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Integral Geometric Properties of Non-compact Harmonic Spaces

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Abstract On non-compact harmonic manifolds we prove that functions satisfying the mean value property for two generic radii must be harmonic. Moreover, functions with vanishing integrals over all spheres (or balls) of two generic radii must be identically zero. We also prove results about the Cheeger constant and the heat kernel.

Keywords Harmonic manifold · Mean value property · Abel transform · Heat kernel

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1 Introduction

A complete Riemannian manifold (X, g) of dimension $n+1$ is called harmonic if the volume density function in normal coordinates around a point depends only on the distance from this point. Rank one symmetric spaces are harmonic, and Lichnerowicz conjectured that a simply connected harmonic space must be flat or rank one symmetric. For compact simply connected spaces this is true by a theorem of Szabo [22]. However, certain 3-step solvmanifolds, constructed by Damek and Ricci [6], provide examples of non-compact *non-symmetric* homogeneous harmonic spaces. Heber [11] proved that there exist no other simply connected *homogeneous* harmonic spaces.

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Recently, Knieper [14] showed for non-compact simply connected harmonic spaces that (i) having a purely exponential volume density function, (ii) being Gromov-hyperbolic, and (iii) having an Anosov geodesic flow, are all equivalent conditions.

Non-compact harmonic spaces have no conjugate points. Moreover, they are Einstein and therefore analytic by the Kazdan–De Turck theorem. It was shown by Willmore [26] that harmonic manifolds can also be characterized as those analytic spaces for which all harmonic functions f satisfy the mean value property, namely, that the average of f over any geodesic sphere equals the value of f at its center. It is well known that in a harmonic space every function satisfying the mean value property at all points for all radii must be harmonic. This is no longer true if a function only satisfies the mean value property at all points for a single radius $r_1 > 0$. A simple example of a non-harmonic function satisfying the mean value property for the radius $r_1 = 2\pi$ is the cosine function on the real line $X = \mathbb{R}$. We will show in Theorem 4.3 that in arbitrary non-compact harmonic spaces, the mean value property for two generically chosen radii r_1, r_2 implies harmonicity of the function. Similarly, by Theorem 4.2, the vanishing of the integral of a function over all spheres, or over all balls, of radii r_1, r_2 implies vanishing of the function, *if and only if* the pair (r_1, r_2) lies in the generic subset of $\mathbb{R}^+ \times \mathbb{R}^+$ given in Proposition 4.1. In the example of the real line, this set is the set of pairs with irrational quotient.

The paper is organized as follows. In Sect. 2, we introduce basic notions and define convolutions in harmonic spaces following ideas of Szabo [22], and prove useful properties of them.

In Sect. 3, we derive fundamental results for the Abel transform and the spherical Fourier transform, in particular, that the Abel transform and its dual are topological isomorphisms (Theorem 3.8), using finite propagation speed of the wave equation and a D'Alembert type formula for the Klein–Gordon equation.

In Sect. 4, we prove the above-mentioned integral geometric results for all non-compact harmonic manifolds. The arguments there are analogous to our earlier paper [20], where we studied two radius results for Damek–Ricci spaces. This realizes the proposed research direction indicated in [2, Sect. 10]. A crucial step is the reduction of the problem to a classical result of L. Schwartz [21] on mean periodic functions. For a modern treatment of mean periodic functions in symmetric spaces, see [25].

Finally, in Sect. 5, we present some results related to the Cheeger constant (Theorem 5.1) and to the heat kernel (Theorem 5.6) of non-compact harmonic manifolds.

2 Radial Eigenfunctions and Convolutions

Henceforth, (X, g) denotes a non-compact, complete, simply connected harmonic space, $\theta(r)$ the density function of a geodesic sphere of radius $r > 0$, $H \geq 0$ the mean curvature of all horospheres, and $x_0 \in X$ a particular reference point. Let $r(x) := d(x_0, x)$. The closed ball of radius $r > 0$ around $x \in X$ is denoted by $B_r(x) \subset X$. For the inner product, we use the notation

$$\langle f, g \rangle = \int_X f(x) g(x) dx.$$

Let $\mathcal{E}(X)$, resp., $\mathcal{D}(X)$ denote the vector space of smooth functions on X, resp., smooth functions with compact support, equipped with the topology of uniform convergence of all derivatives on compact sets, see [12, Chap. II Sect. 2], for instance.

Definition 2.1 For every $x \in X$, the *spherical projector* $\pi_x : \mathcal{E}(X) \to \mathcal{E}(X)$ is defined by

$$(\pi_x f)(y) := \frac{1}{\mathrm{vol}(S_r(x))} \int_{S_r(x)} f \quad \text{with } r = d(x, y),$$

where $S_r(x)$ denotes the geodesic sphere around x with radius r.

Let $\mathcal{E}_0(X, x) := \pi_x(\mathcal{E}(X))$ and $\mathcal{D}_0(X, x) := \pi_x(\mathcal{D}(X))$. Functions in these spaces are called *radial functions* about x. We simply write $\pi, \mathcal{E}_0(X), \mathcal{D}_0(X)$ for $\pi_{x_0}, \mathcal{E}_0(X, x_0), \mathcal{D}_0(X, x_0)$. A radial function $f \in \mathcal{E}_0(X)$ is of the form

$$f(x) = \tilde{f}\big(r(x)\big) = \tilde{f}\big(d(x, x_0)\big) \tag{1}$$

with some even function $\tilde{f} \in \mathcal{E}_0(\mathbb{R})$. We often do not distinguish between f and $\tilde{f}$ in our notation, i.e., we simply write $f(x) = f(r(x))$.

We now present basic properties of the spherical projector. The first property below is obvious, and the second identity can be found, e.g., in [17, Lemma 2].

Lemma 2.2 *For $x \in X$, the operator π_x has the following properties*:

$$\pi_x^2 = \pi_x, \tag{2}$$

$$\langle \pi_x f, g \rangle = \langle f, \pi_x g \rangle. \tag{3}$$

The Laplacian $\Delta = \mathrm{div} \circ \mathrm{grad}$, applied to a radial function $f \in \mathcal{E}_0(X)$, can be written as

$$(\Delta f)\big(r(x)\big) = f''\big(r(x)\big) + \frac{\theta'(r(x))}{\theta(r(x))} f'\big(r(x)\big). \tag{4}$$

It is well known that $\theta'(r)/\theta(r)$ is the mean curvature of a geodesic sphere $S_r(x)$, and that $\theta'(r)/\theta(r)$ is a monotone decreasing function converging to H (see [16, Cor. 2.1]).

Concerning eigenvalues, we follow the sign convention in [4], and call $f \in \mathcal{E}(X)$ an eigenfunction to the eigenvalue $\mu \in \mathbb{C}$ if $\Delta f + \mu f = 0$.

We now prove uniqueness and existence of radial eigenfunctions of the Laplacian. For positive eigenvalues this is shown in [22].

Proposition 2.3 *For each $\lambda \in \mathbb{C}$ there is a unique smooth function $\varphi_\lambda \in \mathcal{E}_0(X)$ such that*

$$\Delta \varphi_\lambda + \left(\lambda^2 + \frac{H^2}{4}\right)\varphi_\lambda = 0, \quad \textit{and} \quad \varphi_\lambda(x_0) = 1. \tag{5}$$

We obviously have $\varphi_{-\lambda} = \varphi_\lambda$ and $\varphi_{iH/2} = \varphi_{-iH/2} = 1$. Also $\varphi_\lambda(r)$ is holomorphic in λ.

Proof We fix $\lambda \in \mathbb{C}$ and abbreviate $L := -(\lambda^2 + \frac{H^2}{4})$. The eigenvalue equation (5) translates to

$$\varphi_\lambda'' + \frac{\theta'}{\theta}\varphi_\lambda' = \frac{(\theta\varphi_\lambda')'}{\theta} = L\varphi_\lambda \quad \text{and} \quad \varphi_\lambda(0) = 1. \tag{6}$$

Integrating twice we get that this is equivalent to

$$\begin{aligned} \varphi_\lambda(r) &= \varphi_\lambda(0) + L\int_0^r \frac{1}{\theta(r_2)}\int_0^{r_2} \theta(r_1)\varphi_\lambda(r_1)dr_1 dr_2 \\ &= 1 + L\int_0^r q(r, r_1)\varphi_\lambda(r_1)dr_1, \end{aligned} \tag{7}$$

where

$$q(r, r_1) = \int_{r_1}^r \frac{\theta(r_1)}{\theta(r_2)} dr_2.$$

By [16, Prop. 2.2] the function $\theta \geq 0$ increases, hence $0 \leq q(r, r_1) \leq r - r_1$, and the Volterra integral equation of the second kind (7) has a unique solution (see [13, Thm. 5]). In order to obtain a power series in L for φ_λ, we use (7) iteratively, starting with the constant function 1, and obtain

$$\varphi_\lambda(r) = 1 + \sum_{k=1}^\infty a_k(r)L^k$$

with coefficients

$$a_k(r) = \int_{r \geq r_1 \geq r_2 \geq \cdots r_k \geq 0} q(r, r_1)q(r_1, r_2)\cdots q(r_{k-1}, r_k)dr_1\cdots dr_k.$$

Since

$$0 \leq a_k(r) \leq \int_{r \geq r_1 \geq r_2 \geq \cdots r_k \geq 0} (r - r_1)(r_1 - r_2)\cdots(r_{k-1} - r_k)dr_1\cdots dr_k = \frac{r^{2k}}{(2k)!},$$

the power series above converges for all $L \in \mathbb{C}$. □

The following lemma can be found in [22, Lemma 1.1]:

Lemma 2.4 *We have*

$$\pi_x \circ \Delta = \Delta \circ \pi_x.$$

An immediate consequence of Lemma 2.4 is the fact that if $\Delta f + \mu f = 0$, then $g := \pi_x f \in \mathcal{E}(X, x)$ is also an eigenfunction of Δ with eigenvalue μ.

