J Geom Anal (2015) 25:1939–1968
DOI 10.1007/s12220-014-9499-9

A New Resolvent Equation for the S-Functional Calculus

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Received: 26 October 2013 / Published online: 11 June 2014

Abstract The S-functional calculus is a functional calculus for $(n+1)$-tuples of not necessarily commuting operators that can be considered a higher-dimensional version of the classical Riesz–Dunford functional calculus for a single operator. In this last calculus, the resolvent equation plays an important role in the proof of several results. Associated with the S-functional calculus there are two resolvent operators: the left $S_L^{-1}(s,T)$ and the right one $S_R^{-1}(s,T)$, where $s=(s_0,s_1,\dots,s_n)\in\mathbb{R}^{n+1}$ and $T=(T_0,T_1,\dots,T_n)$ is an $(n+1)$-tuple of noncommuting operators. The two S-resolvent operators satisfy the S-resolvent equations $S_L^{-1}(s,T)s-TS_L^{-1}(s,T)=\mathcal{I}$, and $sS_R^{-1}(s,T)-S_R^{-1}(s,T)T=\mathcal{I}$, respectively, where $\mathcal{I}$ denotes the identity operator. These equations allow us to prove some properties of the S-functional calculus. In this paper we prove a new resolvent equation which is the analog of the classical resolvent equation. It is interesting to note that the equation involves both the left and the right S-resolvent operators simultaneously.

Communicated by Der-Chen Edward Chang.

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Keywords n-tuples of noncommuting operators · Quaternionic operators · S-spectrum · Right S-resolvent operator · Left S-resolvent operator · Resolvent equation · Projectors

Mathematics Subject Classification 47S10 · 30G35

1 Introduction

The S-resolvent operators are a key tool in the definition of the higher-dimensional version of the Riesz–Dunford functional calculus called S-functional calculus. This calculus works for $(n+1)$-tuples $(T_0, T_1, \dots, T_n)$ of not necessarily commuting operators and is based on the so-called S-spectrum; see [14,17]. In the case of a single operator the S-functional calculus reduces to the Riesz–Dunford functional calculus (see [21,33]).

When the operators $(T_0, T_1, \dots, T_n)$ commute among themselves, this calculus admits a commutative version called SC-functional calculus. In this case the S-resolvent operator and the S-spectrum have a simpler expression; see [15].

The class of functions on which this calculus is based is the so-called set of slice hyperholomorphic (or slice monogenic) functions which are defined on subsets of the Euclidean space $\mathbb{R}^{n+1}$ and have values in the Clifford algebra $\mathbb{R}_n$.

For more details on the S-functional calculus and the function theory on which it is based, see the monograph [19].

As it happens for the classical theory of monogenic functions (see [10,18,20,25]), also in the class of slice hyperholomorphic functions there is the notion of left as well as of right hyperholomorphicity. But despite what happens in the monogenic case, for slice hyperholomorphic functions the Cauchy formulas for left and for right slice hyperholomorphic functions have two different kernels; moreover, each of these kernels can be written in two different ways.

The calculus admits a quaternionic version, which works for quaternionic linear operators and is based on slice hyperholomorphic (or slice regular) functions defined on subsets of the real algebra of quaternions $\mathbb{H}$ with values in the quaternions; see [11,13]. To explain our new result and its consequences, let us focus, at the moment, on the quaternionic setting which is simpler to illustrate.

Let us denote by V a two-sided quaternionic Banach space and let $T : V \to V$ be a bounded right (or left) linear operator. We recall that the S-spectrum is defined as

$$\sigma_S(T) = \{s \in \mathbb{H} \ : \ T^2 - 2Re(s)T + |s|^2\mathcal{I} \ \text{ is not invertible}\},$$

where $s = s_0 + s_1 i + s_2 j + s_3 k$ is a quaternion, $Re(s) = s_0$, $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$. The left and the right S-resolvent operators are defined as

$$S_L^{-1}(s, T) := -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \quad s \in \mathbb{H} \setminus \sigma_S(T) \qquad (1.1)$$

and

$$S_R^{-1}(s,T) := -(T - \bar{s}\mathcal{I})(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}, \quad s \in \mathbb{H} \setminus \sigma_S(T), \tag{1.2}$$

respectively. The left S-resolvent operator satisfies the equation

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \quad s \in \mathbb{H} \setminus \sigma_S(T), \tag{1.3}$$

and the right S-resolvent operator satisfies

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}, \quad s \in \mathbb{H} \setminus \sigma_S(T). \tag{1.4}$$

Consider the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$, for $I \in \mathbb{S}$, where $\mathbb{S}$ is the unit sphere of purely imaginary quaternions. Observe that $\mathbb{C}_I$ can be identified with a complex plane since $I^2 = -1$ for every $I \in \mathbb{S}$. Let $U \subset \mathbb{H}$ be a suitable domain that contains the S-spectrum of T. We define for left slice hyperholomorphic functions $f : U \to \mathbb{H}$ (see the precise definition in the sequel) the quaternionic functional calculus as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s), \tag{1.5}$$

where $ds_I = -ds I$, and for right slice hyperholomorphic functions $f : U \to \mathbb{H}$, we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,T). \tag{1.6}$$

These definitions are well posed since the integrals do not depend neither on the open set U nor on the complex plane $\mathbb{C}_I$ and can be extended to the case of $(n+1)$-tuples of operators, using slice hyperholomorphic functions with values in a Clifford algebra. Using the S-spectrum in [23] the authors introduce the continuous functional calculus in a quaternionic Hilbert space.

The S-resolvent equations (1.3), (1.4) are useful to prove several properties of the S-functional calculus. However, it is natural to ask if it is possible to obtain an analog of the classical resolvent equation

$$(\lambda I - E)^{-1}(\mu I - E)^{-1} = \frac{(\lambda I - E)^{-1} - (\mu I - E)^{-1}}{\mu - \lambda}, \quad \lambda, \mu \in \mathbb{C} \setminus \sigma(E), \tag{1.7}$$

where E is a complex operator on a Banach space, which might be useful to prove other properties of the calculus. The main goal of this paper is to show that (1.7) can be generalized in this noncommutative setting, but it involves both the left and the right S-resolvent operators. Precisely, we will show that

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \Big[(S_R^{-1}(s,T) - S_L^{-1}(p,T))p - \bar{s}(S_R^{-1}(s,T) - S_L^{-1}(p,T))\Big] \times (p^2 - 2s_0 p + |s|^2)^{-1},$$

for $s, p \in \mathbb{H} \setminus \sigma_S(T)$.

It is also worthwhile to mention that the S-resolvent operator plays an important role in the definition of the quaternionic version of the counterpart of the operator $(I - zA)^{-1}$ in the realization $s(z) = D + zC(I - zA)^{-1}B$ for Schur multipliers; see [2]. The reader is referred to [2–4] for Schur analysis in the slice hyperholomorphic setting and to [1,7] for an overview of Schur analysis in the complex setting.

It is interesting to note that in the literature there are other cases in which the authors consider two resolvent operators. We mention in particular the case of Schur analysis in the setting of upper triangular operators and in the setting of compact Riemann surfaces. In the first case, the role of complex numbers is played by diagonal operators and there are two "point evaluations" of an operator at a diagonal, one left and one right, each corresponding to an associated resolvent operator; see [8, (2.4)–(2.6), p. 256], but the resolvent equation is related with just one resolvent at a time; see [8, Corollary 2.9, p. 266]. In the setting of compact Riemann surfaces (see [30,35] for the general setting) there is a resolvent operator associated with every meromorphic function on the given Riemann surface X (see [9, (4.1), p. 307]), and one needs two such operators, associated with a pair of functions which generate the field of meromorphic functions on X, to study underlying spaces; see [9, §5]. The same resolvent equation is satisfied by all the resolvent operators; see [9, Theorem 4.2, p. 309].

