights. We say that $\chi \in \mathcal{W}_M^+$ if $\exists M > 0$

$$0 \leq |t\chi'(t)| \leq M|\chi(t)| \qquad \text{for all } t \in \mathbb{R}^-.$$

Definition 2.11 Let $\chi \in \mathcal{W} := \mathcal{W}^- \cup \mathcal{W}^+$. We define the χ-energy of a θ-psh function φ as

$$E_{\chi,\theta}(\varphi) := \frac{1}{n+1}\sum_{j=0}^{n}\int_X (-\chi)(\varphi - V_\theta)\langle T^j \wedge \theta_{\min}^{n-j}\rangle \; \in\,]-\infty, +\infty]$$

with $T = \theta + dd^c\varphi$ and $\theta_{\min} = \theta + dd^c V_\theta$. We set

$$\mathcal{E}_\chi(X,\theta) := \{\varphi \in \mathcal{E}(X,\theta) \mid E_{\chi,\theta}(\varphi) < +\infty\}.$$

We denote by $\mathcal{E}_\chi(X,\alpha)$ the set of positive currents in the class α whose global potential has finite χ-energy.

When $\chi \in \mathcal{W}^-$, [5, Proposition 2.8] ensures that the χ-energy is non-increasing and for an arbitrary θ-psh function φ,

$$E_{\chi,\theta}(\varphi) := \sup_{\psi \geq \varphi} E_{\chi,\theta}(\psi) \in]-\infty, +\infty]$$

over all $\psi \geq \varphi$ with minimal singularities. On the other hand, if $\chi \in \mathcal{W}_M^+$, we lose monotonicity of the χ-energy function, but it has been shown in [13, p. 465] that

$$\varphi \in \mathcal{E}_\chi(X,\alpha) \qquad \text{iff} \qquad \sup_{\psi \geq \varphi} E_{\chi,\theta}(\psi) < +\infty$$

over all ψ with minimal singularities. Recall that for all weights $\chi \in \mathcal{W}^-$, $\tilde{\chi} \in \mathcal{W}^+$, we have

$$\mathcal{E}_{\tilde{\chi}}(X,\alpha) \subset \mathcal{E}^1(X,\alpha) \subset \mathcal{E}_\chi(X,\alpha) \subset \mathcal{E}(X,\alpha).$$

For any $p > 0$, we use the notation

$$\mathcal{E}^p(X,\theta) := \mathcal{E}_\chi(X,\theta), \text{ when } \chi(t) = -(-t)^p.$$

3 Bimeromorphic Images of Energy Classes

From now on, X and Y denote arbitrary n-dimensional compact Kähler manifolds. We recall that a bimeromorphic map $f : X \longrightarrow Y$ can be decomposed as

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ X & & Y \end{array}$$

where π_1, π_2 are two holomorphic and bimeromorphic maps and Γ denotes a desingularization of the graph of f. For any positive closed $(1, 1)$-current T on X we set

$$f_\star T := (\pi_2)_\star \pi_1^\star T.$$

For any positive closed (p, p)-current S it is not always possible to define the push-forward under a bimeromorphic map. However, we define $f_\star\langle S\rangle$ in the usual sense in the Zariski open set V, where $f : U \to V$ is a biholomorphism and extending to zero in $Y \setminus V$.

3.1 Bimeromorphic Invariance of the Non-Pluripolar Product

The goal of this section is to show that the non-pluripolar product is a bimeromorphic invariant.

Theorem 3.1 *Let $f : X \longrightarrow Y$ be a bimeromorphic map. Let $\alpha_1, \dots, \alpha_p \in H^{1,1}(Y, \mathbb{R})$ be big classes and fix T_j a positive closed $(1, 1)$-current in α_j. Then*

$$f_\star\langle T_1 \wedge \cdots \wedge T_p\rangle = \langle f_\star T_1 \wedge \cdots \wedge f_\star T_p\rangle. \tag{3.1}$$

Proof By the definition of a bimeromorphic map, f induces an isomorphism between Zariski open subsets U and V of X and Y, respectively. By construction, the non-pluripolar product does not charge pluripolar sets, thus it is enough to check (3.1) on V. Since f induces an isomorphism between U and V, we have

$$\big(f_\star\langle T_1 \wedge \cdots \wedge T_p\rangle\big)|_V = f_\star\big(\langle T_1 \wedge \cdots \wedge T_p\rangle|_U\big) = f_\star\langle T_1|_U \wedge \cdots \wedge T_p|_U\rangle$$

and

$$\langle f_\star T_1 \wedge \cdots \wedge f_\star T_p\rangle|_V = \langle f_\star(T_1|_U) \wedge \cdots \wedge f_\star(T_p|_U)\rangle.$$

Now, let ω be a Kähler form on X. Upon adding a multiple of ω to each T_j we can assume that their cohomology classes are Kähler. Thus we can find Kähler forms ω_j such that $T_j = \omega_j + dd^c\varphi_j$. Fix $p \in U$ and take a small open set B such that

$p \in B \subset U$. In the open set B we can write $\omega_j = dd^c \psi_j$ so that $T_j = dd^c u_j$ on B with $u_j := \psi_j + \varphi_j$. We infer that

$$f_\star \langle \bigwedge_{j=1}^{p} dd^c u_j \rangle = \langle f_\star(dd^c u_1) \wedge \cdots \wedge f_\star(dd^c u_p) \rangle.$$

Indeed, on the plurifine open subset $O_k := \bigcap_j \{u_j > -k\}$ we have

$$\begin{aligned} f_\star\Big(\mathbf{1}_{O_k} \langle \bigwedge_j dd^c u_j \rangle\Big) &= f_\star\Big(\mathbf{1}_{O_k} \bigwedge_j dd^c \max(u_j, -k)\Big) \\ &= \mathbf{1}_{\bigcap_j \{u_j \circ f^{-1} > -k\}} \bigwedge_j f_\star(dd^c \max(u_j, -k)), \end{aligned}$$

where the last equality follows from the fact that for any positive $(1, 1)$-current S with locally bounded potential $(f_\star S)^n = f_\star(S^n)$. □

3.2 Condition $(\mathbb{V})$

Finite energy classes are in general not preserved by bimeromorphic maps (see Example 2.7). We introduce a natural condition to circumvent this problem.

Definition 3.2 Fix α a big class on X. Let $\mathcal{T}_\alpha(X)$ denote the set of positive closed $(1, 1)$-currents in α. We say that Condition $(\mathbb{V})$ is satisfied if

$$f_\star\big(\mathcal{T}_\alpha(X)\big) = \mathcal{T}_{f_\star\alpha}(Y),$$

where $\mathcal{T}_{f_\star\alpha}(Y)$ is the set of positive currents in the image class $f_\star\alpha$.

Theorem A of the Introduction is a consequence of Theorem 3.1 and Proposition 3.3.

Proposition 3.3 *Fix $\alpha \in H^{1,1}_{\mathrm{big}}(X, \mathbb{R})$. If Condition $(\mathbb{V})$ holds, then*

(i) $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$,
(ii) $f_\star(\mathcal{E}(X, \alpha)) = \mathcal{E}(Y, f_\star\alpha)$,
(iii) $f_\star(\mathcal{E}_\chi(X, \alpha)) = \mathcal{E}_\chi(Y, f_\star\alpha)$ *for any weight* $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$.