The *displacement* of a radial function $f \in \mathcal{E}_0(X, x)$ at a point $y \in X$ is denoted by $f_y \in \mathcal{E}_0(X, y)$ and defined by

$$f_y(z) := \widetilde{f}\big(d(y, z)\big),$$

with $\widetilde{f}(d(x, y)) = f(y)$. We have $f_y(z) = f_z(y)$.

Lemma 2.5 *The displacement* $(\varphi_\lambda)_x$ *of an eigenfunction* φ_λ *is again an eigenfunction to the same eigenvalue and*

$$\pi\left((\varphi_\lambda)_x\right) = \varphi_\lambda(x)\varphi_\lambda.$$

Proof Because of the representation (4) of the Laplacian in polar coordinates, which is independent of the center, the displacement $(\varphi_\lambda)_x$ is also an eigenfunction to the eigenvalue $\mu = (\lambda^2 + H^2/4)$. From $\Delta \circ \pi = \pi \circ \Delta$ (Lemma 2.4) we conclude that $\pi((\varphi_\lambda)_x)$ is a radial eigenfunction about x_0 to the eigenvalue μ and, by uniqueness, a multiple of φ_λ. We have

$$\big(\pi\left((\varphi_\lambda)_x\right)\big)(x_0) = (\varphi_\lambda)_x(x_0) = (\varphi_\lambda)_{x_0}(x) = \varphi_\lambda(x). \qquad \square$$

A smooth function $F : X \times X \to \mathbb{C}$ is called a *radial kernel function* if there is a function $\widetilde{f} : [0,\infty) \to \mathbb{C}$ such that

$$F(x,y) = \widetilde{f}\big(d(x,y)\big) \quad \text{for all } x, y \in X.$$

F is called *of compact support* if there is a radius $R > 0$ such that $F(x,y) = 0$ for all $d(x,y) \geq R$.

Proposition 2.6 (See [22, Prop. 2.1]) *Let* $F, G : X \times X \to \mathbb{C}$ *be two radial kernel functions, one of them of compact support. Then the convolution*

$$F * G(x,y) := \int_X F(x,z)G(z,y)dy$$

is, again, a radial kernel function, i.e., $F * G(x,y)$ *depends only on the distance* $d(x,y)$.

Radial functions $f \in \mathcal{E}_0(X)$ are in one-one correspondence with radial kernel functions $F : C^\infty(X \times X)$ via

$$F(y,z) = \widetilde{f}\big(d(y,z)\big),$$
$$f(x) = F(x_0, x),$$

where $\widetilde{f}$ was introduced in (1). This correspondence leads to a natural convolution $f * g$ of radial functions $f, g \in \mathcal{E}_0(X)$, and even to an extension of this notion if only one of the two functions is radial:

Definition 2.7 Let $f, g \in \mathcal{E}(X)$, one of them with compact support, and one of them radial about x_0. If f is the radial function, the convolution $f * g \in \mathcal{E}(X)$ is defined as

$$f * g(y) := \langle f_y, g\rangle = \int_X f_y(z)g(z)dz.$$

Similarly, if g is the radial function, we define

$$f * g(y) := \langle f, g_y\rangle = \int_X f(z)g_y(z)dz.$$

Remark 1 The convolution $f * g$ is well defined, since if both f, g are radial, we have

$$\int_X f_y(z)g(z)dz = F * G(y, x_0) \quad \text{and} \quad \int_X f(z)g_y(z)dz = F * G(x_0, y),$$

where F and G are the radial kernel functions associated with f, g. By Proposition 2.6, we have $F * G(y, x_0) = F * G(x_0, y)$. Moreover, the definition immediately implies commutativity of the convolution.

The following two lemmas are further consequences of Proposition 2.6. The proofs are straightforward, once the statements are reformulated in terms of radial kernel functions.

Lemma 2.8 *Let* $f, g \in \mathcal{D}_0(X)$. *Then* $f * g \in \mathcal{D}_0(X)$.

Lemma 2.9 *Let* $f \in \mathcal{D}(X)$ *and* $g, h \in \mathcal{D}_0(X)$. *Then we have*

$$f * (g * h) = (f * g) * h. \tag{8}$$

Let $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ be the dual spaces of $\mathcal{E}(X)$ and $\mathcal{D}(X)$, the space of distributions and the space of distributions with compact support. For their topologies we refer, again, to [12, Chap. II Sect. 2]. Let $\mathcal{E}'_0(X)$ and $\mathcal{D}'_0(X)$ be the corresponding subspaces of radial distributions. The spherical projector and the convolution are continuous and $\mathcal{D}_0(X)$ embeds canonically into $\mathcal{E}'_0(X)$ via $f \mapsto T_f$, where $\langle T_f, g\rangle := \int_X f(x)g(x)dx$. For $T \in \mathcal{E}'(X)$ and $f \in \mathcal{E}_0(X)$, $T * f$ can be interpreted as a function in $\mathcal{E}(X)$, i.e.,

$$T * f(x) = \langle T, f_x\rangle.$$

Since $\mathcal{D}_0(X)$ lies dense in $\mathcal{E}'_0(X)$ (this follows by using a Dirac sequence $\rho_\epsilon \in \mathcal{D}_0(X)$ and $T * \rho_\epsilon \to T$), all the above properties for functions carry over to distributions (as, for instance, the fact that the convolution of two radial distributions is radial, or the associativity of the convolution of radial distributions).

Since $\int_X g_y(z)f(z)dz = \int_X g(z)(\pi_y f)_{x_0}(z)$ for $g \in \mathcal{D}_0(X)$ and $f \in \mathcal{E}(X)$, the convolution of $T \in \mathcal{E}'_0(X)$ and $f \in \mathcal{E}(X)$ is given by

$$T * f(y) := \big\langle T, (\pi_y f)_{x_0}\big\rangle = \left\langle T, z \mapsto \frac{1}{\mathrm{vol}(S_{r(z)}(y))} \int_{S_{r(z)}(y)} f(x)dx \right\rangle. \tag{9}$$

As an example, consider the distribution $T_r \in \mathcal{E}'_0(X)$, given by $\langle T_r, f\rangle = \int_{S_r} f$. If $f \in \mathcal{E}(X)$ we obtain

$$T_r * f(y) = \left\langle T_r, z \mapsto \frac{1}{\mathrm{vol}(S_{r(z)}(y))} \int_{S_{r(z)}(y)} f \right\rangle = \int_{S_r(y)} f. \tag{10}$$

The convolution of two radial distributions $S, T \in \mathcal{E}'_0(X)$ lies in $\mathcal{E}'_0(X)$ and can be written as follows: If $f \in \mathcal{E}_0(X)$, we have

$$\langle S * T, f\rangle = \big\langle S, x \mapsto \langle T, f_x\rangle\big\rangle.$$

For general $f \in \mathcal{E}(X)$, we have

$$\langle S * T, f \rangle = \langle S * T, \pi f \rangle \ \bigl(= \bigl\langle S, x \mapsto \bigl\langle T, (\pi f)_x \bigr\rangle \bigr\rangle \bigr).$$

Proposition 2.10 *For $T \in \mathcal{E}_0'(X)$ and $f \in \mathcal{E}(X)$ we have*

$$\pi(T * f) = T * (\pi f).$$

Proof In view of (9), the claim means that

$$\begin{aligned}\bigl[\pi(T * f)\bigr](y) &= \pi\bigl[u \mapsto \bigl\langle T, z \mapsto (\pi_u f)_{x_0}(z)\bigr\rangle\bigr](y)\\ &= \bigl\langle T, z \mapsto \pi\bigl[u \mapsto (\pi_u f)_{x_0}(z)\bigr](y)\bigr\rangle\end{aligned}$$

is equal to

$$\bigl[T * (\pi f)\bigr](y) = \bigl\langle T, (\pi_y \pi f)_{x_0}\bigr\rangle$$

for all $y \in X$. To see this, we show that for all $y, z \in X$ we have

$$\pi\bigl[u \mapsto (\pi_u f)_{x_0}(z)\bigr](y) = \bigl[(\pi_y \pi f)_{x_0}\bigr](z). \tag{11}$$

We first show (11) if f is an eigenfunction of the Laplacian. So assume

$$f \in \mathcal{E}(X) \quad \text{with } \Delta f + \mu f = 0.$$

Then $\Delta \pi_u f + \mu \pi_u f = 0$ and $\pi_u f \in \mathcal{E}(X, u)$ is radial. Choose $\lambda \in \mathbb{C}$ such that $\mu = \lambda^2 + H^2/4$. By the uniqueness of the radial eigenfunctions we get

$$(\pi_u f)_{x_0} = f(u)\bigl((\varphi_\lambda)_u\bigr)_{x_0} = f(u)\varphi_\lambda.$$

Therefore,

$$\begin{aligned}\pi\bigl[u \mapsto (\pi_u f)_{x_0}(z)\bigr](y) &= \pi\bigl[u \mapsto f(u)\varphi_\lambda(z)\bigr](y)\\ &= \varphi_\lambda(z)[\pi f](y) = \varphi_\lambda(z)\varphi_\lambda(y) f(x_0)\end{aligned}$$

and

$$\bigl[(\pi_y \pi f)_{x_0}\bigr](z) = \bigl[\bigl(\pi_y\bigl[f(x_0)\varphi_\lambda\bigr]\bigr)_{x_0}\bigr](z) = f(x_0)\varphi_\lambda(y)\varphi_\lambda(z).$$