In this paper both S-resolvent operators enter the resolvent equation.

The plan of the paper is as follows.

In Sect. 2 we recall some preliminary results on slice hyperholomorphic functions.

In Sect. 3 we state and prove the new resolvent equation and we show that there are two possible versions which are equivalent. We prove our results for the S-functional calculus for $(n+1)$-tuples of not necessarily commuting operators and we show some applications of the resolvent equation.

In Sect. 4 we consider the commutative version of the S-functional calculus, the so-called SC-functional calculus, and we reformulate our main results for the quaternionic functional calculus. Since the proofs follow the lines of the corresponding proofs in the case of $(n+1)$-tuples of not necessarily commuting operators, we will omit them.

2 Preliminary Results

In this section we recall the notion of slice hyperholomorphic functions and their Cauchy formulas; see [19].

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_ie_j + e_je_i = 0$, $i \neq j$, $e_i^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = \{i_1 \dots i_r\} \in \mathcal{P}\{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index and $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$, $e_\emptyset = 1$. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_n$ called paravector and the real part x_0 of x will also be denoted by $Re(x)$. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{j=1}^n x_j e_j$.

Let

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \dots + e_n x_n \mid x_1^2 + \dots + x_n^2 = 1\};$$

for $I \in \mathbb{S}$ we obviously have $I^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set $I_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and for any nonreal $x \in \mathbb{R}^{n+1}$ the set

$$[x] := \{y \in \mathbb{R}^{n+1} \ : \ y = x_0 + I|\underline{x}|, \ I \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$ and an element belonging to $\mathbb{C}_I$ will be indicated by $u + Iv$, for $u, v \in \mathbb{R}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector x.

Definition 2.1 (*Slice hyperholomorphic functions*) Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$.

The function f is said to be left slice hyperholomorphic (or slice monogenic) if, for every $I \in \mathbb{S}$, on $U \cap \mathbb{C}_I$ it satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u + Iv) + I\frac{\partial}{\partial v} f_I(u + Iv)\right) = 0.$$

We will denote by $\mathcal{SM}(U)$ the set of left slice hyperholomorphic functions on the open set U or by $\mathcal{SM}^L(U)$ when confusion may arise.

The function f is said to be right slice hyperholomorphic (or right slice monogenic) if, for every $I \in \mathbb{S}$, on $U \cap \mathbb{C}_I$, it satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u + Iv) + \frac{\partial}{\partial v} f_I(u + Iv)I\right) = 0.$$

We will denote by $\mathcal{SM}^R(U)$ the set of right slice hyperholomorphic functions on the open set U.

Slice hyperholomorphic functions possess good properties when they are defined on suitable domains which are introduced in the following definition. We refer the reader to [19] for all the missing details.

Definition 2.2 (*Axially symmetric slice domain*) Let U be a domain in $\mathbb{R}^{n+1}$. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is nonempty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that U is axially symmetric if, for all $x \in U$, the $(n-1)$-sphere $[x]$ is contained in U.

Definition 2.3 (*Cauchy kernel for left slice hyperholomorphic functions*) Let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S_L^{-1}(s, x)$ be the function defined by

$$S_L^{-1}(s, x) := -(x^2 - 2x\mathrm{Re}[s] + |s|^2)^{-1}(x - \overline{s}). \tag{2.1}$$

We say that $S_L^{-1}(s, x)$ is the Cauchy kernel (for left slice hyperholomorphic functions) written in form I.

Proposition 2.4 *Suppose that x and $s \in \mathbb{R}^{n+1}$ are such that $x \notin [s]$. The following identity holds:*

$$-(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}(x - \bar{s}) = (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |s|^2)^{-1}. \tag{2.2}$$

Remark 2.5 By Proposition 2.4 $S_L^{-1}(s, x)$ can also be written as

$$S_L^{-1}(s, x) := (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}. \tag{2.3}$$

In this case, we will say $S_L^{-1}(s, x)$ is written in form II.

Proposition 2.6 *The function $S_L^{-1}(s, x)$ is left slice hyperholomorphic in the variable x and right slice hyperholomorphic in the variable s, for $x \notin [s]$.*

The case of the Cauchy kernel for right slice hyperholomorphic functions is similar.

Definition 2.7 (*Cauchy kernel for right slice hyperholomorphic functions*) Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. The Cauchy kernel $S_R^{-1}(s, x)$ for right slice hyperholomorphic functions is defined by

$$S_R^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}. \tag{2.4}$$

We say that $S_R^{-1}(s, x)$ is written in form I.

Remark 2.8 An analog of Proposition 2.4 holds. In fact,

$$-(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1} = (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}), \tag{2.5}$$

for $x, s \in \mathbb{R}^{n+1}$ such that $x \notin [s]$.

Thus $S_R^{-1}(s, x)$ can be written as

$$S_R^{-1}(s, x) = (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}),$$

and in this case we say that $S_R^{-1}(s, x)$ is written in form II.

Theorem 2.9 (The Cauchy formula with slice hyperholomorphic kernel) *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $\partial(U \cap \mathbb{C}_I)$ is a finite union of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Set $ds_I = -ds I$ for $I \in \mathbb{S}$.*

- *If f is a (left) slice hyperholomorphic function on an open set that contains $\overline{U}$, then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, x) ds_I f(s) \tag{2.6}$$

  and the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

- *If f is a right slice hyperholomorphic function on an open set that contains $\overline{U}$, then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)ds_I S_R^{-1}(s,x) \tag{2.7}$$

and the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

The above Cauchy formulas are the starting point to define the S-functional calculus. A crucial fact of slice hyperholomorphic functions is the representation formula (also called the structure formula). This formula will be used in the sequel to give applications of the new resolvent equation.

Theorem 2.10 (Representation Formula) *Let U be an axially symmetric s-domain in $\mathbb{H}$.*

- *Let f be a (left) slice hyperholomorphic function on U. Choose any $J \in \mathbb{S}$. Then the following equality holds for all $x = u + Iv \in U$:*

$$f(u+Iv) = \frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]+I\frac{1}{2}\Big[J[f(u-Jv)-f(u+Jv)]\Big]. \tag{2.8}$$

Moreover, for all $u, v \in \mathbb{R}$ such that $[u+Iv] \subseteq U$, there exist $\mathbb{R}_n$-valued functions α, β depending on u, v only such that for all $K \in \mathbb{S}$

$$\frac{1}{2}\Big[f(u+Kv)+f(u-Kv)\Big] = \alpha(u,v) \ \ and$$
$$\frac{1}{2}\Big[K[f(u-Kv)-f(u+Kv)]\Big] = \beta(u,v). \tag{2.9}$$

- *Let f be a right slice hyperholomorphic function on U. Choose any $J \in \mathbb{S}$. Then the following equality holds for all $x = u + Iv \in U$:*

$$f(u+Iv) = \frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]+\frac{1}{2}\Big[[f(u-Jv)-f(u+Jv)]J\Big]I. \tag{2.10}$$

Moreover, for all $u, v \in \mathbb{R}$ such that $[u+Iv] \subseteq U$, there exist $\mathbb{R}_n$-valued functions α, β depending on u, v only such that for all $K \in \mathbb{S}$

$$\frac{1}{2}\Big[f(u+Kv)+f(u-Kv)\Big] = \alpha(u,v) \ \ and$$
$$\frac{1}{2}\Big[[f(u-Kv)-f(u+Kv)]K\Big] = \beta(u,v). \tag{2.11}$$

3 The Case of Several Noncommuting Operators

In the sequel, we will consider a Banach space V over $\mathbb{R}$ with norm $\|\cdot\|$. It is possible to endow V with an operation of multiplication by elements of $\mathbb{R}_n$ which gives a two-sided module over $\mathbb{R}_n$. A two-sided module V over $\mathbb{R}_n$ is called a Banach module over

$\mathbb{R}_n$, if there exists a constant $C \geq 1$ such that $\|va\| \leq C\|v\||a|$ and $\|av\| \leq C|a|\|v\|$ for all $v \in V$ and $a \in \mathbb{R}_n$. By V_n we denote $V \otimes \mathbb{R}_n$; V_n turns out to be a two-sided Banach module over $\mathbb{R}_n$.