Observe that in general $\mathrm{vol}(\alpha) \leq \mathrm{vol}(f_\star\alpha)$ (see Example 2.7).

Proof Fix $T_{\min}$ a current with minimal singularities in α. Observe that Condition $(\mathbb{V})$ implies that $f_\star T_{\min}$ is still a current with minimal singularities; thus,

$$\mathrm{vol}(\alpha) = \int_X \langle T^n_{\min} \rangle = \int_Y \langle (f_\star T_{\min})^n \rangle = \mathrm{vol}(f_\star\alpha).$$

Fix $T \in \mathcal{T}_\alpha(X)$. Using Theorem 3.1, the change of variables formula, and the fact that the pluripolar product does not put mass on analytic sets, we get

$$\int_X \langle T^n \rangle = \int_Y \langle (f_\star T)^n \rangle.$$

Hence by (i) it follows that

$$T \in \mathcal{E}(X, \alpha) \Longleftrightarrow f_\star T \in \mathcal{E}(Y, f_\star \alpha).$$

We now want to prove (iii). Let $T = \theta + dd^c\varphi$ and $T_k = \theta + dd^c\varphi^k$, where $\varphi^k = \max(\varphi, V_\theta - k)$ are the canonical approximant (note they have minimal singularities and decrease to φ). We recall that f induces an isomorphism between Zariski opens subsets U and V; thus, by (ii) and the change of variables we get that for any $j = 0, \dots, n$

$$\begin{aligned}\int_X (-\chi)(\varphi^k - V_\theta)\langle T_k^j \wedge {\theta_{\min}}^{n-j} \rangle &= \int_U (-\chi)(\varphi - V_\theta)\langle T_k^j \wedge \theta_{\min}^{n-j} \rangle \\ &= \int_V (-\chi)(\varphi^k \circ f^{-1} - V_\theta \circ f^{-1})\langle (f_\star T_k)^j \wedge (f_\star \theta_{\min})^{n-j} \rangle,\end{aligned}$$

hence the conclusion. □

Condition (V) is easy to understand when f is a blow-up with smooth center:

Remark 3.4 Let $\pi : X \to Y$ be a blow-up with smooth center $\mathcal{Z}$, let $E = \pi^{-1}(\mathcal{Z})$ be the exceptional divisor, and fix a big class α_X on X. There exists a unique $\gamma \in \mathbb{R}$ such that at the level of cohomology classes $\alpha_X = \pi^\star \pi_\star \alpha_X + \gamma\{E\}$. Furthermore, for any (1, 1)-current $S \in \alpha_X$ there exists a (1, 1)-current $T \in \pi_\star \alpha_X$ such that $S = \pi^\star T + \gamma[E]$ and S is positive iff T is positive and $\gamma \geq -\nu(T, \mathcal{Z})$ (consequence of Proposition 8.16 in [11] together with Corollary 1.1.8 in [6]). If Condition (V) holds, then any current $S_{\min}$ with minimal singularities in α_X admits the decomposition

$$S_{\min} = \pi^\star T_{\min} + \gamma[E],$$

where $T_{\min}$ is a current with minimal singularities in $\pi_\star \alpha_X$. When $\gamma \geq 0$, Condition (V) is always satisfied. On the other hand, when $\gamma < 0$ this is not necessarily the case since it could happen that for some positive current T in $\pi_\star \alpha_X$, $\nu(T, \mathcal{Z}) < -\gamma$ (see Example 2.7, where $\gamma = -\varepsilon$ and $\nu(\omega_{FS}, \mathcal{Z}) = 0$).

We observe indeed that Condition (V) is equivalent to requiring that every current $T_Y \in \pi_\star \alpha_X$ is such that $\nu(T_Y, \mathcal{Z}) \geq -\gamma$.

As the first statement of Proposition 3.3 shows, there is a link between Condition (V) and the invariance of the volume under push-forward. For example, if $\mathcal{Z} \not\subseteq X \setminus \mathrm{Amp}\,(\pi_\star \alpha_X)$ then

$$\mathrm{vol}(\alpha_X) = \mathrm{vol}(\pi_\star \alpha_X) \Longleftrightarrow \pi_\star\big(\mathcal{T}_{\alpha_X}(X)\big) = \mathcal{T}_{\pi_\star \alpha_X}(Y).$$

Indeed, ($\Longrightarrow$) is an easy consequence of the fact that under the assumption on the volumes we can decompose any current with minimal singularities $S_{\min} \in \alpha_X$ as $S_{\min} = \pi^\star T + \gamma[E]$ with $T \in \mathcal{E}(Y, \pi_\star\alpha_X)$. Proposition 2.9 implies $\nu(T, \mathcal{Z}) = 0$, hence $\gamma \geq 0$. Let us stress that the assumption on $\mathcal{Z}$ could be removed if we knew that $\nu(T, y) = \nu(T_{\min}, y)$ for any T with full Monge–Ampère mass, for any $T_{\min}$ with minimal singularities in $\pi_\star\alpha_X$ and for any $y \in Y$. It is, however, quite delicate to get such information at points y which lie outside the ample locus.

Proposition 3.5 *Let $f : X - - > Y$ be a bimeromorphic map between compact Kähler manifolds of complex dimension* 2. *Then the following are equivalent:*

(i) $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha)$
(ii) $f_\star\big(\mathcal{T}_\alpha(X)\big) = \mathcal{T}_{f_\star\alpha}(Y)$.

Proof Let us recall that (ii) always implies (i). Furthermore, by Noether's factorization theorem it suffices to consider the case of a blow-up at one point p. We write $\alpha = \pi^\star\pi_\star\alpha + \gamma\{E\}$. We recall that if $\gamma \geq 0$ there is nothing to prove; we can thus assume $\gamma < 0$. Let S be a current with minimal singularities representing α and T a current with minimal singularities representing $\pi_\star\alpha$. By [5, Proposition 1.12], $\pi^* T \in \pi^\star\pi_\star\alpha$ is also with minimal singularities. Note that $\pi^\star T$ is cohomologous to $S - \gamma[E]$. Since α is big, the Siu decomposition of S gives in cohomology the Zariski decomposition of α, and similarly the Siu decomposition of $\pi^\star T$ gives the Zariski decomposition of $\pi^\star\pi_\star\alpha$ (see, e.g., [8]). Furthermore, since $\pi^\star T$ is minimal every divisor appearing in the singular part of the Siu decomposition of $\pi^\star T$ also appears in the singular part of the Siu decomposition of $S - \gamma[E]$ with larger or equal coefficients. Then we write the Siu decomposition of S and of $\pi^\star T$ as

$$S = \theta + \sum_{i=1}^{N} \lambda_i[D_i] + \lambda_0[E], \quad \pi^\star T = \tau + \sum_{i=1}^{N} \eta_i[D_i] + \eta_0[E]$$

with $D_i \neq E$ for all i, $\lambda_i > 0$, $\lambda_0, \eta_i, \eta_0 \geq 0$, where in particular $\eta_0 = \nu(\pi^\star T, E)$ $= \nu(T, p)$. Moreover, $\{\theta\}, \{\tau\}$ are big and nef classes and $\rho_i = \lambda_i - \eta_i \geq 0$, ρ_0 $= \lambda_0 - \gamma - \eta_0 \geq 0$. It follows that

$$\{\theta + A\} = \{\tau\}, \tag{3.2}$$

where $A = \sum_{i=1}^{N} \rho_i[D_i] + \rho_0[E]$ is an effective $\mathbb{R}$ divisor. Observe that if we show $\rho_0 = 0$ then $\lambda_0 = \eta_0 + \gamma = \nu(T, p) + \gamma \geq 0$ and so we are done. Intersecting first with θ and then with τ the relation (3.2), using the assumption on the volumes, i.e., $\{\theta\}^2 = \{\tau\}^2$, the fact that A is effective, and that τ and θ are nef, we find $\{\tau\} \cdot \{A\} = \{\theta\} \cdot \{A\} = 0$. If we develop the square of the left-hand side of (3.2) we conclude $\{A\}^2 = 0$. Since $\{\theta\}^2 > 0$, the Hodge index theorem shows that $\{A\} = 0$ and since A is effective, it is the zero divisor. Hence $\rho_0 = 0$. □