In order to show (11) for arbitrary functions f, note that for fixed $y, z \in X$ the values of both sides of (11) depend on the restriction of f to a compact subset $K \subset X$ only. Let K' be a compact subset of X with smooth boundary containing K in its interior. Since f is smooth, we find linear combinations of Dirichlet eigenfunctions of the Laplacian on K' approximating f uniformly on K. Since both sides of (11) are continuous in f with respect to uniform convergence, this establishes (11) for all functions f. □

3 Abel and Spherical Fourier Transformation

We first introduce some fundamental notions. Let SX be the unit tangent bundle of (X, g). The *Busemann function* associated with a unit tangent vector $v_0 \in S_{x_0}X$ is defined by

$$b(x) = b_{v_0}(x) := \lim_{s\to\infty}\bigl(d\bigl(c(s), x\bigr) - s\bigr),$$

where $c : \mathbb{R} \to X$ is the geodesic with $c(0) = x_0$, $c'(0) = v_0$. $\Delta b = H$ implies that b is an analytic function. The level sets of b are smooth hypersurfaces and are called *horospheres*. They are denoted by

$$\mathcal{H}_s := b^{-1}(s).$$

These horospheres foliate X and we have $x_0 \in \mathcal{H}_0$. We also need the smooth unit vector field

$$N(x) = -\operatorname{grad} b(x),$$

orthogonal to the horospheres $\mathcal{H}_s$ and satisfying $N(x_0) = v_0$. We choose an orientation of $\mathcal{H}_0$ and orientations of $\mathcal{H}_s$ such that the diffeomorphisms

$$\Psi_s : \mathcal{H}_0 \to \mathcal{H}_s, \quad \Psi_s(x) := \exp_x\bigl(-sN(x)\bigr),$$

are orientation preserving. Since HN is the mean curvature vector of the horospheres and thus the variation field of the area functional we have

Proposition 3.1

$$(\Psi_s)^*\omega_s = e^{sH}\omega_0.$$

We combine the diffeomorphisms $\Psi_s : \mathcal{H}_0 \to \mathcal{H}_s$ to construct a global diffeomorphism

$$\Psi : \mathbb{R} \times \mathcal{H}_0 \to X, \quad \Psi(s, x) := \Psi_s(x).$$

We have $D\Psi(\frac{\partial}{\partial s}) = -N$.

We choose an orientation on $\mathbb{R} \times \mathcal{H}_0$ such that every oriented base $v_1, \dots, v_n$ of $\mathcal{H}_0$ induces an oriented base $\frac{\partial}{\partial s}, v_1, \dots, v_n$ on $\mathbb{R} \times \mathcal{H}_0$. This yields also an orientation on X by requiring that Ψ is orientation preserving. An immediate consequence of Proposition 3.1 is

Corollary 3.2 *Let ω denote the volume form of the harmonic space (X, g). Then we have*

$$\Psi^*\omega = e^{sH} ds \wedge \omega_0.$$

Next, we fix a unit vector $v_0 \in S_{x_0}X$, and denote the associated Busemann function b_{v_0} by b, for simplicity. We first consider the following important transform:

Definition 3.3 Let $j : \mathcal{E}_0(\mathbb{R}) \to \mathcal{E}(X)$ be defined as

$$(jf)(x) = e^{-\frac{H}{2}b(x)} f\big(b(x)\big).$$

The transformation $a : \mathcal{E}_0(\mathbb{R}) \to \mathcal{E}_0(X)$ is then defined as

$$a = \pi \circ j.$$

The *Abel transform* $\mathcal{A} : \mathcal{E}_0'(X) \to \mathcal{E}_0'(\mathbb{R})$ is defined as the dual of a, i.e., we have for all $T \in \mathcal{E}_0'(X)$ and $f \in \mathcal{E}_0(\mathbb{R})$:

$$\langle \mathcal{A}T, f\rangle_{\mathbb{R}} = \langle T, af\rangle_X.$$

The functions $\psi_\lambda(s) = \frac{1}{2}(e^{i\lambda s} + e^{-i\lambda s}) = \cos(\lambda s)$ are the radial eigenfunctions of the Laplacian $\Delta f = f''$ on the real line.

Lemma 3.4 *We have*

$$a\psi_\lambda = \varphi_\lambda.$$

Proof We first observe that, under the diffeomorphism $\Psi : \mathbb{R} \times \mathcal{H}_0 \to X$, the Laplacian has the form

$$\Delta = \frac{\partial^2}{\partial^2 s} + H\frac{\partial}{\partial s} + A_s, \tag{12}$$

where A_s is a differential operator with derivatives tangent to $\mathcal{H}_0$. Consequently, the functions $f_\alpha = e^{-\alpha b}$ are eigenfunctions of Δ with

$$\Delta f_\alpha = -\alpha(H - \alpha) f_\alpha.$$

Choosing $\alpha = \frac{H}{2} \pm i\lambda$, we obtain

$$\Delta f_\alpha = -\left(\lambda^2 + \frac{H^2}{4}\right) f_\alpha,$$

and, by uniqueness, πf_α must be a multiple of φ_λ. Since $f_\alpha(x_0) = 1$, we conclude that $\varphi_\lambda = \pi f_\alpha$. Let $\alpha_\pm = \frac{H}{2} \pm i\lambda$. Then one easily checks that

$$j\psi_\lambda = \frac{1}{2}\big(e^{-\alpha_- b} + e^{-\alpha_+ b}\big),$$

and, consequently,

$$a\psi_\lambda = \frac{1}{2}(\pi f_{\alpha_-} + \pi f_{\alpha_+}) = \varphi_\lambda. \qquad \square$$

Let $\psi_{\lambda,k}(s) = \frac{d^k}{d\lambda^k}\psi_\lambda(s) = \frac{s^k}{2}(i^k e^{i\lambda s} + (-i)^k e^{-i\lambda s})$ and $\varphi_{\lambda,k} = \frac{d^k}{d\lambda^k}\varphi_\lambda$. Lemma 3.4 implies that we also have

$$a\psi_{\lambda,k} = \varphi_{\lambda,k}, \tag{13}$$

for all $k \geq 1$.

Proposition 3.5 *The Abel transform of a function* $f \in \mathcal{D}_0(X) \subset \mathcal{E}_0'(X)$ *is* $\mathcal{A}f \in \mathcal{D}_0(\mathbb{R}) \subset \mathcal{E}_0'(\mathbb{R})$ *given by*

$$(\mathcal{A}f)(s) = e^{-\frac{H}{2}s} \int_{\mathcal{H}_s} f(z)\omega_s(z) = e^{\frac{H}{2}s} \int_{\mathcal{H}_0} f\big(\Psi_s(z)\big)\omega_0(z). \tag{14}$$

Proof Let $f \in \mathcal{D}_0(X)$ and

$$g(s) = e^{-\frac{H}{2}s} \int_{\mathcal{H}_s} f(z)\omega_s(z).$$

Since f has compact support, there is $T > 0$ such that $\mathcal{H}_s \cap \operatorname{supp} f = \emptyset$ for all $|s| \geq T$, i.e., g also has compact support. Moreover, by Proposition 3.1, we obtain

$$\begin{aligned} g(s) &= e^{-\frac{H}{2}s} \int_{\Psi_s(\mathcal{H}_0)} f(z)\omega_s(z) = e^{-\frac{H}{2}s} \int_{\mathcal{H}_0} f\big(\Psi_s(z)\big)\big(\Psi_s^*\omega_s\big)(z) \\ &= e^{\frac{H}{2}s} \int_{\mathcal{H}_0} f\big(\Psi_s(z)\big)\omega_0(z). \end{aligned}$$

Next, we show $\langle g, h\rangle = \langle f, ah\rangle$ for all $h \in \mathcal{E}_0(\mathbb{R})$:

$$\begin{aligned} \langle g, h\rangle &= \int_{-\infty}^{\infty} g(s)h(s)ds = \int_{-\infty}^{\infty} e^{-\frac{H}{2}s}h(s) \int_{\mathcal{H}_s} f(z)\omega_s(z)ds \\ &= \int_{-\infty}^{\infty}\int_{\mathcal{H}_s} f(z)e^{-\frac{H}{2}b(z)}h\big(b(z)\big)\omega_s(z)ds = \int_X f(z)e^{-\frac{H}{2}b(z)}h\big(b(z)\big)dz \\ &= \langle f, jh\rangle = \langle \pi f, jh\rangle = \langle f, ah\rangle. \end{aligned}$$

Finally, we show that g is an even function: Using $\langle g, h\rangle = \langle f, ah\rangle$ and Lemma 3.4, we derive

$$\int_{-\infty}^{\infty} g(s)e^{\pm i\lambda s}ds = \langle f, \varphi_\lambda\rangle,$$

which implies that

$$\int_{-\infty}^{\infty} e^{i\lambda s}\big(g(s) - g(-s)\big)ds = 0,$$

for all $\lambda \in \mathbb{C}$. This yields $g(s) = g(-s)$, i.e., g is an even function. This finishes the proof of the proposition. □

Lemma 3.6 *For* $f \in \mathcal{E}_0'(X)$, *we have*

$$\mathcal{A}(\Delta f) = \left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)\mathcal{A}f.$$

Proof Let $u \in \mathcal{E}_0(\mathbb{R})$. Since $ju \in \mathcal{E}(X)$ is constant on the horospheres, from (12) we compute

$$[\Delta ju](x) = \left(\frac{\partial^2}{\partial s^2} + H\frac{\partial}{\partial s}\right)e^{-sH/2}u(s)\Big|_{s=b(x)} = \left[j\left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)u\right](x);$$

hence,

$$\begin{aligned}\left\langle \mathcal{A}(\Delta f), u\right\rangle &= \langle \Delta f, au\rangle = \langle f, \Delta\pi ju\rangle = \langle f, \pi\Delta ju\rangle \\ &= \left\langle f, \pi j\left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)u\right\rangle = \left\langle f, a\left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)u\right\rangle \\ &= \left\langle \mathcal{A}f, \left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)u\right\rangle = \left\langle \left(\frac{d^2}{ds^2} - \frac{H^2}{4}\right)\mathcal{A}f, u\right\rangle. \end{aligned}$$

□

The next result will be used in the proof of Theorem 3.8 below; namely, to establish the local injectivity of $a : \mathcal{E}_0(\mathbb{R}) \to \mathcal{E}_0(X)$.