An element in V_n is of the type $\sum_A v_A \otimes e_A$ (where $A = i_1 \dots i_r, i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index). The multiplications of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined by $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. For simplicity, we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Finally, we define $\|v\|^2_{V_n} = \sum_A \|v_A\|^2_V$. We denote by $\mathcal{B}(V)$ the space of bounded $\mathbb{R}$-homomorphisms of the Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$. Given $T_A \in \mathcal{B}(V)$, we can introduce the operator $T = \sum_A T_A e_A$ and its action on $v = \sum v_B e_B \in V_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The operator T is a module homomorphism which is a bounded linear map on V_n.

In the sequel, we will consider operators of the form $T = T_0 + \sum_{j=1}^n e_j T_j$ where $T_j \in \mathcal{B}(V)$ for $j = 0, 1, \dots, n$. The subset of such operators in $\mathcal{B}(V_n)$ will be denoted by $\mathcal{B}^{0,1}(V_n)$. We define $\|T\|_{\mathcal{B}^{0,1}(V_n)} = \sum_{j=0}^n \|T_j\|_{\mathcal{B}(V)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}^{0,1}(V_n)$ in the norm of an operator. Note also that $\|TS\| \leq \|T\|\|S\|$.

Definition 3.1 Let $T \in \mathcal{B}^{0,1}(V_n)$. We define the left Cauchy kernel operator series or S-resolvent operator series as

$$S_L^{-1}(s, T) = \sum_{n \geq 0} T^n s^{-1-n}, \tag{3.1}$$

and the right Cauchy kernel operator series as

$$S_R^{-1}(s, T) = \sum_{n \geq 0} s^{-1-n} T^n, \tag{3.2}$$

for $\|T\| < |s|$.

The Cauchy kernel operator series are the power series expansion of the S-resolvent operators. Their sum is computed in the following result:

Theorem 3.2 *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \mathbb{H}$. Then, for $\|T\| < |s|$, we have*

$$\sum_{m \geq 0} T^m s^{-1-m} = -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{3.3}$$

$$\sum_{m \geq 0} s^{-1-m} T^m = -(T - \overline{s}\mathcal{I})(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}. \tag{3.4}$$

We observe that the sum of the above series is independent of the fact that the components of the paravector operator T commute. Moreover, the operators on the right-hand sides of (3.3) and (3.4) are defined on a subset of $\mathbb{R}^{n+1}$ that is larger than $\{s \in \mathbb{R}^{n+1} \; : \; \|T\| < |s|\}$. This fact suggests the definition of S-spectrum, of S-resolvent set and of S-resolvent operators.

Definition 3.3 (*The S-spectrum and the S-resolvent set*) Let $T \in \mathcal{B}^{0,1}(V_n)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} \ : \ T^2 - 2\ Re(s)T + |s|^2\mathcal{I} \quad \textit{is not invertible}\}.$$

The S-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

Definition 3.4 (*The S-resolvent operators*) Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. We define the left S-resolvent operator as

$$S_L^{-1}(s,T) := -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{3.5}$$

and the right S-resolvent operator as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}. \tag{3.6}$$

The operators $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ satisfy the equations below; see [19]:

Theorem 3.5 *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \rho_S(T)$. Then, the left S-resolvent operator satisfies the equation*

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \tag{3.7}$$

and the right S-resolvent operator satisfies the equation

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{3.8}$$

Our goal is to establish the analog of the classical resolvent equation. To this end, we need some preliminary results. A crucial fact is the following Theorem 3.6 that will give us the hint to discover what is the structure of the resolvent equation in this noncommutative setting, at least in the case where the S-resolvent operators are expressed in power series, see also [6].

Theorem 3.6 *Let $A, B \in \mathcal{B}(V_n)$ and let $s, p \in \mathbb{R}^{n+1}$. Then, for $|p| < |s|$, we have*

$$\sum_{m\geq 0} p^m A s^{-1-m} = -(p^2 - 2Re(s)p + |s|^2)^{-1}(pA - A\overline{s}), \tag{3.9}$$

and

$$\sum_{m\geq 0} s^{-1-m} B p^m = -(Bp - \overline{s}B)(p^2 - 2Re(s)p + |s|^2)^{-1}. \tag{3.10}$$

Moreover, (3.10) *can be written as*

$$\sum_{m\geq 0} s^{-1-m} B p^m = (s^2 - 2Re(p)s + |p|^2)^{-1}(sB - B\overline{p}). \tag{3.11}$$

Proof To verify (3.9) define

$$X := (p^2 - 2Re(s)p + |s|^2) \sum_{m\geq 0} p^m A s^{-1-m}$$

and observe that

$$\begin{aligned} X &= \sum_{m\geq 0} (p^2 - 2Re(s)p + |s|^2) p^m A s^{-1-m} \\ &= p^2 A s^{-1} - 2Re(s) p A s^{-1} + |s|^2 A s^{-1} \\ &\quad + p^3 A s^{-2} - 2Re(s) p^2 A s^{-2} + |s|^2 p A s^{-2} \\ &\quad + p^4 A s^{-3} - 2Re(s) p^3 A s^{-3} + |s|^2 p^2 A s^{-3} + \cdots \\ &= -(pA - A\overline{s}) + \sum_{m\geq 2} p^m A (s^2 - 2Re(s)s + |s|^2) s^{-1-m}. \end{aligned} \tag{3.12}$$

Since any paravector s satisfies

$$s^2 - 2Re(s)s + |s|^2 = 0$$

we deduce that

$$X = (p^2 - 2Re(s)p + |s|^2) \sum_{m\geq 0} p^m A s^{-1-m} = -(pA - A\overline{s})$$

and the statement follows. The equality in (3.10) can be verified by setting

$$Y := \sum_{m\geq 0} s^{-1-m} B p^m (p^2 - 2Re(s)p + |s|^2)$$

and observing that

$$Y = -(Bp - \overline{s}B) + \sum_{m\geq 0} s^{-1-m} B p^m (p^2 - 2Re(p)p + |p|^2) = -(Bp - \overline{s}B).$$

With similar computations one can verify equality (3.11). □

Corollary 3.7 *Let $A, B \in \mathcal{B}(V_n)$ and let s, p be paravectors. Then, for $|p| < |s|$, the following equations hold:*

$$\begin{aligned} \sum_{j=0}^{m} p^j A s^{-1-j} &= -(p^2 - 2Re(s)p + |s|^2)^{-1} (pA - A\overline{s}) \\ &\quad + p^{m+1} (p^2 - 2Re(s)p + |s|^2)^{-1} (pA - A\overline{s}) s^{-1-m}, \end{aligned} \tag{3.13}$$

and

$$\sum_{j=0}^{m} s^{-1-j} B p^j = -(Bp - \overline{s}B)(p^2 - 2Re(s)p + |s|^2)^{-1} + s^{-1-m}(Bp - \overline{s}B)(p^2 - 2Re(s)p + |s|^2)^{-1} p^{m+1}. \quad (3.14)$$

Moreover, (3.14) *can also be written as*

$$\sum_{j=0}^{m} s^{-1-j} B p^j = (s^2 - 2Re(p)s + |p|^2)^{-1}(sB - B\overline{p}) - s^{-1-m}(s^2 - 2Re(p)s + |p|^2)^{-1}(sB - B\overline{p})p^{m+1}. \quad (3.15)$$

Proof Identity (3.13) follows from

$$\sum_{j=0}^{m} p^j A s^{-1-j} = \sum_{j=0}^{\infty} p^j A s^{-1-j} - \sum_{j=m+1}^{\infty} p^j A s^{-1-j},$$

which can be written as

$$\sum_{j=0}^{m} p^j A s^{-1-j} = \sum_{j=0}^{\infty} p^j A s^{-1-j} - p^{m+1} \left(\sum_{j=0}^{\infty} p^j A s^{-1-j} \right) s^{-1-m},$$

but now we use (3.9) to get the result. Identities (3.14) and (3.15) follow with similar computations. □

We now prove the new S-resolvent equation. In the proof we first consider the case in which the S-resolvent operators admit the power series expansion

$$S_L^{-1}(s,T) = \sum_{m \geq 0} T^m s^{-1-m}, \qquad S_R^{-1}(s,T) = \sum_{m \geq 0} s^{-1-m} T^m,$$

which is for $\|T\| < |s|$. Then, we verify that the equation holds in general.

