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Explicit Reconstruction of Riemann Surface with Given Boundary in Complex Projective Space

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Abstract In this paper we propose a numerically realizable method for reconstruction of a complex curve with known boundary and without compact components in complex projective space.

Keywords Riemann surface · Reconstruction algorithm · Burgers equation · Cauchy-type formulas

Mathematics Subject Classification 30F · 32H · 35Q · 58Z · 65E · 76L.

1 Introduction

Let us denote by $\mathbb{C}P^2$ the complex projective space with homogeneous coordinates $(w_0 : w_1 : w_2)$. Let a real closed rectifiable, oriented curve γ in $\mathbb{C}P^2$ be the boundary of a complex curve $X \subset \mathbb{C}P^2$ with notation $\gamma = bX$. Without restriction of generality we suppose that the following conditions of general position hold:

$$(0 : 1 : 0) \notin X, \quad w_0|_\gamma \neq 0.$$

Put $\mathbb{C}^2 = \{w \in \mathbb{C}P^2 \colon w_0 \neq 0\}$ with coordinates $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$. For almost all $\xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*$ the points of intersection of X with complex line $\mathbb{C}^1_\xi = \{z \in \mathbb{C}^2 \colon \xi_0 + \xi_1 z_1 + z_2 = 0\}$ form a finite set of points

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$$\left(z_1^{(j)}(\xi), z_2^{(j)}(\xi)\right) = \left(h_j(\xi_0, \xi_1), -\xi_0 - \xi_1 h_j(\xi_0, \xi_1)\right), \quad j = 1, \ldots, N_+(\xi).$$

By Darboux's lemma [3,5] functions $\{h_j\}$ satisfy the equations

$$\frac{\partial h_j(\xi_0, \xi_1)}{\partial \xi_1} = h_j(\xi_0, \xi_1) \frac{\partial h_j(\xi_0, \xi_1)}{\partial \xi_0}, \quad j = 1, \ldots, N_+(\xi), \tag{1}$$

which are often called shock-wave equations or Riemann–Burgers equations. In this interpretation ξ_1 is the time variable and ξ_0 is the space variable.

The following Cauchy-type formula from [5] plays the essential role in reconstruction of X through γ:

$$G_m(\xi_0, \xi_1) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_\gamma z_1^m (\xi_0 + \xi_1 z_1 + z_2)^{-1} d(\xi_0 + \xi_1 z_1 + z_2)$$
$$= \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad m = 0, 1, \ldots, \tag{2}$$

where $N_+(\xi) = N_+(\xi_0, \xi_1)$ is the number of points of intersection (multiplicities taken into account) of X with complex line $\mathbb{C}^1_\xi$, $P_m(\xi_0, \xi_1)$ is a polynomial of degree at most m with respect to ξ_0. In addition, $P_0(\xi_0, \xi_1) = -N_-$, where N_- is the number of points of intersection of X with infinity $\{w \in \mathbb{C}\mathrm{P}^2 : w_0 = 0\}$,

$$P_1(\xi_0, \xi_1) = \sum\nolimits_{k=1}^{N_-(\xi)} \frac{a_k \xi_0 - b_k}{a_k \xi_1 + 1}, \tag{3}$$

$a_k = w_2(q_k)$, $b_k = \frac{dw_2}{dw_0}(q_k)$, where q_k, $k = 1, \ldots, N_-$, are the points of intersection of X with infinity $\{w \in \mathbb{C}\mathrm{P}^2 : w_0 = 0\}$. In particular, the following corollary of (2) holds:

$$G_0(\xi_0, \xi_1) = \frac{1}{2\pi i} \int_\gamma \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \tag{4}$$

Let further $\xi_1 = 0$ and let $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$ be the projection on the second factor: $\pi_2(z_1, z_2) = -z_2$. We have $\pi_2\gamma \subset \mathbb{C}$, $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^{L} \Omega_l$, where $\{\Omega_l\}$ are the connected components of $\mathbb{C} \setminus \pi_2\gamma$. For every component Ω_l the number of points of intersection of X with line $z_2 = -\xi_0$, $\xi_0 \in \Omega_l$, multiplicities taken into account, will be denoted by $\mu_l = N_+(\xi_0, 0)$. Let Ω_0 denote the unbounded component of set $\mathbb{C} \setminus \pi_2\gamma$. From the definition of $N_\pm$ it follows that

$$\mu_0 = N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0. \tag{5}$$

Assume that complex curve X does not contain compact components, or equivalently, satisfies the following condition of minimality:

For arbitrary complex curve $\widetilde{X} \subset \mathbb{C}\mathrm{P}^2$ *with condition* $b\widetilde{X} = bX = \gamma$ *and for almost all* $\xi \in (\mathbb{C}^2)^*$ *the number of points of intersection* $\widetilde{N}_+(\xi)$ *of* $\widetilde{X}$ *with line* $\mathbb{C}^1_\xi$, *multiplicities taken into account, is not less than the number* $N_+(\xi)$ *for curve* X. (*)

Condition of minimality (*) is a condition of general position and is fulfilled for X if, for example, every irreducible component of X is a transcendental complex curve. Note that from theorems of Chow [2] and Harvey, Shiffman [7] it follows that an arbitrary complex curve $\widetilde{X} \subset \mathbb{C}\mathrm{P}^2$ with condition $b\widetilde{X} = bX$ admits the unique representation $\widetilde{X} = X \cup V$, where X is a curve with condition of minimality (*), and V is a compact algebraic curve, possibly with multiple components.

The main result of [4] gives a solution of the important problem of J. King [9], when a real curve $\gamma \subset \mathbb{C}\mathrm{P}^2$ is the boundary of a complex curve $X \subset \mathbb{C}\mathrm{P}^2$. Let $\gamma \subset \mathbb{C}^2 \subset \mathbb{C}\mathrm{P}^2$. Then $\gamma = bX$ for some open connected complex curve X in $\mathbb{C}\mathrm{P}^2$ if and only if on a neighborhood W_{ξ^*} of some point $\xi^* \in (\mathbb{C}^2)^*$ one can find mutually distinct holomorphic functions $h_1, \ldots, h_p$ satisfying shock-wave Eq. (1) and also the equation

$$\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi_0, \xi_1) - \sum\nolimits_{j=1}^{p} h_j(\xi_0, \xi_1)\big) = 0, \quad \xi = (\xi_0, \xi_1) \in W_{\xi^*}.$$

In this work in development of [4,5] we obtained a numerically realizable algorithm for reconstruction of complex curve $X \subset \mathbb{C}P^2$ with known boundary and with condition of minimality. This algorithm permits, in particular, making applicable the result of [8] about the principal possibility to reconstruct topology and conformal structure of a two-dimensional bordered surface X in $\mathbb{R}^3$ with constant scalar conductivity from measurements on bX of electric current densities, being created by three potentials in general position.

Our algorithm depends on parameter $\mu_0 = N_\pm(\xi_0, 0)$, $\xi_0 \in \Omega_0$. It was tested on many examples and admits simple and complete justification for $\mu_0 = 0, 1, 2$. Despite a cumbersome description for $\mu_0 \geq 3$, the algorithm shows that there are no obstacles for its justification and numerical realization for any $\mu_0 \geq 0$. Moreover, in Theorem 3.2 we propose a method for finding parameter μ_0 in terms of γ. This makes the algorithm much more applicable.

The preliminary version [1] of this work appeared in HAL (http://hal.archives-ouvertes.fr/hal-00912925) 2013, 2014.

2 Cauchy-Type Formulas and Riemann–Burgers Equations

Let us give at first a new proof of the Cauchy-type formula (2) from [5], permitting us to obtain explicit expressions for functions $P_m(\xi_0, \xi_1)$.

Theorem 2.1 *Let* $X \subset \mathbb{C}\mathrm{P}^2 \setminus [0 : 1 : 0]$ *be a complex curve without compact components,* $\gamma = bX \subset \mathbb{C}^2$ *be a real rectifiable oriented curve. Suppose that for* *almost all* $\xi \in (\mathbb{C}^2)^*$ *all the points of intersection of* X *with* $\mathbb{C}\mathrm{P}^1_\xi$ *have multiplicity at most one. Then the following equalities are fulfilled:*

$$G_m(\xi_0, \xi_1) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad \xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*, \ m \geq 1, \tag{6}$$

where $P_m(\xi_0, \xi_1)$ *is a polynomial of degree at most* m *with respect to* ξ_0 *of the following form:*

$$P_m(\xi_0, \xi_1) = \sum_{s=1}^{\mu_0} \sum_{k=0}^{m-1} \sum_{i_1+\cdots+i_m=k} \frac{\frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)} \frac{d^{m-k}}{dw_0^{m-k}} \times \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} - \sum_{s=1}^{\mu_0} \sum_{i_1+\cdots+i_m=m} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s),$$

where $q_s \in X \cap \{w \in \mathbb{C}\mathrm{P}^2 \colon w_0 = 0\}$. *In particular, if* $\mu_0 = 0$ *then* $P_m \equiv 0$.

Remark 2.1 In the exceptional case, when $[0:1:0] \in X$, the term $P_m(\xi_0, \xi_1)$ in (6) need not be a polynomial with respect to ξ_0 in general.

