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Fundamental Groups of Spaces with Bakry–Emery Ricci Tensor Bounded Below

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Abstract We first extend Cheeger–Colding's Almost Splitting Theorem (Ann Math 144:189–237, 1996) to smooth metric measure spaces. Arguments utilizing this extension show that if a smooth metric measure space has almost nonnegative Bakry–Emery Ricci curvature and a lower bound on volume, then its fundamental group is almost abelian. Second, if the smooth metric measure space has Bakry–Emery Ricci curvature bounded from below then the number of generators of the fundamental group is uniformly bounded. These results are extensions of theorems which hold for Riemannian manifolds with Ricci curvature bounded from below. The first result extends a result of Yun (Proc Amer Math Soc 125:1517–1522, 1997), while the second extends a result of Kapovitch and Wilking (Structure of fundamental groups of manifolds with Ricci curvature bounded below, 2011).

Keywords Bakry–Emery Ricci curvature · Fundamental groups · Smooth metric measure spaces

Mathematics Subject Classification 53C20

1 Introduction

A smooth metric measure space is a triple $(M^n, g, e^{-f}\mathrm{dvol}_g)$, where M^n is a complete n-dimensional Riemannian manifold equipped with metric g and volume density dvol_g. The potential function $f : M^n \to \mathbb{R}$ is smooth. Smooth metric measure

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spaces occur naturally as collapsed measured Gromov–Hausdorff limits of sequences of warped products $(M^n \times F^m, g_\epsilon, \widetilde{\mathrm{dvol}_{g_\epsilon}})$ as $\epsilon \to 0$, where $\widetilde{\mathrm{dvol}_{g_\epsilon}}$ is the renormalized Riemannian measure and $g_\epsilon = g_M + (\epsilon e^{\frac{-f}{m}})^2 g_F$ is the warped product metric with g_M and g_F the metrics on M and F, respectively. The Ricci curvature of the warped product metric g_ϵ in the M direction is given by

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df.$$

This leads to the definition of the m-Bakry–Emery Ricci tensor on the limit space as

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df, \quad 0 < m \leq \infty.$$

When $m = \infty$, we have the Bakry–Emery Ricci tensor on $(M^n, g, e^{-f}\mathrm{dvol}_g)$ given by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f.$$

Thus, the Bakry–Emery Ricci tensor is a natural analogue to Ricci curvature on $(M^n, g, e^{-f}\mathrm{dvol}_g)$. This tensor has appeared in the work on diffusion processes by Dominique Bakry and Michel Emery. Moreover, it occurs in the study of Ricci flow, which was utilized most notably by Grigori Perelman in his proof of the Poincaré conjecture.

Since topological and geometric information can be obtained for manifolds with Ricci curvature bounded from below and $\mathrm{Ric}_f = \mathrm{Ric}$ when f is constant, it is natural to ask if the same information holds true for smooth metric measure spaces with Bakry–Emery Ricci tensor bounded from below. Indeed, this very question has been at the center of an active field of research studied by many. In particular, Guofang Wei and Will Wylie have shown that when Ric_f is bounded from below and, in addition, either f is bounded or $\partial_r f \geq -a$ for $a \geq 0$ along minimal geodesics from a fixed $p \in M$, then the mean curvature and volume comparison theorems can be extended to the smooth metric measure space setting [19].

One important result which has already been extended to the smooth metric measure space setting is the Cheeger–Gromoll Splitting Theorem. Lichnerowicz [14] showed that if $\mathrm{Ric}_f \geq 0$ on M for bounded f, and M contains a line, then $M = N^{n-1} \times \mathbb{R}$ and f is constant along the line. See also Wei–Wylie [19]. In fact, Fang et al. [9] have shown that this holds when f is only bounded from above.

In this paper, we discuss the extension of the quantitative version of this theorem, the Cheeger–Colding Almost Splitting Theorem, to the smooth metric measure space setting.

A crucial step in the proof of the Cheeger–Gromoll Splitting Theorem is to construct a function b such that $|\nabla b| = 1$ and $\mathrm{Hess}\, b \equiv 0$. In the proof of the Cheeger–Colding Almost Splitting Theorem, one constructs a harmonic function $\overline{b}$ whose Hessian is small in the L^2-sense ([4, Proposition 6.60]). In order to extend the Almost Splitting Theorem to smooth metric measure spaces following Cheeger–Colding's proof, we also construct f-harmonic functions $\overline{b}_{\pm}$ and obtain for these functions an L^2-Hessian estimate with respect to the conformally changed volume density $e^{-f}\,\mathrm{dvol}_g$.

Theorem 1.1 *Given $R > 0, L > 2R + 1$ and $\epsilon > 0$, let $p, q_+, q_- \in M^n$. If $(M^n, g, e^{-f} dvol_g)$ satisfies*

$$|f| \leq k, \tag{1.1}$$
$$Ric_f \geq -(n-1)H \quad (H \geq 0), \tag{1.2}$$
$$\min\{d(p, q_+), d(p, q_-)\} \geq L, \tag{1.3}$$
$$e(p) = d(p, q_+) + d(p, q_-) - d(q_+, q_-) \leq \epsilon, \tag{1.4}$$

then

$$\fint_{B(p,\frac{R}{2})} |Hess\,\overline{b}_{\pm}|^2 e^{-f} dvol_g \leq \Psi(H, L^{-1}, \epsilon | k, n, R). \tag{1.5}$$

The functions Ψ and $\overline{b}_{\pm}$ in Theorem 1.1 are defined in (2.11), and (2.12), respectively. With the L^2-Hessian estimate (1.5), one may then obtain a type of Pythagorean Theorem from which the Almost Splitting Theorem will follow.

Feng Wang and Xiaohua Zhu also have an L^2-Hessian estimate and Almost Splitting Theorem for Bakry–Emery Ricci curvature bounded from below [20]. In Wang and Zhu's version, the Hessian estimate assumes that the gradient of the potential function, rather than the potential function itself, is bounded. Wang and Zhu's Almost Splitting Theorem [20, Theorem 3.1] assumes that both the potential functions and their gradients are bounded, whereas our Theorem 2.10 assumes only that the potential functions are bounded.

The hypotheses for the Hessian estimates and Almost Splitting Theorem of these two papers differ due in part to the gradient estimates used by the authors. In Cheeger–Colding's proof of the L^2-Hessian estimate in the Riemannian setting, a cutoff function ϕ is used. This cutoff function ϕ has the property that $|\nabla\phi|$ and $|\Delta\phi|$ are bounded by constants depending only on n, H, R. The construction of this cutoff function in the Riemannian case relies on the gradient of a function being bounded away from the boundary of a ball. This boundedness is guaranteed by the Cheng–Yau gradient estimate. The gradient estimate obtained by Wang and Zhu requires that the gradient of the potential function is bounded. The gradient estimate obtained here requires that the potential function itself is bounded. More precisely, we prove:

Theorem 1.2 *Let $(M^n, g, e^{-f} dvol)$ be a complete smooth metric measure space with $|f| \leq k$ and $Ric_f \geq -(n-1)H^2$ where $H \geq 0$. If u is a positive function defined on $\overline{B(q, 2R)}$ with $\Delta_f u = c$, for a constant $c \geq 0$, then for any $q_0 \in \overline{B(q, R)}$, we have*

$$|\nabla u| \leq \sqrt{c_1(n, k, H, R) \sup_{p \in B(q;2R)} u(p)^2 + c_2(c, n) \sup_{p \in B(q;2R)} u(p)}.$$

Theorem 1.2 follows directly from Theorem 2.2, which is an extension of Kevin Brighton's gradient estimate for f-harmonic functions in [3] to functions u such that $\Delta_f u = c$ for any nonnegative constant c. We are not sure if the estimate holds true when $c < 0$.

The Almost Splitting theorem will allow us to generalize the arguments in the proofs of two results on the fundamental group of Riemannian manifolds to the smooth metric measure space setting. The first result is an extension of a theorem of Yun [21]. Yun's result asserts that the fundamental group of a Riemannian manifold with almost nonnegative Ricci curvature, diameter bounded from above, and volume bounded from below is almost abelian. This result is a strengthening of a theorem of Wei [16] which shows that under the same conditions $\pi_1(M)$ has polynomial growth. In order to extend Yun's theorem, we first develop an absolute volume comparison, Proposition 3.2, which allows us to extend Wei's theorem to smooth metric measure spaces. We then follow Yun's argument, utilizing the Almost Splitting Theorem, to obtain the following result.

Theorem 1.3 *For any constants $D, k, v > 0$, there exists $\epsilon = \epsilon(D, k, n, v) > 0$ such that if a smooth metric measure space $(M^n, g, e^{-f} dvol_g)$ with $|f| \leq k$ admits a metric under which it satisfies the conditions*

$$Ric_f \geq -\epsilon, \tag{1.6}$$

$$diam(M) \leq D, \tag{1.7}$$

$$Vol_f(M) \geq v, \tag{1.8}$$

then $\pi_1(M)$ is almost abelian, i.e., $\pi_1(M)$ contains an abelian subgroup of finite index.

The next result is an extension of Kapovitch and Wilking's Theorem 3 of [13] which gives a uniform bound on the number of generators of $\pi_1(M)$ for the class of n-dimensional manifolds M^n with $\mathrm{Ric} \geq -(n-1)$ and $\mathrm{diam}(M, g) \leq D$, for given n and D. A uniform bound had been given previously in the case when the conjugate radius is bounded from below [17]. An extension of this theorem to the smooth metric measure space setting is as follows.

Theorem 1.4 *Given n, D, and k, there is a constant $C = C(n, D, k) > 0$ such that the following holds. Let $(M^n, g, e^{-f} dvol_g)$ be a smooth metric measure space with*

$$|f| \leq k, \tag{1.9}$$

$$diam\, M \leq D, \tag{1.10}$$

$$Ric_f \geq -(n-1). \tag{1.11}$$

Then $\pi_1(M)$ is generated by at most C elements.

Remark When $m \neq \infty$ and Ric_f^m is bounded from below, comparison theorems hold with no additional assumptions on f; see Qian [15]; see also Wei–Wylie [18]. Due to this fact, there are versions of our theorems for Ric_f^m bounded from below with no additional assumptions on f.

The remaining sections of this paper are structured as follows. In Sect. 2, we discuss the extension of the Almost Splitting Theorem, along with the essential tools which allow one to extend to the smooth metric measure space setting. These essential tools include the f-Laplacian Comparison, Gradient Estimate, Segment Inequality, and the Quantitative Maximum Principle for smooth metric measure spaces. We will also provide a proof of the key Hessian estimate, Theorem 1.1. In Sect. 3, we develop an absolute volume comparison which is used to extend Wei's theorem to the smooth metric measure space setting. In Sects. 4 and 5 we discuss the proofs of Theorems 1.3 and 1.4, respectively.

2 The Almost Splitting Theorem for Smooth Metric Measure Spaces

As indicated in the Introduction, an essential tool in establishing the Almost Splitting Theorem for smooth metric measure spaces will be the Hessian estimate, Theorem 1.1. In order to obtain the original estimate, other essential tools, including the Laplacian Comparison, Cheng–Yau Gradient Estimate [8], and Abresch–Gromoll Inequality [1], were utilized. Cheeger and Colding also developed key tools, such as the Segment Inequality [4, Theorem 2.11]. With the extension of such tools to the smooth metric measure space setting, we can generalize the arguments of Cheeger and Colding [4]; see also [7].