Theorem 3.8 *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let s and $p \in \rho_S(T)$. Then we have the resolvent equation*

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = ((S_R^{-1}(s,T) - S_L^{-1}(p,T))p - \overline{s}(S_R^{-1}(s,T) - S_L^{-1}(p,T)))(p^2 - 2s_0 p + |s|^2)^{-1}. \quad (3.16)$$

Moreover, the resolvent equation can also be written as

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = (s^2 - 2p_0 s + |p|^2)^{-1}(s(S_R^{-1}(s,T) - S_L^{-1}(p,T)) - (S_R^{-1}(s,T) - S_L^{-1}(p,T))\overline{p}). \quad (3.17)$$

Proof We prove the theorem in two steps.
Step I First we assume that the S-resolvent operators are expressed in power series. If $\|T\| < |p| < |s|$ then the S-resolvent operators have power series expansion and so

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \left(\sum_{j\geq 0} s^{-1-j}T^j\right)\left(\sum_{j\geq 0} T^j p^{-1-j}\right). \tag{3.18}$$

By setting

$$\Lambda_m(s,p;T) := \sum_{j=0}^{m} s^{-1-j}\left(T^m p^{-1-m}\right)p^j$$

(3.18) can be written as

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \sum_{m\geq 0}\Lambda_m(s,p;T).$$

Formula (3.14) with $B = T^m p^{-1-m}$ and some computations give

$$\begin{aligned}
&\Lambda_m(s,p;T)\\
&\quad = -\left((T^m p^{-1-m})p - \overline{s}(T^m p^{-1-m})\right)\left(p^2 - 2Re(s)p + |s|^2\right)^{-1}\\
&\qquad + s^{-1-m}\left((T^m p^{-1-m})p - \overline{s}(T^m p^{-1-m})\right)\left(p^2 - 2Re(s)p + |s|^2\right)^{-1}p^{m+1}\\
&\quad = -\left[(T^m p^{-1-m})p - \overline{s}(T^m p^{-1-m}) + (s^{-1-m}T^m)p - \overline{s}(s^{-1-m}T^m)\right]\\
&\qquad \times\left(p^2 - 2Re(s)p + |s|^2\right)^{-1}.
\end{aligned} \tag{3.19}$$

From the chain of equalities

$$\begin{aligned}
S_R^{-1}(s,T)S_L^{-1}(p,T) &= \sum_{m\geq 0}\Lambda_m(s,p;T)\\
&= -\left[\left(\sum_{m\geq 0}(T^m p^{-1-m})p - \overline{s}\sum_{m\geq 0}(T^m p^{-1-m})\right)\right]\\
&\quad + \left[\left(\sum_{m\geq 0}s^{-1-m}T^m\right)p - \overline{s}\sum_{m\geq 0}(s^{-1-m}T^m)\right]\\
&\quad \times\left(p^2 - 2Re(s)p + |s|^2\right)^{-1}
\end{aligned} \tag{3.20}$$

(3.16) follows.

To prove that the resolvent equation can be written in the second form (3.17) observe that $\Lambda_m(s,p;T)$ can also be written using (3.11) as

$$\begin{aligned}\Lambda_m(s,p;T) &= (s^2-2Re(p)s+|p|^2)^{-1}(s(T^m p^{-1-m})-(T^m p^{-1-m})\overline{p})\\ &\quad - s^{-1-m}(s^2-2Re(p)s+|p|^2)^{-1}(s(T^m p^{-1-m})\\ &\quad -(T^m p^{-1-m})\overline{p})p^{m+1},\end{aligned} \tag{3.21}$$

so taking the sum $\sum_{m\geq 0}\Lambda_m(s,p;T)$ we get the second version of the resolvent equation.

Step II We prove that, for s and $p\in\rho_S(T)$, (3.16) and (3.17) hold with $S_R^{-1}(s,T)$ and $S_L^{-1}(p,T)$ defined in (3.5) and (3.6), respectively.

Let us verify (3.16). Since s and $p\in\rho_S(T)$ the left and right S-resolvent operators defined by (3.5) and (3.6) satisfy the left and the right resolvent equations (3.7) and (3.8), respectively. To verify (3.16) we have to show that $S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)$ equals

$$(S_R^{-1}(s,T)-S_L^{-1}(p,T))p-\overline{s}(S_R^{-1}(s,T)-S_L^{-1}(p,T)).$$

To do this we use the left and the right S-resolvent equations (3.7), (3.8). Indeed, using the left S-resolvent equation, written as

$$S_L^{-1}(p,T)p = TS_L^{-1}(p,T)+\mathcal{I},$$

we have

$$\begin{aligned}S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2) &= S_R^{-1}(s,T)[S_L^{-1}(p,T)p]p\\ &\quad -2s_0S_R^{-1}(s,T)S_L^{-1}(p,T)p+|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T)\\ &= S_R^{-1}(s,T)[TS_L^{-1}(p,T)+\mathcal{I}]p-2s_0S_R^{-1}(s,T)[TS_L^{-1}(p,T)+\mathcal{I}]\\ &\quad +|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T)\end{aligned} \tag{3.22}$$

and using again the left S-resolvent equation

$$\begin{aligned}&S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)\\ &\quad = S_R^{-1}(s,T)T[TS_L^{-1}(p,T)+\mathcal{I}]+S_R^{-1}(s,T)p\\ &\qquad -2s_0S_R^{-1}(s,T)[TS_L^{-1}(p,T)+\mathcal{I}]\\ &\qquad +|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T)\end{aligned} \tag{3.23}$$

we obtain

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)\\ &\quad=[S_R^{-1}(s,T)T]TS_L^{-1}(p,T)+\ S_R^{-1}(s,T)T+S_R^{-1}(s,T)p\\ &\quad\quad-2s_0\Big[[S_R^{-1}(s,T)T]S_L^{-1}(p,T)+S_R^{-1}(s,T)\Big]\\ &\quad\quad+|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T). \end{aligned} \tag{3.24}$$

Now we use the right S-resolvent equation

$$S_R^{-1}(s,T)T=sS_R^{-1}(s,T)-\mathcal{I}$$

and we obtain

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)\\ &\quad=[sS_R^{-1}(s,T)-\mathcal{I}]TS_L^{-1}(p,T)+\ sS_R^{-1}(s,T)-\mathcal{I}+S_R^{-1}(s,T)p\\ &\quad\quad-2s_0[[sS_R^{-1}(s,T)-\mathcal{I}]S_L^{-1}(p,T)+S_R^{-1}(s,T)]\\ &\quad\quad+|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T). \end{aligned} \tag{3.25}$$