We expect that $\nu(T, x) = \nu(T_{\min}, x)$ for all $x \in X$ whenever $T \in \mathcal{E}(X, \alpha)$. We show the following partial result in this direction:

Proposition 3.6 *Let X be a compact Kähler surface, α be a big class on X and $T \in \mathcal{E}(X, \alpha)$. Then the set $\{x \mid \nu(T, x) > \nu(T_{\min}, x)\}$ is at most countable.*

Proof We write the Siu decomposition of the current T as $T = R + \sum_{j=1}^{N} \lambda_i [D_i]$. Note that the set $E_+(T) := \{x \in X \mid \nu(T, x) > 0\}$ contains at most finitely many divisors (Proposition 2.9). We claim that $\{R\}$ is big and nef. Indeed, by construction the current R has no positive Lelong number along curves and so any current with minimal singularities $R_{\min} \in \{R\}$ has the same property. Thus the Zariski decomposition of $\{R\}$ is of the type $\{R\} = \{R\} + 0$. Furthermore,

$$\mathrm{vol}(\{R\}) \leq \mathrm{vol}(\alpha) = \int_X \langle T^2 \rangle = \int_X \langle R^2 \rangle \leq \mathrm{vol}(\{R\}),$$

which implies $\mathrm{vol}(\alpha) = \{R\}^2 > 0$. Then $T = R + \sum_{j=1}^{N} \rho_i [D_i] + \sum_{j=1}^{N} \eta_i [D_i]$, where $\eta_i = \nu(T_{\min}, D_i)$ with $T_{\min} \in \alpha$. Clearly $\rho_i \geq 0$, for any i. We want to show that $\rho_i = 0$. Set $S := R + \sum_{j=1}^{N} \rho_i [D_i]$ and write the Zariski decomposition of α as $\alpha = \alpha_1 + \sum_{j=1}^{N} \eta_i \{D_i\}$. Then $\alpha_1 = \{S\}$. This means that $\{S\}$ is big and nef and $\mathrm{vol}(\alpha) = \alpha_1^2 = \{S\}^2$. Now, $\{R + A\} = \{S\}$, where $A = \sum_{j=1}^{N} \rho_i [D_i]$ is an effective $\mathbb{R}$ divisor. Using the same arguments as in the proof of Proposition 3.5, we get $\{A\} \cdot \{R\} = \{A\} \cdot \{S\} = \{A\}^2 = 0$, and using the Hodge index theorem we conclude. □

4 Sums of Finite Energy Currents

Let X be a compact Kähler manifold of complex dimension n and let α and β be big classes on X. Given two positive currents $T \in \alpha$ and $S \in \beta$ with full Monge–Ampère mass, it is natural to wonder whether $T + S$ has full Monge–Ampère mass in $\alpha + \beta$, and conversely.

4.1 Stability of Energy Classes

We start proving Theorem B of the Introduction.

Theorem 4.1 *Fix $T \in \mathcal{T}_\alpha(X)$, $S \in \mathcal{T}_\beta(X)$ and $\chi \in \mathcal{W}^- \cup \mathcal{W}_M^+$. Then*

(i) $T + S \in \mathcal{E}(X, \alpha + \beta)$ implies $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$,
(ii) $T + S \in \mathcal{E}_\chi(X, \alpha + \beta)$ implies $T \in \mathcal{E}_\chi(X, \alpha)$ and $S \in \mathcal{E}_\chi(X, \beta)$.

If α, β are Kähler classes, then conversely

(iii) $T \in \mathcal{E}(X, \alpha)$ and $S \in \mathcal{E}(X, \beta)$ implies $T + S \in \mathcal{E}(X, \alpha + \beta)$,
(iv) $T \in \mathcal{E}_\chi(X, \alpha)$ and $S \in \mathcal{E}_\chi(X, \beta)$ implies $T + S \in \mathcal{E}_\chi(X, \alpha + \beta)$.

Proof Pick θ_α and θ_β smooth representatives in α and β, so that $\tilde{\theta} := \theta_\alpha + \theta_\beta$ is a smooth form representing $\alpha + \beta$. We decompose $T = \theta_\alpha + dd^c \varphi$ and $S = \theta_\beta + dd^c \psi$.

We assume $\varphi + \psi \in \mathcal{E}(X, \tilde{\theta})$, and first prove that φ has full mass, which is equivalent to showing

$$m_k := \int_{\{\varphi \le \varphi_{\min} - k\}} \langle (\theta_\alpha + dd^c \max(\varphi, \varphi_{\min} - k))^n \rangle \longrightarrow 0 \qquad \text{as} \quad k \to +\infty,$$

where $T_{\min} = \theta_\alpha + dd^c \varphi_{\min}$ has minimal singularities in α ([5, p. 229]). First, observe that on $X \setminus \{\psi = -\infty\}$ we have

$$\{\varphi \le \varphi_{\min} - k\} \subseteq \{\varphi + \psi \le \varphi_{\min} + \psi - k\} \subseteq \{\varphi + \psi \le \phi_{\min} - k\},$$

where $S_{\min} = \tilde{\theta} + dd^c \phi_{\min}$ has minimal singularities in $\alpha + \beta$. Since the non-pluripolar product does not charge pluripolar sets, we infer

$$\begin{aligned} 0 \le m_k &\le \int_{\{\varphi + \psi \le \phi_{\min} - k\}} \langle (\theta_\alpha + dd^c \max(\varphi, \varphi_{\min} - k))^n \rangle \\ &\le \int_{\{\varphi + \psi \le \phi_{\min} - k\} \setminus \{\psi = -\infty\}} \langle (\tilde{\theta} + dd^c \max(\varphi + \psi, \varphi_{\min} + \psi - k))^n \rangle \\ &\le \int_{\{\varphi + \psi \le \phi_{\min} - k\}} \langle (\tilde{\theta} + dd^c \max(\varphi + \psi, \phi_{\min} - k))^n \rangle, \end{aligned}$$

where the last inequality follows from the fact that $\phi_{\min}$ is less singular than $\varphi_{\min} + \psi$ (see [5, Proposition 2.14]). But, by assumption, the last term goes to 0 as k tends to $+\infty$, hence the conclusion. Changing the role of φ and ψ one can prove similarly that also ψ is with full Monge–Ampère mass.