In order to extend the essential tools mentioned above, one must integrate with respect to the measure $e^{-f}\,\mathrm{dvol}_g$. In addition, one must replace the Laplace–Beltrami operator on Riemannian manifolds with its natural analog on smooth metric measure spaces. This analog is the f-Laplacian, defined for functions $u \in C^2(M)$ by

$$\Delta_f(u) = \Delta(u) - \langle \nabla u, \nabla f \rangle.$$

This operator is natural in the sense that it is self-adjoint with respect to the measure $e^{-f}\,\mathrm{dvol}_g$.

From the Wei–Wylie Mean Curvature Comparison [19, Theorem 1.1] and definition of the f-Laplacian, one immediately obtains the following f-Laplacian comparison.

Proposition 2.1 (f-Laplacian Comparison) *Suppose $Ric_f \geq (n-1)H$ with $|f| \leq k$. Let Δ_H^{n+4k} denote the Laplacian of the simply connected model space of dimension $n+4k$ with constant sectional curvature H. Then for radial functions u,*

(1) $\Delta_f(u) \leq \Delta_H^{n+4k} u$ *if* $u' \geq 0$,
(2) $\Delta_f(u) \geq \Delta_H^{n+4k} u$ *if* $u' \leq 0$.

Using the definition of the f-Laplacian and the classical Bochner formula on Riemannian manifolds, one also immediately obtains a Bochner formula for smooth metric measure spaces

$$\frac{1}{2}\Delta_f(|\nabla u|^2) = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f(\nabla u, \nabla u), \tag{2.1}$$

for any $u \in C^3(M)$. This Bochner formula allows us to obtain the following gradient estimate.

Proposition 2.2 (Gradient Estimate) *Let $(M^n, g, e^{-f} dvol)$ be a complete smooth metric measure space with $Ric_f \geq -(n-1)H^2$ where $H \geq 0$. If u is a positive function defined on $\overline{B(q, 2R)}$ with $\Delta_f u = c, c \geq 0$, then for any $q_0 \in \overline{B(q, R)}$, we have*

$$|\nabla u| \leq \sqrt{c_1(\alpha, n, H, R) \sup_{p \in B(q;2R)} u(p)^2 + c_2(c, n) \sup_{p \in B(q;2R)} u(p)}$$

where $\alpha = \max_{p \in p: d(p,q) = r_0} \Delta_f r(p)$ for any $r_0 \leq R$ and $r(p) = d(p, q)$.

Note that in the above gradient estimate we make no assumption on the potential function f. We include a sketch of the proof below, which modifies the proof of Brighton's gradient estimate for f-harmonic functions in [3] to consider the case of $\Delta_f u = c$, for a positive constant c.

Proof Let $h = u^\epsilon$ where $\epsilon \in (0, 1)$. Applying (2.1) to h gives

$$\frac{1}{2} \Delta_f |\nabla h|^2 = |\text{Hess}\, h|^2 + \langle \nabla h, \nabla(\Delta_f h) \rangle + \text{Ric}_f(\nabla h, \nabla h).$$

Using the Schwarz inequality, we have

$$\begin{aligned}
|\text{Hess}\, h|^2 &\geq \frac{|\Delta h|^2}{n} \\
&= \frac{1}{n} (\Delta_f h + \langle \nabla f, \nabla h \rangle)^2 \\
&= \frac{1}{n} \left(\epsilon u^{\epsilon - 1} \Delta_f u + \frac{(\epsilon - 1)|\nabla h|^2}{\epsilon h} + \langle \nabla f, \nabla h \rangle \right)^2 \\
&= \frac{1}{n} \left(\epsilon u^{\epsilon - 1} c + \frac{(\epsilon - 1)|\nabla h|^2}{\epsilon h} + \langle \nabla f, \nabla h \rangle \right)^2
\end{aligned}$$

where in the last equality we used the fact that $\Delta_f u = c$. This, together with the lower bound on the Bakry–Emery Ricci tensor, gives

$$\begin{aligned}
\frac{1}{2} \Delta_f |\nabla h|^2 \geq\ & \frac{(\epsilon - 1)^2}{\epsilon^2 h^2 n} |\nabla h|^4 + \frac{2c(\epsilon - 1)}{h^{1/\epsilon} n} |\nabla h|^2 + \frac{2(\epsilon - 1)}{\epsilon h n} |\nabla h|^2 \langle \nabla f, \nabla h \rangle \\
& + \frac{\epsilon^2 c^2}{n} (h^{2 - 2/\epsilon}) + \frac{2c\epsilon}{n} (h^{1 - 1/\epsilon}) \langle \nabla f, \nabla h \rangle + \frac{1}{n} \langle \nabla f, \nabla h \rangle^2 \\
& + \frac{(\epsilon - 1)}{\epsilon h} \langle \nabla h, \nabla |\nabla h|^2 \rangle - \frac{(\epsilon - 1)}{\epsilon h^2} |\nabla h|^4 + \frac{c(\epsilon - 1)}{h^{1/\epsilon}} |\nabla h|^2 \\
& - (n - 1) H^2 |\nabla h|^2. \qquad (2.2)
\end{aligned}$$

In order to control the mixed term $2 \frac{(\epsilon - 1)}{\epsilon h n} |\nabla h|^2 \langle \nabla f, \nabla h \rangle$ in (2.2), we consider two cases according to whether $|\nabla h|^2$ dominates over$\langle \nabla h, \nabla f \rangle$, or vice versa. In the first case, suppose that $p \in \overline{B(q,2R)}$ such that $\langle \nabla h, \nabla f \rangle \leq a\frac{|\nabla h|^2}{h}$ for some $a > 0$ to be determined. At this point we have

$$\begin{aligned}
\frac{1}{2}\Delta_f |\nabla h|^2 a \geq & \left[\frac{(\epsilon-1)^2 + 2\epsilon(\epsilon-1)a - \epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} \\
& + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
& + \frac{1}{n}(\epsilon c h^{1-1/\epsilon} + \langle \nabla f, \nabla h \rangle)^2 + \frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla |\nabla h|^2 \rangle - (n-1)H^2|\nabla h|^2 \\
\geq & \left[\frac{(\epsilon-1)^2 + 2\epsilon(\epsilon-1)a - \epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} \\
& + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
& + \frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla |\nabla h|^2 \rangle - (n-1)H^2|\nabla h|^2. \qquad (2.3)
\end{aligned}$$

In the case that $p \in \overline{B(q,2R)}$ such that $\langle \nabla h, \nabla f \rangle \geq a\frac{|\nabla h|^2}{h}$, we have

$$\begin{aligned}
\frac{1}{2}\Delta_f |\nabla h|^2 \geq & \left[\frac{(\epsilon-1)^2 - \epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
& + \left[\frac{2(\epsilon-1)+\epsilon a}{\epsilon n a}\right]\langle \nabla f, \nabla h \rangle^2 + \frac{\epsilon^2 c^2}{n}(h^{2-2/\epsilon}) + \frac{2c\epsilon a}{n h^{1/\epsilon}}|\nabla h|^2 \\
& + \frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla |\nabla h|^2 \rangle - (n-1)H^2|\nabla h|^2 \\
\geq & \left[\frac{(\epsilon-1)^2 - \epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
& + \left[\frac{2(\epsilon-1)+\epsilon a}{\epsilon n a}\right]\langle \nabla f, \nabla h \rangle^2 + \frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla |\nabla h|^2 \rangle - (n-1)H^2|\nabla h|^2. \\
& \qquad (2.4)
\end{aligned}$$

Note that in (2.4) the assumption that $c \geq 0$ is necessary to have $\frac{2c\epsilon a}{nh^{1/\epsilon}}|\nabla h|^2 \geq 0$ which allows us to obtain the second inequality.

As in Brighton's proof, we see that choosing $\epsilon = \frac{7}{8}$ and $a = \frac{1}{2}$ will make the coefficient of the $\frac{|\nabla h|^4}{h^2}$ term positive in both cases. This choice also gives a positive coefficient of the $\langle \nabla f, \nabla h \rangle^2$ term in the second case. With this choice of ϵ and a, we see that for every $p \in \overline{B(q,2R)}$, we have

$$\begin{aligned}
\frac{1}{2}\Delta_f |\nabla h|^2 \geq & \frac{7n-6}{49n}\frac{|\nabla h|^4}{h^2} - \frac{c(2+n)}{8n}\frac{|\nabla h|^2}{h^{8/7}} - \frac{1}{7h}\langle \nabla h, \nabla |\nabla h|^2 \rangle \\
& - (n-1)H^2|\nabla h|^2. \qquad (2.5)
\end{aligned}$$

Let $g : [0, 2R] \to [0, 1]$ have the properties

- $g|_{[0,R]} = 1$
- $\mathrm{supp}(g) \subseteq [0, 2R)$
- $\frac{-K}{R}\sqrt{g} \leq g' \leq 0$
- $|g''| \leq \frac{K}{R^2}$

where the last two properties hold for some $K > 0$. Define $\phi : \overline{B(q, 2R)} \to [0, 1]$ by $\phi(x) = g(d(x, q))$. Set $G = \phi|\nabla h|^2$. Then (2.5) can be written as

$$\frac{1}{\phi}\Delta_f G \geq \frac{G}{\phi^2}\Delta_f\phi + 2\langle\frac{\nabla\phi}{\phi}, \frac{\nabla G}{\phi} - \frac{\nabla\phi}{\phi^2}G\rangle + \frac{14n-12}{49nh^2}\frac{G^2}{\phi^2} - \frac{c(2+n)}{4nh^{8/7}}\frac{G}{\phi} - \frac{2}{7h}\langle\nabla h, \frac{\nabla G}{\phi} - \frac{\nabla\phi}{\phi^2}G\rangle - 2(n-1)H^2\frac{G}{\phi}. \quad (2.6)$$

Next, we consider the point $q_0 \in \overline{B(q, 2R)}$ at which G achieves its maximum. At such a point, (2.6) can be rewritten as

$$\frac{14n-12}{49nh^2}G \leq -\Delta_f\phi + 2\langle\frac{\nabla\phi}{\phi}, \nabla\phi\rangle + \frac{c(2+n)}{4nh^{8/7}}\phi - \frac{2}{7h}\langle\nabla h, \nabla\phi\rangle + 2(n-1)H^2\phi. \quad (2.7)$$

If $q_0 \in B(q, R)$, then (2.7) can be rewritten as

$$|\nabla u|^2 \leq \frac{8c(2+n)}{7n-6}u + \frac{64n(n-1)}{7n-6}H^2u^2 \quad (2.8)$$

when evaluated at q_0.

If $q_0 \in \overline{B(q, 2R)} \setminus B(q, R)$, one uses the mean curvature comparison [19, Theorem 2.1], the properties of ϕ, and (2.7) to obtain

$$|\nabla u|^2 \leq \frac{64}{13n-12}\left[\frac{K\alpha R + K + 3K^2}{R^2} + 2(K+1)(n-1)H^2\right]u^2 + \frac{16c(2+n)}{n(13n-12)}u \quad (2.9)$$

at the point q_0.