Iterating the use of the above right S-resolvent equation we get

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)\\ &\quad=s[sS_R^{-1}(s,T)-\mathcal{I}]S_L^{-1}(p,T)\\ &\quad\quad-TS_L^{-1}(p,T)+sS_R^{-1}(s,T)-\mathcal{I}+S_R^{-1}(s,T)p\\ &\quad\quad-2s_0[sS_R^{-1}(s,T)S_L^{-1}(p,T)-S_L^{-1}(p,T)+S_R^{-1}(s,T)]\\ &\quad\quad+|s|^2S_R^{-1}(s,T)S_L^{-1}(p,T), \end{aligned} \tag{3.26}$$

which leads to

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)\\ &\quad=(s^2-2s_0s+|s|^2)S_R^{-1}(s,T)S_L^{-1}(p,T)\\ &\quad\quad+[S_R^{-1}(s,T)-S_L^{-1}(p,T)]p-\bar{s}[S_R^{-1}(s,T)-S_L^{-1}(p,T)], \end{aligned} \tag{3.27}$$

and since $s^2-2s_0s+|s|^2=0$ we obtain (3.16). With similar computations we can show that also (3.17) holds. □

In the commutative case besides the resolvent equation, we have also the validity of the following relation between the resolvent operators:

$$(\lambda I-E)^{-1}(\mu I-E)^{-1}=(\mu I-E)^{-1}(\lambda I-E)^{-1},\quad \text{for}\quad \lambda,\mu\in\rho(E).$$

In the noncommutative case we cannot expect the validity of such a relation; however, we will show that an analogous equation holds for the so-called pseudo S-resolvent operators defined below.

Definition 3.9 Let $T \in \mathcal{B}^{0,1}(V_n)$. For $s \in \rho_S(T)$, the pseudo S-resolvent operator of T is defined as

$$Q_s(T) := (T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}.$$

With the above definition the resolvents $S_L^{-1}(s, T)$ and $S_R^{-1}(s, T)$ can be written as

$$S_L^{-1}(s, T) := -Q_s(T)(T - \overline{s}\mathcal{I}), \quad s \in \rho_S(T), \tag{3.28}$$

and

$$S_R^{-1}(s, T) := -(T - \overline{s}\mathcal{I})Q_s(T), \quad s \in \rho_S(T). \tag{3.29}$$

We now prove the following:

Theorem 3.10 *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s, p \in \rho_S(T)$. Then*

$$(T - \overline{s}\mathcal{I})Q_s(T)Q_p(T)(T - \overline{p}\mathcal{I}) = (T - \overline{s}\mathcal{I})Q_p(T)Q_s(T)(T - \overline{p}\mathcal{I}).$$

Proof It follows from the fact that

$$(T^2 - 2Re(s)T + |s|^2\mathcal{I})(T^2 - 2Re(p)T + |p|^2\mathcal{I}) = (T^2 - 2Re(p)T + |p|^2\mathcal{I}) \\ (T^2 - 2Re(s)T + |s|^2\mathcal{I}). \tag{3.30}$$

Since $s, p \in \rho_S(T)$ we can take the inverse and the statement follows. □

Remark 3.11 Observe that the function $F_T(s, p)$ defined by

$$F_T(s, p) := S_R^{-1}(s, T)S_L^{-1}(p, T)$$

is left slice hyperholomorphic in s and right slice hyperholomorphic in p with values in $\mathcal{B}(V_n)$. The function

$$G_T(s, p) := S_L^{-1}(p, T)S_R^{-1}(s, T)$$

is not slice hyperholomorphic neither in p nor in s.

Remark 3.12 Using the star products left and right in the variables s, p, which will be denoted by $\star_{s,left}$, $\star_{p,right}$ respectively, see [6], the resolvent equation (3.16) can be written as

$$S_R^{-1}(s, T)S_L^{-1}(p, T) = [S_R^{-1}(s, T) - S_L^{-1}(p, T)] \star_{s,left} \\ \times (p - \overline{s})(p^2 - 2Re(s)p + |s|^2)^{-1}\mathcal{I},$$

or

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = (s-\overline{p})(s^2-2Re(p)s+|p|^2)^{-1}\mathcal{I}\star_{p,right} \times[S_R^{-1}(s,T)-S_L^{-1}(p,T)].$$

3.1 Some Applications

Here we recall the formulations of the S-functional calculus, and then we use the resolvent equation to deduce some results. We first recall two important properties of the S-spectrum.

Theorem 3.13 (Structure of the S-spectrum) *Let $T \in \mathcal{B}^{0,1}(V_n)$ and suppose that $p = p_0 + \underline{p}$ belongs $\sigma_S(T)$ with $\underline{p} \neq 0$. Then all the elements of the $(n-1)$-sphere $[p]$ belong to $\sigma_S(T)$.*

This result implies that if $p \in \sigma_S(T)$ then either p is a real point or the whole $(n-1)$-sphere $[p]$ belongs to $\sigma_S(T)$.

Theorem 3.14 (Compactness of the S-spectrum) *Let $T \in \mathcal{B}^{0,1}(V_n)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set. Moreover, $\sigma_S(T)$ is contained in $\{s \in \mathbb{R}^{n+1} : |s| \leq \|T\|\}$.*

Definition 3.15 Let V_n be a two-sided Banach module, $T \in \mathcal{B}^{0,1}(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$ such that $\partial(U \cap \mathbb{C}_I)$ is the union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. In this case we say that U is a T-admissible open set.

We now introduce the class of functions for which we can define the two versions of the S-functional calculus.

Definition 3.16 Let V_n be a two-sided Banach module, $T \in \mathcal{B}^{0,1}(V_n)$ and let W be an open set in $\mathbb{R}^{n+1}$.

(i) A function $f \in \mathcal{SM}^L(W)$ is said to be locally left hyperholomorphic on $\sigma_S(T)$ if there exists a T-admissible domain $U \subset \mathbb{R}^{n+1}$ such that $\overline{U} \subset W$, on which f is left slice hyperholomorphic. We will denote by $\mathcal{SM}^L_{\sigma_S(T)}$ the set of locally left hyperholomorphic functions on $\sigma_S(T)$.

(ii) A function $f \in \mathcal{SM}^R(W)$ is said to be locally right regular on $\sigma_S(T)$ if there exists a T-admissible domain $U \subset \mathbb{R}^{n+1}$ such that $\overline{U} \subset W$, on which f is right slice hyperholomorphic. We will denote by $\mathcal{SM}^R_{\sigma_S(T)}$ the set of locally right slice hyperholomorphic functions on $\sigma_S(T)$.

Definition 3.17 (*The S-functional calculus*) Let V_n be a two-sided Banach module and $T \in \mathcal{B}^{0,1}(V_n)$. Let $U \subset \mathbb{R}^{n+1}$ be a T-admissible domain and set $ds_I = -ds I$. We define

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s), \quad for \quad f \in \mathcal{SM}^L_{\sigma_S(T)}, \tag{3.31}$$

and

$$f(T)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\, S_R^{-1}(s,T),\ \ for\ \ f\in\mathcal{SM}^R_{\sigma_S(T)}. \tag{3.32}$$

We now define the Riesz projectors for the S-functional calculus. We begin with a preliminary lemma.

Lemma 3.18 *Let $B\in\mathcal{B}(V_n)$ and let G be an axially symmetric s-domain such that $p\in G$. Then*

$$(\overline{s}B-Bp)(p^2-2s_0p+|s|^2)^{-1}=(s^2-2p_0s+|p|^2)^{-1}(sB-B\overline{p}),\quad p\notin[s], \tag{3.33}$$

and

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(\overline{s}B-Bp)(p^2-2s_0p+|s|^2)^{-1}=B. \tag{3.34}$$

Proof Formula (3.33) is obtained by direct computation. Let us prove (3.34). So we write

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(\overline{s}B-Bp)(p^2-2s_0p+|s|^2)^{-1}\\
&=\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(s^2-2p_0s+|p|^2)^{-1}(sB-B\overline{p})\\
&=\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(s^2-2p_0s+|p|^2)^{-1}(s-\overline{p})B\\
&\quad+\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(s^2-2p_0s+|p|^2)^{-1}(\overline{p}B-B\overline{p})
\end{aligned}$$

but observe that

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(s^2-2p_0s+|p|^2)^{-1}(s-\overline{p})B=\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I S_R^{-1}(s,p)B=B$$

and moreover by the residue theorem it is

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} ds_I(s^2-2p_0s+|p|^2)^{-1}=0$$

so we get the statement. □

Theorem 3.19 *Let $T\in\mathcal{B}^{0,1}(V_n)$ and let $\sigma_S(T)=\sigma_{1S}(T)\cup\sigma_{2S}(T)$, with*

$$\text{dist}\,(\sigma_{1S}(T),\sigma_{2S}(T))>0.$$

Let U_1 and U_2 be two axially symmetric s-domains such that $\sigma_{1S}(T) \subset U_1$ and $\sigma_{2S}(T) \subset U_2$, with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Set

$$P_j := \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I, \qquad j = 1, 2, \tag{3.35}$$

$$T_j := \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, s, \qquad j = 1, 2. \tag{3.36}$$

Then P_j are projectors and $T_j = T P_j = P_j T$ for $j = 1, 2$.