We now prove the second statement. By assumption, $\varphi + \psi \in \mathcal{E}_\chi(X, \tilde{\theta})$ with χ a convex weight, and so from above we know that φ and ψ both have full Monge–Ampère mass. It suffices to check that $\varphi \in \mathcal{E}_\chi(X, \theta_\alpha)$. By [5],

$$E_{\chi,\theta}(\varphi) < +\infty \quad \text{iff} \quad \sup_k \int_X (-\chi)(\varphi_k - \varphi_{\min}) MA(\varphi_k) < +\infty,$$

for any sequence φ_k of θ_α-psh functions with full Monge–Ampère mass decreasing to φ. Since $T_1 \le T_2$ implies $\langle T_1^n \rangle \le \langle T_2^n \rangle$, we obtain

$$\begin{aligned} &\int_X (-\chi)(\varphi_k - \varphi_{\min}) \langle (\theta_\alpha + dd^c \varphi_k)^n \rangle \\ &\qquad \le \int_{X \setminus \{\psi = -\infty\}} (-\chi)(\varphi_k - \varphi_{\min}) \langle (\tilde{\theta} + dd^c(\varphi_k + \psi))^n \rangle \\ &\qquad \le \int_{X \setminus \{\psi = -\infty\}} (-\chi)(\varphi_k + \psi - \phi_{\min}) \mathrm{MA}\,(\varphi_k + \psi), \end{aligned}$$

where the last inequality follows from the monotonicity of χ and the fact that on $X \setminus \{\psi = -\infty\}$

$$\varphi_k - \varphi_{\min} = (\varphi_k + \psi) - (\varphi_{\min} + \psi) \geq (\varphi_k + \psi) - \phi_{\min}.$$

Therefore, $E_{\chi,\tilde{\theta}}(\varphi + \psi) < +\infty$ implies $E_{\chi,\theta_\alpha}(\varphi) < +\infty$, as desired.

Assume now that α, β are both Kähler classes and choose Kähler forms $\omega_\alpha \in \alpha$, $\omega_\beta \in \beta$ as smooth representatives. We want to prove that if $\varphi \in \mathcal{E}(X, \omega_\alpha)$ and $\psi \in \mathcal{E}(X, \omega_\beta)$ then $\varphi + \psi \in \mathcal{E}(X, \omega_\alpha + \omega_\beta)$. Let ω be another Kähler form on X. We first show that $\varphi \in \mathcal{E}(X, \omega_\alpha)$ (resp., $\varphi \in \mathcal{E}_\chi(X, \omega_\alpha)$) if and only if $\varphi \in \mathcal{E}(X, \omega)$ (resp., $\varphi \in \mathcal{E}_\chi(X, \omega)$) whenever $\varphi \in PSH(X, \omega)$. We recall that, since ω_α and ω are Kähler forms, there exists a constant $C > 0$ such that $\frac{1}{C}\omega \leq \omega_\alpha \leq C\omega$. Thus,

$$\begin{aligned}\int_{\{\varphi \leq -k\}} (\omega_\alpha + dd^c\varphi_k)^n &\leq \int_{\{\varphi \leq -k\}} (C\omega + dd^c\varphi_k)^n \\ &\leq \tilde{C} \sum_{j=0}^n \int_{\{\varphi \leq -k\}} \omega^j \wedge (\omega + dd^c\varphi_k)^{n-j},\end{aligned}$$

where $\varphi_k := \max(\varphi, -k)$. And so $\varphi \in \mathcal{E}(X, \omega)$ implies $\varphi \in \mathcal{E}(X, \omega_\alpha)$. Analogously, one can prove the reverse. Similarly, for any weight $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$,

$$\int_X -\chi(\varphi_k)(\omega_\alpha + dd^c\varphi_k)^n \leq \tilde{C} \sum_{j=0}^n \int_X -\chi(\varphi_k)(\omega + dd^c\varphi_k)^j \wedge \omega^{n-j}.$$

Thus, if $\varphi \in \mathcal{E}_\chi(X, \omega)$ then $\varphi \in \mathcal{E}_\chi(X, \omega_\alpha)$. With the same argument we get the reverse. Now, let ω be a Kähler form such that $\omega_\alpha, \omega_\beta \leq \omega$. From above we have that $\varphi, \psi \in \mathcal{E}(X, \omega)$ (resp., $\varphi, \psi \in \mathcal{E}_\chi(X, \omega)$) and since the energy classes are convex ([13, Propositions 1.6, 2.10 and 3.8]), it follows $\varphi + \psi \in \mathcal{E}(X, 2\omega)$ (resp., $\varphi + \psi \in \mathcal{E}_\chi(X, 2\omega)$). From the previous observation we can deduce $\varphi + \psi \in \mathcal{E}(X, \omega_\alpha + \omega_\beta)$.□

Examples 4.5 and 4.7 below show the reverse implication is not true in general. This is particularly striking if the following condition is not satisfied:

Definition 4.2 We say that pseudoeffective classes $\alpha_1, \ldots, \alpha_p$ satisfy Condition $\mathcal{MS}$ if the sum $T_1 + \cdots + T_p$ of positive currents $T_i \in \alpha_i$ with minimal singularities has minimal singularities in $\alpha_1 + \cdots + \alpha_p$.

Note that if $\alpha_1, \ldots, \alpha_p$ satisfy Condition $\mathcal{MS}$ the positive intersection class $\langle \alpha_1 \cdots \alpha_p \rangle$ turns out to be multi-linear, while it is not so in general ([5, p. 219]).

Proposition 4.3 *Let $T \in \mathcal{T}_\alpha(X)$ and $\chi \in \mathcal{W}^- \cup \mathcal{W}^-_M$. Assume that α is a Kähler class and β is a semipositive class. Fix $\theta_\beta \in \beta$ a semipositive form. Then*

(i) $T + \theta_\beta \in \mathcal{E}(X, \alpha + \beta)$ if and only if $T \in \mathcal{E}(X, \alpha)$,
(ii) $T + \theta_\beta \in \mathcal{E}_\chi(X, \alpha + \beta)$ if and only if $T \in \mathcal{E}_\chi(X, \alpha)$.

We will exhibit an Example 4.5 such that α is semipositive, β is Kähler, θ_β is a Kähler form in β, $T \in \mathcal{E}^1(X,\alpha)$ but $T+\theta_\beta \notin \mathcal{E}^1(X,\alpha+\beta)$.

Proof We will first prove the second statement. Fix ω, θ_β smooth representatives of α and β, respectively, and denote $\tilde{\omega} := \omega + \theta_\beta$. Note that ω can be chosen to be Kähler. Let $T := \omega + dd^c\varphi \in \mathcal{E}_\chi(X,\alpha)$. By [5] we have

$$E_{\chi,\omega}(\varphi) \Longleftrightarrow \sup_k E_{\chi,\omega}(\varphi_k) < +\infty,$$

where $\varphi_k := \max(\varphi, -k)$. We now show that $E_{\chi,\tilde{\omega}}(\varphi_k)$ is uniformly bounded from above. Fix A such that $\tilde{\omega} \leq (A+1)\omega$. Then

$$\begin{aligned} &\int_X -\chi(\varphi_k)\left(\tilde{\omega} + dd^c\varphi_k\right)^j \wedge \tilde{\omega}^{n-j} \\ &\leq (A+1)^{n-j}\int_X -\chi(\varphi_k)\left(A\omega + \omega + dd^c\varphi_k\right)^j \wedge \omega^{n-j} \\ &\leq C\sum_{l=0}^{j}\int_X -\chi(\varphi_k)\left(\omega + dd^c\varphi_k\right)^{j-l} \wedge \omega^{n-j+l} \leq C'\,E_{\chi,\omega}(\varphi_k). \end{aligned}$$

The first statement is an easy consequence of the second one, recalling that

$$\mathcal{E}(X,\alpha) = \bigcup_{\chi\in\mathcal{W}^-} \mathcal{E}_\chi(X,\alpha).$$

The reverse inclusion is Theorem 4.1. □

Remark 4.4 Let us stress that the first statement of Proposition 4.3 could be proved in great generality (α, β big classes such that Condition $\mathcal{MS}$ holds, θ_β current with minimal singularities) if given $\alpha_1,\ldots,\alpha_n$ big classes and $T_1 \in \mathcal{E}(X,\alpha_1)$; the following would hold

$$\int_X \langle T_1 \wedge \theta_{2,\min} \wedge \cdots \wedge \theta_{n,\min}\rangle = \int_X \langle \theta_{1,\min} \wedge \cdots \wedge \theta_{n,\min}\rangle,$$

where $\theta_{i,\min} := \theta_i + dd^c V_{\theta_i} \in \alpha_i$.