Taking the supremum of u and u^2 over $\overline{B(q, 2R)}$ in (2.8) and (2.9) yields the desired form of the gradient estimate. □

Remark Since for our purposes the potential function is bounded, that is $|f| \leq k$, we may use the f-Laplacian comparison, Proposition 2.1, to modify the above gradient estimate, Proposition 2.2, slightly. By setting $r_0 = R/2$ we may apply the f-Laplacian comparison directly to α to obtain

$$\alpha \leq (n + 4k - 1)H\coth(HR/2).$$

Alternatively, we may use Proposition 2.1 in place of the Wei–Wylie mean curvature comparison [19, Theorem 2.1] when obtaining (2.9). In that case,

$$\Delta_f r(q_0) \leq (n + 4k - 1)H \coth(Hr(q_0)) \leq (n + 4k - 1)H \coth(HR)$$

since $R \leq r(q_0) \leq 2R$. In either case, one will obtain a gradient estimate as in Theorem 1.2 which is no longer dependent upon α.

Note that Proposition 2.2 only holds for nonnegative c. If we consider the case $c < 0$, the term $\frac{2c\epsilon}{n}(h^{1-1/\epsilon})\langle \nabla f, \nabla h\rangle$ in (2.2) becomes problematic. In particular, when $p \in \overline{B(q, 2R)}$ is such that $\langle \nabla h, \nabla f\rangle$ dominates over $|\nabla h|^2$, we replace the term $\frac{2(\epsilon-1)}{\epsilon hn}|\nabla h|^2\langle \nabla f, \nabla h\rangle$ by $\frac{2(\epsilon-1)}{\epsilon hna}\langle \nabla f, \nabla h\rangle^2$. In order to control this term, we group it with $\frac{1}{n}\langle \nabla f, \nabla h\rangle^2$. Then we can no longer group the $\langle \nabla f, \nabla h\rangle$ term with other terms to create a perfect square, as in (2.3). Moreover, since its coefficient is negative, we must keep this term in the estimate. Thus, without any additional assumptions, such as a bound on $|\nabla f|$, there is no way to control this term. As noted in the Introduction, the assumption of a bound on $|f|$ rather than a bound on $|\nabla f|$ in this gradient estimate is one of the reasons for the difference in hypotheses of the Almost Splitting Theorem of Wang and Zhu [20, Theorem 3.1] and Theorem 2.10.

Finally, in order to convert estimates of integrals of functions over a ball to estimates of integrals of functions along a geodesic segment, we need a Segment Inequality similar to that developed by Cheeger and Colding in [4, Theorem 2.11].

Proposition 2.3 (Segment Inequality) *Let $(M^n, g, e^{-f}dvol_g)$ be a smooth metric measure space with $Ric_f \geq (n-1)H$ and $|f(x)| \leq k$. Let $A_1, A_2 \in M^n$ be open sets and assume for all $y_1 \in A_1$, $y_2 \in A_2$, there is a minimal geodesic, γ_{y_1,y_2} from y_1 to y_2, such that for some open set, W,*

$$\bigcup_{y_1,y_2} \gamma_{y_1,y_2} \subset W.$$

If v_i is a tangent vector at y_i, $i = 1, 2$, and $|v_i| = 1$, set

$$I(y_i, v_i) = \{t|\gamma(t) \in A_{i+1}, \gamma|[0,t] \text{ is minimal}, \gamma'(0) = v_i\}.$$

Let $|I(y_i, v_i)|$ denote the measure of $I(y_i, v_i)$ and put

$$\mathcal{D}(A_i, A_{i+1}) = \sup_{y_1,y_2} |I(y_i, v_i)|.$$

Here $A_{2+1} := A_1$. Let h be a nonnegative integrable function on M. Let $D = \max d(y_1, y_2)$. Then

$$\int_{A_1\times A_2} \int_0^{d(y_1,y_2)} h(\gamma_{y_1,y_2}(s))ds(e^{-f}dvol_g)^2 \leq c(n+4k, H, D)[\mathcal{D}(A_1, A_2)Vol_f(A_1)$$

$$+ \mathcal{D}(A_2, A_1) Vol_f(A_2)] \times \int_W he^{-f} dvol_g$$

where $c(n+4k, H, D) = \sup_{0<s/2\le u\le s} \mathcal{A}_H^{n+4k}(s)/\mathcal{A}_H^{n+4k}(u)$, *where* $\mathcal{A}_H^{n+4k}(r)$ *denotes the area element on* $\partial B(r)$ *in the model space with constant curvature* H *and dimension* $n + 4k$.

To obtain this result for smooth metric measure spaces one may follow the arguments of the proof in the original setting as given by Cheeger and Colding in [4, Theorem 2.11]. In the smooth metric measure space setting, integrals are computed with respect to the conformally changed volume element, $e^{-f}\,\mathrm{dvol}_g$, and we use Wei–Wylie's volume element comparison which follows from [19, Theorem 1.1].

Finally, the Abresch–Gromoll Quantitative Maximal Principle was necessary in the proof of the Abresch–Gromoll inequality and also in obtaining an appropriate cutoff function needed to prove the Hessian estimate. Since this proof varies slightly from the exposition contained in Abresch–Gromoll [1] or Cheeger's [7] works, we retain the proof here.

Proposition 2.4 (Quantitative Maximal Principle) *If* $Ric_f \geq (n-1)H, (H \leq 0), |f| \leq k$ *and* $U : \overline{B(y, R)} \subset M^n \to \mathbb{R}$ *is a Lipschitz function with*

(1) $Lip(U) \leq a, U(y_0) = 0$ *for some* $y_0 \in B(y, R)$,
(2) $\Delta_f U \leq b$ *in the barrier sense,* $U|_{\partial B(y,R)} \geq 0$.

Then $U(y) \leq ac + bG_R(c)$ *for all* $0 < c < R$, *where* $G_R(r(x))$ *is the smallest function on the model space* M_H^{n+4k} *such that:*

(1) $G_R(r) > 0, G'_R(r) < 0$ *for* $0 < r < R$
(2) $\Delta_H G_R \equiv 1$ *and* $G_R(R) = 0$.

Proof Let $G_R(r)$ be the comparison function in the model space M_H^{n+4k} as given in the statement of the theorem. By the f-Laplacian Comparison, one has

$$\Delta_f G_R \geq \Delta_H^{n+4k} G_R = 1.$$

Consider the function $V = bG_R - U$. Then

$$\Delta_f V = b\Delta_f G_R - \Delta_f U \geq b\Delta_H^{n+4k} G_R - \Delta_f U = b - \Delta_f U \geq 0.$$

Then the maximal principle on $V : \overline{A(y, c, R)} \to \mathbb{R}$ gives

$$V(x) \leq \max\{V|_{\partial B(y,R)}, V|_{\partial B(y,c)}\}$$

for all $x \in \overline{A(y, c, R)}$. By assumption, we have

$$V|_{\partial B(y,R)} = bG_R|_{\partial B(y,R)} - U|_{\partial B(y,R)} \leq 0$$

and

$$V(y_0) = bG_R(y_0) - U(y_0) = bG_R(y_0) > 0.$$

Then there are two cases.

If $y_0 \in A(y, c, R)$, then max $V|_{\partial B(y,c)} > 0$ *so* $V(y') > 0$ for some $y' \in \partial B(y, c)$. Since

$$U(y) - U(y') \leq a \cdot d(y, y') = ac$$

and

$$bG_R(c) - U(y') = V(y') > 0,$$

it follows that

$$U(y) \leq ac + U(y') \leq ac + bG_R(c).$$

On the other hand, if $d(y, y_0) \leq c$, we may use the Lipschitz condition directly:

$$U(y) = U(y) - U(y_0) \leq a \cdot d(y, y_0) \leq ac \leq ac + bG_R(c).$$

In either case, we have $U(y) \leq ac + bG_R(c)$ for all $0 < c < R$, as desired. □

For any point $x \in M$, the excess function at x is given by

$$e(x) = d(x, q_+) + d(x, q_-) - d(q_+, q_-), \tag{2.10}$$

where $q_+, q_- \in M$ are fixed. For the excess function, we have the following Abresch–Gromoll Inequality, which gives an upper bound on the excess function in terms of a function

$$\Psi = \Psi(\epsilon_1, \ldots, \epsilon_k | c_1, \ldots, c_N) \tag{2.11}$$

such that $\Psi \geq 0$ and for any fixed $c_1, \ldots, c_N$,

$$\lim_{\epsilon_1, \ldots, \epsilon_k \to 0} \Psi = 0.$$

Such ϵ_i, c_i will be given explicitly below.

Proposition 2.5 (Abresch–Gromoll Inequality) *Given* $R > 0$, $L > 2R+1$ *and* $\epsilon > 0$, *for any* $p, q_+, q_- \in M^n$, *if* (1.1)–(1.4) *hold, then*

$$e(x) \leq \Psi(H, L^{-1}, \epsilon | n, k, R)$$

on $B(p, R)$.

Proof The proof of the Abresch–Gromoll Inequality for smooth metric measure spaces runs parallel to the proof one finds in Cheeger [7, Theorem 9.1], with the modification that one uses the Quantitative Maximal Principle 2.4 together with the f-Laplacian Comparison 2.1 in place of their Riemannian counterparts. □

We note that an excess estimate for smooth metric measure spaces with $\mathrm{Ric}_f \geq 0$ and $|f| \leq k$ is given in Theorem 6.1 of Wei–Wylie [19].

For fixed $p, q_+, q_- \in M$, define the function $b_\pm : M \to \mathbb{R}$ by

$$b_\pm(x) = d(x, q_\pm) - d(p, q_\pm).$$

Let $\overline{b}_\pm : M \to \mathbb{R}$ be the function such that

$$\Delta_f \overline{b}_\pm = 0 \quad \text{and} \quad b_\pm\big|_{\partial B(p,R)} = \overline{b}_\pm\big|_{\partial B(p,R)}. \tag{2.12}$$

Lemma 2.6 *Given $R > 0, L > 2R + 1$ and $\epsilon > 0$, for any $p, q_+, q_- \in M^n$, if (1.1)–(1.4) hold then on $B(p, R)$,*

$$|b_\pm - \overline{b}_\pm| \leq \Psi(H, L^{-1}, \epsilon | n, k, R). \tag{2.13}$$

Proof The Abresch–Gromoll Inequality 2.5 along with the f-Laplacian Comparison 2.1 and the Maximal Principle 2.4 allow one to follow the proof of [4, Lemma 6.15] to obtain the above. □

Lemma 2.7 *Given $R > 0, L > 2R + 1$ and $\epsilon > 0$, for any $p, q_+, q_- \in M^n$, if (1.1)–(1.4) hold then*

$$\fint_{B(p,R)} |\nabla b_\pm - \nabla \overline{b}_\pm|^2 \, e^{-f} dvol \leq \Psi(H, L^{-1}, \epsilon | n, k, R). \tag{2.14}$$

Proof Use the above pointwise estimate on $\overline{b}_\pm$, Lemma 2.6, along with the Gradient Estimate, Theorem 2.2, and integration by parts to obtain (2.14). □

Lemmas 2.6 and 2.7 now allow one to obtain the key estimate for $\mathrm{Hess}\,\overline{b}_\pm$. The proof of the Hessian estimate is retained below for completeness. We will now prove Theorem 1.1.

Proof of Theorem 1.1 Applying the Bochner formula (2.1) to the f-harmonic function $\overline{b}_\pm$ yields

$$\frac{1}{2}\Delta_f |\nabla \overline{b}_\pm|^2 = |\mathrm{Hess}\,\overline{b}_\pm|^2 + \mathrm{Ric}_f(\nabla \overline{b}_\pm, \nabla \overline{b}_\pm).$$

Multiply by a cutoff function ϕ that has the following properties:

- $\phi\big|_{B(p,\frac{R}{2})} \equiv 1$,
- $supp(\phi) \subset B(p, R)$,

- $|\nabla\phi| \le C(n, H, R, k)$,
- $|\Delta_f\phi| \le C(n, H, R, k)$.