Proof Put $\widetilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$ and $g = \frac{\widetilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$. Consider differential forms

$$\omega_m \stackrel{\text{def}}{=} z_1^m \frac{dg}{g} = \frac{w_1^m}{w_0^m} \frac{w_0}{\widetilde{g}} d\left(\frac{\widetilde{g}}{w_0}\right) = \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \ldots$$

Then $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$. Let us compute this integral explicitly. Denote by p_j, $j = 1, \ldots, N_+(\xi)$ the points of intersection of X with $\mathbb{C}\mathrm{P}^1_\xi$, and by q_s, $s = 1, \ldots, \mu_0$ the points of intersection of X with infinity $\{w \in \mathbb{C}\mathrm{P}^2 \colon w_0 = 0\}$. Denote by B_j^ε the intersection of X with the ball of radius ε in $\mathbb{C}\mathrm{P}^2$ centered at p_j and by D_s^ε the intersection of X with the ball of radius ε centered at q_s. The restriction of form ω_m on X is meromorphic with poles at points p_j and q_s. Thus the following equality is valid:

$$G_m(\xi) = \frac{1}{2\pi i} \int\limits_\gamma \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int\limits_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{\mu_0} \frac{1}{2\pi i} \int\limits_{bD_s^\varepsilon} \omega_m.$$

If $\mu_0 = 0$, then the second group of terms is absent. The integral $\int_{bB_j^\varepsilon} \omega_m$ can be calculated as a residue at the first order pole:

$$\int\limits_{bB_j^\varepsilon} \omega_m = \int\limits_{bB_j^\varepsilon} z_1^m \frac{d\widetilde{g}}{\widetilde{g}} - \int\limits_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int\limits_{bB_j^\varepsilon} z_1^m \frac{d\widetilde{g}}{\widetilde{g}} = 2\pi i \, h_j^m(\xi).$$

Let $\mu_0 > 0$. Computation of integral $\int_{bD_s^\varepsilon} \omega_1$ will be done in two steps. Let us calculate first $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$. Consider the expansion of $w_1(w_0)$ into power series in w_0 in the neighborhood of point q_s:

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2 w_1}{dw_0^2}(q_s)w_0^2 + \cdots .$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) w_0^k.$$

The coefficient near w_0^m can be presented in the form

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\cdots+i_m=m} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}}$. Using relation $d\widetilde{g} = \frac{d\widetilde{g}}{dw_0} dw_0$ and expansion of $w_1(w_0)$ into power series in w_0 we obtain:

$$\begin{aligned}
\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} &= \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left(w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2 w_1}{dw_0^2}(q_s)w_0^2 + \cdots \right)^m \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 \\
&= \sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 \\
&= \sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 \\
&= \sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) \\
&\quad \times \lim_{w_0 \to 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left(\frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} \right).
\end{aligned}$$

From here, taking into account the relation $\frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} = \frac{d \ln \widetilde{g}}{dw_0}$, we obtain, finally

$$\begin{aligned}
\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} &= 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\cdots+i_m=k} \frac{\frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)} \frac{d^{m-k}}{dw_0^{m-k}} \\
&\quad \times \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.
\end{aligned}$$

It is a polynomial of degree at most m with respect to ξ_0. □

We will also use the following result, giving the effective characterization of functions $\{h_j(\xi_0, \xi_1)\}$ and $\{P_m(\xi_0, \xi_1)\}$ satisfying Riemann–Burgers equations for $\{h_j\}$ and system (6) for $\{h_j\}$ and $\{P_m\}$. Denote by $\Omega_l^{(k)}$ the infinitesimal neighborhood of order k of the set Ω_l in $\{(z_1, z_2)\colon z_1 \in \Omega_l,\ z_2 \in \mathbb{C}\}$.

Theorem 2.2 *Let $X \subset \mathbb{C}\mathrm{P}^2 \setminus [0:1:0]$ be a complex curve without compact components, $\gamma = bX \subset \mathbb{C}^2$. Fix $l \in \{0, \ldots, L\}$. Suppose that functions $\widehat{h}_j$, $j = 1$, ..., μ_l, are mutually distinct and analytic in $\Omega_l^{(1)}$ and satisfy the Riemann–Burgers equation in $\xi \in \Omega_l^{(1)}$:*

$$\frac{\partial \widehat{h}_j}{\partial \xi_1}(\xi) = \widehat{h}_j(\xi)\frac{\partial \widehat{h}_j}{\partial \xi_0}(\xi), \quad j = 1, \ldots, \mu_l. \tag{7}$$

Then the functions $\widehat{h}_j$, $j = 1, \ldots, \mu_l$, satisfy the system

$$G_m(\xi) = \sum_{j=1}^{\mu_l} \widehat{h}_j^m(\xi) + \widehat{P}_m(\xi), \quad \xi \in \Omega_l^{(1)}, \quad m = 1, 2, \ldots, \tag{8}$$

where $\widehat{P}_m$, $m = 1, 2, \ldots$, are some analytic functions in $\Omega_l^{(1)}$, being polynomials of degree at most m with respect to ξ_0, if and only if the functions $\widehat{h}_j$, $j = 1, \ldots, \mu_l$, satisfy the equation

$$0 = \frac{\partial^2}{\partial \xi_0^2}\left(G_1(\xi) - \sum_{j=0}^{\mu_l} \widehat{h}_j(\xi)\right), \quad \xi \in \Omega_l^{(1)}. \tag{9}$$

Moreover, for minimal $\{\mu_l\}$ with properties (7)–(9) there exists the unique set of functions $\widehat{h}_j$, $j = 1, \ldots, \mu_l$, satisfying the equivalent conditions (8)–(9) and the unique set of functions $\widehat{P}_m$, $m = 1, 2, \ldots$, from condition (8). Furthermore, $\widehat{h}_j = h_j$, $\widehat{P}_m = P_m$ for $j = 1, \ldots, \mu_l$, $m = 1, \ldots$, where functions h_j and P_m are defined in Theorem (2.1).

Proof *Necessity.* From (8) it follows that

$$G_1(\xi) - \sum_{j=1}^{\mu_l} \widehat{h}_j(\xi) = \widehat{P}_1(\xi), \quad \xi \in \Omega_l^{(1)}.$$

Differentiating the latter equality two times with respect to ξ_0 and taking into account that $\widehat{P}_1$ is a polynomial in ξ_0 of degree at most 1 we obtain (9).

Sufficiency. Suppose that mutually distinct functions $\{\widehat{h}_j(\xi)\}$ on $\Omega_l^{(1)}$, $l = 0, 1, \ldots, L$, are holomorphic and satisfy the Eqs. (7), (9). In particular, for any $\xi_0 \in \Omega_l^{(0)}$ we have

$$\frac{\partial \widehat{h}_j}{\partial \xi_1}(\xi_0, 0) = \widehat{h}_j(\xi_0, 0)\frac{\partial \widehat{h}_j}{\partial \xi_0}(\xi_0, 0).$$

By Cauchy–Kowalewski's theorem in a neighborhood of arbitrary $\xi^* \in \Omega_l^{(1)}$ there exist unique holomorphic functions $\{\widetilde{h}_j(\xi_0, \xi_1)\}$, satisfying the Riemann–Burgers equation (7) and such that $\widetilde{h}_j|_{\Omega_l^{(1)}} = \widehat{h}_j$.

From here and from Proposition 3.3.3 of [5] we obtain existence and uniqueness of holomorphic functions $\{\widehat{P}_m(\xi)\}$, being polynomials of degree at most m in ξ_0, such that $\{\widehat{h}_j(\xi)\}$ and $\{\widehat{P}_m(\xi)\}$ satisfy the system (8) for $m = 1, 2, \ldots, j = 1, \ldots, \mu_l$, $\xi \in \Omega_l^{(1)}$.

Existence and uniqueness. Existence of functions $\{\widehat{h}_j\}$ and $\{\widehat{P}_m\}$ with necessary properties follows from Theorem 2.1. More precisely, $\widehat{h}_j = h_j$ and $\widehat{P}_m = P_m$, $j = 1, \ldots, \mu_l$, $m \geq 1$. Uniqueness of functions $\{\widehat{h}_j\}$ for minimal $\{\mu_l\}$ with properties (8)–(9) follows from Theorem II of [5] and from Theorem 3 of [8]. Uniqueness of polynomials $\{\widehat{P}_m\}$ for minimal $\{\mu_l\}$ with properties (8)–(9) follows from the proof of sufficiency. □

3 Reconstruction Algorithm

Consider now the reconstruction algorithm of complex curve $X \subseteq \mathbb{C}\mathrm{P}^2$ with given boundary bX and with condition of minimality (*). Let us consider the cases $\mu_0 = 0$, 1, 2.