Lemma 3.7 *Let* $g \in \mathcal{D}(\mathbb{R})$. *Then the Klein–Gordon equation*

$$\frac{\partial^2}{\partial t^2}v(t,s) = \frac{\partial^2}{\partial s^2}v(t,s) - \frac{H^2}{4}v(t,s) \tag{15}$$

$$v(0,s) = g(s) \quad \textit{and} \quad \frac{\partial}{\partial t}v(0,s) = 0 \tag{16}$$

has a solution of the form

$$v(t,s) = \frac{g(s-t) + g(s+t)}{2} + \int_{s-t}^{s+t} W\big(t, s-s'\big)g\big(s'\big)ds', \tag{17}$$

with $W \in \mathcal{E}(\mathbb{R}^2)$. *This solution is unique in the sense that if* $\tilde{v}$ *is another solution of* (15) *and* (16) *and so that for all* t *the function* $\tilde{v}_t : s \mapsto \tilde{v}(t,s)$ *has compact support, then* $\tilde{v} = v$.

Proof The function W is explicitly given by

$$W(t,s) = t\sum_{k=0}^{\infty}\left(\frac{-H^2}{16}\right)^{k+1}\frac{(t^2-s^2)^k}{k!(k+1)!}, \tag{18}$$

but we will only need that W is smooth. A straightforward computation shows that (17) actually solves (15) and (16). The function W is even in the second argument s, and solves the equations

$$W(t,t) = -\frac{H^2}{16}t, \qquad W_{tt} = W_{ss} - \frac{H^2}{4}W.$$

Uniqueness of the solution follows from conservation of energy as, for instance, in [23, p. 145]. To see this, assume that $\tilde{v}$ is another solution of (15) and (16). Then the difference $\omega = v - \tilde{v}$ solves (15) and (16) with g replaced by 0. We now look at the energy

$$E_\omega(t) := \int_{-\infty}^{\infty} \omega_s(t,s)^2 + \omega_t(t,s)^2 + \frac{H^2}{4}\omega(t,s)^2 ds,$$

and compute, integrating by parts,

$$\begin{aligned}\frac{d}{dt}E_\omega(t) &= 2\int_{-\infty}^{\infty} \omega_s\omega_{st} + \omega_{tt}\omega_t + \frac{H^2}{4}\omega_t\omega ds\\ &= 2\int_{-\infty}^{\infty} -\omega_{ss}\omega_t + \omega_{tt}\omega_t + \frac{H^2}{4}\omega\omega_t ds\\ &= 2\int_{-\infty}^{\infty} \left(-\omega_{ss} + \omega_{tt} + \frac{H^2}{4}\omega\right)\omega_t ds = 0,\end{aligned}$$

because ω satisfies (15). Since $E_\omega(0) = 0$ we have $E_\omega(t) = 0$ for all t, which forces $\omega = 0$. □

Theorem 3.8 *The maps $a : \mathcal{E}_0(\mathbb{R}) \to \mathcal{E}_0(X)$ and $\mathcal{A} = a'$ are topological isomorphisms.*

Proof We first show local injectivity of a. For all $R \geq 0$ the map a induces a well-defined map

$$a^R\colon \mathcal{E}_0\big([-R,R]\big) \to \mathcal{E}_0\big(B_R(x_0)\big),$$

i.e., for $x \in X$ the value of $au(x)$ depends only on the restriction of u to $[-r(x), r(x)]$. Local injectivity now is the fact that for all $R \geq 0$, the maps a^R are injective. Thus for $u \in \mathcal{E}_0(\mathbb{R})$ we have

$$au|_{B_R(x_0)} = 0 \quad\Longrightarrow\quad u|_{[-R,R]} = 0.$$

The proof is based on the fact that $\mathcal{A}\colon \mathcal{E}_0'(X) \to \mathcal{E}_0'(\mathbb{R})$ transforms the fundamental solution of the radial wave equation to the fundamental solution of the Klein–Gordon equation. Since we need $\mathcal{A}$ on compactly supported distributions, finite propagation speed of the solution of the wave equation is essential here.

For all $\epsilon > 0$ we choose a function $q_\epsilon \in \mathcal{E}_0(X)$ so that

$$q_\epsilon(x) = 0 \quad \text{if } r(x) > \epsilon \quad \text{and} \quad \int_X q_\epsilon = 1.$$

Let $w \in \mathcal{E}_0(\mathbb{R}\times X)$, $w = w(t, r(x)) = w_t(r(x))$, be the solution of the wave equation starting with q_ϵ, i.e.,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}w\big(t,r(x)\big) = \Delta_x w_t\big(r(x)\big) &= \frac{\partial^2}{\partial r^2}w\big(t,r(x)\big) + \frac{\theta'}{\theta}\big(r(x)\big)\frac{\partial}{\partial r}w\big(t,r(x)\big),\\ w(0,r) &= q_\epsilon(r),\\ \frac{\partial}{\partial t}w(0,r) &= 0.\end{aligned}$$

By the finite propagation speed of the wave equation, the support of w_t is compact (in fact, contained in $B_{\epsilon+t}(0)$).

Let $v \in \mathcal{E}(\mathbb{R}^2)$ be so that $v_t := \mathcal{A}w_t$, i.e.,

$$v(t,s) = e^{-sH/2} \int_{\mathcal{H}_s} w\big(t, r(x)\big) d\omega_s(x).$$

Then by Lemma 3.6

$$\frac{\partial^2}{\partial t^2} v_t = \frac{\partial^2}{\partial t^2} \mathcal{A} w_t = \mathcal{A} \frac{\partial^2}{\partial t^2} w_t = \mathcal{A}\Delta w_t = \left(\frac{\partial^2}{\partial s^2} - \frac{H^2}{4}\right) v_t.$$

It follows that v solves the Klein–Gordon equation (15) with initial conditions (in place of (16))

$$v_0 = g_\epsilon := \mathcal{A} q_\epsilon \text{ and } \frac{\partial}{\partial t} v(0,s) = 0.$$

If $|s| > \epsilon$ then $v(0,s) = g_\epsilon(s) = 0$. Also

$$\int_{-\infty}^{\infty} e^{sH/2} g_\epsilon(s) ds = \int_{-\infty}^{\infty} \int_{\mathcal{H}_s} q_\epsilon\big(r(x)\big) d\omega_s ds = \int_X q_\epsilon = 1.$$

Now, to prove local injectivity of a, let $R \geq 0$ and $u \in \mathcal{E}_0(\mathbb{R})$ with $au|_{B_R(x_0)} = 0$. For all $t \in [-(R-\epsilon), R-\epsilon]$ we then have from (17)

$$\begin{aligned} 0 &= \langle w_t, au \rangle = \langle \mathcal{A} w_t, u \rangle = \langle v_t, u \rangle \\ &= \int_{-\infty}^{\infty} \frac{g_\epsilon(s-t) + g_\epsilon(s+t)}{2} u(s) ds + \int_{-\infty}^{\infty} \int_{s-t}^{s+t} W\big(t, s-s'\big) g_\epsilon\big(s'\big) ds' u(s) ds. \end{aligned}$$

Since W, u and g_ϵ are smooth, we can take the limit $\epsilon \to 0$ here to get the identity

$$\frac{u(t) + u(-t)}{2} = -\int_{-t}^{t} W(t,s) u(s) ds.$$

Since u and W are even (in s), we can write this as a fixed-point equation,

$$u(t) = -2 \int_0^t W(t,s) u(s) ds.$$

Since $\pi f(x_0) = f(x_0)$ for all $f \in \mathcal{E}(X)$ we have $u(0) = 0$. Let

$$M := \max_{t \in [0,R], s \in [0,t]} \big|W(t,s)\big|.$$

Hence, if $u|_{[0,R]} \neq 0$ there is some $T \in [0,R]$ with the following properties: (i) $u(T) \neq 0$ and (ii) $T < \frac{1}{4M}$ or $u|_{[0, T - \frac{1}{4M}]} = 0$. Now for all $t \in [0,T]$ we estimate

$$\begin{aligned}|u(t)| &\leq 2\int_0^t |W(t,s)||u(s)|ds \\ &\leq 2\int_{\max\{0,t-\frac{1}{4M}\}}^t |W(t,s)||u(s)|ds \leq \frac{1}{2}\sup_{t'\in[0,T]}|u(t')|,\end{aligned}$$

contradicting our assumption about T and $u(T) \neq 0$.

Now we prove surjectivity of a: Let $f \in \mathcal{E}_0(X)$ and $R > 0$ be fixed. Choose a function $\phi \in \mathcal{D}_0(X)$ with $\phi = 1$ on $B_R(x_0)$, $0 \leq \phi \leq 1$ on X, and $\phi = 0$ on $X \backslash B_{R+1}(x_0)$. We will first show that $\phi f|_{B_R(x_0)}$ is in $a(\mathcal{E}_0(\mathbb{R}))|_{B_R(x_0)}$.