Proof Let $\sigma_{jS}(T) \subset G_1$ and G_2 be two T-admissible open sets such that $G_1 \cup \partial G_1 \subset G_2$ and $G_2 \cup \partial G_2 \subset U_j$, for $j = 1$ or 2. Thanks to the structure of the S-spectrum we will assume that G_1 and G_2 are axially symmetric and s-domains.

Take $p \in \partial(G_1 \cap \mathbb{C}_I)$ and $s \in \partial(G_2 \cap \mathbb{C}_I)$ and observe that, for $I \in \mathbb{S}$, we have

$$P_j := \frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I S_R^{-1}(s, T)$$

but we can also write P_j as

$$P_j = \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_I)} S_L^{-1}(p, T) dp_I.$$

Now consider P_j^2 written as

$$P_j^2 = \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} S_R^{-1}(s, T) S_L^{-1}(p, T) dp_I.$$

Using the resolvent equation we write:

$$\begin{aligned} P^2 = {} & \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \\ & \times \int_{\partial(G_1 \cap \mathbb{C}_I)} [S_R^{-1}(s, T) - S_L^{-1}(p, T)] p (p^2 - 2s_0 p + |s|^2)^{-1} dp_I \\ & - \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \\ & \times \int_{\partial(G_1 \cap \mathbb{C}_I)} \overline{s} [S_R^{-1}(s, T) - S_L^{-1}(p, T)] (p^2 - 2s_0 p + |s|^2)^{-1} dp_I. \end{aligned}$$

Now observe that

$$\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)} ds_I S_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_I)} p(p^2-2s_0p+|s|^2)^{-1}dp_I=0$$

and

$$-\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)} ds_I\bar{s}S_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_I)} (p^2-2s_0p+|s|^2)^{-1}dp_I=0$$

since the functions

$$p\mapsto p(p^2-2s_0p+|s|^2)^{-1}, \qquad p\mapsto (p^2-2s_0p+|s|^2)^{-1}$$

are slice hyperholomorphic and do not have singularities inside $\partial(G_1\cap\mathbb{C}_I)$. So P_j^2 can be written as

$$\begin{aligned}
P_j^2 &= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)} ds_I\int_{\partial(G_1\cap\mathbb{C}_I)} -S_L^{-1}(p,T)p(p^2-2s_0p+|s|^2)^{-1}dp_I\\
&\quad -\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)} ds_I\int_{\partial(G_1\cap\mathbb{C}_I)} -\bar{s}S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)^{-1}dp_I,\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)}\int_{\partial(G_1\cap\mathbb{C}_I)} ds_I\\
&\quad\times(\bar{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p)(p^2-2s_0p+|s|^2)^{-1}dp_I.
\end{aligned}$$

Applying now Lemma 3.18 with $B:=S_L^{-1}(p,T)$ and observing that $p\in G_2$, we finally have

$$P_j^2=\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_I)} S_L^{-1}(p,T)dp_I=P_j.$$

Let us now prove that $TP_j=P_jT$. Observe that the functions $f(s)=s^m$, for $m\in\mathbb{N}_0$ are both right and left slice hyperholomorphic. So the operator T can be written as

$$T=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} s\,ds_I\,S_R^{-1}(s,T);$$

analogously, as already observed, for the projectors P_j we have

$$P_j=\frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I=\frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I\,S_R^{-1}(s,T).$$

From the identity

$$T_j = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, s = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s\, ds_I\, S_R^{-1}(s,T)$$

we can compute $T P_j$ as:

$$T P_j = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} T S_L^{-1}(s,T)\, ds_I$$

and using the resolvent equation (3.7) it follows that

$$\begin{aligned} T P_j &= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} [S_L^{-1}(s,T)\, s - \mathcal{I}]\, ds_I \\ &= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} S_L^{-1}(s,T)\, s\, ds_I \\ &= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, s \\ &= T_j. \end{aligned} \tag{3.37}$$

Now consider

$$P_j T = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} ds_I\, S_R^{-1}(s,T) T$$

and using the resolvent equation (3.8) we obtain

$$\begin{aligned} P_j T &= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} ds_I\, [s\, S_R^{-1}(s,T) - \mathcal{I}] \\ &= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} ds_I\, s\, S_R^{-1}(s,T) \\ &= T_j, \end{aligned}$$

so the equality $P_j T = T P_j$ holds. □

Remark 3.20 The property that the Riesz projectors commute with the operator T has been proved for the quaternionic version of the S-functional calculus in [4], while the property that $P^2 = P$ given in [19] is obtained heuristically. The new resolvent equation allows us to prove this last property rigorously.

As is well known, the pointwise product of two hyperholomorphic functions is not in general hyperholomorphic, but for the class of functions defined below this property holds.

Definition 3.21 Let $f : U \to \mathbb{R}_n$ be a slice hyperholomorphic function, where U is an open set in $\mathbb{R}^{n+1}$. We define

$$\mathcal{N}(U) = \{f \in \mathcal{SM}(U) \ : \ f(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I, \ \ \forall I \in \mathbb{S}\}.$$

Proposition 3.22 *Let U be an open set in $\mathbb{R}^{n+1}$. Let $f \in \mathcal{N}(U)$, $g \in \mathcal{SM}(U)$. Then $fg \in \mathcal{SM}(U)$.*

Let us observe that functions in the subclass $\mathcal{N}(U)$ are both left and right slice hyperholomorphic. When we take the power series expansion of this class of functions at a point on the real line the coefficients of the expansion are real numbers.

Now observe that for functions in $f \in \mathcal{N}(U)$ we can define $f(T)$ using the left but also the right S-functional calculus. It is

$$\begin{aligned} f(T) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s, T). \end{aligned}$$

Lemma 3.23 *Let $B \in \mathcal{B}(V_n)$. Let G be an axially symmetric s-domain and assume that $f \in \mathcal{N}(G)$. Then, for $p \in G$, we have*

$$\frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (\overline{s} B - Bp)(p^2 - 2s_0 p + |s|^2)^{-1} = Bf(p). \tag{3.38}$$

Proof Recalling formula (3.33) we write

$$\begin{aligned} &\frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (\overline{s} B - Bp)(p^2 - 2s_0 p + |s|^2)^{-1} \\ &= \frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1} (sB - B\overline{p}) \\ &= \frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1} (s - \overline{p}) B \\ &\quad + \frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1} (\overline{p} B - B\overline{p}) \\ &:= \mathcal{J}_1 + \mathcal{J}_2 \end{aligned}$$

but observe that

$$\mathcal{J}_1 = \frac{1}{2\pi} \int\limits_{\partial(G\cap\mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1}(s - \overline{p})B$$
$$= \frac{1}{2\pi} \int\limits_{\partial(G\cap\mathbb{C}_I)} f(s) ds_I S_R^{-1}(s, p)B = f(p)B.$$