To construct such a cutoff function, one begins with a function $h : A(p, \frac{R}{2}, R) \to \mathbb{R}$ such that $\Delta_f h \equiv 1$, $h|_{\partial B(p,\frac{R}{2})} = G_R(R/2)$, $h|_{\partial B(p,R)} = 0$, where G_R is as specified in Proposition 2.4. Then let $\psi : [0, G_R(R/2)] \to [0, 1]$ such that ψ is 1 near $G_R(R/2)$ and ψ is 0 near 0. The function $\phi = \psi(h)$ extended to all of M by setting $\phi = 1$ inside $B(p, R/2)$ and $\phi = 0$ outside of $B(p, R)$, is the cutoff function desired. The gradient estimate 1.2 guarantees that $|\nabla\phi|$ and $|\Delta_f\phi|$ are bounded away from the boundary of the annulus on which h was originally defined.

Then the above equation may be rewritten as

$$\phi|\mathrm{Hess}\,\overline{b}_\pm|^2 = \frac{1}{2}\phi\Delta_f|\nabla\overline{b}_\pm|^2 - \phi\mathrm{Ric}_f(\nabla\overline{b}_\pm, \nabla\overline{b}_\pm).$$

Integrating both sides of this equality over $B(p, R)$ gives

$$\begin{aligned}\int\limits_{B(p,R)} \phi|\mathrm{Hess}\,\overline{b}_\pm|^2 e^{-f}dvol_g &= \int\limits_{B(p,R)} \left(\frac{1}{2}\phi\Delta_f|\nabla\overline{b}_\pm|^2 - \phi\mathrm{Ric}_f(\nabla\overline{b}_\pm, \nabla\overline{b}_\pm)\right)e^{-f}dvol_g\\ &\le \int\limits_{B(p,R)} \left(\frac{1}{2}\phi\Delta_f|\nabla\overline{b}_\pm|^2 + (n-1)H|\nabla\overline{b}_\pm|^2\right)e^{-f}dvol_g\\ &= \frac{1}{2}\int\limits_{B(p,R)} \phi\Delta_f|\nabla\overline{b}_\pm|^2 e^{-f}dvol_g\\ &\quad + \int\limits_{B(p,R)} (n-1)H|\nabla\overline{b}_\pm|^2 e^{-f}dvol_g.\end{aligned}$$

For the first integrand, we have

$$\begin{aligned}\int\limits_{B(p,R)} \phi\Delta_f|\nabla\overline{b}_\pm|^2 e^{-f}dvol_g &= \int\limits_{B(p,R)} \phi\Delta_f(|\nabla\overline{b}_\pm|^2 - 1)e^{-f}dvol_g\\ &= \int\limits_{B(p,R)} \Delta_f\phi(|\nabla\overline{b}_\pm|^2 - 1)e^{-f}dvol_g.\end{aligned}$$

Thus

$$\begin{aligned}\int\limits_{B(p,R)} \phi|\mathrm{Hess}\,\overline{b}_\pm|^2 e^{-f}dvol_g &\le \int\limits_{B(p,R)} \left[\frac{1}{2}\Delta_f\phi(|\nabla\overline{b}_\pm|^2 - 1) + (n-1)H|\nabla\overline{b}_\pm|^2\right]e^{-f}dvol_g\\ &e^{-f}dvol_g\\ &\le \int\limits_{B(p,R)} \left[\frac{1}{2}|\Delta_f\phi|\,||\nabla\overline{b}_\pm|^2 - 1| + (n-1)H|\nabla\overline{b}_\pm|^2\right]e^{-f}dvol_g.\end{aligned}$$

Since $|\nabla b_\pm| = 1$,

$$||\nabla \overline{b}_\pm|^2 - 1| = ||\nabla \overline{b}_\pm| - |\nabla b_\pm||(|\nabla \overline{b}_\pm| + 1) \leq |\nabla \overline{b}_\pm - \nabla b_\pm|(|\nabla \overline{b}_\pm| + 1),$$

we have

$$\fint_{B(p,R)} \phi |\mathrm{Hess}\, \overline{b}_\pm|^2 e^{-f} dvol_g \leq \Psi.$$

□

This Hessian estimate is important because it, together with the Segment Inequality, Proposition 2.3, allows us to extend the Quantitative Pythagorean Theorem, stated as Lemma 9.16 in Cheeger [7], to the smooth metric measure space setting as follows.

Proposition 2.8 (Quantitative Pythagorean Theorem) *Given* $R > 0$, $L > 2R + 1$ *and* $\epsilon > 0$, *for any* $p, q_+, q_- \in M^n$, *assume* (1.1)–(1.4) *hold. Let* $x, z, w \in B(p, \frac{R}{8})$, *with* $x \in \overline{b}_+^{-1}(a)$, *and* z *a point on* $\overline{b}_+^{-1}(a)$ *closest to* w. *Then* $|d(x,z)^2 + d(z,w)^2 - d(x,w)^2| \leq \Psi$.

From this Quantitative Pythagorean Theorem for smooth metric measure spaces, one may establish the following Almost Splitting Theorem.

Theorem 2.9 (Almost Splitting Theorem) *Given* $R > 0$, $L > 2R + 1$ *and* $\epsilon > 0$, *let* $p, q_+, q_- \in M^n$. *If* $(M^n, g, e^{-f} dvol_g)$ *satisfies* (1.1)–(1.4), *then there is a length space* X *such that for some ball* $B((0,x), \frac{R}{4}) \subset \mathbb{R} \times X$ *with the product metric, we have*

$$d_{GH}\left(B\left(p, \frac{R}{4}\right), B\left((0,x), \frac{R}{4}\right)\right) \leq \Psi(H, L^{-1}, \epsilon | k, n, R).$$

From this Almost Splitting Theorem for smooth metric measure spaces, it follows that the splitting theorem extends to the limit of a sequence of smooth metric measure spaces in the following manner.

Theorem 2.10 *Let* $(M_i^n, g_i, e^{-f_i} dvol_{g_i})$ *be a sequence satisfying the following:* $M_i^n \to Y$, $Ric_{f_i} M_i \geq -(n-1)\delta_i$, *where* $\delta_i \to 0$, $|f_i| \leq k$. *If* Y *contains a line, then* Y *splits as an isometric product,* $Y = \mathbb{R} \times X$ *for some length space* X.

Again, we note that a splitting theorem for limit spaces of sequences of smooth metric measure spaces has been proven by Wang–Zhu; see [20, Theorem 3.1]. The gradient estimate, Proposition 2.2, used for the proof of our Theorem 2.10 allows us to relax the conditions on the potential functions in the sequence, requiring only that the $|f_i|$ for each i are bounded, rather than both $|f_i|$ and $|\nabla f_i|$ as in Theorem 3.1 of Wang–Zhu.

3 Polynomial Growth of the Fundamental Group

As mentioned in the Introduction, the first theorem which we wish to extend to smooth metric measure spaces, Yun's theorem [21, Main Theorem], is actually a strengthening of Wei's theorem [16].

Theorem 3.1 *[16, Theorem 1] For any constant $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if a complete manifold M^n admits a metric satisfying the conditions $Ric \geq -\epsilon, diam(M) = 1$, and $Vol(M) \geq v$, then the fundamental group of M is of polynomial growth with degree $\leq n$.*

Yun uses the existence of such ϵ to construct a contradicting sequence of Riemannian manifolds M_i such that $\mathrm{Ric}(M_i) \geq -\epsilon_i \to 0$, where $\epsilon_i \leq \epsilon$, $\mathrm{Vol}(M_i) \geq v$, and $\mathrm{diam}(M_i) \leq D$ but $\pi_1(M_i)$ is not almost abelian. It is with this sequence that Yun utilizes the Almost Splitting Theorem. If we wish to generalize his arguments to the smooth metric measure space setting, we should also establish the existence of such an ϵ for smooth metric measure spaces.

Wei's proof of Theorem 3.1 requires use of the Bishop–Gromov absolute volume comparison. The relative volume comparison on smooth metric measure spaces formulated in [19, Theorem 1.2b] only yields a volume growth estimate for $R > 1$ since, as noted by Wei and Wylie, the right-hand side of

$$\frac{\mathrm{Vol}_f(B(p, R))}{V_H^{n+4k}(R)} \leq \frac{\mathrm{Vol}_f(B(p, r))}{V_H^{n+4k}(r)}$$

blows up as $r \to 0$. Using this type of estimate to extend Wei's proof methods to the smooth metric measure space setting would require additional assumptions. Moreover, using this comparison will yield polynomial growth of degree $n + 4k$. In order to improve the degree with only the additional assumption that $|f| \leq k$, we formulate the following volume estimate.

Proposition 3.2 *Let $(M^n, g, e^{-f} dvol_g)$ be a smooth metric measure space with $Ric_f \geq (n-1)H$, $H < 0$, and $|f| \leq k$. Let $p \in M$. Then*

$$Vol_f(B(p, R)) \leq k \int_0^R \mathcal{A}_H(r) e^{2k[\cosh(2\sqrt{-H}r)+1]} dr, \tag{3.1}$$

where $\mathcal{A}_H(r)dr$ denotes the volume element on the model space with constant curvature H.

Proof Let $sn_H(r)$ be the solution to $sn''_H + H sn_H = 0$ such that $sn_H(0) = 0$ and $sn'_H(0) = 1$. When $H < 0$, this solution is given by

$$\frac{1}{\sqrt{-H}} \sinh \sqrt{-H} r. \tag{3.2}$$

From the proof [Theorem 1.1, inequality (2.17)] [19] we get

$$sn_H^2(r)m_f(r) \le sn_H^2(r)m_H(r) - f(r)(sn_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \quad (3.3)$$

Then integrating both sides of (3.3) from $r = r_1$ to r_2 gives

$$\begin{aligned}
\int_{r_1}^{r_2} m_f(r)dr &\le \int_{r_1}^{r_2} m_H(r)dr - \int_{r_1}^{r_2} f(r)\frac{(sn_H^2(r))'}{sn_H^2(r)}dr \\
&\quad + \int_{r_1}^{r_2} \frac{1}{sn_H^2(r)} \left\{ \int_0^r f(t)(sn_H^2)''(t)dt \right\} dr \\
&= \int_{r_1}^{r_2} m_H(r)dr - 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}r dr \\
&\quad +2(-H)\int_{r_1}^{r_2} \operatorname{csch}^2\sqrt{-H}r \left\{ \int_0^r f(t)[\sinh^2\sqrt{-H}t \right. \\
&\quad \left. + \cosh^2\sqrt{-H}t]dt \right\} dr \\
&= \int_{r_1}^{r_2} m_H(r)dr - 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}r dr \\
&\quad +2(-H)\int_{r_1}^{r_2} \operatorname{csch}^2\sqrt{-H}r \left\{ \int_0^r f(t)\cosh 2\sqrt{-H}t dt \right\} dr \\
&= \int_{r_1}^{r_2} m_H(r)dr - 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}r dr \\
&\quad +2(-H)\left[-\frac{\coth\sqrt{-H}r}{\sqrt{-H}} \int_0^r f(t)\cosh 2\sqrt{-H}t dt \right]_{r_1}^{r_2} \\
&\quad +4(-H)\int_{r_1}^{r_2} \frac{\coth\sqrt{-H}r}{\sqrt{-H}} f(r)\sinh^2\sqrt{-H}r dr \\
&\quad +2(-H)\int_{r_1}^{r_2} \frac{\coth\sqrt{-H}r}{\sqrt{-H}} f(r)dr
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{r_1}^{r_2} m_H(r)dr + k\coth\sqrt{-H}r_2 \sinh 2\sqrt{-H}r_2 \\
&\quad + k\coth\sqrt{-H}r_1 \sinh 2\sqrt{-H}r_1 \\
&\quad + 2k[\sinh^2\sqrt{-H}r_2 - \sinh^2\sqrt{-H}r_1] \\
&= \int_{r_1}^{r_2} m_H(r)dr + 2k[\cosh(2\sqrt{-H}r_2) + 1],
\end{aligned}
$$

where the first equality is obtained by substituting (3.2) for sn_H, and the third equality is obtained through integration by parts.