Let φ_k be an orthonormal basis of Dirichlet eigenfunctions of the Laplacian on $B_{R+1}(x_0) \subset X$ with corresponding eigenvalues $0 \leq \mu_k \nearrow \infty$. We have $\pi\varphi_k = \varphi_k(x_0)\varphi_{\lambda_k}$ with $\lambda_k \in \mathbb{C}$ such that $\mu_k = \lambda_k^2 + \frac{H^2}{4}$. Let

$$\phi f = \sum_{k=0}^{\infty} a_k \varphi_k$$

be the Fourier expansion of ϕf. Therefore,

$$\phi f = \pi(\phi f) = \sum_{k=0}^{\infty} a_k \varphi_k(x_0)\varphi_{\lambda_k}.$$

Our first goal is to show that the series

$$g_R(s) = \sum_{k=0}^{\infty} a_k \varphi_k(x_0)\cos(\lambda_k s)$$

converges uniformly with all its derivatives to a smooth function $g_R \in \mathcal{E}_0([-R, R])$. Then we have $a^R(g_R) = \phi f|_{B_R(x_0)}$, by the continuity of a. We prove this by showing that

$$\sum_{k=0}^{\infty} |a_k||\varphi_k(x_0)||\lambda_k|^m < \infty \quad \text{for all } m \in \mathbb{N}. \tag{19}$$

By the Sobolev imbedding theorem, there is a constant C_0 such that for all $u \in C(B_{R+1}(x_0))$ we have

$$\|u\|_\infty \leq C_0\big(\|u\|_2 + \|\Delta^{n+1}u\|_2\big),$$

where $\|\cdot\|_\infty$ denotes the supremum norm and $\|\cdot\|_2$ denotes the L^2-norm, and $n+1$ is the dimension of X. This implies that

$$\|\varphi_k(x_0)| \leq \|\varphi_k\|_\infty \leq C_0\big(1 + \mu_k^{n+1}\big).$$

By Weyl's law, the eigenvalues μ_k grow with an exponent $2/(n+1)$, which implies that there is a $k_0 \in \mathbb{N}$ and a $C_1 > 0$ such that, for all $k \geq k_0$:

$$|\varphi_k(x_0)| \leq \|\varphi_k\|_\infty \leq C_1 k^2. \tag{20}$$

The Fourier expansion of $\Delta^\nu(\phi f)$ is given by

$$\Delta^\nu(\phi f) = \sum_{k=0}^{\infty} a_k \mu_k^\nu \varphi_k,$$

and, since $\Delta^\nu(\phi f) \in L^2(B_{R+1}(x_0))$, we have

$$\sum_{k=1}^{\infty} |a_k|^2 |\mu_k|^{2\nu} < \infty,$$

which implies that

$$\sum_{k=1}^{\infty} |a_k|^2 |\lambda_k|^{4\nu} < \infty, \tag{21}$$

for every $\nu \in \mathbb{N}$. We have

$$\begin{aligned}
\sum_{k=0}^{\infty} |a_k||\varphi_k(x_0)||\lambda_k|^m &\overset{(20)}{\leq} C_1 \sum_{k=0}^{\infty} |a_k|\, k^2\, |\lambda_k|^m \\
&= C_1 \sum_{k=0}^{\infty} \left(|a_k|\, k^2\, |\lambda_k|^{m+l}\right) |\lambda_k|^{-l} \\
&\leq C_1 \left(\sum_{k=0}^{\infty} |a_k|^2 k^4\, |\lambda_k|^{2m+2l} \right)^{1/2} \left(\sum_{k=0}^{\infty} |\lambda_k|^{-2l} \right)^{1/2} \\
&\overset{\text{Weyl}}{\leq} C_2 \left(\sum_{k=0}^{\infty} |a_k|^2\, |\lambda_k|^{2(m+l+2n+2)} \right)^{1/2} \left(\sum_{k=0}^{\infty} |\lambda_k|^{-2l} \right)^{1/2}.
\end{aligned}$$

The required finiteness (19) now follows from (21) and Weyl's law, for the choice $l = n + 1$.

Hence g_R defines a smooth function with $a^R g_R = f$ on $B_R(x_0)$. Now for a given $f \in \mathcal{E}_0(X)$ and each $N \in \mathbb{N}$, construct a function $g_N \in \mathcal{E}_0([-N, N])$ as above. We will have $a^N(g_N) = f|_{B_N(x_0)}$. By local injectivity of a, $g_{N+1}|_{[-N,N]} = g_N$, and the functions g_N patch together to define a function $g \in \mathcal{E}_0(\mathbb{R})$ with $a(g) = f$.

This shows that a is a bijective linear continuous map. By the open mapping theorem [24, Thm. 17.1], a is a topological isomorphism. Using the corollary of Proposition 19.5 in [24], we conclude that its dual $\mathcal{A} : \mathcal{E}_0'(X) \to \mathcal{E}_0'(\mathbb{R})$ is also a topological isomorphism. □

Definition 3.9 The *spherical Fourier transformation* $\mathcal{F}T$ of a radial distribution $T \in \mathcal{E}_0'(X)$ is the function $\mathcal{F}T : \mathbb{C} \to \mathbb{C}$ with

$$\mathcal{F}T(\lambda) = \langle T, \varphi_\lambda \rangle \quad \text{for all } \lambda \in \mathbb{C}.$$

Next, we will see that there is a close relationship between the Abel transform $\mathcal{A}$ and the spherical Fourier transform $\mathcal{F}$. By the classical Paley–Wiener theorem for distributions (see, e.g., [8, p. 211] and [9, Thm. 5.19]), the Euclidean Fourier transform $\mathcal{E}_0'(\mathbb{R}) \ni S \mapsto \hat{S}$, with $\hat{S}(\lambda) := \langle S, \psi_\lambda \rangle$ is a topological isomorphism $\mathcal{E}_0'(\mathbb{R}) \to \mathbf{E}_0'$, where $\mathbf{E}_0'$ is the space of all even entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type which are polynomially bounded on $\mathbb{R}$, endowed with a suitable topology.

Proposition 3.10 *We have*

$$\widehat{\mathcal{A}T} = \mathcal{F}T.$$

Proof We have for $\lambda \in \mathbb{C}$,

$$\widehat{\mathcal{A}T}(\lambda) = \langle \mathcal{A}T, \psi_\lambda \rangle = \langle T, a\psi_\lambda \rangle = \langle T, \varphi_\lambda \rangle = \mathcal{F}T(\lambda).$$ □

Proposition 3.11 *For $S, T \in \mathcal{E}_0'(X)$ we have*

$$\mathcal{A}(S * T) = \mathcal{A}S * \mathcal{A}T.$$

Proof Note that $\mathcal{A}S, \mathcal{A}T, \mathcal{A}(S * T) \in \mathcal{E}_0'(\mathbb{R})$. It was shown in [24, formula (30.1)] that we have, for these distributions on the real line,

$$\widehat{\mathcal{A}S} \cdot \widehat{\mathcal{A}T} = \widehat{\mathcal{A}S * \mathcal{A}T}.$$

We now show that $\widehat{\mathcal{A}(S * T)} = \widehat{\mathcal{A}S} \cdot \widehat{\mathcal{A}T}$:

$$\begin{aligned}
\widehat{\mathcal{A}(S * T)}(\lambda) &= \big\langle \mathcal{A}(S * T), \psi_\lambda \big\rangle = \langle S * T, \varphi_\lambda \rangle \\
&= \big\langle S, x \mapsto \big\langle T, (\varphi_\lambda)_x \big\rangle\big\rangle = \big\langle S, x \mapsto \big\langle T, \pi\big((\varphi_\lambda)_x\big)\big\rangle\big\rangle \\
&= \big\langle S, x \mapsto \big\langle T, \varphi_\lambda(x)\varphi_\lambda \big\rangle\big\rangle = \big\langle S, \langle T, \varphi_\lambda \varphi_\lambda \rangle\big\rangle \\
&= \langle S, a\psi_\lambda \rangle \langle T, a\psi_\lambda \rangle = \langle \mathcal{A}S, \psi_\lambda \rangle \cdot \langle \mathcal{A}T, \psi_\lambda \rangle \\
&= \widehat{\mathcal{A}S}(\lambda) \cdot \widehat{\mathcal{A}T}(\lambda).
\end{aligned}$$

Putting both results together, we conclude that

$$\widehat{\mathcal{A}(S * T)} = \widehat{\mathcal{A}S * \mathcal{A}T}.$$

Since the Euclidean Fourier transform $\mathcal{E}_0'(\mathbb{R}) \to \mathbf{E}_0'$, $T \mapsto \hat{T}$, is a topological isomorphism, we finally obtain

$$\mathcal{A}(S * T) = \mathcal{A}S * \mathcal{A}T,$$

finishing the proof. □

An immediate consequence of Paley–Wiener for radial distributions in Euclidean space, Theorem 3.8, and Propositions 3.10 and 3.11 is the following:

Theorem 3.12 (Paley–Wiener for radial distributions) *The spherical Fourier transform*

$$\mathcal{F}T(\lambda) = \langle T, \varphi_\lambda \rangle$$

defines a topological isomorphism

$$\mathcal{F} : \mathcal{E}_0'(X) \to \mathbf{E}_0'.$$

Furthermore, for radial distributions $S, T \in \mathcal{E}_0'(X)$, *we have*

$$\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T.$$

Proposition 3.13 *The following diagram commutes*:

$$\begin{array}{ccc} \mathcal{E}_0'(X) \times \mathcal{E}_0(X) & \xrightarrow{*_X} & \mathcal{E}_0(X) \\ \Big\downarrow{\scriptstyle \mathcal{A}\times a^{-1}} & & \Big\downarrow{\scriptstyle a^{-1}} \\ \mathcal{E}_0'(\mathbb{R}) \times \mathcal{E}_0(\mathbb{R}) & \xrightarrow{*_\mathbb{R}} & \mathcal{E}_0(\mathbb{R}) \end{array} \tag{22}$$