4.2 Counterexamples

The following example shows that given two currents $T \in \mathcal{E}^1(X,\alpha)$ and $S \in \mathcal{E}^1(X,\beta)$, we cannot expect that $T+S \in \mathcal{E}^1(X,\alpha+\beta)$, even if α is semipositive and β is Kähler.

Example 4.5 Let $\pi : X \to \mathbb{P}^2$ be the blow-up at one point p and set $E := \pi^{-1}(p)$. Fix $\alpha = \pi^\star\{\omega_{FS}\}$ and $\beta = 2\pi^\star\{\omega_{FS}\} - \{E\}$ so that $\alpha+\beta = 3\pi^\star\{\omega_{FS}\} - \{E\}$. We pick $\tilde{\omega} \in \alpha+\beta$ a Kähler form of the type $\tilde{\omega} = \pi^\star\omega_{FS} + \omega$, where $\omega \in \beta$ is a Kähler form. We will show that

$$\mathcal{E}^1(X,\alpha) \nsubseteq \mathcal{E}^1(X,\alpha+\beta) \cap \mathcal{T}_\alpha(X).$$

The goal is to find a ω_{FS}-psh function φ on $\mathbb{P}^2$ such that $\pi^\star\varphi \in \mathcal{E}^1(X, \pi^\star\omega_{FS})$ but $\pi^\star\varphi \notin \mathcal{E}^1(X, \tilde{\omega})$. Let U be a local chart of $\mathbb{P}^2$ such that $p \to (0,0) \in U$. We define

$$\varphi_\delta := \frac{1}{C}\chi \cdot u_\delta - K_\delta,$$

where $u_\delta := -(-\log\|z\|)^\delta$, χ is a smooth cut-off function such that $\chi \equiv 1$ on $\mathbb{B}$ and $\chi \equiv 0$ on $U \setminus \mathbb{B}(2)$, K_δ is a positive constant such that $\varphi_\delta \le 1$ and $C > 0$. Choosing C big enough φ_δ induces a ω_{FS}-psh function on $\mathbb{P}^2$, say $\tilde{\varphi}_\delta$. Note that by [9, Corollary 2.6] $\tilde{\varphi}_\delta \in \mathcal{E}(\mathbb{P}^2, \omega_{FS})$ if $0 \le \delta < 1$. We let the reader check that $\tilde{\varphi}_\delta \in W^{1,2}(\mathbb{P}^2, \omega_{FS})$ for all $0 \le \delta < 1$. Therefore, $\tilde{\varphi}_\delta \in \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$ iff

$$\int_{\mathbb{P}^2} -\tilde{\varphi}_\delta (dd^c\tilde{\varphi}_\delta)^2 < +\infty.$$

We claim this is the case iff $0 \le \delta < \frac{2}{3}$.
Note that $\tilde{\varphi}_\delta$ is smooth outside p; therefore, we have to check that

$$\int_{\mathbb{B}(\frac{1}{2})} -u_\delta (dd^c u_\delta)^2 < +\infty. \tag{4.1}$$

Set $\chi(t) = -(-t)^\delta$ so that $u_\delta = \chi(\log\|z\|)$. Then $(dd^c u_\delta)^2 = C_1 \frac{1}{8\|z\|^4}\chi'' \cdot \chi'(\log\|z\|) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ on $\mathbb{B}(\frac{1}{2}) \setminus \{(0,0)\}$, hence the convergence of the integral in (4.1) is equivalent to the convergence of

$$\int_{\mathbb{B}(\frac{1}{2})\setminus\{(0,0)\}} \frac{-\chi(\log\|z\|)\cdot\chi''(\log\|z\|)\cdot\chi'(\log\|z\|)}{\|z\|^4} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$
$$= \int_0^{\frac{1}{2}} \frac{-\chi(\log\rho)\cdot\chi''(\log\rho)\cdot\chi'(\log\rho)}{\rho} d\rho = \delta(1-\delta)\int_{-\log\frac{1}{2}}^{+\infty} \frac{1}{(s)^{3-3\delta}} ds,$$

which is finite iff $0 \le \delta < \frac{2}{3}$, as claimed. Therefore, by Proposition 3.3 we get $\pi^\star\tilde{\varphi}_\delta \in \mathcal{E}^1(X, \pi^\star\omega_{FS})$. But $\pi^\star\tilde{\varphi}_\delta \notin \mathcal{E}^1(X,\tilde{\omega})$ if $\frac{1}{2} \le \delta < \frac{2}{3}$ since

$$\left|\nabla(\pi^\star\tilde{\varphi}_\delta)\right| \notin L^2(X, (\tilde{\omega})^2) \quad if \quad \delta \ge \frac{1}{2}.$$

Indeed, let $z = (z_1, z_2) \in \mathbb{B}$ and fix a coordinate chart in X; then $\pi(s,t) = (z_1, z_2) = (s, st)$. Therefore, on $\pi^{-1}(\mathbb{B})$,

$$\varphi_\delta \circ \pi(s,t) = \frac{1}{C} u_\delta(s, st) = -\frac{1}{C}\left(-\log|s| - \log\sqrt{1+|t|^2}\right)^\delta.$$

Hence,

$$\int_{\pi^{-1}(\mathbb{B})}\left|\frac{\partial(\varphi_\delta\circ\pi)}{\partial s}\right|^2 ds\wedge d\bar{s}\wedge dt\wedge d\bar{t}\geq\left(\frac{\delta}{2C}\right)^2\int_{\pi^{-1}(\mathbb{B})}\frac{ds\wedge d\bar{s}\wedge dt\wedge d\bar{t}}{|s|^2(-\log|s|)^{2-2\delta}},$$

which is not finite if $\delta\geq\frac{1}{2}$. The conclusion follows from [13, Theorem 3.2].