The reconstruction algorithm is based on formulas (6) with polynomials P_m, $m = 0$, 1, …. The next theorem permits calculating these polynomials. If i, j, k, l are non-negative integers we will use the notation

$$æ_{kl}^{ij} = \frac{1}{2\pi i} \int\limits_{\gamma} \left(z_1^i z_2^j \, dz_1 + z_1^k z_2^l \, dz_2\right). \tag{10}$$

Theorem 3.1 *Let $X \subset \mathbb{C}P^2$ be a complex curve without algebraic subdomains, $\gamma \subset \mathbb{C}^2$ be its boundary. Let mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$, $l = 0, 1, \ldots, L$, functions $\{h_j(\xi)\}$ and holomorphic in $\xi \in \Omega_l^{(1)}$, $l = 0, \ldots, L$, functions $P_m(\xi_0, \xi_1)$, being polynomials of degree at most m in ξ_0, satisfy the system (1), (6) for $\xi \in \Omega_l^{(1)}$, $j = 1, \ldots, N_+(\xi)$ with minimal $N_+(\xi)$ (existence and uniqueness of such functions follow from Theorem 2.2). Then the following statements are valid:*

1. *If $\mu_0 = 0$, then $P_m(\xi_0, 0) \equiv 0$ for all m. Besides, $G_1(\xi_0, \xi_1) = 0$, if $|\xi_0| \geq \mathrm{const}(X)(1 + |\xi_1|)$.*
2. *If $\mu_0 = 1$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, where constants c_{11} and c_{12} satisfy the identity in $\xi_0 \in \Omega_0$:*

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + c_{12}\left(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + G_1(\xi_0, 0)\right) = G_1(\xi_0, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0). \tag{11}$$

3. *If $\mu_0 = 2$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$, where constants c_{11}, c_{12}, c_{21}, c_{22}, c_{23} satisfy the identity in $\xi_0 \in \Omega_0$:*

$$æ_{10}^{00}(c_{12}^2 + c_{23}) = \frac{\partial G_2}{\partial \xi_1} - 2\frac{\partial G_1}{\partial \xi_1}(G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0)$$

$$+\frac{\partial G_1}{\partial \xi_0}\cdot\big((G_1-c_{11}-c_{12}\xi_0)^2-G_2+c_{21}+c_{22}\xi_0+c_{23}\xi_0^2\big)$$
$$+\big(G_1^2-2c_{11}G_1-2c_{12}G_1\xi_0-G_2\big)\cdot(-c_{12}), \tag{12}$$

where all the functions are evaluated at point $(\xi_0, 0)$.

Proof By Theorem 2.2 functions P_m from the condition of this theorem are defined in Theorem 2.1.

1. By Theorem 2.1 $P_m \equiv 0$, if $\mu_0 = 0$.
2. For $\xi \in \Omega_l^{(1)}$ we have equality $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$. We need to find constants $c_{11} = C_{11}(0)$ and $c_{12} = C_{12}(0)$.
   Differentiate equation (6) with respect to ξ_0, ξ_1 and restrict this equation and its differentiated versions to $\xi \in \Omega_l^{(0)}$:

$$h_1(\xi_0,0) = G_1(\xi_0,0) - c_{11} - c_{12}\xi_0,$$
$$\frac{\partial h_1}{\partial \xi_1}(\xi_0,0) = \frac{\partial G_1}{\partial \xi_1}(\xi_0,0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0,$$
$$\frac{\partial h_1}{\partial \xi_0}(\xi_0,0) = \frac{\partial G_1}{\partial \xi_0}(\xi_0,0) - c_{12},$$

   where $\xi_0 \in \Omega_0$. By (1) for $\xi_0 \in \Omega_0$ function $h_1(\xi_0, 0)$ satisfies the equality $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0)\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$. If we substitute in this equality the expressions for $h_1(\xi_0, 0)$, $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$ and $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$, we will obtain the equation

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0,0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0$$
$$= \Big(G_1(\xi_0,0) - c_{11} - c_{12}\xi_0\Big)\Big(\frac{\partial G_1}{\partial \xi_0}(\xi_0,0) - c_{12}\Big). \tag{13}$$

   This equation is valid for $\xi_0 \in \Omega_0$. Let us divide it into ξ_0 and tend $\xi_0 \to \infty$. We obtain equality $\dot{C}_{12}(0) = -c_{12}^2$. Taking into account this equality, we can rewrite the Eq. (13) in the form

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0,0) - \dot{C}_{11}(0)$$
$$= \Big(G_1(\xi_0,0) - c_{11} - c_{12}\xi_0\Big)\frac{\partial G_1}{\partial \xi_0}(\xi_0,0) - \Big(G_1(\xi_0,0) - c_{11}\Big)c_{12}. \tag{14}$$

   Taking into account that $\xi_0\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \to 0$ as $\xi_0 \to \infty$ and passing $\xi_0 \to \infty$ in (14), we obtain the equality $\dot{C}_{11}(0) = -c_{11}c_{12}$. Due to the just-obtained equality, Eq. (14) takes the desired form.

3. By (1) functions $h_1(\xi)$ and $h_2(\xi)$ satisfy the Riemann–Burgers equation for $\xi \in \Omega_l^{(1)}$. So, the following equalities are valid:

$$\frac{\partial(h_1h_2)}{\partial\xi_1} = h_1\frac{\partial h_2}{\partial\xi_1} + \frac{\partial h_1}{\partial\xi_1}h_2 = h_1h_2\frac{\partial(h_1+h_2)}{\partial\xi_0}, \tag{15}$$

$$\frac{\partial(h_1^2+h_2^2)}{\partial\xi_0} = 2h_1\frac{\partial h_1}{\partial\xi_0} + 2h_2\frac{\partial h_2}{\partial\xi_0} = 2\frac{\partial(h_1+h_2)}{\partial\xi_1}. \tag{16}$$

Note that $h_1h_2 = \frac{1}{2}\big(h_1+h_2\big)^2 - \frac{1}{2}\big(h_1^2+h_2^2\big)$. Therefore the system (15)–(16) is equivalent to the system

$$\frac{\partial(h_1+h_2)^2}{\partial\xi_1} - \frac{\partial(h_1^2+h_2^2)}{\partial\xi_1} = \Big(\big(h_1+h_2\big)^2 - \big(h_1^2+h_2^2\big)\Big)\frac{\partial(h_1+h_2)}{\partial\xi_0}, \tag{17}$$

$$\frac{\partial(h_1^2+h_2^2)}{\partial\xi_0} = 2\frac{\partial(h_1+h_2)}{\partial\xi_1}. \tag{18}$$

We substitute into this system $h_1^2+h_2^2$ and h_1+h_2 from Eq. (6), using the notation $P_1(\xi_0,\xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$, $P_2(\xi_0,\xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$. Equation (18) restricted to $\Omega_l^{(0)}$ takes the form

$$\frac{\partial G_2}{\partial\xi_0}(\xi_0,0) - c_{22} - 2c_{23}\xi_0 = 2\bigg(\frac{\partial G_1}{\partial\xi_1}(\xi_0,0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0\bigg). \tag{19}$$

Divide this equation into ξ_0 and tend $\xi_0 \to \infty$. We obtain the equality $\dot{C}_{12}(0) = c_{23}$. Taking this equality into account and passing $\xi_0 \to \infty$ in (19) we obtain the equality $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$.

Now substitute the expressions for $h_1^2+h_2^2$ and h_1+h_2 into (17) and restrict the obtained formula to $\Omega_l^{(0)}$. We obtain the equality

$$\begin{aligned}
&2\big(G_1 - c_{11} - c_{12}\xi_0\big)\bigg(\frac{\partial G_1}{\partial\xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0\bigg)\\
&\quad - \frac{\partial G_2}{\partial\xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2\\
&= \Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\bigg(\frac{\partial G_1}{\partial\xi_0} - c_{12}\bigg).
\end{aligned} \tag{20}$$

Divide this equation into ξ_0^2 and pass $\xi_0 \to \infty$. This leads to the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = -\big(c_{12}^2 + c_{23}\big)c_{12}.$$

Using the latter equality, divide (20) into ξ_0 and pass $\xi_0 \to \infty$ to obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = -\big(2c_{11}c_{12} + c_{22}\big)c_{12}.$$

Taking into account the obtained equalities one can rewrite (20) in the form

$$\begin{aligned}
&2\big(G_1 - c_{11} - c_{12}\xi_0\big)\frac{\partial G_1}{\partial \xi_1} - 2G_1\big(\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0\big)\\
&\quad +2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0)\\
&= \Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\frac{\partial G_1}{\partial \xi_0}\\
&\quad +\Big(\big(G_1 - c_{11}\big)^2 - 2G_1c_{12}\xi_0 - G_2 + c_{21}\Big)(-c_{12}). \qquad (21)
\end{aligned}$$

Pass $\xi_0 \to \infty$ in this equality and note that the following relations are valid:

$$\lim_{\xi_0\to\infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} = \lim_{\xi_0\to\infty} \xi_0 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i}\int_\gamma z_1\,dz_1 = \langle b\gamma, \frac{1}{4\pi i} z_1^2\rangle = 0,$$

$$\lim_{\xi_0\to\infty} \xi_0 G_1 = \lim_{\xi_0\to\infty} \xi_0 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i}\int_\gamma z_1\,dz_2 = æ_{10}^{00},$$

$$\lim_{\xi_0\to\infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} = -\lim_{\xi_0\to\infty} \xi_0^2 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_2}{(\xi_0 + z_2)^2} = -\frac{1}{2\pi i}\int_\gamma z_1\,dz_2 = -æ_{10}^{00}.$$

We obtain

$$-2æ_{10}^{00}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -\big(c_{12}^2 + c_{23}\big)æ_{10}^{00} - \big(c_{11}^2 - 2c_{12}æ_{10}^{00} + c_{21}\big)c_{12}.$$

Express constants $\dot{C}_{ij}(0)$ through c_{ij} in the obtained equations:

$$\begin{aligned}
&\dot{C}_{11}(0) = \frac{1}{2}c_{22},\\
&\dot{C}_{12}(0) = c_{23},\\
&\dot{C}_{23}(0) = -c_{12}^3 - 3c_{12}c_{23},\\
&\dot{C}_{22}(0) = -2\big(c_{11}c_{12}^2 + c_{12}c_{22} + c_{11}c_{23}\big),\\
&\dot{C}_{21}(0) = æ_{10}^{00}(c_{12}^2 + c_{23}) - c_{12}(c_{11}^2 + c_{21}) - c_{11}c_{22}.
\end{aligned} \qquad (22)$$

Substituting these constants into (21), we obtain the third statement of Theorem 3.1.