Proof Since $a : \mathcal{E}_0(\mathbb{R}) \to \mathcal{E}_0(X)$ and $\mathcal{A} : \mathcal{E}_0'(X) \to \mathcal{E}_0'(\mathbb{R})$ are topological isomorphisms, in view of Proposition 3.11, it only remains to show that $a(\mathcal{A}T *_\mathbb{R} f) = T *_X af$: For $g, h \in \mathcal{D}_0(X)$ we obtain

$$\begin{aligned} \langle g * af, h \rangle &= \big((g * af) * h\big)(x_0) = \big((af * g) * h\big)(x_0) \\ &= \big(af * (g * h)\big)(x_0) = \langle af, g * h \rangle = \langle f, \mathcal{A}g * \mathcal{A}h \rangle \\ &= \big(f * (\mathcal{A}g * \mathcal{A}h)\big)(x_0) = \big((\mathcal{A}g * f) * \mathcal{A}h\big)(x_0) \\ &= \langle \mathcal{A}g * f, \mathcal{A}h \rangle = \big\langle a(\mathcal{A}g * f), h \big\rangle. \end{aligned}$$

Since $\mathcal{D}_0(X)$ is dense in $\mathcal{E}_0'(X)$ and $a, \mathcal{A}$ are continuous, we conclude the required identity. □

4 Spectral Analysis/Synthesis and Two Radius Theorems

In this section, we discuss the proofs of the integral geometric results mentioned in the Introduction. Since the proofs are very similar to the ones given in [20] for Damek–Ricci spaces, we give the ideas and outlines of the proofs, and refer to that paper for more details.

The following proposition is a consequence of the holomorphicity of the map $\lambda \mapsto \varphi_\lambda(r)$, and guarantees that the integral geometric results hold for two *generic radii*, as claimed in the Introduction.

Proposition 4.1 *For each $r_1 > 0$ there is an at most countable set of $r_2 > 0$ such that there exists $\lambda \in \mathbb{C}$ with $\varphi_\lambda(r_1) = \varphi_\lambda(r_2) = 0$. In particular, the set*

$$\left\{(r_1, r_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \forall \lambda \in \mathbb{C} : \phi_\lambda(r_1) \neq 0 \text{ or } \phi_\lambda(r_2) \neq 0\right\}$$

is generic.

Proof Let $r_1 > 0$. By Proposition 2.3, the set $S_{r_1} = \{\lambda \in \mathbb{C} \mid \varphi_\lambda(r_1) = 0\}$ is at most countable. The zero set $\varphi_\lambda^{-1}(0)$ of φ_λ, is also at most countable because φ_λ satisfies the differential equation (6) $\varphi_\lambda'' = -(\lambda^2 + \frac{H^2}{4})\varphi_\lambda - \frac{\theta'}{\theta}\varphi_\lambda$, the solution of which is determined by the values of $\varphi_\lambda(r_0)$ and $\varphi_\lambda'(r_0)$ for any $r_0 > 0$. Thus φ_λ cannot have a limit point of zeros. It follows that the set

$$\left\{r_2 > 0 \mid \exists \lambda \in \mathbb{C} : \varphi_\lambda(r_1) = \varphi_\lambda(r_2) = 0\right\} = \bigcup_{\lambda \in S_{r_1}} \varphi_\lambda^{-1}(0)$$

is an at most countable union of at most countable sets, hence itself at most countable. □

The same reasoning applies to the function $\varphi_\lambda - 1$ (we must exclude here $\lambda = \pm i\frac{H}{2}$, since then $\varphi_\lambda \equiv 1$), and the function Φ_λ given by

$$\Phi_\lambda(r) = \int_0^r \theta(\rho)\varphi_\lambda(\rho)d\rho \tag{23}$$

where θ denotes the volume density function.

Analogously to [20], we have *spectral analysis* and *spectral synthesis* in $\mathcal{E}_0(X)$. Let us briefly explain this.

A *variety* $V \subset \mathcal{E}_0(X)$ is a closed subspace satisfying $\mathcal{E}_0'(X) * V \subset V$, which is proper ($V \neq \mathcal{E}_0(X)$) and contains a non-zero function.

It follows from Propositions 3.13 and 3.11, together with the isomorphism Theorem 3.8, that the transformation a maps varieties of $\mathcal{E}_0(\mathbb{R})$ to varieties of $\mathcal{E}_0(X)$, and that every variety in $\mathcal{E}_0(X)$ is of the form $a(W)$ with a variety $W \subset \mathcal{E}_0(\mathbb{R})$. By Schwartz's theorem on varieties in $\mathcal{E}_0(\mathbb{R})$ (see Thm. 2.4 in [20]), we have

$$W = \overline{\text{span}\{\psi_{\lambda,k} \in W\}},$$

that is, each variety is the closure of the span of its spectrum, where the spectrum of a variety is the set of all those $\psi_{\lambda,k}$ contained in V. Because of (13) this carries over to X, i.e., we have for any variety $V \subset \mathcal{E}_0(X)$ that

$$V = \overline{\text{span}\{\phi_{\lambda,k} \in V\}}.$$

This property is called *spectral synthesis*. For varieties $V \subset \mathcal{E}_0(X)$, it was shown in [20, Lemma 4.2] that $\varphi_{\lambda,k} \in V$ implies $\varphi_{\lambda,l} \in V$, for all $0 \leq l \leq k$. There X was a Damek–Ricci space, but the arguments carry over verbatim for general harmonic manifolds. Therefore, spectral synthesis implies that every variety $V \subset \mathcal{E}_0(X)$ contains a radial eigenfunction φ_λ. This latter property is called *spectral analysis*.

As an immediate application, we can prove the analogues of the Two-Radius theorems in [20] for general non-compact harmonic spaces.

Theorem 4.2 *Let (X, g) be a simply connected, non-compact harmonic manifold. Then we have the following facts.*

(1) *Let $r_1, r_2 > 0$ be such that the equations*

$$\varphi_\lambda(r_j) = 0, \quad j = 1, 2,$$

have no common solution $\lambda \in \mathbb{C}$. Suppose $f \in C(X)$ and

$$\int_{S_r(x)} f = 0$$

for $r = r_1, r_2$ and all $x \in X$. Then $f = 0$.

(2) *For $\lambda \in \mathbb{C}$, let Φ_λ be given by (23). Let $r_1, r_2 > 0$ be such that the equations*

$$\Phi_\lambda(r_j) = 0, \quad j = 1, 2,$$

have no common solution $\lambda \in \mathbb{C}$. Suppose $f \in C(X)$ and

$$\int_{B_r(x)} f = 0$$

for $r = r_1, r_2$ and all $x \in X$. Then $f = 0$.

Proof Let us start with the proof of the first assertion. Let $g \in C(X)$ be a *non-zero* function, satisfying $\int_{S_{r_i}(x)} g = 0$ for all $x \in X$ and $i = 1, 2$. We choose our reference point $x_0 \in X$ such that $g(x_0) \neq 0$. Consider the distributions $T_r \in \mathcal{E}_0'(X)$, given by $T_r f = \int_{S_r(x_0)} f$. First recall (10), namely, $\int_{S_r(x)} f = (T_r * f)(x)$. This implies that we have $T_{r_1} * g = T_{r_2} * g = 0$. Using the extension of Proposition 2.10 to continuous functions $f \in C(X)$, we conclude that $g_0 := \pi g$ also satisfies $T_{r_1} * g_0 = T_{r_2} * g_0 = 0$. Without loss of generality, we can assume that g_0 is a smooth radial function, since every continuous radial function g_0 can be approximated, uniformly on compact sets, by functions $g_0^\epsilon := g_0 * \rho_\epsilon \in \mathcal{E}_0(X)$ (via a Dirac sequence $\rho_\epsilon \in \mathcal{D}_0(X)$), such that we still have $T_{r_1} * g_0^\epsilon = T_{r_2} * g_0^\epsilon = 0$. Therefore, we can now assume that $g_0 \in \mathcal{E}_0(X)$. Then

$$\begin{aligned} V &= \left\{ f \in \mathcal{E}_0(X) \,\Big|\, \int_{S_{r_1}(x)} f = 0 = \int_{S_{r_2}(x)} f \text{ for all } x \in X \right\} \\ &= \left\{ f \in \mathcal{E}_0(X) \mid T_{r_1} * f = 0 = T_{r_2} * f \right\} \end{aligned}$$

contains g_0 and is a variety in $\mathcal{E}_0(X)$, since for all $T \in \mathcal{E}_0'(X)$ and all $f \in V$:

$$T_{r_i} * (T * f) = T * (T_{r_i} * f) = 0, \quad \text{with } i = 1, 2,$$

by commutativity and associativity of the convolution for radial distributions. By spectral analysis, V must contain a φ_λ. But $T_r(\varphi_\lambda) = \mathrm{vol}(S_r(x_0))\varphi_\lambda(r)$, hence we have $\varphi_\lambda(r_1) = 0 = \varphi_\lambda(r_2)$.

For the second assertion we work with the distributions $T_r \in \mathcal{E}_0'(X)$ given by $T_r f = \int_{B_r(x_0)} f$. As before, $T_r * f(x) = \int_{B_r(x)} f$ for all $x \in X$. Now, the proof proceeds as above, with the variety

$$V = \left\{ f \in \mathcal{E}_0(X) \,\middle|\, \int_{B_{r_1}(x)} f = 0 = \int_{B_{r_2}(x)} f \text{ for all } x \in X \right\}$$
$$= \left\{ f \in \mathcal{E}_0(X) \mid T_{r_1} * f = 0 = T_{r_2} * f \right\}.$$

Again, we conclude the existence of a φ_λ satisfying

$$0 = \int_{B_{r_i}(x)} \varphi_\lambda(x)dx = \omega_n \int_0^{r_i} \varphi_\lambda(\rho)\theta(\rho)d\rho = \omega_n \Phi_\lambda(r_i),$$

where $\dim(X) = n+1$ and ω_n is the volume of the standard unit sphere of dimension n. As before, this contradicts the choice of the r_i. □

Also, harmonicity of a function follows from the mean value property for two suitably chosen radii:

Theorem 4.3 *Let $r_1, r_2 > 0$ be such that the equations*

$$\varphi_\lambda(r_j) = 1, \quad j = 1, 2,$$

have no common solution $\lambda \in \mathbb{C}\backslash\{\pm iH/2\}$.