Using exponential polar coordinates around p, we may write the volume element of M as $\mathcal{A}(r,\theta)\wedge d\theta_{n-1}$ where $d\theta_{n-1}$ is the standard volume element of the unit sphere $\mathbb{S}^{n-1}$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$ and $\mathcal{A}_H(r)$ denotes the volume element for the model space with constant curvature H. The mean curvatures on the smooth metric measure space and on the model space can be written, respectively, as

$$
m_f(r) = (\ln(\mathcal{A}_f(r,\theta))' \quad \text{and} \quad m_H(r) = (\ln(\mathcal{A}_H(r))'.
$$

Then we may rewrite the above inequality as

$$
\ln\left(\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)}\right) \leq \ln\left(\frac{\mathcal{A}_H(r_2)}{\mathcal{A}_H(r_1)}\right) + 2k[\cosh(2\sqrt{-H}r_2) + 1].
$$

Hence

$$
\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)} \leq \frac{\mathcal{A}_H(r_2)}{\mathcal{A}_H(r_1)} e^{2k[\cosh(2\sqrt{-H}r_2)+1]}.
$$

Then

$$
\mathcal{A}_f(r_2,\theta)\mathcal{A}_H(r_1) \leq \mathcal{A}_H(r_2)\mathcal{A}_f(r_1,\theta)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}.
$$

Integrating both sides of the inequality over $\mathbb{S}^{n-1}$ with respect to θ yields

$$
\mathcal{A}_H(r_1)\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_2,\theta)d\theta \leq \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_1,\theta)d\theta.
$$

Then we integrate both sides of the inequality with respect to r_1 from $r_1 = 0$ to $r_1 = R_1$:

$$
Vol_H(B(R_1))\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_2,\theta)d\theta \leq Vol_f(B(p,R_1))\mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}.
$$

Finally, we integrate both sides of the inequality with respect to r_2 from $r_2 = 0$ to $r_2 = R_2$:

$$Vol_H(B(R_1))Vol_f(B(p, R_2)) \leq Vol_f(B(p, R_1)) \int\limits_0^{R_2} \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}dr_2,$$

thus yielding a new volume inequality:

$$\frac{Vol_H(B(R_1))}{Vol_f(B(p, R_1)} \leq \frac{\int_0^{R_2} \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}dr}{Vol_f(B(p, R_2))}.$$

Note that the left-hand side of the inequality tends to $\frac{1}{f(p)}$ as $R_1 \to 0$. Then

$$Vol_f(B(p, R_2)) \leq f(p) \int\limits_0^{R_2} \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}dr_2.$$

□

Using Prop 3.2, we may extend M. Anderson's Theorem 2.1 and Theorem 2.2 of [2] to the smooth metric measure space setting.

Proposition 3.3 *Let $(M^n, g, e^{-f}dvol_g)$ be a smooth metric measure space with $|f| \leq k$ satisfying the bounds $Ric_f \geq (n-1)H$, $diam(M) \leq D$ and $Vol_f(M) \geq v$. If γ is a loop in M such that $[\gamma]^p \neq 0$ for $p \leq N = \frac{k}{v} \int_0^{2D} \mathcal{A}_H e^{2k[\cosh(2\sqrt{-H}r)+1]}dr$, then*

$$l(\gamma) \geq \frac{D}{N}.$$

The proof of Proposition 3.3 follows Anderson's proof of [2, Theorem 2.1], where in place of the absolute volume comparison of Bishop and Gromov, one uses the volume comparison (3.1). A sketch of the proof is provided below.

Proof Consider the subgroup $\Gamma = \langle\gamma\rangle$ of $\pi_1(M) = \pi_1(M, x_0)$ where elements act as deck transformations on the universal cover $\tilde{M}$ of M. Choose $\tilde{x_0} \in \tilde{M}$ such that $\tilde{x_0} \to x_0$ under the covering map. Then, choose $F \subseteq \tilde{M}$ to be a fundamental domain of $\pi_1(M)$ containing $\tilde{x_0}$.

Let $U(r) = \{g \in \Gamma | g = \gamma^i, |i| \leq r\}$. Since $[\gamma]^p \neq 0$ in $\pi_1(M)$ for $p \leq N$, we have $|\Gamma| \geq N$ and we may choose the smallest $r = r_0$ such $\#U(r_0) > N$. Note now that

$$\bigcup_{g \in U(r_0)} g(B(\tilde{x_0}, D) \cap F) \subseteq B(\tilde{x_0}, Nl(\gamma) + D).$$

Then, by (3.1), we have

$$N \cdot \mathrm{Vol}_f M \leq \mathrm{Vol}_f(B(\tilde{x_0}, Nl(\gamma) + D)) \leq k \int\limits_0^{Nl(\gamma)+D} \mathcal{A}_H(r)e^{2k[\cosh(2\sqrt{-H}r)+1]}dr. \quad (3.4)$$

Seeking contradiction, suppose that $l(\gamma) \leq \frac{D}{N}$. Then by (3.4), we have

$$N < \frac{k}{v} \int_0^{2D} \mathcal{A}_H(r) e^{2k[\cosh(2\sqrt{-H}r)+1]} dr,$$

contradicting the definition of N. $\square$

Proposition 3.3 is used directly in the proof of Theorem 1.3. It is also used to prove the extension of Anderson's theorem [2, Theorem 2.2] to the smooth metric measure space setting.

Proposition 3.4 *For the class of manifolds M^n with $Ric_f \geq (n-1)H, Vol_f \geq v, diam(M) \leq D$ and $|f| \leq k$, there are only finitely many isomorphism types of $\pi_1(M)$.*

Just as in Anderson's proof of [2, Theorem 2.2], we use Proposition 3.3 to show that there is a bound on the number of generators of $\pi_1(M)$. This is sufficient due to a theorem of Gromov [12, Proposition 5.28] which guarantees a set of generators $g_1, \ldots, g_l$ of $\pi_1(M)$ such that $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 3D$ and every relation is of the form $g_i g_j = g_k$. Proposition 3.4 is also stated in Wei and Wylie's work without proof; see [19, Theorem 4.7].

Now, with Proposition 3.2 and Proposition 3.4, we may extend Wei's theorem about polynomial growth of the fundamental group [16] to smooth metric measure spaces.

Theorem 3.5 *For any constant $v \geq 0$, there exists $\epsilon = \epsilon(n, v, k, H, D) > 0$ such that if a smooth metric measure space $(M^n, g, e^{-f} dvol_g)$ with $|f| \leq k$ satisfies the conditions (1.6)–(1.8), then the fundamental group of M is of polynomial growth of degree $\leq n$.*

Proof Let $\Gamma(s) = \{$distinct words in $\pi_1(M)$ of length $\leq s\}$. Let us assume by means of contradiction that $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. It follows that for any set of generators of $\pi_1(M)$, we can find real numbers s_i for all i, such that

$$\#\Gamma(s_i) > i s_i^n. \tag{3.5}$$

Choose a base point $\tilde{x}_0$ in the universal covering $p : \tilde{M} \to M$, and let $x_0 = p(\tilde{x}_0)$. By Proposition 3.4, there are only finitely many isomorphism types of $\pi_1(M)$. For each isomorphism type, choose a set of generators $g_1, \ldots, g_N$ of $\pi_1(M)$ such that $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 3D$ and every relation is of the form $g_i g_j = g_k$. Again, such a set of generators is guaranteed by a theorem of Gromov [12, Proposition 5.28]. By the proof of Proposition 3.4, we know that N is uniformly bounded. Having chosen generators in this manner, we are guaranteed that (3.5) is independent of ϵ. View this set of generators of the fundamental group $\pi_1(M)$ as deck transformations in the isometry group of $\tilde{M}$.

Now, choose a fundamental domain F of $\pi_1(M)$ containing $\tilde{x}_0$. Then

$$\bigcup_{g \in \Gamma(s)} g(F) \subseteq B(\tilde{x}_0, D(3s+1)),$$

which implies

$$\#\Gamma(s) \leq \frac{1}{v}\mathrm{Vol}_f(B(\tilde{x}_0, D(3s+1))).$$

Then, by Proposition 3.2, it follows that

$$\#\Gamma(s) \leq \frac{k}{v}\int_0^{D(3s+1)} \frac{\sinh\sqrt{\epsilon}r}{\epsilon} e^{2k[\cosh(2\sqrt{\epsilon}r)+1]}dr.$$

For any fixed, sufficiently large s_0, there exists $\epsilon_0 = \epsilon(s_0)$ such that for all $\epsilon \leq \epsilon_0$, we have

$$\#\Gamma(s) \leq \frac{2^{3n}e^4k}{nv}s^n. \tag{3.6}$$

Let $i_0 > \frac{2^{3n}e^4k}{nv}$. Then $\epsilon < \epsilon(s_{i_0})$ together with (3.5) and (3.6) yields a contradiction. □

4 Proof of Theorem 1.3

With Theorems 2.10, 3.5, and Proposition 3.3, one may generalize the arguments in [21] to the smooth metric measure space setting. Before continuing to the proof, which we retain here for completeness, we review the notion of equivariant Hausdorff convergence, which is instrumental in the proof of Theorem 1.3 as well as Theorem 5.9.

Definition 4.1 [10, Definition 3.1] Let $\mathcal{M}$ denote the set of all isometry classes of pointed metric spaces (X, p) such that for each D, the ball $B(p, D, X)$ is relatively compact and such that X is a length space.

Let $\mathcal{M}_{eq}$ denote the set of triples (X, Γ, p) where $(X, p) \in \mathcal{M}$ and Γ is a closed group of isometries of X. Set

$$\Gamma(D) = \{\gamma \in \Gamma | d(\gamma p, p) < D\}.$$

Definition 4.2 [10, Definition 3.3] Let $(X, \Gamma, p), (Y, \Lambda, q) \in \mathcal{M}_{eq}$. An ϵ-equivariant pointed Hausdorff approximation is a triple of maps (f, ϕ, ψ)

$$f : B(p, 1/\epsilon) \to Y,$$
$$\phi : \Gamma(1/\epsilon) \to \Lambda(1/\epsilon),$$
$$\psi : \Lambda(1/\epsilon) \to \Gamma(1/\epsilon),$$

such that

(1) $f(p) = q$;

(2) the ϵ-neighborhood of $f(B(p, 1/\epsilon))$ contains $B(q, 1/\epsilon)$;
(3) $x, y \in B(p, 1/\epsilon) \Rightarrow |d(f(x), f(y)) - d(x, y)| < \epsilon$;
(4) $\gamma \in \Gamma(1/\epsilon), x \in B(p, 1/\epsilon), \gamma x \in B(p, 1/\epsilon) \Rightarrow d(f(\gamma x), \phi(\gamma)(f(x))) < \epsilon$;
(5) $\mu \in \Lambda(1/\epsilon), x \in B(p, 1/\epsilon), \psi(\mu)(x) \in B(p, 1/\epsilon) \Rightarrow d(f(\psi(\mu)(x)), \mu(f(x))) < \epsilon$.