Complement 3.1 Statement of Theorem 3.1 admits a development for the case $\mu_0 \geq 3$. In this case

$$P_k(\xi_0, \xi_1) = C_{k1}(\xi_1) + C_{k2}(\xi_1)\xi_0 + \cdots + C_{k,k+1}(\xi_1)\xi_0^k, \quad k = 1, \ldots, \mu_0.$$

Define $\dot{C}_{ij}(0) = \frac{\partial C_{ij}}{\partial \xi_1}(0)$ and $c_{ij} = C_{ij}(0)$ for $i = 1, \ldots, \mu_0$ and $j = 1, \ldots, i + 1$.

Let us indicate the following general procedure for finding constants c_{ij}. Due to the Riemann–Burgers equations (1) the following identities in $\xi_0 \in \Omega_0$ hold for $k = 1, \dots, \mu_0 - 1$:

$$-\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) + \dot{C}_{k1}(0) + \dot{C}_{k2}(0)\xi_0 + \cdots + \dot{C}_{k,k+1}(0)\xi_0^k$$
$$= \frac{k}{k+1}\left(-\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) + c_{k+1,2} + 2c_{k+1,3}\xi_0 + \cdots + (k+1)c_{k+1,k+2}\xi_0^k\right).$$

Taking into account that $\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) \to 0$ and $\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) \to 0$ as $\xi_0 \to +\infty$ we obtain the equalities

$$\dot{C}_{k,m}(0) = \frac{km}{k+1}c_{k+1,m+1}, \quad k = 1, \dots, \mu_0 - 1, \quad m = 1, \dots, k+1.$$

Due to the Riemann–Burgers equations (1) the following identity in $\xi_0 \in \Omega_0$ holds:

$$\frac{\partial e_{\mu_0}}{\partial \xi_1}(\xi_0, 0) = e_{\mu_0}(\xi_0, 0)\frac{\partial p_1}{\partial \xi_0}(\xi_0, 0), \tag{23}$$

where functions e_k are given by the following formulas:

$$\begin{aligned} ke_k(\xi_0, \xi_1) &= \sum_{i=1}^{k-1}(-1)^{i+1}e_{k-i}(\xi_0, \xi_1)p_i(\xi_0, \xi_1) + (-1)^{k+1}p_k(\xi_0, \xi_1), \\ p_k(\xi_0, \xi_1) &= G_k(\xi_0, \xi_1) - C_{k1}(\xi_1) - C_{k2}(\xi_1)\xi_0 - \cdots - C_{k,k+1}(\xi_1)\xi_0^k, \end{aligned} \tag{24}$$

where $k = 1, \dots, \mu_0$.

Equality (23) allows us to represent constants $\{\dot{C}_{\mu_0,j}(0)\}$ as functions of constants $\{c_{ij}\}$. Finally, substituting the obtained expressions for constants $\{\dot{C}_{ij}(0)\}$ via constants $\{c_{ij}\}$ into Eq. (23) we obtain the identity in $\xi_0 \in \Omega_0$ for computation of constants $\{c_{ij}\}$.

For example, in the case $\mu_0 = 3$ the identity (23) in $\xi_0 \in \Omega_0$ for finding constants c_{ij} takes the form

$$\dot{C}_{31}(0) + \dot{C}_{32}(0)\xi_0 + \dot{C}_{33}(0)\xi_0^2 + \dot{C}_{34}(0)\xi_0^3$$
$$= \frac{\partial G_3}{\partial \xi_1} + \frac{3}{4}(p_1^2 - p_2)\frac{\partial p_2}{\partial \xi_0} - p_1\frac{\partial p_3}{\partial \xi_0} - \frac{1}{2}\left(p_1^3 - 3p_1p_2 + 2p_3\right)\frac{\partial p_1}{\partial \xi_0},$$

where all functions are evaluated at point $(\xi_0, 0)$, the functions p_k are defined in formula (24) and the constants $\dot{C}_{31}(0)$, $\dot{C}_{32}(0)$, $\dot{C}_{33}(0)$, $\dot{C}_{34}(0)$ are given by formulas

$$\begin{aligned} \dot{C}_{31}(0) = &\frac{1}{2}æ_{10}^{00}\left(3c_{11}c_{12}^2 + 3c_{12}c_{22} + 3c_{11}c_{23} + 2c_{33}\right) \\ &- \frac{1}{2}æ_{11}^{00}\left(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\right) - \frac{3}{2}æ_{00}^{11}\left(c_{12}^2 + c_{23}\right) \\ &- \frac{1}{2}c_{11}^3c_{12} - \frac{3}{2}c_{11}c_{12}c_{21} - \frac{3}{4}c_{11}^2c_{22} - \frac{3}{4}c_{21}c_{22} - c_{12}c_{31} - c_{11}c_{32}, \end{aligned}$$

$$\dot{C}_{32}(0) = æ_{10}^{00}\left(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\right) - \frac{3}{2}c_{11}^2c_{12}^2 - \frac{3}{2}c_{12}^2c_{21} - 3c_{11}c_{12}c_{22}$$
$$- \frac{3}{4}c_{22}^2 - \frac{3}{2}c_{11}^2c_{23} - \frac{3}{2}c_{21}c_{23} - 2c_{12}c_{32} - 2c_{11}c_{33},$$
$$\dot{C}_{33}(0) = -\frac{3}{2}c_{11}c_{12}^3 - \frac{9}{4}c_{12}^2c_{22} - \frac{9}{2}c_{11}c_{12}c_{23} - \frac{9}{4}c_{22}c_{23} - 3c_{12}c_{33} - 3c_{11}c_{34}$$
$$\dot{C}_{34}(0) = -\frac{1}{2}c_{12}^4 - 3c_{12}^2c_{23} - \frac{3}{2}c_{23}^2 - 4c_{12}c_{34},$$

where $æ_{10}^{00}$, $æ_{00}^{11}$ and $æ_{11}^{00}$ are defined in formula (10).

The next theorem permits us to find μ_0 through γ.

Theorem 3.2 *Let $X \subset \mathbb{C}\mathrm{P}^2 \setminus [0:1:0]$ be a complex curve without algebraic subdomains, $\gamma = bX \subset \mathbb{C}^2$ be the boundary of X. Let functions $G_m(\xi_0, \xi_1)$, $m \geq 1$, be defined by formula (2) and number μ_0 defined by formula (5). Then the following statements are valid:*

1. *If $G_1(\xi_0, \xi_1) = 0$ for $|\xi_0| \geq \mathrm{const}(X)(1 + |\xi_1|)$, then $\mu_0 = 0$.*
2. *If there exist such complex constants c_{11}, c_{12} that for any $\xi \in \Omega_0^{(1)}$ the following equality is valid:*

$$c_{11}\frac{\partial G_1}{\partial \xi_0}(\xi) + c_{12}\left(\xi_0\frac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi)\right) = G_1(\xi)\frac{\partial G_1}{\partial \xi_0}(\xi) - \frac{\partial G_1}{\partial \xi_1}(\xi), \quad (25)$$

then $\mu_0 \leq 1$.

3. *If there exist such complex constants $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}$ that the following identity in $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$ is valid:*

$$æ_{10}^{00}(c_{12}^2 + c_{23}) = \frac{\partial G_2}{\partial \xi_1} - 2\frac{\partial G_1}{\partial \xi_1}(G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0)$$
$$+ \frac{\partial G_1}{\partial \xi_0} \cdot \left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\right)$$
$$+ \left(G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2\right) \cdot (-c_{12}), \quad (26)$$

where all functions are evaluated at point ξ, then $\mu_0 \leq 2$.

Complement 3.2 The statement of Theorem 3.2 for $\mu_0 \geq 3$ in the spirit of cases $\mu_0 \leq 2$ will be developed in a separate paper together with statement of Theorem 3.1 for $\mu_0 \geq 3$, indicated in Complement 3.1.

Proof 1. Equality $G_1(\xi_0, \xi_1) = 0$ for $|\xi_0| \geq \mathrm{const}(X)(1 + |\xi_1|)$ implies according to [5] the moment condition

$$\int_\gamma z_1^{k_1} z_2^{k_2}\, dz_2 = 0 \quad \text{for all } k_1, k_2 \in \mathbb{N}.$$

From here according to [10] and [6] it follows that for an appropriate choice of orientation γ is the boundary of a complex curve in $\mathbb{C}^2$. Hence either $\mu_0 = 0$ or X is a domain on an algebraic curve in $\mathbb{C}\mathrm{P}^2$. But X cannot be a domain on an algebraic curve in $\mathbb{C}\mathrm{P}^2$ because X does not contain algebraic subdomains.