Then $f \in C^\infty(X)$ is harmonic if and only if

$$\frac{1}{\mathrm{vol}(S_r(x))} \int_{S_r(x)} f = f(x)$$

for $r = r_1, r_2$ and all $x \in X$.

Proof We now use the distributions $T_r f = \frac{1}{\mathrm{vol}(S_r(x_0))}(\int_{S_r(x_0)} f) - f(x_0)$ and assume, as above, the existence of a function $g \in \mathcal{E}_0(X)$ with $T_{r_1} * g = T_{r_2} * g = 0$ and $\Delta g \neq 0$ (i.e., g not harmonic). As in the proof of Theorem 1.3 of [20], we consider the variety $V_0^g = \overline{\{T * g \mid T \in \mathcal{E}_0'(X)\}} \subset \mathcal{E}_0(X)$, and show that the only non-zero functions $\varphi_{\lambda,k} \in V_0^g$ are $\varphi_{\pm iH/2} = 1$. Therefore, V_0^g consists only of constant functions, contradicting that $g \in V_0^g$ and $\Delta g \neq 0$. □

5 Cheeger Constant and Heat Kernel

The *Cheeger constant* $h(X)$ of a non-compact, complete n-dimensional Riemannian manifold (X, g) is defined as

$$h(X) := \inf_{K \subset X} \frac{\mathrm{area}(\partial K)}{\mathrm{vol}(K)}, \tag{24}$$

where K ranges over all connected, open submanifolds of X with compact closure and smooth boundary. The *volume growth exponent* of X is defined by

$$\mu(X) := \limsup_{r\to\infty} \frac{\log \operatorname{vol}(B_r(x))}{r}. \tag{25}$$

One easily checks that $\mu(X)$ does not depend on the choice $x \in X$. The following result states that several fundamental constants of non-compact harmonic spaces agree.

Theorem 5.1 *Let (X, g) be a non-compact, simply connected harmonic space and $H \geq 0$ be the mean curvature of its horospheres. Then we have the equalities*

$$h(X) = H = \mu(X) = \lim_{r\to\infty} \frac{\log \operatorname{vol}(B_r(x))}{r}.$$

Proof Our first goal is to prove $h(X) \geq H$. The proof is very similar to the proof of Theorem 3 in [19]. We refer the reader to this reference for more details. Let $\Psi : \mathbb{R} \times \mathcal{H}_0 \to X$ be the diffeomorphism introduced in Sect. 3. We work in the space $X' = \mathbb{R} \times \mathcal{H}_0$ with the induced Riemannian metric $g' = \Psi^* g$. We know from Corollary 3.2 that the volume element on X' is given by $e^{sH} dt \wedge \omega_0$.

Without loss of generality, we can assume $H > 0$, for otherwise there is nothing to prove. Let $P : X' \to \mathcal{H}_0$ be the canonical projection and $K \subset X'$ be an admissible set of (24). Let U be the projection of K without the critical points of $P|_{\partial K}$. By Sard's theorem, U has full measure in $P(K)$. For $x \in U$, let $f^{\pm}(x)$ be the maximum, resp., minimum of the set $\{t \in \mathbb{R} \mid (t, x) \in K\}$. Let $\widetilde{K} := \{(x, t) \mid x \in U, f^-(x) \leq t \leq f^+(x)\}$. Then

$$\operatorname{vol}(K) \leq \operatorname{vol}(\widetilde{K}) = \frac{1}{H} \int_U \big(e^{f^+(x)H} - e^{f^-(x)H}\big)\omega_0(x).$$

Now we introduce the sets $\partial K^{\pm} := \{(u, f^{\pm}(u)) \mid u \in U\}$. Obviously, we have $\operatorname{area}(\partial K) \geq \operatorname{area}(\partial K^+) + \operatorname{area}(\partial K^-)$ and, analogously as in [19], we obtain the estimate

$$\operatorname{area}\big(\partial K^{\pm}\big) \geq \int_U e^{f^{\pm}(x)H} \omega_0(x).$$

This yields the desired estimate

$$\frac{\operatorname{area}(\partial K)}{\operatorname{vol}(K)} \geq \frac{\operatorname{area}(\partial K^+) + \operatorname{area}(\partial K^-)}{\operatorname{vol}(K)} \geq H.$$

Let $f(r) = \log \operatorname{vol}(B_r(x))$. Then, for all $r > 0$,

$$f'(r) = \frac{\operatorname{area}(S_r(x))}{\operatorname{vol}(B_r(x))} \geq h(X).$$

It was shown in [16] that $A(r) := \operatorname{area}(S_r(x))$ is strictly increasing in r and that $\frac{A'}{A}$ is monotone decreasing with limit $H \geq 0$. Applying l'Hôpital's rule twice, we conclude that

$$\lim_{r\to\infty}\frac{f(r)}{r}=\lim_{r\to\infty}f'(r)=\lim_{r\to\infty}\frac{A(r)}{\mathrm{vol}(B_r(x))}=\lim_{r\to\infty}\frac{A'(r)}{A(r)}=H.$$

Thus we have

$$h(X)\le\mu(X)=H=\lim\frac{\log\mathrm{vol}(B_r(x))}{r}.$$

Both estimates together prove the theorem. □

Remark 2 C. Connell proved in [5] that the Cheeger constant and the exponential volume growth of simply connected strictly negatively curved homogeneous spaces agree. This fails without the curvature condition: horospheres $\mathcal{H}$ with barycentric normal directions in higher rank symmetric spaces of non-compact type are unimodular solvable groups with $h(\mathcal{H})=0$ and $\mu(\mathcal{H})>0$ (see [18]). The simplest example of this type is Solv(3), the diagonal horosphere in the product of two hyperbolic planes. Note that our Theorem 5.1 does not contain any curvature condition.

Applying Cheeger's inequality and Brooks's result $\lambda_0^{\mathrm{ess}}\le\mu(X)^2/4$ (see [3]), we obtain

Corollary 5.2 *Let* (X,g) *be a non-compact, simply connected harmonic space and* $H\ge 0$ *be the mean curvature of its horospheres. Then the bottom of the spectrum and of the essential spectrum agree, and*

$$\lambda_0(X)=\lambda_0^{\mathrm{ess}}(X)=\frac{H^2}{4}.$$

Applying [14, Prop. 2.4], we obtain

Corollary 5.3 *Let* (X,g) *be a non-compact, simply connected harmonic space of dimension* n. *If* X *has vanishing Cheeger constant, then* X *is isometric to the flat Euclidean space* $\mathbb{R}^n$.

Finally, we consider the Abel transform of the heat kernel on non-compact harmonic manifolds. We first state a useful lemma.

Lemma 5.4 *Let* $\mathcal{H}\subset X$ *be a horosphere and* $x\in\mathcal{H}$. *Then there is* $C>0$ *so that*

$$\mathrm{vol}_{\mathcal{H}}\big(B_r(x)\cap\mathcal{H}\big)\le Ce^{Cr}$$

for all $r\ge 0$.

Proof Without loss of generality, we can assume that $x=x_0$ and $\mathcal{H}_0=\mathcal{H}$. We use the diffeomorphisms $\Psi_s:\mathcal{H}\to\mathcal{H}_s$, introduced earlier.

Let $R>0$ be fixed. Let $A_r=B_r(x_0)\cap\mathcal{H}$ and

$$G_{R,r}=\bigcup_{s\in[-R,R]}\Psi_s(A_r).$$

We compute

$$\begin{aligned}\mathrm{vol}(G_{R,r}) &= \int_{-R}^{R} \mathrm{vol}_{\mathcal{H}_s}\big(\Psi_s(A_r)\big)ds \\ &= \int_{-R}^{R} e^{sH}\, \mathrm{vol}_{\mathcal{H}}(A_r)ds = \frac{2\sinh(RH)}{H}\,\mathrm{vol}_{\mathcal{H}}(A_r).\end{aligned}$$

By the triangle inequality, $G_{R,r} \subset B_{r+R}(x_0)$, and, since

$$\mathrm{vol}\big(B_{r+R}(x_0)\big) \le C' e^{C'(r+R)}$$

with some constant $C' > 0$, by Bishop's volume comparison theorem, we have

$$\mathrm{vol}_{\mathcal{H}}(A_r) = \frac{H\,\mathrm{vol}(G_{R,r})}{2\sinh(RH)} \le \frac{HC'e^{C'(r+R)}}{2\sinh(RH)} = \frac{HC'e^{C'R}}{2\sinh(RH)} e^{C'r}. \qquad \square$$

It is a well-known fact that a general complete Riemannian manifold (X, g) with Ricci curvature bounded from below has a unique heat kernel $p_t^X(x, y)$ (see, e.g., [4, Thm. VIII.3]). In the case that (X, g) is harmonic, the heat kernel is a *radial* kernel function (see, e.g., [22, Thm. 1.1]), and is therefore uniquely determined by the function $k_t^X(x) := p_t^X(x_0, x)$, where $x_0 \in X$ is a fixed reference point. Our main result states that the Abel transform of the heat kernel on a non-compact harmonic space agrees, up to the factor $e^{-H^2 t/4}$, with the Euclidean heat kernel $k_t^{\mathbb{R}}(s) = p_t^{\mathbb{R}}(0, s) = \frac{1}{\sqrt{4\pi t}} e^{-s^2/(4t)}$. Since the heat kernel of a non-compact harmonic manifold does not have compact support, one has to guarantee that its Abel transform (centered at x_0), evaluated via the integral (14) over the horospheres $\mathcal{H}_s = \Psi_s(\mathcal{H}_0)$, is well defined. This follows from the following result.