Then, a sequence of pointed triples $(X_i, \Gamma_i, p_i) \in \mathcal{M}_{eq}$ converges in the equivariant Hausdorff sense to $(X_\infty, \Gamma_\infty, p_\infty)$ if there exist ϵ_i-equivariant pointed Hausdorff approximations between (X_i, Γ_i, p_i) and $(X_\infty, \Gamma_\infty, p_\infty)$ with $\epsilon_i \to 0$ as $i \to \infty$.

Proof of Theorem 1.3 By Theorem 3.5, there exists $\epsilon_0 = \epsilon_0(n, v, k, H, D) > 0$ such that if a smooth metric measure space $(M, g, e^{-f}\mathrm{dvol}_g)$ with $|f| < k$, satisfies (1.6)–(1.8), then $\pi_1(M)$ is a finitely generated group of polynomial growth of order $\leq n$.

Assume Theorem 1.3 is not true. Then there exists a contradicting sequence of smooth metric measure spaces $(M_i, g_i, e^{-f_i}\mathrm{dvol}_{g_i})$ with $|f_i| \leq k$ and

$$\mathrm{Ric}_{f_i}(M_i) \geq -\epsilon_i \to 0, \quad \epsilon_i \leq \epsilon_0, \quad \mathrm{Vol}_{f_i}(M_i) \geq v, \quad \mathrm{diam}(M_i) \leq D,$$

such that $\pi_1(M_i)$ is not almost abelian for each i. Note, however, $\pi_1(M_i)$ is of polynomial growth for each i.

Since $\pi_1(M_i)$ is of polynomial growth, [21, Lemma 1.3] implies it contains a torsion free nilpotent subgroup Γ_i of finite index. Since Γ_i has finite index in $\pi_1(M)$, it must be nontrivial. Furthermore, Γ_i cannot be almost abelian.

Consider the action Γ_i on the universal cover $\widetilde{M}_i$. For $p_i \in \widetilde{M}_i$ consider the sequence $(\widetilde{M}_i, \Gamma_i, p_i)$. There exists a length space (Y, q) and a closed subgroup G of $Isom(Y)$ such that $(\widetilde{M}_i, \Gamma_i, p_i)$ subconverges to a triple (Y, G, q) with respect to the pointed equivariant Gromov–Hausdorff distance [10, Theorem 3.6].

Using the Almost Splitting Theorem 2.10, we know Y splits as an isometric product $Y = \mathbb{R}^k \times Y_0$ for some k and length space Y_0 which contains no lines. By Proposition 3.4 it follows $[\pi_1(M_i) : \Gamma_i]$ is uniformly bounded, say $[\pi_1(M_i) : \Gamma_i] \leq m$. Hence $\mathrm{diam}(\widetilde{M}_i/\Gamma_i) \leq Dm$. Then since $(\widetilde{M}_i, \Gamma_i, p_i) \to (\mathbb{R}^k \times Y_0, G, q)$, it follows that $\mathrm{diam}(\mathbb{R}^k \times Y_0/G) \leq Dm$. Then Y_0 must be compact. Otherwise, it would contain a line. Thus we may consider the projection

$$\phi : G \to Isom(\mathbb{R}^k).$$

By [10, Theorem 6.1], for every $\delta > 0$ there exists a normal subgroup G_δ of G such that G/G_δ contains a finite index, free abelian group of rank not greater than $\dim(\mathbb{R}^k/\phi(G))$. Since Γ_i is torsion free, Proposition 3.3 gives that for all nontrivial $\gamma \in \Gamma_i$, we have $l(\gamma) \geq \frac{D}{N}$ where $N = \frac{k}{v}\int_0^{2D} \mathcal{A}_{\epsilon_0} e^{2k[\cosh(2\sqrt{\epsilon_0}r)+1]}dr$. Choose $\delta = \frac{D}{N}$ and set $\delta_0 = \delta/2$.

Define

$$\Gamma_i(\delta) = \{\gamma \in \Gamma_i : d(p_i, \gamma(p_i)) < \delta\}.$$

Similarly, define

$$G(\delta) = \{\gamma \in G : d(q, \gamma(q)) < \delta\}.$$

Then

$$\Gamma_i(\delta) = \{1\}.$$

Since $(\widetilde{M_i}, \gamma_i, p_i) \to (\mathbb{R}^k \times Y_0, G, q)$, it follows that

$$G(\delta_0) = \{1\}.$$

Let K denote the kernel of ϕ. Since $\delta_0 > 0$ was chosen so that $G(\delta_0) = \{1\}$, it follows that

$$\{\gamma \in K | d(\gamma(x), x) < \delta_0 \text{ for all } x \in Y\} = \{1\}.$$

Thus the subgroup generated by this set is trivial. That is,

$$K_{\delta_0} = \langle \{\gamma \in K | d(\gamma(x), x) < \delta \text{ for all } x \in Y\} \rangle = \{1\}.$$

Then, the quotient map

$$\pi : G \to G/K_{\delta_0}$$

is simply the identity map. The subgroup G_{δ_0} of G which has the properties we seek is defined by

$$G_{\delta_0} = \pi^{-1}([1]),$$

where [1] denotes the coset containing the identity element of G/K_{δ_0}. But since K_{δ_0} is trivial and π is the identity map, it follows that $G_{\delta_0} = \{1\}$. Thus by [10, Lemma 6.1], $G/G_{\delta_0} = G$ contains a finite index free abelian subgroup of rank $\leq k$; that is, G is almost abelian. Moreover, by [10, Theorem 3.10] we have that Γ_i is isomorphic to G for i sufficiently large. But this contradicts the fact that Γ_i is not almost abelian for each i. □

5 Bound on Number of Generators of the Fundamental Group

In order to obtain a uniform bound on the number of generators of the fundamental group, Kapovitch and Wilking require two results closely related to the Almost Splitting Theorem. The first of these results is due to Cheeger and Colding [6, see Sect. 1]; see also [7, Theorem 9.29].

Theorem 5.1 *Given $R > 0$ and $L > 2R + 1$, let $Ric_{M^n} \geq -(n-1)\delta$ and $d_{GH}(B(p, L), B(0, L)) \leq \delta$, where $B(0, L) \subset \mathbb{R}^n$. Then there exist harmonic functions $\overline{b}_1, \ldots, \overline{b}_n$ on $B(p, R)$ such that in the Gromov–Hausdorff sense $d(e_i, \overline{b}_i) \leq \Psi$, where $\{e_i\}$ denote the standard coordinate functions on $\mathbb{R}^n$ and*

$$\fint_{B(p,R)} \sum_i |\nabla \overline{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \overline{b}_i, \nabla \overline{b}_j \rangle| + \sum_i |Hess\, \overline{b}_i|^2 \leq \Psi. \tag{5.1}$$

In the smooth metric measure space setting, a similar statement may be made:

Theorem 5.2 *Given $R > 0$ and $L > 2R + 1$, let $Ric_f \geq -(n-1)\delta$, with $|f| \leq k$ and $d_{GH}(B(p, L), B(0, L)) \leq \delta$, where $B(0, L) \subset \mathbb{R}^n$. Then there exist f-harmonic functions $\overline{b}_1, \ldots, \overline{b}_n$ on $B(p, R)$ such that in the Gromov–Hausdorff sense $d(e_i, \overline{b}_i) \leq \Psi$, where $\{e_i\}$ denote the standard coordinate functions on $\mathbb{R}^n$ and*

$$\fint_{B(p,R)} \left(\sum_i |\nabla \overline{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \overline{b}_i, \nabla \overline{b}_j \rangle| + \sum_i |Hess\, \overline{b}_i|^2 \right) e^{-f} dvol_g \leq \Psi. \tag{5.2}$$

Proof The manner in which the harmonic functions $\overline{b}_i$ are constructed for Theorems 5.1 and 5.2 is similar to the manner in which the harmonic functions $\overline{b}_\pm$ are constructed in the proof of the Almost Splitting Theorem in both the Riemannian and smooth metric measure space settings. Since the two L-balls are δ-close in the Gromov–Hausdorff sense, there exists a δ-Gromov–Hausdorff approximation

$$F : B(0, L) \to B(p, L).$$

For each $i = 1, \ldots, n$, set

$$q_i = F(Le_i)$$

and define $b_i : M \to \mathbb{R}$ by

$$b_i(x) = d(x, q_i) - d(p, q_i).$$

For the smooth metric measure space version, let $\overline{b}_i$ be the f-harmonic function such that $\overline{b}_i\big|_{\partial B(p,L)} = b_i\big|_{\partial B(p,L)}$. Integrating each term separately, we see that the first term can be controlled by (2.14) and the third term by (1.5). One can show a Ψ-upper bound for the middle term of the integrand (5.2) by noting that

$$\begin{aligned} \langle \nabla \overline{b}_i, \nabla \overline{b}_j \rangle &= \langle \nabla \overline{b}_i - \nabla b_i + \nabla b_i, \nabla \overline{b}_j - \nabla b_j + \nabla b_j \rangle \\ &= \langle \nabla \overline{b}_i - \nabla b_i, \nabla \overline{b}_j \rangle + \langle \nabla \overline{b}_j - \nabla b_j, \nabla \overline{b}_i \rangle + \langle \nabla b_i, \nabla b_j \rangle. \end{aligned}$$

Using integration by parts and (2.13), one can show that the average value of each of the first two terms of the summand is bounded from above by Ψ. Moreover, $\langle \nabla b_i, \nabla b_j \rangle \to 0$ when $L \to \infty$. □

The Product Lemma of Kapovitch and Wilking, stated below, can be viewed as another type of splitting result.

Theorem 5.3 *[13, Lemma 2.1] Let M_i be a sequence of manifolds with $Ric_{M_i} > -\epsilon_i \to 0$ satisfying*

- *$\overline{B(p_i, r_i)}$ compact for all i with $r_i \to \infty$, $p_i \in M_i$,*
- *for all i and $j = 1, \ldots, k$ there exist harmonic functions $\overline{b}^i_j : B(p_i, r_i) \to \mathbb{R}$ which are L-Lipschitz and fulfill*

$$\fint_{B(p_i,R)} \left(\sum_{j,l=1}^{k} |\langle \nabla \overline{b}^i_j, \nabla \overline{b}^i_l \rangle - \delta_{jl}| + \sum_{j=1}^{k} |Hess\, \overline{b}^i_j|^2 \right) d\mu_i \to 0 \qquad \text{for all } R > 0,$$

then $(B(p_i, r_i), p_i)$ subconverges in the pointed Gromov–Hausdorff topology to a metric product $(\mathbb{R}^k \times X, p_\infty)$, for some metric space X.