Consider now the second integral. Taking $s = u + Iv$ then the solutions of the equation $s^2 - 2p_0 s + |p|^2 = 0$ are $s_1 = \alpha$ and $s_2 = \overline{\alpha}$ where $\alpha = p_0 + I|\underline{p}|$, so

$$\mathcal{J}_2 = \frac{1}{2\pi} \int\limits_{\partial(G\cap\mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1}(\overline{p}B - B\overline{p})$$
$$= \frac{1}{2\pi} \int\limits_{\partial(G\cap\mathbb{C}_I)} \frac{f(s)}{(s-\alpha)(s-\overline{\alpha})} ds_I (\overline{p}B - B\overline{p}),$$

by the residue theorem we get

$$\mathcal{J}_2 = \frac{1}{2\pi} \int\limits_{\partial(G\cap\mathbb{C}_I)} f(s) ds_I (s^2 - 2p_0 s + |p|^2)^{-1}(\overline{p}B - B\overline{p})$$
$$= \frac{I}{2|\underline{p}|}[f(p_0 - I|\underline{p}|) - f(p_0 + I|\underline{p}|)](\overline{p}B - B\overline{p}).$$

Now we recall the structure formula that shows that a slice hyperholomorphic function can be written as

$$f(p) = \alpha(p_0, |\underline{p}|) + I_p \beta(p_0, |\underline{p}|)$$

where

$$\alpha(p_0, |\underline{p}|) = \frac{1}{2}[f(p_0 - I|\underline{p}|) + f(p_0 + I|\underline{p}|)],$$
$$\beta(p_0, |\underline{p}|) = \frac{I}{2}[f(p_0 - I|\underline{p}|) - f(p_0 + I|\underline{p}|)]$$

and in the case of functions $f \in \mathcal{N}(G)$ the functions α and β are real valued. Observe that

$$\mathcal{J}_1 + \mathcal{J}_2 = f(p)B + \frac{I}{2|\underline{p}|}[f(p_0 - I|\underline{p}|) - f(p_0 + I|\underline{p}|)](\overline{p}B - B\overline{p})$$
$$= \alpha(p_0, |\underline{p}|)B + I_p \beta(p_0, |\underline{p}|)B + \frac{\beta(p_0, |\underline{p}|)}{|\underline{p}|}(\overline{p}B - B\overline{p})$$
$$= \alpha(p_0, |\underline{p}|)B + I_p \beta(p_0, |\underline{p}|)B$$

$$+\frac{\beta(p_0,|\underline{p}|)}{|\underline{p}|}((p_0-I_p|\underline{p}|)B-B(p_0-I_p|\underline{p}|))$$
$$=B(\alpha(p_0,|\underline{p}|)+I_p\beta(p_0,|\underline{p}|))$$
$$=Bf(p),$$

so we get the statement. □

Remark 3.24 If we assume that $f\in\mathcal{N}(\mathbf{B}(0,r))$ where $\mathbf{B}(0,r)$ is the open ball in $\mathbb{R}^{n+1}$ centered at 0 and of radius $r>0$ and $s\in\mathbf{B}(0,r)$, then the proof of the above theorem follows in a shorter way. Indeed, we have

$$(\overline{s}B-Bp)(p^2-2s_0p+|s|^2)^{-1}=\sum_{m\geq 0}s^{-1-m}Bp^m,\quad |p|<|s|.$$

So the left-hand side of (3.38) rewrites as

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)}f(s)ds_I\sum_{m\geq 0}s^{-1-m}Bp^m,\quad |p|<|s|,$$

but

$$\sum_{m\geq 0}\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)}f(s)ds_Is^{-1-m}Bp^m=\sum_{m\geq 0}\frac{1}{m!}f^{(m)}(0)Bp^m$$

and for functions in $\mathcal{N}(\mathbf{B}(0,r))$ the derivatives $f^{(m)}(0)$ are real numbers and so they commute with B. We get

$$\sum_{m\geq 0}\frac{1}{m!}f^{(m)}(0)Bp^m=B\sum_{m\geq 0}\frac{1}{m!}f^{(m)}(0)p^m=Bf(p).$$

We now offer a different proof of the theorem that shows that $(fg)(T)=f(T)g(T)$, under suitable assumptions of f,g. Originally, see [19], the proof was based on the Cauchy formula and the resolvent equations (3.7), (3.8).

Theorem 3.25 *Let $T\in\mathcal{B}^{0,1}(V_n)$ and assume $f\in\mathcal{N}_{\sigma_S(T)}$ and $g\in\mathcal{SM}_{\sigma_S(T)}$. Then we have*

$$(fg)(T)=f(T)g(T).$$

Proof Let $\sigma_S(T)\subset G_1$ and G_2 be two T-admissible open sets such that $G_1\cup\partial G_1\subset G_2$ and $G_2\cup\partial G_2\subset U$. Take $p\in\partial(G_1\cap\mathbb{C}_I)$ and $s\in\partial(G_2\cap\mathbb{C}_I)$ and observe that, for $I\in\mathbb{S}$, we have

$$
\begin{aligned}
f(T)g(T) = &\frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,T) \\
&\times \int_{\partial(G_1\cap\mathbb{C}_I)} S_L^{-1}(p,T)\, dp_I\, g(p) \\
= &\frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \\
&\times \int_{\partial(G_1\cap\mathbb{C}_I)} S_R^{-1}(s,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p) \\
&- \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \\
&\times \int_{\partial(G_1\cap\mathbb{C}_I)} S_L^{-1}(p,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p) \\
&- \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \\
&\times \int_{\partial(G_1\cap\mathbb{C}_I)} \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p) \\
&+ \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \\
&\times \int_{\partial(G_1\cap\mathbb{C}_I)} \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p)
\end{aligned}
$$

where we have used the resolvent equation. But now observe that

$$
\frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \int_{\partial(G_1\cap\mathbb{C}_I)} S_R^{-1}(s,T) p (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p) = 0
$$

and

$$
-\frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \int_{\partial(G_1\cap\mathbb{C}_I)} \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1} dp_I\, g(p) = 0.
$$

So it follows that

$$f(T)g(T) = -\frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \times \int_{\partial(G_1\cap\mathbb{C}_I)} S_L^{-1}(p,T)p(p^2-2s_0p+|s|^2)^{-1} dp_I\, g(p) + \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \times \int_{\partial(G_1\cap\mathbb{C}_I)} \bar{s} S_L^{-1}(p,T)(p^2-2s_0p+|s|^2)^{-1} dp_I\, g(p)$$

which can be written as

$$f(T)g(T) = \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_I)} f(s)\, ds_I \int_{\partial(G_1\cap\mathbb{C}_I)} \left[\bar{s} S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right] \times (p^2-2s_0p+|s|^2)^{-1} dp_I\, g(p).$$

Using Lemma 3.23 we get

$$f(T)g(T) = \frac{1}{2\pi} \int_{\partial(G_1\cap\mathbb{C}_I)} S_L^{-1}(p,T) dp_I\, f(p)\, g(p)$$

which gives the statement. □

In the original proof of the above theorem we have used the fact that for functions $f \in \mathcal{N}_{\sigma_S(T)}$ the left S-resolvent equation gives

$$f(T)T^m = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(p,T) dp_I\, f(p)p^m$$

from which one obtains

$$f(T)T^m t^{-1-m} = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(p,T) dp_I\, f(p)p^m t^{-1-m}.$$

By taking the sum and considering $t \in \rho_S(T)$, we have

$$f(T)S_L^{-1}(t,T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(p,T) dp_I\, f(p) S_L^{-1}(t,p).$$

Using this equality and the Cauchy formula we obtain the statement.

4 The Case of Commuting Operators and the Quaternionic Case

In this last section we state the resolvent equation in the case of commuting operators and for the quaternionic functional calculus. We also take the occasion to make some comments that show how the S-functional calculus turns out to be a natural extension of the Riesz–Dunford functional calculus.