Lemma 5.5 *Let $t > 0$ be fixed, $x_0 \in \mathcal{H}_0$, and $\Psi_s : \mathcal{H}_0 \to \mathcal{H}_s$ be the diffeomorphisms introduced earlier. Let $x_s = \Psi_s(x_0) \in \mathcal{H}_s$. For all $\epsilon > 0$, there exists an $r_0 > 0$ such that we have for all $s \in \mathbb{R}$:*

$$0 \le \int_{\mathcal{H}_s \setminus B_{|s|+r_0}(x_s)} k_t^X(x) d\omega_s(x) \le \epsilon. \tag{26}$$

Proof Since the Ricci curvature of the non-compact harmonic manifold (X, g) is bounded below, there exist constants $C_t, \alpha_t > 0$ such that

$$0 \le k_t^X(x) \le C_t e^{-\alpha_t r(x)^2} \qquad \text{for all } x \in X, \tag{27}$$

by a classical result of Li and Yau [15]. Using Lemma 5.4, we derive for arbitrary $r = r_0 + |s| > 0$:

$$\begin{aligned}
\int_{\mathcal{H}_s \setminus B_r(x_s)} k_t^X(x) d\omega_s(x) &= \sum_{j=0}^{\infty} \int_{(\mathcal{H}_s \cap B_{r+j+1}(x_s)) \setminus B_{r+j}(x_s)} k_t^X(x) d\omega_s(x) \\
&\leq \sum_{j=0}^{\infty} \mathrm{vol}_{\mathcal{H}_s}\big(\mathcal{H}_s \cap B_{r+j+1}(x_s)\big) C_t e^{-\alpha_t (r+j-|s|)^2} \\
&\leq C C_t \sum_{j=0}^{\infty} e^{C(r+j+1) - \alpha_t (r+j-|s|)^2} \\
&= C C_t \sum_{j=0}^{\infty} e^{C(r_0+|s|+j+1) - \alpha_t (r_0+j)^2} \\
&\leq C C_t \sum_{j=0}^{\infty} e^{C(r_0+|s|+j+1) - \alpha_t (r_0^2+j^2)} \\
&\leq C C_t \left(\sum_{j=0}^{\infty} e^{Cj - \alpha_t j^2} \right) e^{C(|s|+1)} e^{C r_0 - \alpha_t r_0^2}.
\end{aligned}$$

Since α_t is positive, the sum over j converges. By choosing r_0 sufficiently large, we can make the rightmost factor and thus the whole expression as small as we wish. □

Theorem 5.6 *Let (X, g) be a non-compact, simply connected harmonic space and $H \geq 0$ be the mean curvature of its horospheres. Then the Abel transform $\mathcal{A}k_t^X$ of the heat kernel $k_t^X(x) = p_t^X(x_0, x)$ is*

$$\big(\mathcal{A}k_t^X\big)(s) = e^{-H^2 t/4} \frac{1}{\sqrt{4\pi t}} e^{-s^2/4t}.$$

Proof In the case $H = 0$, (X, g) is the Euclidean space and there is nothing to prove. So we can assume that $H > 0$.

Since $\mathcal{A}k_t^X : \mathbb{R} \to \mathbb{R}$ is an even function, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{A}k_t^X(s) ds &= 2 \int_0^{\infty} e^{-\frac{H}{2}s} \int_{\mathcal{H}_s} k_t^X(z) d\omega_s(z) ds \\
&\leq 2 \int_0^{\infty} \int_{\mathcal{H}_s} k_t^X(z) d\omega_s(z) ds \\
&\leq 2 \int_X k_t^X(x) dx = 2,
\end{aligned}$$

by the heat conservation property $\int_X k_t^X(x) dx = 1$ for all $t > 0$ (see, e.g., [4, Thm. 8.5]). This shows that $\mathcal{A}k_t^X \in L^1(\mathbb{R})$. Next, we show that $\mathcal{A}k_t^X$ is continuous. Let $s_0 \in \mathbb{R}$ and $\epsilon > 0$ be given. We conclude from Lemma 5.5 that there is an $r_0 > 0$

such that

$$e^{-\frac{H}{2}s}\int_{\mathcal{H}_s\setminus B_{2s+r_0}(x_s)} k_t^X(x)d\omega_s(x)\le \epsilon/3,$$

for all $s\in(s_0-1,s_0+1)$. Since the map

$$s\mapsto F(s):=e^{-\frac{H}{2}s}\int_{\mathcal{H}_s\cap B_{r_0+2|s|}(x_s)} k_t^X(x)d\omega_s(x)$$

is obviously continuous, we can find $0<\delta<1$ such that

$$\left|F(s)-F(s_0)\right|\le\epsilon/3,$$

for all $s\in(s_0-\delta,s_0+\delta)$. This implies that

$$\begin{aligned}&\left|\mathcal{A}k_t^X(s)-\mathcal{A}k_t^X(s_0)\right|\\ &\quad\le\left|e^{-\frac{H}{2}s}\int_{\mathcal{H}_s\setminus B_{2s+r_0}(x_s)} k_t^X(x)d\omega_s(x)\right|+\left|e^{-\frac{H}{2}s_0}\int_{\mathcal{H}_{s_0}\setminus B_{2s_0+r_0}(x_{s_0})} k_t^X(x)d\omega_{s_0}(x)\right|\\ &\qquad+\left|F(s)-F(s_0)\right|\le\epsilon,\end{aligned}$$

for all $s\in(s_0-\delta,s_0+\delta)$. This shows that $\mathcal{A}k_t^X\in C(\mathbb{R})\cap L^1(\mathbb{R})$.

For the proof of the theorem, it only remains to show that the Fourier transforms of the L^1-functions $\mathcal{A}k_t^X$ and $e^{-H^2t/4}k_t^{\mathbb{R}}$ agree. To show this, we need some growth information of k_t^X and φ_λ and their derivatives.

Let us first consider φ_λ for $\lambda\in\mathbb{R}$. An immediate consequence of $\varphi_\lambda=a\psi_\lambda$ is

$$\left|\varphi_\lambda(r)\right|\le Ce^{\frac{H}{2}r},$$

with a suitable constant $C>0$. Moreover, $\varphi''+\frac{\theta'}{\theta}\varphi'=L\varphi$ with $L=-(\lambda^2+H^2/4)$ implies that

$$\left\|\nabla\varphi_\lambda(r)\right\|\le|L|\int_0^r\underbrace{\left|\frac{\theta(t)}{\theta(r)}\right|}_{\le 1}\left|\varphi_\lambda(t)\right|dt,$$

which shows that $\|\nabla\varphi_\lambda\|$ grows also at most exponentially in the radius.

Next, we derive superexponential decay of the derivatives $\frac{\partial}{\partial t}k_t^X$ and $\|\nabla k_t^X\|$. Since the Ricci curvature of (X,g) is bounded from below and all balls of the same radius have the same volume, we conclude from [10, Prop. 1.1] that

$$p(x,x,t)\le C\begin{cases}e^{-\lambda_0(X)t}, & \text{if } t\ge 1,\\ t^{-n/2}, & \text{if } t\le 1,\end{cases}$$

with a suitable constant $C>0$. Since we have $\lambda_0(X)=H^2/4>0$, we can find another constant $C'>0$ such that

$$p(x,x,t)\le\frac{C'}{t^{n/2}},$$

for all $x \in X$ and $t > 0$. Then we are in Case 1 of [7], and Theorems 2 and 6 in [7] imply that, for any fixed time $t > 0$, the above heat kernel derivatives decay at the rate $q(r)e^{-r^2/4t}$, with a suitable polynomial q.

To finish the proof, let $\lambda \in \mathbb{R}$. We need to show that

$$f(t) := \widehat{\mathcal{A}k_t^X}(\lambda) = \langle \mathcal{A}k_t^X, \psi_\lambda \rangle = \langle k_t^X, a\psi_\lambda \rangle = \int_X p_t^X(x_0, x)\varphi_\lambda(x)dx$$

and

$$g(t) := e^{-\frac{H^2}{4}t}\widehat{k_t^{\mathbb{R}}}(\lambda) = e^{-\frac{H^2}{4}t}e^{-\lambda^2 t}$$

agree.

Obviously, both functions satisfy $\lim_{t\to 0} f(t) = \lim_{t\to 0} g(t) = 1$. So it only remains to show that we have $f'(t) = g'(t)$ for all $t > 0$. Now,

$$\begin{aligned}
\frac{d}{dt}\int_X p_t^X(x_0, x)\varphi_\lambda(x)dx &= \int_X \left(\frac{\partial}{\partial t} p_t^X(x_0, x)\right)\varphi_\lambda(x)dx \\
&= \int_X \left(\Delta_x p_t^X(x_0, x)\right)\varphi_\lambda(x)dx \\
&= -\int_X \langle \nabla k_t^X(x), \nabla\varphi_\lambda(x)\rangle dx \\
&= \int_X p_t^X(x_0, x)\Delta\varphi_\lambda(x)dx \\
&= -\left(\lambda^2 + \frac{H^2}{4}\right)\varphi_\lambda(x_0) = g'(t).
\end{aligned}$$

All steps in this calculation are justified by the growth properties derived above. □

The result corresponding to Theorem 5.6 in the special case of Damek–Ricci spaces can be found, e.g., in [1, (5.6)].

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