Without the assumption that a line exists in the limit space, Kapovitch and Wilking instead show that each of the functions $\overline{b}^i_j$, as in the hypothesis of Theorem 5.3, limit to a submetry $\overline{b}^\infty_j$ as $i \to \infty$. The submetry $\overline{b}^\infty_j$ then lifts lines to lines, which allows one to apply the Almost Splitting Theorem to show that the limit indeed splits. Their argument may be modified to the smooth metric measure space setting by using the volume comparison [19, Theorem 1.2], the Segment Inequality 2.3, and the fact that gradient flow of an f-harmonic function is measure preserving with respect to the measure $e^{-f}\,\mathrm{dvol}_g$. Augmenting their arguments in this manner yields the following extension.

Theorem 5.4 *Let $(M_i, g_i, e^{-f_i} dvol_{g_i})$ be a sequence of smooth metric measure spaces with $|f_i| \le k$ and $Ric_{f_i} > -\epsilon_i \to 0$. Suppose that $r_i \to \infty$ and for every i and $j = 1, \ldots, m$, there are harmonic functions $\overline{b}^i_j : B(p_i, r_i) \to \mathbb{R}$ which are L-Lipschitz and fulfill*

$$\fint_{B(p_i,R)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}^i_j, \nabla \overline{b}^i_l \rangle - \delta_{jl}| + \sum_{j=1}^{m} |Hess\, \overline{b}^i_j|^2 \right) e^{-f_i} dvol_{g_i} \to 0 \, \text{for all } R > 0.$$

Then $(B(p_i, r_i), p_i)$ subconverges in the pointed Gromov–Hausdorff topology to a metric product $(\mathbb{R}^m \times X, p_\infty)$ for some metric space X.

The following lemma of Kapovitch and Wilking requires only an inner metric space structure and hence may be applied to smooth metric measure spaces.

Lemma 5.5 *[13, Lemma 2.2] Let $(Y_i, \tilde{p}_i)$ be an inner metric space endowed with an action of a closed subgroup G_i of its isometry group, $i \in \mathbb{N} \cup \{\infty\}$. Suppose $(Y_i, G_i, \tilde{p}_i) \to (Y_\infty, G_\infty, \tilde{p}_\infty)$ in the equivariant Gromov–Hausdorff topology. Let $G_i(r)$ denote the subgroup generated by those elements that displace $\tilde{p}_i$ by at most r, $i \in \mathbf{N} \cup \{\infty\}$. Suppose there are $0 \leq a < b$ with $G_\infty(r) = G_\infty(\frac{a+b}{2})$ for all $r \in (a, b)$. Then there is some sequence $\epsilon_i \to 0$ such that $G_i(r) = G_i(\frac{a+b}{2})$ for all $r \in (a + \epsilon_i, b - \epsilon_i)$.*

For more on equivariant Gromov–Hausdorff convergence, see [10]. Lemma 5.5 and the Almost Splitting Theorem 2.10 allow us to modify arguments of the proof of [13, Lemma 2.3] to show that the following holds for smooth metric measure spaces.

Lemma 5.6 *Suppose (M_i^n, q_i) is a pointed sequence of smooth metric measure spaces where $(M_i^n, g_i, e^{-f_i} dvol_{g_i})$ has $|f_i| \leq k$ and $Ric_{f_i}(M_i) \geq -1/i$. Moreover, assume $(M_i^n, q_i) \to (\mathbb{R}^m \times K, q_\infty)$ where K is compact, and the action of $\pi_1(M_i)$ on the universal cover $(\widetilde{M}_i, \tilde{q}_i)$ converges to a limit action of a group G on some limit space $(Y, \tilde{q}_\infty)$. Then $G(r) = G(r')$ for all $r, r' > 2diam(K)$.*

We will also need the following result on the dimension of the limit space.

Lemma 5.7 *Let $(M_i^n, g_i, e^{-f_i} dvol_g)$ be a sequence of smooth metric measure spaces such that $|f_i| \leq k$, $diam(M_i^n) \leq D$, and $Ric_f \geq -(n-1)H$, $H > 0$. If M_i^n converges to the length space Y^m in the Gromov–Hausdorff sense, then for the Hausdorff dimension we have $m \leq n + 4k$.*

Proof Begin by noting that for any $(M^n, g, e^{-f} \mathrm{dvol}_g)$ with $\mathrm{Ric}_f \geq -(n-1)H$, $H > 0$, and fixed $x \in M$ and $R > 0$, the f-volume comparison [19, Theorem 1.2b] gives a bound on the number of disjoint ϵ-balls contained in $B(x, R)$: Let $B(x_1, \epsilon), \ldots, B(x_l, \epsilon) \subset B(x, R)$ be disjoint. Let $B(x_i, \epsilon)$ denote the ball with the smallest f-volume. Then

$$l \leq \frac{\mathrm{Vol}_f B(x, R)}{\mathrm{Vol}_f B(x_i, \epsilon)} \leq \frac{\mathrm{Vol}_f B(x_i, 2R)}{\mathrm{Vol}_f B(x_i, \epsilon)} \leq \frac{\mathrm{Vol}_H^{n+4k} B(2R)}{\mathrm{Vol}_H^{n+4k} B(\epsilon)} = C(n + 4k, H, R, \epsilon).$$

Thus $\mathrm{Cap}_{M_i}(\epsilon)$, the maximum number of disjoint $\epsilon/2$-balls which can be contained in M_i^n, is bounded above by $C = C(n + 4k, H, D, \frac{\epsilon}{2})$ for each i. Moreover, $\mathrm{Cov}_{M_i}(\epsilon)$, the minimum number of ϵ-balls covering M_i^n less than or equal to $\mathrm{Cap}_{M_i}(\epsilon)$, so $\mathrm{Cov}_{M_i} \leq C$.

Since $M_i^n \to Y$ in the Gromov–Hausdorff sense, there exists a sequence $\delta_i > 0$ such that $d_{GH}(M_i, Y) < \delta_i \to 0$ as $i \to \infty$. Then $\mathrm{Cov}_Y(\epsilon) \leq \mathrm{Cov}_{M_i}(\epsilon - 2\delta_i) \leq C$. As $i \to \infty$, we have $\mathrm{Cov}_Y(\epsilon) \leq C$.

To see that the Hausdorff dimension is bounded above by $n + 4k$, recall that the d-dimension Hausdorff measure of Y is defined by

$$H^d(Y) = \lim_{\epsilon \to 0} H_\epsilon^d(Y),$$

where

$$H^d_\epsilon(Y) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}U_i)^d \,\middle|\, \bigcup_{i=1}^{\infty} U_i \supset Y, \mathrm{diam}U_i \le \epsilon\right\}.$$

Since $\mathrm{Cov}_Y(\epsilon) \le C$, it follows that $H^d_\epsilon(Y) \le \sum_{i=1}^{C}(2\epsilon)^d$. Notice

$$C = \frac{\mathrm{Vol}_H^{n+4k} B(D)}{\mathrm{Vol}_H^{n+4k} B(\epsilon/2)} \sim (\epsilon/2)^{-(n+4k)}$$

as $\epsilon \to 0$. Thus as $\epsilon \to 0$

$$\sum_{i=1}^{C}(2\epsilon)^d = C(2\epsilon)^d \to 0$$

for all $d > n + 4k$. Thus the Hausdorff dimension of Y, defined by $\dim_H(Y) = \inf\{d \ge 0 | H^d(Y) = 0\}$ is at most $n + 4k$. □

The final tool we will use to extend [13, Theorem 2.5] to smooth metric measure spaces is a type of Hardy–Littlewood maximal inequality for smooth metric measure spaces.

Proposition 5.8 (Weak 1-1 Inequality) *Suppose* $(M^n, g, e^{-f}dvol_g)$ *with* $|f| < k$ *has* $Ric_f \ge -(n-1)H$ *and* $h : M \to \mathbb{R}$ *is a nonnegative function. Define* $Mx_\rho h(p) = \sup_{r\le\rho} \fint_{B(p,r)} he^{-f}dvol_g$ *for* $\rho \in (0, 1]$. *Then if* $h \in L^1(M)$*, we have*

$$Vol_f\{x|Mx_\rho h(x) > c\} \le \frac{C(n+4k, H)}{c}\int_M he^{-f}dvol_g$$

for any $c > 0$.

As in the proof of the Hardy–Littlewood maximal inequality for Euclidean spaces, one utilizes the Vitali Covering Lemma which states that for an arbitrary collection of balls $\{B(x_j, r_j) : j \in J\}$ in a metric space, there exists a subcollection of balls $\{B(x_j, r_j) : j \in J'\}$ with $J' \subseteq J$ from the original collection which are disjoint and satisfy

$$\bigcup_{j\in J} B(x_j, r_j) \subseteq \bigcup_{j\in J'} B(x_j, 5r_j).$$

We also note that the f-Volume Comparison [19, Theorem 1.2] gives a type of doubling estimate. In particular, for all $r \le 1$, we have

$$\mathrm{Vol}_f(B(x, 5r)) \le C(n+4k, H)\mathrm{Vol}_f(B(x, r)).$$

Proof Let $J = \{x | \text{Mx}_\rho h(x) > c\}$. For all $x \in J$ there exists a ball $B(x, r_x)$ centered at x with radius $r_x \leq 1$ such that

$$\int_{B(x,r_x)} he^{-f}\text{dvol}_g \geq c\text{Vol}_f B(x, r_x). \tag{5.3}$$

Then by the Vitali Covering Lemma, we have

$$J \subseteq \bigcup_{x \in J} B(x, r_x) \subseteq \bigcup_{x \in J'} B(x, 5r_x)$$

where $J' \subseteq J$. Then

$$\text{Vol}_f\{x|\text{Mx}_\rho h(x) > c\} \leq \text{Vol}_f\left(\bigcup_{x \in J'} B(x, 5r_x)\right) \leq C(n + 4k, H) \sum_{x \in J'} \text{Vol}_f B(x, r_x). \tag{5.4}$$

Combining (5.3) and (5.4) yields the desired result. □

Before continuing to the proof of the theorem, we take a moment to recall Gromov's short generator system and the notion of a regular point. As in Gromov [11, 2.1], to construct a Gromov short generator system of the fundamental group $\pi_1(p, M)$, we represent each element of $\pi_1(p, M)$ by a shortest geodesic loop γ in that homotopy class. A minimal γ_1 is chosen so that it represents a nontrivial homotopy class of $\pi_1(M)$. If $\langle\gamma_1\rangle = \pi_1(M)$, then $\{\gamma_1\}$ is a Gromov short generator system of $\pi_1(M)$. If not, consider $\pi_1(M) \setminus \langle\gamma_1\rangle$. Choose $\gamma_2 \in \pi_1(M) \setminus \langle\gamma_1\rangle$ to be of minimal length. If $\langle\gamma_1, \gamma_2\rangle = \pi_1(M)$, then $\{\gamma_1, \gamma_2\}$ is a Gromov short generator system of $\pi_1(M)$. If not, choose $\gamma_3 \in \pi_1(M) \setminus \langle\gamma_1, \gamma_2\rangle$ such that γ_3 is of minimal length. Continue in this manner until $\pi_1(M)$ is generated. By this construction, we obtain a sequence of generators $\{\gamma_1, \gamma_2, \dots\}$ such that $|\gamma_i| \leq |\gamma_{i+1}|$ for all i. The short generators have the property $|\gamma_i| \leq |\gamma_j^{-1}\gamma_i|$ for $i > j$. Although this sequence of generators is not unique, the sequence of lengths of generators $\{|\gamma_1|, |\gamma_2|, \dots\}$ is unique.