4.1 The Case of Several Commuting Operators

We denote by $\mathcal{BC}^{0,1}(V_n)$ the subset of $\mathcal{B}^{0,1}(V_n)$ consisting of operators in paravector form $T = T_0 + e_1T_1 + \cdots + e_nT_n$ with commuting components T_j. Given an operator T in paravector form, its so-called conjugate $\overline{T}$ is defined by $\overline{T} = T_0 - e_1T_1 - \cdots - e_nT_n$. When $T \in \mathcal{BC}^{0,1}(V_n)$ the operator $T\overline{T}$ is well defined and $T\overline{T} = \overline{T}T = T_0^2 + T_1^2 + \cdots + T_n^2$ and $T + \overline{T} = 2T_0$.

Theorem 4.1 *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$ be such that $|s| < \|T\|$. Then*

$$\sum_{m\geq 0} T^m s^{-1-m} = (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}, \tag{4.1}$$

$$\sum_{m\geq 0} s^{-1-m} T^m = (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}(s\mathcal{I} - \overline{T}). \tag{4.2}$$

The above theorem follows from the fact that the Cauchy kernels for slice hyperholomorphic functions can be written in two possible ways; see Sect. 2 and [15]. In the case of commuting operators the two expressions are equivalent. The advantage of this approach is that one can work with the so-called F-spectrum which is easier to compute than the S-spectrum. In fact it can be computed over a complex plane $\mathbb{C}_I$, taking $s = u + Iv$, and then extended to $\mathbb{R}^{n+1}$. This is a consequence of the fact that the F-spectrum takes into account the commutativity of the operators T_j, $j = 0, 1, ..., n$. The F-spectrum is suggested by Theorem 4.1 and it is described below.

Definition 4.2 (*The F-spectrum and the F-resolvent sets*) Let $T \in \mathcal{BC}^{0,1}(V_n)$. We define the F-spectrum of T as:

$$\sigma_F(T) = \{s \in \mathbb{R}^{n+1} \ : \ s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \ \ \textit{is not invertible}\}.$$

The F-resolvent set of T is defined by

$$\rho_F(T) = \mathbb{R}^{n+1} \setminus \sigma_F(T).$$

The main properties of the F-spectrum are similar to those of the S-spectrum as is proved in the next results:

Theorem 4.3 (Structure of the F-spectrum) *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let $p = p_0 + p_1I \in [p_0 + p_1I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R}$, such that $p \in \sigma_F(T)$. Then all the elements of the $(n-1)$-sphere $[p_0 + p_1I]$ belong to $\sigma_F(T)$.*

Theorem 4.4 (Compactness of the F-spectrum) *Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then the F-spectrum $\sigma_F(T)$ is a compact nonempty set. Moreover, $\sigma_F(T)$ is contained in $\{s \in \mathbb{R}^{n+1} : |s| \leq \|T\| \}$.*

The relation between the S-spectrum and the F-spectrum is contained in the following result:

Proposition 4.5 *Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then $\sigma_F(T) = \sigma_S(T)$.*

Definition 4.6 (*The S_C-resolvent operator*) Let $T \in \mathcal{BC}^{0,1}(V)$ and $s \in \rho_F(T)$. We define the S_C-resolvent operator as

$$S_{C,L}^{-1}(s,T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}. \tag{4.3}$$

$$S_{C,R}^{-1}(s,T) := (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}(s\mathcal{I} - \overline{T}). \tag{4.4}$$

Theorem 4.7 *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s, p \in \rho_F(T)$. Then $S_{C,L}^{-1}(s,T)$ satisfies the left S-resolvent equation*

$$S_{C,L}^{-1}(s,T)s - TS_{C,L}^{-1}(s,T) = \mathcal{I}, \tag{4.5}$$

and $S_{C,R}^{-1}(s,T)$ satisfies the right S-resolvent equation

$$sS_{C,R}^{-1}(s,T) - S_{C,R}^{-1}(s,T)T = \mathcal{I}.$$

Moreover, for $p \notin [s]$, we have the resolvent equation

$$\begin{aligned} S_{C,R}^{-1}(s,T)S_{C,L}^{-1}(p,T) = &((S_{C,R}^{-1}(s,T) - S_{C,L}^{-1}(p,T))p \\ &- \overline{s}(S_{C,R}^{-1}(s,T) - S_{C,L}^{-1}(p,T)))(p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned}$$

which can be written as

$$\begin{aligned} S_{C,R}^{-1}(s,T)S_{C,L}^{-1}(p,T) = &(s^2 - 2p_0s + |p|^2)^{-1}(s(S_{C,R}^{-1}(s,T) - S_{C,L}^{-1}(p,T)) \\ &- (S_{C,R}^{-1}(s,T) - S_{C,L}^{-1}(p,T))\overline{p}). \end{aligned} \tag{4.6}$$

We conclude this subsection with a couple of considerations on the case of unbounded operators.

(I) Suppose that T is a closed operator with domain $D(T)$. As one can clearly see, the noncommutative version of $S_L^{-1}(s,T)$, which is

$$S_L^{-1}(s,T) := -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}),$$

is defined on the domain of T, and not on V_n as it is in the classical case. So we have to consider the extension to V_n, writing $S_L^{-1}(s,T)$ as follows:

$$\hat{S}_L^{-1}(s,T) := -(T(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1} - (T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}\overline{s}\mathcal{I}).$$

In this case $\hat{S}_L^{-1}(s,T)$ turns out to be defined on V_n. Observe now that if T is a closed operator with domain $D(T)$ and with commuting components, the left S-resolvent operator $S_{C,L}^{-1}(s,T)$ turns out to be already defined on V_n. For a more detailed discussion, see the original papers [13,15]. In the case of the right S-resolvent we have the opposite situation. With the above consideration the new resolvent equation remains the same also for unbounded operators.

(II) The F-spectrum is also useful to define the so-called F-functional calculus; see [5,16]. This calculus is defined using the Fueter–Sce–Qian mapping theorem in integral form; see [22,32,34]. It is a hyperholomorphic functional calculus in the spirit of the works of A. McIntosh, B. Jefferies and their coauthors who first used the theory of hyperholomorphic functions, see [10,18,20,25], to define a hyperholomorphic functional calculus for n-tuples of operators; see [27–29,31], the monograph [26] and the references therein.

4.2 The Quaternionic Setting

The results proved in the paper can be rephrased also for the quaternionic functional calculus. We point out that, in this case, slice hyperholomorphic functions are defined on an open set $U \subseteq \mathbb{H}$ and have values in the quaternions $\mathbb{H}$. The resolvent operators are defined as in the Introduction of this paper. Here it is important to consider right linear operators as well as left linear operators T. The possible formulations of the quaternionic functional calculus have been carried out in [13]. The resolvent equations in Theorem 3.8 hold in this setting, where instead of the paravector operator $T = T_0 + T_1e_1 + \cdots + T_ne_n$ we replace a quaternionic linear operator. We finally mention one more analogy with the classical case. As is well known, the Laplace transform of a semigroup e^{tE} (where for simplicity we take a bounded operator E defined on a Banach space X) is the resolvent operator $(\lambda I - E)^{-1}$. In the quaternionic case, we have the analog result for the two S-resolvent operators. Let $T \in \mathcal{B}(V)$ where V is two-sided quaternionic Banach space and let $s_0 > \|T\|$. Then the left S-resolvent operator $S_L^{-1}(s,T)$ is given by

$$S_L^{-1}(s,T) = \int_0^{+\infty} e^{tT}\, e^{-ts}\, dt,$$

and $S_R^{-1}(s,T)$ is given by

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts} e^{t\,T}\, dt.$$

The theory of the quaternionic evolution operators is developed in [12] where is also studied the case in which the generator is unbounded. Recently, the case of sectorial operators has been treated in [24].

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