Remark 4.6 Observe that α,β satisfy Condition $\mathcal{MS}$ in the previous example and also that $\pi^\star\tilde{\varphi}_\delta\in\mathcal{E}(X,\tilde{\omega})$. Indeed, let $T:=\pi^\star\omega_{FS}+dd^c(\tilde{\varphi}_\delta\circ\pi)$; we need to check that $T+\omega\in\mathcal{E}(X,\alpha+\beta)$. Since $T\in\mathcal{E}(X,\alpha)$ and

$$\langle(T+\omega)^2\rangle=\langle T^2\rangle+2\langle T\rangle\wedge\omega+(\omega)^2,$$

it suffices to show that

$$\{\langle T\rangle\wedge\omega\}=\{\pi^\star\omega_{FS}\}\cdot\{\omega\},$$

which is equivalent to

$$\{(T-\langle T\rangle)\wedge\omega\}=0.$$

Hence, what we need to show is that $T-\langle T\rangle=0$. The $(1,1)$-current $T-\langle T\rangle$ is positive and is supported by the exceptional divisor E. Therefore, using [11, Corollary 2.14], it results that

$$T=\langle T\rangle+\gamma[E],$$

where $\gamma=\nu(T,E)=\nu(\pi_\star T,p)=0$ since $\delta<1$. And so the conclusion.

The previous remark could let us think that whenever $T\in\mathcal{E}(X,\alpha)$ and $S\in\mathcal{E}(X,\beta)$ then $T+S\in\mathcal{E}(X,\alpha+\beta)$, but this is not true either, as the following example shows:

Example 4.7 Let $\pi:X\to\mathbb{P}^2$ be the blow-up at one point p and set $E:=\pi^{-1}(p)$. Consider $\alpha=\pi^\star\{\omega_{FS}\}+\{E\}$ and $\beta=2\pi^\star\{\omega_{FS}\}-\{E\}$. Thus $\alpha+\beta=3\pi^\star\{\omega_{FS}\}$. Since β is a Kähler class we can choose $S=\omega$ with ω a Kähler form. Observe that currents with minimal singularities in α are of the type $\pi^\star S_{\min}+[E]$, where $S_{\min}$ is a current with minimal singularities in $\{\omega_{FS}\}$ (Remark 3.4). By Lemma 2.10,

$$\mathrm{vol}(\alpha)=\int_X\langle(\pi^\star S_{\min}+[E])^2\rangle=\int_X\langle(\pi^\star S_{\min})^2\rangle=\int_X\pi^\star\langle S^2_{\min}\rangle=1,$$

while $\mathrm{vol}(\alpha+\beta)=(\alpha+\beta)^2=9$.

Let now $T\in\mathcal{E}(X,\alpha)$ and recall that any positive $(1,1)$-current in α is of the form $T=\pi^\star S+[E]$ with $S\in\mathcal{T}_{\{\omega_{FS}\}}(\mathbb{P}^2)$. In particular, we choose $T:=\pi^\star\omega_{FS}+[E]$. We want to show that $T+\omega\notin\mathcal{E}(X,\alpha+\beta)$. Now, from the multilinearity of the non-pluripolar product, we get

$$\int_X\langle(T+\omega)^2\rangle=\int_X\langle(\pi^\star\omega_{FS}+[E]+\omega)^2\rangle=\int_X\langle(\pi^\star\omega_{FS}+\omega)^2\rangle=8.$$

Hence $\int_X \langle (T+\omega)^2 \rangle = 8 < 9 = \mathrm{vol}(\alpha+\beta)$.

The same type of computations show that if we pick $T \in \mathcal{E}(X,\alpha)$, then, for any $0 < \varepsilon \le 1$, $T + \varepsilon\omega \notin \mathcal{E}(X, \alpha + \varepsilon\omega)$.

Remark 4.8 Note that in the latter example α, β do not satisfy Condition $\mathcal{MS}$.

5 Comparison of Capacities

Let X be a compact Kähler manifold of complex dimension n and let α be a big class on X. Set $\theta \in \alpha$ a smooth form and $\theta_{\min} := \theta + dd^c V_\theta$ the positive (1, 1)-current in α with "canonical" minimal singularities.

5.1 Intrinsic Capacities

We introduce the space of "$\theta_{\min}$-plurisubharmonic" functions

$$PSH(X, \theta_{\min}) := \{\psi \mid \psi + V_\theta \quad \text{is a } \ \theta\text{−psh function}\}.$$

Note that a $\theta_{\min}$-psh function ψ is not upper-semi-continuous, but $\psi + V_\theta$ is.

5.1.1 Monge–Ampère Capacity

Following [5] we introduce the Monge–Ampère capacity with respect to a big class.

Definition 5.1 We define the capacity of a Borel set $K \subseteq X$ as

$$\mathrm{Cap}_{\theta_{\min}}(K) := \sup\left\{\int_K \langle(\theta_{\min} + dd^c\psi)^n\rangle,\ \psi \in PSH(X, \theta_{\min}) \mid -1 \le \psi \le 0\right\}.$$

Observe that the above one is the same definition as [5, Definition 4.3], just taking $\psi = \varphi - V_\theta$, where φ is a θ-psh function. Here we introduce this equivalent formulation since we need the positivity of the reference current $\theta_{\min}$.

5.1.2 The Relative Extremal Function

We introduce the notion of the relative extremal function with respect to $\theta_{\min}$. If E is a Borel subset of X, we set

$$h_{E,\theta_{\min}}(x) := \sup\left\{\psi(x) \mid \psi \in PSH(X, \theta_{\min}),\ \psi \le 0 \text{ and } \psi_{|E} \le -1\right\},$$

and

$$h^*_{E,\theta_{\min}} := (h_{E,\theta_{\min}} + V_\theta)^* - V_\theta.$$

It is a standard matter to show that, as in the Kähler case (see [12]), the $\theta_{\min}$-psh function $h^*_{E,\theta_{\min}}$ satisfies

$$\mathrm{Cap}_{\theta_{\min}}(K) = \int_K \mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}}) = \int_X (-h^*_{K,\theta_{\min}})\mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}}),$$

where $K \subset X$ is a compact set (for details, see [4, Lemma 1.5]).

5.1.3 Capacities of Sublevel Sets

We now generalize [13, Lemma 5.1].

Lemma 5.2 *Fix $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$, $M \geq 1$. If $\varphi \in \mathcal{E}_\chi(X,\theta)$, then*

$$\exists C_\varphi > 0, \forall t > 1,\ \mathrm{Cap}_{\theta_{\min}}(\varphi < V_\theta - t) \leq C_\varphi |t\,\chi(-t)|^{-1}.$$

Conversely, if there exists C_φ, $\varepsilon > 0$ such that for all $t > 1$,

$$\mathrm{Cap}_{\theta_{\min}}(\varphi < V_\theta - t) \leq C_\varphi |t^{n+\varepsilon}\,\chi(-t)|^{-1},$$

then $\varphi \in \mathcal{E}_\chi(X,\theta)$.