2. Let the conditions of general position be fulfilled. Put

$$\begin{aligned} h(\xi_0,\xi_1) &= G_1(\xi_0,\xi_1) - C_{11}(\xi_1) - C_{12}(\xi_1)\xi_0, \\ C_{11}(\xi_1) &= c_{11} - c_{11}c_{12}\xi_1, \\ C_{12}(\xi_1) &= c_{12} - c_{12}^2\xi_1, \end{aligned} \tag{27}$$

where $\xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}$.
Taking into account (27), we can rewrite equality (25) in the form

$$\frac{\partial h}{\partial \xi_1}(\xi_0,\xi_1) = h(\xi_0,\xi_1)\frac{\partial h}{\partial \xi_0}(\xi_0,\xi_1), \quad \xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}. \tag{28}$$

From definition (27) we obtain the following equality:

$$\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi_0,\xi_1) - h(\xi_0,\xi_1)\big) = 0, \quad \xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}. \tag{29}$$

From equalities (28), (29) due to Theorems 2.1, 2.2 and Theorem 3 from [8] we obtain

$$X \cap \{z_2 = -\xi_0\} = \big\{\big(h(\xi_0,0), -\xi_0\big)\big\}, \quad \xi_0 \in \Omega_0. \tag{30}$$

From here it follows that $\mu_0 \le 1$.

3. Let the conditions of general position be fulfilled. Let us define functions h_1 and h_2 by the following relations:

$$\begin{aligned} h_1(\xi_0,\xi_1) + h_2(\xi_0,\xi_1) &= G_1(\xi_0,\xi_1) - C_{11}(\xi_1) - C_{12}(\xi_1)\xi_0, \\ h_1^2(\xi_0,\xi_1) + h_2^2(\xi_0,\xi_1) &= G_2(\xi_0,\xi_1) - C_{21}(\xi_1) - C_{22}(\xi_1)\xi_0 - C_{23}(\xi_1)\xi_0^2, \end{aligned} \tag{31}$$

$$C_{ij}(\xi_1) = c_{ij} + \dot{C}_{ij}(0)\xi_1, \quad i = 1,\ j = 1,2 \text{ and } i = 2,\ j = 1,2,3, \tag{32}$$

where $\xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}$ and constants $\dot{C}_{ij}(0)$ are defined by formulas (22).

Taking into account definitions (32), identity (26) is equivalent to identity (20), where $\xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}$.

Taking into account definitions (31), identity (20) for $\xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}$ is equivalent to identity (17) for $\xi \in \Omega_0^{(1)}$.

By Lemma 3.2.1 from paper [4] for $\xi = (\xi_0,\xi_1) \in \Omega_0^{(1)}$ the following equality is valid:

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0,\xi_1) = 2\frac{\partial G_1}{\partial \xi_1}(\xi_0,\xi_1). \tag{33}$$

From definitions (31), (32) and from equality (33) we obtain equality (18) for $\xi \in \Omega_0^{(1)}$.

Equalities (17), (18) mean that functions h_1 and h_2 satisfy the Riemann–Burgers equations:

$$\frac{\partial h_j}{\partial \xi_1}(\xi_0, \xi_1) = h_j(\xi_0, \xi_1)\frac{\partial h_j}{\partial \xi_0}(\xi_0, \xi_1), \quad \xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}. \tag{34}$$

Further, because of definition (31) the following equality is valid:

$$\frac{\partial^2}{\partial \xi_0}\big(G_1(\xi_0, \xi_1) - h_1(\xi_0, \xi_1) - h_2(\xi_0, \xi_1)\big) = 0. \tag{35}$$

From equalities (34), (35) and using Theorems 2.1, 2.2 and Theorem 3 of the paper [8] we obtain:

$$X \cap \{z_2 = -\xi_0\} = \{(h_j(\xi_0, 0), -\xi_0) \mid j = 1, 2\}, \quad \xi_0 \in \Omega_0.$$

From here it follows that $\mu_0 \le 2$. □

Let us describe the algorithm of reconstruction of a complex curve X in $\mathbb{C}P^2$ without compact components (satisfying minimality condition (*)). As above, $\gamma = bX$ is a compact real curve.

The algorithm for reconstruction of curve X permits us to find a curve coinciding with the original curve in the given finite number of points and obtained by interpolation in other points. Let $\{\xi_0^k\}_{k=1}^N$, $\xi_0^k \in \mathbb{C}$ be an arbitrary grid on $\mathbb{C}$, $\xi_0^i \ne \xi_0^j$, $i \ne j$, and $\xi_0^k \notin \pi_2\gamma$, $k = 1, \ldots, N$. Complex curve X intersects complex line $\{z_2 = -\xi_0^k\}$ in $N_+(\xi_0^k, 0)$ points. The algorithm allows us to find these points.

The algorithm takes as input points $\{\xi_0^k\}_{k=1}^N$ and a curve γ (for example, represented as a finite number of points belonging to γ). On the output of the algorithm we obtain a set of points $(h_s(\xi_0^k, 0), -\xi_0^k)$, $k = 1, \ldots, N$; $s = 1, \ldots, N_+(\xi_0^k, 0)$, belonging to the complex curve X.

3.1 The Case of $\mu_0 = 0$

1. *Calculation of* μ_l. By formula (4) for every domain Ω_l, $l = 1, \ldots, L$, the number μ_l is equal to the winding number of curve $\pi_2\gamma$ with respect to point $\xi_0 \in \Omega_l$:

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{dz_2}{z_2 + \xi_0} \equiv \frac{1}{2\pi i}\int\limits_{\pi_2\gamma} \frac{dz}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Computation of power sums.* If $\mu_0 = 0$ then for every point $\xi_0^k \in \Omega_l$, $l = 1, \ldots, L$, by Theorem 3.1 we have equalities $P_m(\xi_0^k, 0) \equiv 0$. By formula (6) we have the following formulas for the power sums:

$$s_m(\xi_0^k) \equiv h_1^m(\xi_0^k, 0) + \cdots + h_{\mu_l}^m(\xi_0^k, 0) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{z_1^m\, dz_2}{z_2 + \xi_0^k}, \quad m = \overline{1, \mu_l}.$$

By Theorem 2.1 the points $(h_s(\xi_0^k, 0), -\xi_0^k)$, $s = 1, \ldots, N_+(\xi_0^k, 0)$; $k = 1, \ldots, N$, are the desired points of X.

3. *Computation of symmetric functions.* For every point $\xi_0^k \in \Omega_l$, $l = 1, \ldots, L$, the Newton identities

$$k\sigma_k(\xi_0^k) = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i}(\xi_0^k) s_i(\xi_0^k), \quad k = 1, \ldots, N_+(\xi_0^k, 0).$$

allow us to reconstruct the elementary symmetric functions:

$$\begin{aligned} \sigma_1(\xi_0^k) &= h_1(\xi_0^k, 0) + \cdots + h_{\mu_l}(\xi_0^k, 0), \\ \cdots &= \cdots \\ \sigma_{\mu_l}(\xi_0^k) &= h_1(\xi_0^k, 0) \times \cdots \times h_{\mu_l}(\xi_0^k, 0). \end{aligned}$$

4. *Desymmetrization.* For every point $\xi_0^k \in \Omega_l$ using Vieta formulas one can find complex numbers $h_1(\xi_0^k, 0), \ldots, h_{\mu_l}(\xi_0^k, 0)$. The points $\big(h_s(\xi_0^k, 0), -\xi_0^k\big)$, $s = 1, \ldots, N_+(\xi_0^k, 0)$; $k = 1, \ldots, N$, are the required points of complex curve X.

3.2 The Cases of $\mu_0 = 1, 2$

These cases are reduced to the case $\mu_0 = 0$ in the following way. Since $\pi_2 \gamma \subset \mathbb{C}$ is a compact real curve, there exists such $R > 0$, such that the set $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geqslant R\}$ belongs to Ω_0. Without restriction of generality, one can suppose that $|\xi_0^k| < R$ for all $k = 1, \ldots, N$. Otherwise one can increase R.

Let us define the auxiliary complex curve $X_R = \{(z_1, z_2) \in X \mid |z_2| \leqslant R\}$. Its boundary γ_R consists of two disjoint parts (possibly, multiconnected): the first part is γ and the second is a real curve obtained by lifting the circle $S_R = \{z \in \mathbb{C} \mid |z| = R\}$ on surface X by inversion of projection $\pi_2 \colon X \to \mathbb{C}$. Complex curve X_R does not intersect infinity and, as a consequence $\mu_0(X_R) = 0$. Moreover, every point $\big(z_1(\xi_0^k), -\xi_0^k\big)$, $k = 1, \ldots, N$, belongs to X if and only if it belongs to X_R. Therefore, in order to reconstruct the complex curve X_R it is sufficient to reconstruct the real curve obtained by lifting S_R on X and to solve the reconstruction problem for surface X_R, being in the conditions of the case of $\mu_0 = 0$. Finally, we come to the following algorithm:

1. *New boundary.* Choose a sufficiently large constant R, so that the exterior of the disk of radius R centered at origin belongs to Ω_0 and all ξ_0^k belong to this disk. Denote the boundary of this disk by S_R. In the case of $\mu_0 = 1$ by virtue of formulas (6) the points $\xi_0 \in S_R$ satisfy the equality

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1 \, dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0).$$

In the case of $\mu_0 = 2$ we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1\, dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0),$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1^2\, dz_2}{z_2 + \xi_0} - P_2(\xi_0, 0),$$

where the polynomials can be found using Theorem 3.1. In the case of $\mu_0 = 1$ by lifting the curve S_R on X we obtain at once the real curve of the form $\{(h_1(\xi_0, 0), -\xi_0) \mid \xi_0 \in S_R\}$. In the case of $\mu_0 = 2$ we have to apply Newton identities and Vieta formulas in order to obtain h_1 and h_2 from functions $h_1 + h_2$ and $h_1^2 + h_2^2$.