To review the notion of a regular point, we first recall that for a Riemannian manifold (M^n, g) a tangent cone C_pM at $p \in M$ is a pointed Gromov–Hausdorff limit of rescaled spaces $(M, p, r_i g)$ for $r_i \to \infty$. Note that tangent cones may depend on the choice of convergent subsequence and hence may not be unique. As defined in Cheeger–Colding [5, Definition 0.1], a point $p \in M$ is regular if for some k, every tangent cone at p is isometric to $\mathbb{R}^k$. We note that in the case of Ricci curvature bounded from below, Cheeger and Colding have shown that the set of regular points has full measure [5, Theorem 2.1].

We now have the necessary tools which will allow us to modify the argument of Kapovitch and Wilking to obtain a bound for the number of generators of $\pi_1(M)$ in the smooth metric measure space setting. As in [13], we prove a more general statement from which Theorem 1.4 is a consequence. This general statement, as well as its proof, is parallel to the statement and proof of [13, Theorem 2.5]. The argument is included in its entirety below for completeness.

Theorem 5.9 *Given n, k, and R, there is a constant C such that the following holds. Suppose $(M^n, g, e^{-f} dvol_g)$ is a smooth metric measure space with $|f| \leq k$, $p \in M$ and $Ric_f \geq -(n-1)$ on $B(p, 2R)$. Suppose also that $\pi_1(M, p)$ is generated by loops of length $\leq R$. Then $\pi_1(M, p)$ can be generated by C loops of length $\leq R$.*

Proof of Theorem 5.9 In order to prove Theorem 5.9 we begin, as in Kapovitch and Wilking's argument, by showing that there is a point $q \in B(p, \frac{R}{4})$ such that any Gromov short generator system of $\pi_1(M, q)$ has at most C elements.

For $q \in B(p, \frac{R}{4})$ consider a Gromov short generator system $\{\gamma_1, \gamma_2, \ldots\}$ of $\pi_1(M, q)$. By assumption, $\pi_1(M, p)$ is generated by loops of length $\leq R$. In choosing generators for any Gromov short generator system of $\pi_1(M, q)$, loops of the form $\sigma \circ g \circ \sigma^{-1}$, where σ is a minimal geodesic from q to p and g is a generator of length $\leq R$ of $\pi_1(M, p)$, are contained in each of the homotopy classes of $\pi_1(M, q)$. Such a loop has length $\leq \frac{3R}{2}$ and hence the minimal length representative of that class, γ_i must have the property that $|\gamma_i| \leq \frac{3R}{2}$. Moreover, there are a priori bounds on the number of short generators of length $\geq r$. To see this, let us only consider the short generators such that $|\gamma_i| \geq r$. In the universal cover $\widetilde{M}$ of M, if $\tilde{q} \in \pi_1^{-1}(q)$, we have

$$r \leq d(\gamma_i \tilde{q}, \tilde{q}) \leq d(\gamma_j^{-1} \gamma_i \tilde{q}, \tilde{q}) = d(\gamma_i \tilde{q}, \gamma_j \tilde{q})$$

for $i > j$. Thus the balls $B(\gamma_i \tilde{q}, r/2)$ are pairwise disjoint for all γ_i such that $|\gamma_i| \geq r$. Then,

$$\bigcup_{\{\gamma_i : |\gamma_i| \geq r\}} B(\gamma_i \tilde{q}, \frac{r}{2}) \subset B(\tilde{q}, 2R + \frac{r}{2})$$

implies that

$$\#\{\gamma_i : |\gamma_i| \geq r\} \mathrm{Vol}_f B(q, \frac{r}{2}) \leq \mathrm{Vol}_f B(q, 2R + \frac{r}{2}).$$

And hence by the volume comparison [19, Theorem 1.2(a)], it follows that $\#\{\gamma_i : |\gamma_i| \geq r\} \leq C(n, k, r, R)$. Since one can control the number of short generators of length between r and $\frac{3R}{2}$ for $r < R$, one need only show that the number of short generators of $\pi_1(M, q)$ with length $< r$ can also be controlled. This argument proceeds by contradiction. We assume the existence of a contradicting pointed sequence of smooth metric measure spaces (M_i, p_i) such that $(M_i, g_i, e^{-f_i} \mathrm{dvol}_g)$ has the property that

- $|f_i| \leq k$
- $\mathrm{Ric}_{f_i} \geq -(n-1)$ on $B(p_i, 3)$
- for all $q_i \in B(p_i, 1)$ the number of short generators of $\pi_1(M_i, q_i)$ of length ≤ 4 is larger than 2^i.

By the Gromov compactness theorem, we may assume that $(B(p_i, 3), p_i)$ converges to a limit space (X, p_∞). Set

$$\dim(X) = \max\{k : \text{ there is a regular } x \in B(p_\infty, 1/4) \text{ with } C_x X \simeq \mathbb{R}^k\}$$

where $C_x X$ denotes a tangent cone of X at x.

We prove that there is no such contradicting sequence by reverse induction on $\dim(X)$. For the base case, let $m > n+4k+1$. By Lemma 5.7, $\dim(X) \leq n+4k+1$, so there is nothing to prove here. Suppose then that there is no contradicting sequence with $\dim(X) = j$ where $j \in \{m+1, \ldots, n+4k\}$ but that there exists a contradicting sequence with $\dim(X) = m$. The induction step is divided into two substeps.

Step 1 For any contradicting sequence (M_i, p_i) converging to (X, p_∞) there is a new contradicting sequence converging to $(\mathbb{R}^{\dim X}, 0)$.

Suppose (M_i, p_i) is a contradicting sequence converging to (X, p_∞). By definition of $\dim(X)$, there exists $q_\infty \in B(p_\infty, \frac{1}{4})$ such that $C_{q_\infty} X \simeq \mathbb{R}^m$. Let $q_i \in B(p_i, \frac{1}{2})$ such that $q_i \to q_\infty$ as $i \to \infty$. Since this is a contradicting sequence, it follows that the Gromov short generator systems of $\pi_1(M_i, x_i)$ for all $x_i \in B(q_i, \frac{1}{4})$ contain at least 2^i generators of length ≤ 4. As noted earlier, for each fixed $\epsilon < 4$, the number of short generators of $\pi_1(M_i, x)$ of length $\in [\epsilon, 4]$ is bounded by a constant $C(n, k, \epsilon, 4)$. Then we can find a rescaling $\lambda_i \to \infty$ such that for every $x_i \in B(q_i, \frac{1}{\lambda_i})$, the number of generators of $\pi_1(M_i, x)$ of length $\leq 4/\lambda_i$ is at least 2^i. Moreover, $(\lambda_i M_i, q_i) \to (\mathbb{R}^m, 0)$, where $\lambda_i M_i$ denotes the smooth metric measure space $(M_i, \lambda_i g_i, e^{-f_i} \mathrm{dvol}_{\lambda_i g_i})$. Thus the sequence $(\lambda_i M_i, q_i)$ is the new contradicting sequence desired.

Step 2 If there is a contradicting sequence converging to $(\mathbb{R}^m, 0)$, then we can find a contradicting sequence converging to a space whose dimension is larger than m.

Let (M_i, q_i) denote the contradicting sequence converging to $(\mathbb{R}^m, 0)$ as obtained in Step 1 above. Without loss of generality, assume that for some $r_i \to \infty$ and $\epsilon_i \to 0$, $\mathrm{Ric}_f \geq -\epsilon_i$ on $B(p_i, r_i)$. By Theorem 5.2 there exist f-harmonic functions $(\overline{b}_1^i, \ldots, \overline{b}_m^i) : B(q_i, 1) \to \mathbb{R}^m$ such that

$$\fint_{B(q_i,1)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}_l^i, \nabla \overline{b}_j^i \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}_l^i)||^2 \right) e^{-f} \mathrm{dvol}_g < \delta_i \to 0.$$

Claim There exists $z_i \in B(q_i, \frac{1}{2})$, $c > 0$ such that for any $r \leq \frac{1}{4}$,

$$\fint_{B(z_i,r)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}_l^i, \nabla \overline{b}_j^i \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}_l^i)||^2 \right) e^{-f} \mathrm{dvol}_g \leq c\delta_i \to 0.$$

Let $h(x)$ denote $\sum_{j,l=1}^m |\langle \nabla \overline{b}_l^i, \nabla \overline{b}_j^i \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}_l^i)||^2$ evaluated at x. Seeking contradiction, suppose that for all $c > 0$, $r \leq 1/2$, and $z \in B(q_i, \frac{1}{2})$

$$\fint_{B(z,r)} he^{-f}\mathrm{dvol}_g > c\delta_i,$$

then it follows that $\mathrm{Mx}_{1/2}h(z) = \sup_{r\le 1/2}\fint_{B(z,r)} he^{-f}\mathrm{dvol}_g \ge c\delta_i$. Hence

$$\mathrm{Vol}_f\{x|\mathrm{Mx}_{1/2}h(x) \ge c\delta_i\} \ge \mathrm{Vol}_f(B(q_i, \frac{1}{2})). \tag{5.5}$$

By Proposition 5.8, we also have that for all $c \ge 0$,

$$\mathrm{Vol}_f\{x|\mathrm{Mx}_{1/2}h(x) \ge c\delta_i\} \le \frac{C(n+4k,-1)}{c}. \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$1 \le \frac{\mathrm{Vol}_f\{x|\mathrm{Mx}_{1/2}h(x) \ge c\delta_i\}}{\mathrm{Vol}_f(B(q_i,\frac{1}{2}))} \le \frac{C(n+4k,-1)}{c\cdot \mathrm{Vol}_f(B(q_i,\frac{1}{2}))}.$$

Choosing $c > C(n+4k,-1)/\mathrm{Vol}_f(B(q_i,\frac{1}{2}))$ yields a contradiction and hence the claim is proven.

By Lemmas 5.5 and 5.6, there exists a sequence $\delta_i \to 0$ such that for all $z_i \in B(p_i, 2)$ the Gromov short generator system of $\pi_1(M_i, z_i)$ does not contain any elements of length in $[\delta_i, 4]$. Choose $r_i \le 1$ maximal with the property that there is $y_i \in B(z_i, r_i)$ such that the short generators of $\pi_1(M_i, y_i)$ contain a generator of length r_i. Then $r_i < \delta_i \to 0$.

Rescaling by $\frac{1}{r_i}$ gives that $\pi_1(\frac{1}{r_i}M_i, y_i)$ has at least 2^i short generators of length ≤ 1 for all $y_i \in B(z_i, 1)$. By the choice of rescaling, there is at least one $y_i \in B(z_i, r_i)$ such that the Gromov short generator system at that y_i contains a generator of length 1. Moreover, the above claim together with the Product Lemma 5.4 gives $(\frac{1}{r_i}M_i, z_i) \to (\mathbb{R}^k \times Z, z_\infty)$. Moreover, by Lemmas 5.5 and 5.6, Z is nontrivial and thus $\dim(\mathbb{R}^m \times Z) \ge m+1$, a contradiction. So, we have completed the induction step.

Thus there exists $q \in B(p, \frac{R}{4})$ such that number of generators of $\pi_1(M, q)$ has at most C elements. Thus the subgroup of $\pi_1(M, p)$ generated by loops of length $< 3R/5$ can be generated by C elements. Moreover, the number of short generators of $\pi_1(M, p)$ with length in $[3R/5, R]$ is bounded by some a priori constant. □

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