Proof Fix $\varphi \in \mathcal{E}_\chi(X,\theta)$ and $u \in PSH(X,\theta)$ such that $-1 \leq u - V_\theta \leq 0$. For $t \geq 1$, observe that by [5, Proposition 2.14], $\frac{\varphi}{t} + \left(1 - \frac{1}{t}\right) V_\theta \in \mathcal{E}(X,\theta)$ and

$$(\varphi - V_\theta < -2t) \subseteq \left(\frac{\varphi - V_\theta}{t} < -1 + u - V_\theta\right) \subseteq (\varphi - V_\theta < -t).$$

It therefore follows from the generalized comparison principle and from the multilinearity of the non-pluripolar product ([5, Propositions 2.2 and 1.4]) that

$$\begin{aligned}
&\int_{(\varphi - V_\theta < -2t)} MA(u) \\
&\quad \leq \int_{(\varphi - V_\theta < -t)} MA\left(\frac{\varphi}{t} + \left(1 - \frac{1}{t}\right) V_\theta\right) \\
&\quad \leq \left(1 - \frac{1}{t}\right)^n \int_{(\varphi - V_\theta < -t)} \langle \theta^n_{\min} \rangle + t^{-1} \sum_{k=1}^n \binom{n}{k} \int_{(\varphi - V_\theta < -t)} \langle T^k \wedge \theta^{n-k}_{\min} \rangle,
\end{aligned}$$

where $T := \theta + dd^c\varphi$. Furthermore, since

$$MA(V_\theta) = \mathbf{1}_{\{V_\theta = 0\}} \theta^n$$

(see [3, Corollary 2.5]), we get

$$\int_{(\varphi - V_\theta < -t)} \langle \theta^n_{\min} \rangle = \int_{(\varphi - V_\theta < -t) \cap D} \theta^n = \mathbf{1}_D \theta^n(\varphi < -t) \leq C\omega^n(\varphi < -t),$$

where $D := \{V_\theta = 0\}$, ω is a Kähler form on X and $C > 0$. We recall that $\mathrm{vol}_\omega(\varphi < -t)$ decreases exponentially fast (see [12]) and observe that for all $1 \leq k \leq n$,

$$\int_{(\varphi - V_\theta < -t)} \langle T^k \wedge \theta_{\min}^{n-k} \rangle \leq \frac{1}{|\chi(-t)|} \int_X (-\chi) \circ (\varphi - V_\theta) \langle T^k \wedge \theta_{\min}^{n-k} \rangle \leq \frac{1}{|\chi(-t)|} E_\chi(\varphi).$$

This yields the first assertion.

The second statement follows from similar arguments as in the Kähler case, working with the θ-psh function $u := \frac{1}{t}\varphi_t + \left(1 - \frac{1}{t}\right) V_\theta$, where $\varphi_t := \max(\varphi, V_\theta - t)$ for any $\varphi \in PSH(X, \theta)$. Let us stress that this is the only place where the assumption on the weight, $\chi \in \mathcal{W}^- \cup \mathcal{W}_M^+$, is used. □

5.1.4 Alexander Capacity

For K a Borel subset of X, we set

$$V_{K,\theta} := \sup\{\varphi \mid \varphi \in PSH(X, \theta),\ \varphi \leq 0 \text{ on } K\}.$$

Note that

$$V_\theta = V_{X,\theta} \leq V_{K,\theta}$$

by definition. It follows from standard arguments (see [12, Theorem 4.2]) that the usc regularization $V_{K,\theta}^*$ of $V_{K,\theta}$ is either a θ-psh function with minimal singularities (when K is non-pluripolar) or identically $+\infty$ (when K is pluripolar).

Definition 5.3 (*Alexander–Taylor capacity*) Let K be a Borel subset of X. We set

$$T_\theta(K) := \exp(-\sup_X V_{K,\theta}^*).$$

As in the Kähler case, the capacities T_θ and $\mathrm{Cap}_{\theta_{\min}}$ compare as follows:

Proposition 5.4 *There exists $A > 0$ such that for all Borel subsets $K \subset X$,*

$$\exp\left[-\frac{A}{\mathrm{Cap}_{\theta_{\min}}(K)}\right] \leq T_\theta(K) \leq e \cdot \exp\left[-\left(\frac{\mathrm{vol}(\alpha)}{\mathrm{Cap}_{\theta_{\min}}(K)}\right)^{\frac{1}{n}}\right]$$

Proof It suffices to treat the case of compact sets. The second inequality is [5, Lemma 4.2]. We prove the first inequality. We can assume that $M := M_\theta(K) \geq 1$; otherwise, it is sufficient to adjust the value of A. Let φ be a θ-psh function such that $\varphi \leq 0$ on K. Then $\varphi \leq M$ on X, hence $w := M^{-1}(\varphi - M - V_\theta) \in PSH(X, \theta_{\min})$ satisfies $\sup_X w \leq 0$ and $w \leq -1$ on K. We infer $w \leq h_{K,\theta_{\min}}^*$ and

$$w_K := \frac{V_{K,\theta}^* - M - V_\theta}{M} \leq h_{K,\theta_{\min}}^* \leq 0.$$

Then we get

$$\begin{aligned}\mathrm{Cap}_{\theta_{\min}}(K) &= \int_X \left(-h^*_{K,\theta_{\min}}\right) \mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}})\\ &\le \frac{1}{M}\int_X -(V^*_{K,\theta} - M - V_\theta)\,\mathrm{MA}\,(V_\theta + h^*_{K,\theta_{\min}})\\ &\le \frac{C_1}{M}\end{aligned}$$

with $C_1 > 0$. The last estimate follows from the lemma below, together with [12, Proposition 1.7], since $\sup_X(V^*_{K,\theta} - M - V_\theta) = 0$ and by [3, Corollary 2.5], $\langle(\theta + dd^c V_\theta)^n\rangle = \mathbf{1}_{\{V_\theta=0\}}\theta^n \le C\omega^n$. □

The following lemma is a straightforward generalization of [12, Corollary 2.3] (see also [4, Lemma 3.2]).

Lemma 5.5 *Let ψ, φ be θ-psh functions with minimal singularities with φ normalized in such a way that $0 \le \varphi - V_\theta \le 1$. Then we have*

$$\int_X -(\psi - V_\theta)\langle(\theta + dd^c\varphi)^n\rangle \le \int_X -(\psi - V_\theta)\langle(\theta + dd^c V_\theta)^n\rangle + n\,\mathrm{vol}(\alpha).$$

5.2 Comparing Capacities

We introduce a slightly different notion of big capacity that is comparable with respect to the usual one. For any Borel set $K \subset X$ we define

$$\mathrm{Cap}^\lambda_{\theta_{\min}}(K) := \sup\left\{\int_K \langle(\theta_{\min} + dd^c\psi)^n\rangle,\ \psi \in PSH(X, \theta_{\min}) \mid -\lambda \le \psi \le 0\right\},$$

where $\lambda \ge 1$. We let the reader check that

$$\mathrm{Cap}_{\theta_{\min}}(K) \le \mathrm{Cap}^\lambda_{\theta_{\min}}(K) \le \lambda^n\,\mathrm{Cap}_{\theta_{\min}}(K). \tag{5.1}$$

We now compare the Monge–Ampère capacities with respect to different big classes (Theorem D of the Introduction).

Theorem 5.6 *Let α_1 and α_2 be big classes on X such that $\alpha_1 \le \alpha_2$. We assume that $\{\alpha_1, \alpha_2 - \alpha_1\}$ satisfies Condition $\mathcal{MS}$ and that there exists a positive $(1,1)$-current $T_0 \in \alpha_2 - \alpha_1$ with bounded potentials. Then there exist $C > 0$ such that for any Borel set $K \subset X$,*

$$\frac{1}{C}\,\mathrm{Cap}_{\theta_{1,\min}}(K) \le \mathrm{Cap}_{\theta_{2,\min}}(K) \le C\left(\mathrm{Cap}_{\theta_{1,\min}}(K)\right)^{\frac{1}{n}}.$$

Note that in case of Kähler forms the result is stronger and the proof much simpler (see [5, Proposition 2.5]) but we cannot expect better in the general case of big classes. The following Example 5.7 shows that the exponent at the right-hand side is necessary.