2. *Reduction.* In order to find the complex curve X_R with boundary $bX_R = \gamma_R$ we apply the algorithm of reconstruction for the case of $\mu_0 = 0$. The discussion before the description of the algorithm shows that we will obtain the desired points.

4 Visualization

Let us describe in a few words the algorithm of visualization of complex curves that we have used in our examples. Denote by $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ the projection into the first factor: $\pi_1(z_1, z_2) = z_1$. Suppose that X is a complex curve in $\mathbb{C}^2$ such that the covering $\pi_1 : X \setminus \{\text{ramification points}\} \to \mathbb{C}$ has multiplicity L. Consider, for simplicity, a rectangular grid Λ in $\mathbb{C}$:

$$\Lambda = \left\{z_1^{ij} : \operatorname{Re} z_1^{ij} = \frac{i}{N},\ \operatorname{Im} z_1^{ij} = \frac{j}{N},\ i, j = 0, \ldots, N\right\},$$

where N is a natural number. Suppose now that we are given the set $X_\Lambda = \pi_1^{-1}(\Lambda) \cap X$ and we need to visualize the part of X lying above the rectangle $0 \le \operatorname{Re} z_1 \le 1$, $0 \le \operatorname{Im} z_1 \le 1$.

Let us introduce some terminology. We define a path in Λ as a map $\gamma : \{1, \ldots, M\} \to \Lambda$ such that $|\gamma(k+1) - \gamma(k)| = \frac{1}{N}$ for all admissible k, where M is some natural number.

Let $\gamma : \{1, \ldots, M\} \to \Lambda$ be a path in Λ and let $i : \{1, \ldots, M\} \to [1, M]$ be the inclusion map. Define the function $i_*\gamma : [1, M] \to \mathbb{C}$ such that $i_*\gamma(k) = \gamma(k)$ for integer k and $i_*\gamma|_{[k,k+1]}$ is linear for all admissible k. It is clear that $i_*\gamma$ is a continuous function and hence it can be lifted to X by the map π_1.

We define a path in X_Λ as a map $\Gamma : \{1, \ldots, M\} \to X_\Lambda$ such that $\gamma = \pi_1 \circ \Gamma$ is a path in Λ and $\Gamma = i^* L(i_*\gamma)$, where i^* is the pullback map with respect to i and $L(i_*\gamma)$ is some lift of $i_*\gamma$ to X by π_1, i.e., $L(i_*\gamma)$ is a continuous map from $[1, M]$ to X such that $\pi_1 \circ L(i_*\gamma) = i_*\gamma$. We also say that Γ is obtained by lifting γ.

We will call subsets of Λ and X_Λ path-connected if every two points of these sets can be connected by a path in Λ and X_Λ, respectively.

Let us describe the practical way to lift paths in Λ to paths in X_Λ. Suppose that N is sufficiently large. Let $\gamma\colon \{1, \dots, M\} \to \Lambda$ be a path in Λ and let $\Gamma(1) \in \pi_1^{-1}(\gamma(1)) \cap X$ be an arbitrary point. We select $\Gamma(k) \in \pi_1^{-1}(\gamma(k)) \cap X$ in such a way that

$$|\Gamma(k) - \Gamma(k-1)| = \min\big\{|z - \Gamma(k-1)| \colon z \in \pi_1^{-1}(\gamma(k)) \cap X\big\}, \quad k = 2, \dots, M.$$

Then Γ is a path in X_Λ obtained by lifting γ. All possible lifts of γ may be obtained by varying $\Gamma(1)$. Note that if γ is closed, i.e., $\gamma(1) = \gamma(M)$, Γ need not to be closed.

Finding Ramification Points and Making Branch Cuts. The first step in the visualization procedure consists in finding ramification points of X with respect to projection π_1. Since we have only a finite number of points on X we can find ramification points only approximately. More precisely, we will localize them in small circles.

Without restriction of generality we suppose that all ramification points are projected by π_1 into interior points of Λ. Take any interior point $z_1 \in \Lambda$ and select a small closed path $\gamma\colon \{1, \dots, M\} \to \Lambda$ around z_1 so that there is at most one ramification point inside the polygon $\overline{\gamma(1) \dots \gamma(M)}$. For example, one can take as γ the following path:

$$z_1 + \frac{1}{N} \to z_1 + \frac{1+i}{N} \to z_1 + \frac{i}{N} \to \cdots \to z_1 + \frac{1-i}{N} \to z_1 + \frac{1}{N},$$

where i is the imaginary unit.

Now consider different lifts of γ to X_Λ. If at least one lift is not closed, mark z_1 as a possible ramification point (meaning that it is situated near the projection of some ramification point of X). Now vary z_1 and mark all possible ramification points. The resulting set will consist of several path-connected components each of which localizes the position of one ramification point of X with respect to π_1.

Now connect each of the obtained connected components of possible ramification points by path with boundary of the grid Λ in such a way that different paths do not intersect. Denote the union of the set of possible ramification points with images of these paths by Λ_c. An important observation is that every closed path in $\Lambda \setminus \Lambda_c$ always lifts to a closed path in X_Λ since it does not contain π_1-projections of ramification points inside.

Visualization. Now denote $\Lambda \setminus \Lambda_c = \cup_{s=1}^{S} \Lambda_s$, where Λ_s are different path-connected components. Take any $z_1^s \in \Lambda_s$ and $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$. Now take other $z_1 \in \Lambda_s$ and connect z_1^s with z_1 by some path γ. Then γ lifts to a path Γ with $\Gamma(1) = (z_1^s, z_2^s)$ and $\Gamma(2) = (z_1, z_2)$ for some $z_2 \in \pi_1^{-1}(z_1) \cap X$ and z_2 does not depend on γ. Varying z_1 we thus obtain the map $\Sigma(z_1^s, z_2^s)\colon \Lambda_s \to X_\Lambda$ which allows us to visualize the part of X.

Varying $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$ (the latter is the finite set, namely, it consists of L elements) we obtain the other maps $\Sigma(z_1^s, z_2^s)$ which allow us to visualize other parts of X. Clearly, the set of obtained maps does not depend on the choice of $z_1^s \in \Lambda_s$. Hence we can denote the obtained maps by Σ_s^l, $l = 1, \dots, L$. It is clear that $\cup_{l=1}^{L} \Sigma_s^l(\Lambda_s) = \pi^{-1}(\Lambda_s) \cap X$. Now vary s to visualize

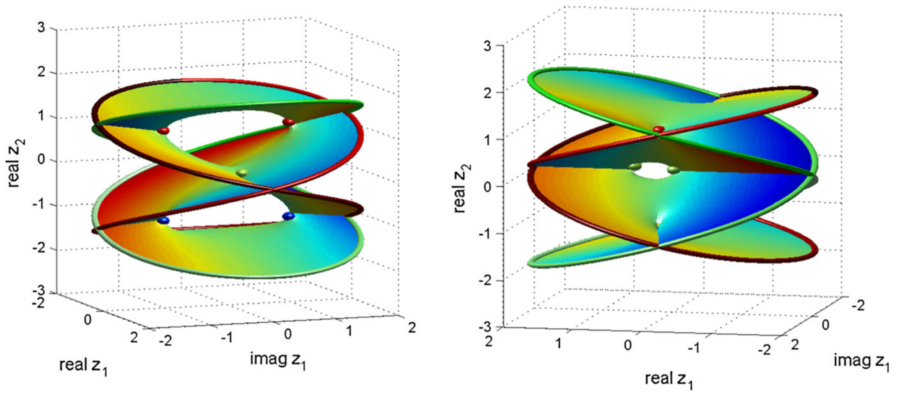

Fig. 1 Riemann surface of function $f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2+1}}$, $|z| \leqslant 2$, obtained by the visualization algorithm. *Red* and *green curves* represent two connected components of the surface boundary, *colored small balls* represent ramification points (Color figure online)

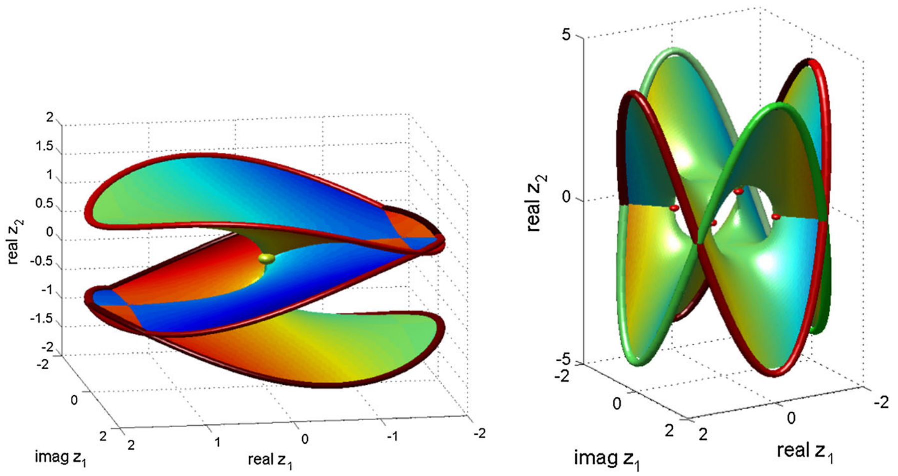

Fig. 2 Riemann surfaces of functions $f(z) = \sqrt{\sin(z)}$, $|z| \leqslant 2$ (*left*) and $f(z) = \sqrt{z^4+1}$, $|z| \leqslant 2$ (*right*) obtained by the visualization algorithm

$$\cup_{s=1}^{S} \cup_{l=1}^{L} \Sigma_s^l(\Lambda_s) = \pi^{-1}(\cup_{s=1}^{S} \Lambda_s) \cap X = X_\Lambda \setminus \pi^{-1}(\Lambda_c).$$

The part $\pi^{-1}(\Lambda_c) \cap X_\Lambda$ consists of cuts and preimages of possible ramification points. The cuts can be visualized as the already-visualized part of the surface. The only problem is the visualization of π_1-preimages of possible ramification points. But the latter take a little part of the surface when N is large and one can just forget about their visualization. On the other hand, in our examples they were visualized using a low-level graphics approach.

Examples of application of this algorithm are given in Figs. 1 and 2. The visualization algorithm can be easily generalized to the case of general grids. For instance, in our examples we have used a modification with periodic grid.

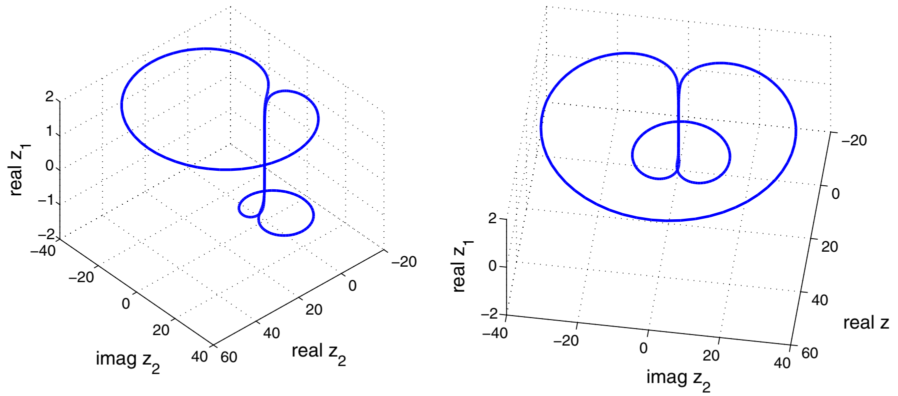

Fig. 3 Boundary $\gamma_1 = bX_1$ of the surface X_1

5 Examples

5.1 The Case of $\mu_0 = 1$

Consider an example of reconstruction of a Riemann surface with given boundary. The simplest case is the case of $\mu_0 = 0$ but it follows from the discussion of the reconstruction algorithm that this case is directly included in any other case. Therefore, we begin with the next simplest case, namely, the case of $\mu_0 = 1$.

Let us reconstruct the surface

$$X_1 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 - 1)z_2 = \exp(z_1^2), \quad |z_1| \leqslant 2\right\}.$$

We suppose that the boundary $\gamma_1 = bX_1$ is given in the form of a discrete number of points (see further Fig. 3).

Note that if point $(z_1, z_2) \in X_1$ is such that z_1 approaches 1, then z_2 approaches infinity. Choose R large enough, e.g., $R = 60$, and reconstruct the real curve $\Gamma = \{(z_1, z_2) \in X_1 \mid |z_2| = R\}$. At first, compute for two different points ξ_0^1, ξ_0^2, $|\xi_0^1| = |\xi_0^2| = R$, the values of functions

$$G_1(\xi_0, 0) = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{z_1 dz_2}{z_2 + \xi_0},$$

$$\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) = -\frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2},$$

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i}\int\limits_{\gamma_1} \left(\frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2}\right)$$

and find from the linear system for c_{11} and c_{12}

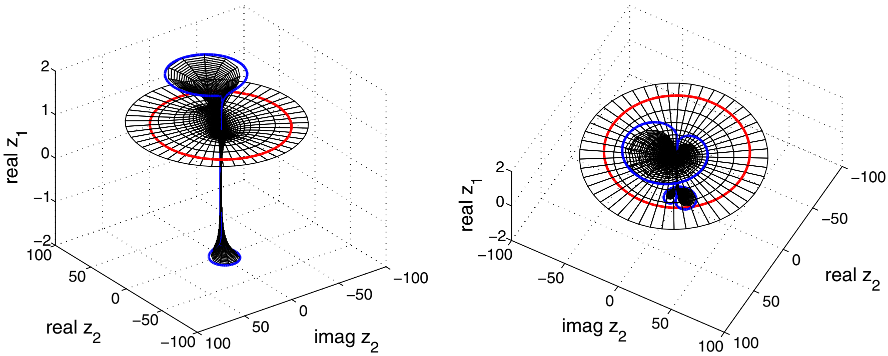

Fig. 4 The contour of surface X_1 (*black*), boundary γ_1 of X_1 (*blue*), reconstructed curve Γ, belonging to X_1 (*red*) (Color figure online)

$$c_{11}\frac{\partial G_1}{\partial \xi_0}(\xi_0^1,0)+c_{12}\left(\xi_0^1\frac{\partial G_1}{\partial \xi_0}(\xi_0^1,0)+G_1(\xi_0^1,0)\right)$$
$$=G_1(\xi_0^1,0)\frac{\partial G_1}{\partial \xi_0}(\xi_0^1,0)-\frac{\partial G_1}{\partial \xi_1}(\xi_0^1,0),$$
$$c_{11}\frac{\partial G_1}{\partial \xi_0}(\xi_0^2,0)+c_{12}\left(\xi_0\frac{\partial G_1}{\partial \xi_0}(\xi_0^2,0)+G_1(\xi_0^2,0)\right)$$
$$=G_1(\xi_0^2,0)\frac{\partial G_1}{\partial \xi_0}(\xi_0^2,0)-\frac{\partial G_1}{\partial \xi_1}(\xi_0^2,0)$$

values $c_{11}=1, c_{12}=0$. Now calculate the values of functions $G_1(\xi_0,0)$ on the circle $|\xi_0|=R$ and find function $h_1(\xi_0,0)=G_1(\xi_0,0)-c_{11}-c_{12}\xi_0, |\xi_0|=R$. This allows us to reconstruct the real curve $\Gamma=\{(h_1(\xi_0,0),-\xi_0)\mid |\xi_0|=R\}\subseteq X_1$ (see Fig. 4).

We apply further the reconstruction algorithm for the case of $\mu_0=0$ to the surface $X_1^R=\{z\in X_1\mid |z_2|\leqslant 60\}$ with boundary $bX_1^R=\gamma_1+\Gamma$. We can compute the values of function

$$\sigma_0(\xi_0)=\frac{1}{2\pi i}\int\limits_{\gamma_1+\Gamma}\frac{dz_2}{z_2+\xi_0}.$$

The value of function $\sigma_0(\xi_0)$ at point ξ_0 is equal to the number $N_+(\xi_0,0)$ of points of surface X_1^R projected onto the point ξ_0 under projection $(z_1,z_2)\mapsto -z_2$. Further, for every point with $N_+(\xi_0,0)>0$ we compute functions

$$s_k(\xi_0)=\frac{1}{2\pi i}\int\limits_{\gamma_1+\Gamma}\frac{z_1^k dz_2}{z_2+\xi_0},\quad k=1,\dots,N_+(\xi_0,0).$$

From functions $s_k(\xi_0)$ we can find functions $\sigma_k(\xi_0)$ using Newton identities:

$$k\sigma_k(\xi_0)=\sum\nolimits_{i=1}^{k}(-1)^{i-1}\sigma_{k-i}(\xi_0)s_i(\xi_0),\quad k=1,\dots,N_+(\xi_0,0).$$

After, we find roots $h_1(\xi_0,0)$, …, $h_{\sigma_0(\xi_0)}(\xi_0,0)$ of polynomial

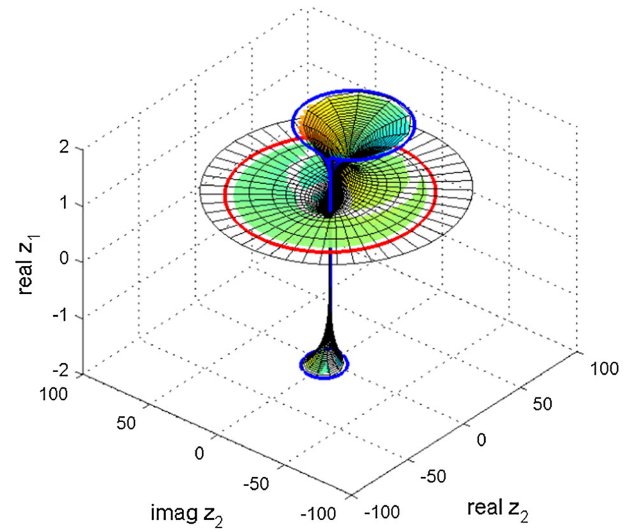

Fig. 5 The contour of surface X_1 (*black*), boundary γ_1 of X_1 (*blue*), reconstructed curve Γ, belonging to X_1 (*red*), *colored* domains represent the reconstructed leaves of surface X_1 (Color figure online)

$$t^{N_+(\xi_0)} - \sigma_1(\xi_0)t^{N_+(\xi_0)-1} + \cdots + (-1)^{N_+(\xi_0)}\sigma_{N_+(\xi_0,0)}(\xi_0) = 0.$$

The points $\{(h_k(\xi_0, 0), -\xi_0) \mid k = 1, \ldots, N_+(\xi_0, 0)\}$ represent the set of all points of X_1^R projected onto ξ_0 by projection $(z_1, z_2) \to -z_2$. Visualization of the obtained set of points $\{(h_k(\xi_0, 0), -\xi_0)\}$ corresponding to varying ξ_0 can be realized by the visualization algorithm described in the previous section. The reconstructed surface is represented in Fig. 5.