Proof Fix $\theta_1 \in \alpha_1, \theta_2 \in \alpha_2$ smooth forms. Write $T_0 = (\theta_2 - \theta_1) + dd^c f_0$, where f_0 is a bounded potential. Let φ be a θ_1-psh function such that $-1 \leq \varphi - V_{\theta_1} \leq 0$. Then $\varphi + f_0$ is a θ_2-psh function. Condition $\mathcal{MS}$ ensures that the potential $V_{\theta_1} + f_0$ has minimal singularities, thus there exists a positive constant C such that $|V_{\theta_2} - V_{\theta_1} - f_0| \leq C$. Therefore, $-\lambda \leq \varphi + f_0 - C - V_{\theta_2} \leq 0$, where $\lambda = 1 + 2C$. Now, using (5.1) and the fact that $T_1 \leq T_2$ implies $\langle T_1^n \rangle \leq \langle T_2^n \rangle$, we get

$$\int_K \langle (\theta_1 + dd^c \varphi)^n \rangle \leq \int_K \langle (\theta_2 + dd^c (\varphi + f_0)^n \rangle,$$

namely, $\mathrm{Cap}_{\theta_{1,\min}}(K) \leq \mathrm{Cap}^{\lambda}_{\theta_{2,\min}}(K) \leq \lambda^n \, \mathrm{Cap}_{\theta_{2,\min}}(K)$, hence the left inequality. In order to prove the other inequality, we have to go through the Alexander capacity. Since $V^*_{\theta_1,K} + f_0 \leq V^*_{\theta_2,K}$,

$$\sup_X (V^*_{\theta_2,K}) \geq \sup_X (V^*_{\theta_1,K}) + \inf_X f_0,$$

and so

$$T_{\theta_2}(K) \leq T_{\theta_1}(K) \cdot e^{-\inf_X f_0}.$$

Furthermore, using Proposition 5.4 we get

$$\begin{aligned} \exp\left[-\frac{A}{\mathrm{Cap}_{\theta_{2,\min}}(K)} \right] &\leq T_{\theta_2}(K) \\ &\leq T_{\theta_1}(K) \cdot e^{-\inf_X f_0 + 1} \\ &\leq e^{-\inf_X f_0 + 1} \cdot \exp\left[-\left(\frac{\mathrm{vol}(\alpha_1)}{\mathrm{Cap}_{\theta_{1,\min}}(K)} \right)^{\frac{1}{n}} \right] \end{aligned}$$

with A a positive constant. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned} \mathrm{Cap}_{\theta_{2,\min}}(K) &\leq A \left[\left(\frac{\mathrm{vol}(\alpha_1)}{\mathrm{Cap}_{\theta_{1,\min}}(K)} \right)^{\frac{1}{n}} + \inf_X f_0 - 1 \right]^{-1} \\ &\leq C \ \mathrm{Cap}_{\theta_{1,\min}}(K)^{\frac{1}{n}}. \end{aligned}$$

Hence the conclusion. □

Example 5.7 Let $\pi : X \to \mathbb{P}^2$ be the blow-up at one point p and set $E := \pi^{-1}(p)$. Consider $\alpha_1 = \{\pi^\star \omega_{FS}\}$ and $\alpha_2 = \{\tilde{\omega}\}$, where $\tilde{\omega}$ is a Kähler form on X. Let Δ_r be the polydisc of radius $r < 1$ on $\mathbb{P}^2$. By [12, Proposition 2.10] and [14, Lemma 4.5.8] we know that

$$\mathrm{Cap}_{\pi^\star \omega_{FS}}(\pi^{-1}(\Delta_r)) = \mathrm{Cap}_{\omega_{FS}}(\Delta_r) \sim \frac{1}{(-\log r)^2}.$$

Fix now a local chart $U \subset X$ such that $p \in U$ and consider $K_r \subset U$, $K_r := \{(s,t) \in U \mid 0 < \|s\| < r,\ 0 < \|t\| < 1\}$. Then

$$\mathrm{Cap}_{\tilde{\omega}}(\pi^{-1}(\Delta_r)) \geq \mathrm{Cap}_{\tilde{\omega}}(K_r) \sim C\frac{1}{-\log r},$$

with C a positive constant.

5.3 Energy Classes with Homogeneous Weights

As Example 4.5 shows, we cannot hope to get stability of weighted energy classes $\mathcal{E}_\chi$ by only adding Condition $\mathcal{MS}$. We nevertheless establish a partial stability property with a gap for energy classes with respect to homogeneous weights $\chi(t) = -(-t)^p$. We recall that the functions $\chi(t) = -(-t)^p$ belong to $\mathcal{W}^-$ if $0 < p \leq 1$, while they belong to $\mathcal{W}_M^+$ when $p \geq 1$.

Proposition 5.8 *Let α, β be big classes. Assume that $S \in \beta$ has bounded potential and the couple (α, β) satisfies Condition $\mathcal{MS}$. If $p > n^2 - 1$ then*

$$T \in \mathcal{E}^p(X,\alpha) \Longrightarrow T + S \in \mathcal{E}^q(X, \alpha+\beta),$$

where $0 < q < p - n^2 + 1$.

Proof Fix $\theta_\alpha, \theta_\beta$ smooth representatives of α, β, respectively, and set $\tilde{\theta} := \theta_\alpha + \theta_\beta$. Write $S = \theta_\beta + dd^c\psi$ and denote $\theta_{\alpha,\min} := \theta_\alpha + dd^c V_{\theta_\alpha}$ and $\tilde{\theta}_{\min} := \tilde{\theta} + dd^c V_{\tilde{\theta}}$. We want to show that there exists a positive number $q < p$ such that given a θ_α-psh function $\varphi \in \mathcal{E}^p(X,\theta_\alpha)$ then $\varphi + \psi \in \mathcal{E}^q(X,\tilde{\theta})$. By the first claim of Lemma 5.2, for any $t > 1$ there exists a constant $C_\varphi > 0$ such that

$$\mathrm{Cap}_{\theta_{\alpha,\min}}(\varphi - V_{\theta_\alpha} < -t) \leq C_\varphi t^{-(p+1)}. \tag{5.2}$$

The goal is to find a similar estimate from above of the quantity $\mathrm{Cap}_{\tilde{\theta}_{\min}}(\varphi + \psi - V_{\tilde{\theta}} < -t)$. Set $K := \{\varphi - V_{\theta_\alpha} < -t\}$ and $\tilde{K} := \{\varphi + \psi - V_{\tilde{\theta}} < -t\}$. We infer that Condition $\mathcal{MS}$ implies $\tilde{K} \subseteq K$. Thus $\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq \mathrm{Cap}_{\tilde{\theta}_{\min}}(K)$. Now, by Theorem 5.6 we know that there exists $A > 0$ such that

$$\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq A\ \mathrm{Cap}_{\theta_{\alpha,\min}}(K)^{\frac{1}{n}} \leq \tilde{C}_\varphi\ t^{-\frac{p+1}{n}},$$

where the last inequality follows from (5.2). This means that there exist $C_\varphi, \varepsilon > 0$ such that

$$\mathrm{Cap}_{\tilde{\theta}_{\min}}(\tilde{K}) \leq C_\varphi t^{-(n+\varepsilon+q)}$$

with $0 < q < p - n^2 + 1 - n\varepsilon$. Hence by Lemma 5.2 we get $\varphi + \psi \in \mathcal{E}^q(X,\tilde{\theta})$. □

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