5.2 The Case of $\mu_0 = 2$

Consider an example of the reconstruction of the Riemann surface for the case of $\mu_0 = 2$. We are going to reconstruct the surface

$$X_2 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2(z_1^2 - 1) = z_1 \exp(z_1^2), \quad |z_1| \leqslant 2\right\}$$

given its boundary γ_2 represented as an array of a finite number of equidistributed points on γ_2 (see Fig. 6).

Choose R large enough, e.g., $R = 60$, and consider the circle C_R of radius R in the z_2-plane centered at the origin. Compute for $\xi_0 \in C_R$ the values of functions

$$G_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0},$$

$$G_2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1^2 dz_2}{z_2 + \xi_0},$$

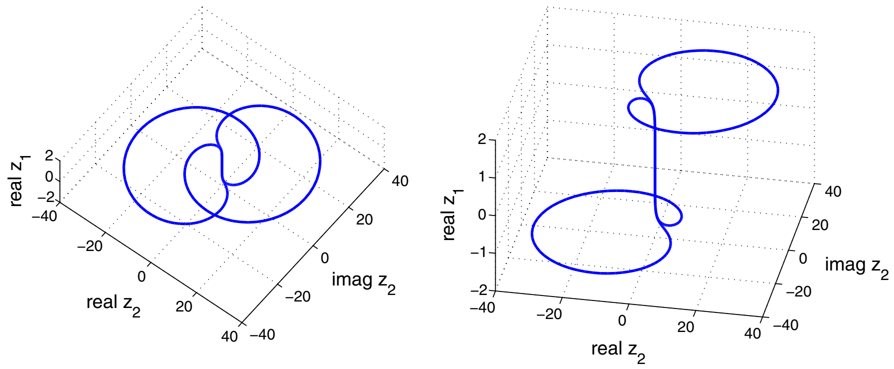

Fig. 6 Boundary γ_2 of the surface X_2

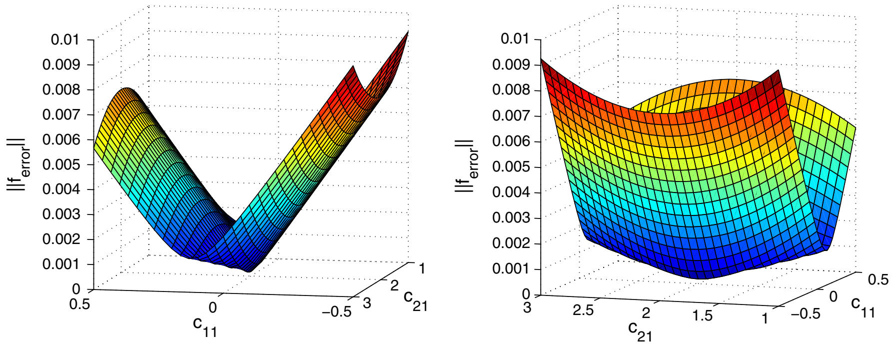

Fig. 7 Graph of function $\|f_{\text{error}}\|_{L^2(C_R)}(c_{11}, c_{21})$ in a neighborhood of the point of global minimum

$$\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) = -\frac{1}{2\pi i}\int\limits_{\gamma_2}\frac{z_1 dz_2}{(z_2+\xi_0)^2},$$

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i}\int\limits_{\gamma_2}\left(\frac{z_1 dz_1}{z_2+\xi_0} - \frac{z_1^2 dz_2}{(z_2+\xi_0)^2}\right),$$

$$\frac{\partial G_2}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i}\int_{\gamma_2}\left(\frac{z_1^2 dz_1}{z_2+\xi_0} - \frac{z_1^3 dz_2}{(z_2+\xi_0)^2}\right)$$

and the value of constant $æ_{10}^{00} = \frac{1}{2\pi i}\int_{\gamma_2} z_1 dz_2$, for example, using the method of rectangles.

In order to find constants $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}$ we solve numerically the problem of minimization of the $L^2(C_R)$-norm of function

$$f_{\text{error}}(\xi_0) = \frac{\partial G_2}{\partial \xi_1} - 2\frac{\partial G_1}{\partial \xi_1}(G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0) + \frac{\partial G_1}{\partial \xi_0}\cdot\left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\right)$$

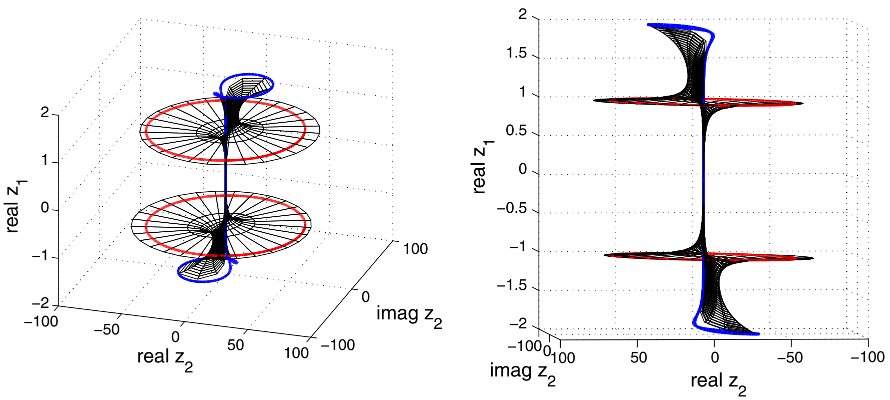

Fig. 8 The contour of surface X_2 (*black*), boundary γ_2 of X_2 (*blue*), reconstructed curves $\Gamma_{1,2}$, belonging to X_2 (*red*) (Color figure online)

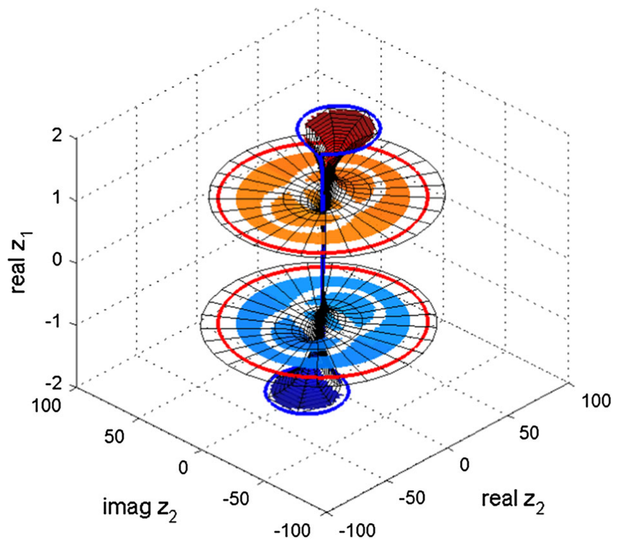

Fig. 9 The contour of surface X_2 (*black*), boundary γ_2 of X_2 (*blue*), reconstructed curves $\Gamma_{1,2}$, belonging to X_2 (*red*), reconstructed leaves of the surface are represented by *dark-blue, orange, red* and *blue* domains (Color figure online)

$$+\big(G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2\big)\cdot(-c_{12}) - æ_{10}^{00}(c_{12}^2 + c_{23}),$$

in variables $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}$. As a result of solving of this minimization problem we find $c_{11} = 0$, $c_{12} = 0$, $c_{21} = 2$, $c_{22} = 0$, $c_{23} = 0$ (see Fig. 7).

Then we compute the power sums $s_1 = G_1 - c_{11} - c_{12}\xi_0$, $s_2 = G_2 - c_{21} - c_{22}\xi_0 - c_{23}\xi_0^2$ and symmetric functions σ_1, σ_2 on the circle C_R. Further, we desymmetrize functions σ_1, σ_2 to obtain functions $h_1(\xi_0, 0)$ and $h_2(\xi_0, 0)$ on the circle C_R, which perform lifting of the circle C_R to the surface X_2. Denote by $\Gamma_{1,2} = \{(h_{1,2}(z_2, 0), -z_2) \mid z_2 \in C_R\}$ the curves, obtained by the corresponding lifting of C_R to X_2 (see Fig. 8).

Now we consider the curve $\gamma_2+\Gamma_1+\Gamma_2$ as a new initial curve and we reconstruct the surface $X_2^R = \{z \in X_2 \mid |z_2| \leqslant R\}$, $bX_2^R = \gamma_2 + \Gamma_1 + \Gamma_2$. Further, our considerations are similar to those for the case of $\mu_0 = 1$. At first, we compute functions s_k. Then, we find symmetric functions σ_k. Further, we solve the algebraic equation (numerically) and find functions $h_k(\xi_0, 0)$, $k = 1, \dots, N_+(\xi_0, 0)$. The reconstructed surface is given by Fig. 9.

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