J Geom Anal (2014) 24:1346–1367
DOI 10.1007/s12220-012-9376-3

ORIGINAL RESEARCH

Volume Growth, Number of Ends, and the Topology of a Complete Submanifold

Vicent Gimeno · Vicente Palmer

Received: 17 December 2011 / Published online: 27 November 2012
© Mathematica Josephina, Inc. 2012

Abstract Given a complete isometric immersion $\varphi : P^m \longrightarrow N^n$ in an ambient Riemannian manifold N^n with a pole and with radial sectional curvatures bounded from above by the corresponding radial sectional curvatures of a radially symmetric space M^n_w, we determine a set of conditions on the extrinsic curvatures of P that guarantee that the immersion is proper and that P has finite topology in line with the results reported in Bessa et al. (Commun. Anal. Geom. 15(4):725–732, 2007) and Bessa and Costa (Glasg. Math. J. 51:669–680, 2009). When the ambient manifold is a radially symmetric space, an inequality is shown between the (extrinsic) volume growth of a complete and minimal submanifold and its number of ends, which generalizes the classical inequality stated in Anderson (Preprint IHES, 1984) for complete and minimal submanifolds in $\mathbb{R}^n$. As a corollary we obtain the corresponding inequality between the (extrinsic) volume growth and the number of ends of a complete and minimal submanifold in hyperbolic space, together with Bernstein-type results for such submanifolds in Euclidean and hyperbolic spaces, in the manner of the work Kasue and Sugahara (Osaka J. Math. 24:679–704, 1987).

Keywords Volume growth · Minimal submanifold · End · Hessian-Index comparison theory · Extrinsic distance · Total extrinsic curvature · Second fundamental form · Gap theorem · Bernstein-type theorem

Communicated by Marco Abate.

Work partially supported by the Caixa Castelló Foundation, and DGI grant MTM2010-21206-C02-02.

V. Gimeno · V. Palmer (✉)
Departament de Matemàtiques—Institute of New Imaging Technologies, Universitat Jaume I, Castellon, Spain
e-mail: palmer@mat.uji.es

V. Gimeno
e-mail: gimenov@uji.es

Mathematics Subject Classification (2000) Primary 53A20 · 53C40 · Secondary 53C42

1 Introduction

A natural question in Riemannian geometry is to explore the influence of the conduct of the curvature of a complete Riemannian manifold on its geometric and topological properties. Classical results concerning this are the intrinsic gap theorems shown by Greene and Wu in [10] (see Theorems 2 and 3; see also [11]) and the Bernstein-type theorems shown by Anderson in [1] and by Schoen in [25] when minimal submanifolds properly immersed in the Euclidean space $\mathbb{R}^n$ were considered.

Greene and Wu's results state that a Riemannian manifold M^n with a pole and with faster-than-quadratic decay of its sectional curvatures is isometric to Euclidean space, if either its dimension is odd and the curvature does not change its sign or its dimension is even (and $n \neq 2, 4, 8$) and the curvature is everywhere non-negative. On the other hand, Anderson proved, as a corollary of a generalization of the Chern–Osserman theorem on complete and minimal submanifolds of $\mathbb{R}^n$ with finite total (extrinsic) curvature, that any such submanifold having one end is an affine n-plane.

More examples concerning submanifolds immersed in an ambient Riemannian manifold and the analysis of their (intrinsic and extrinsic) curvature behavior are the (Bernstein-type) gap results given by Kasue and Sugahara in [13] (see Theorems A and B) and by Kasue in [14], where an accurate (extrinsic) curvature decay forces minimal (or not) submanifolds with one end of the Euclidean and hyperbolic spaces to be totally geodesic.

The estimation of the number of ends of these submanifolds plays a fundamental role in all the Bernstein-type results mentioned above. In this way, in [1] it is proved that given a complete and minimal submanifold $\varphi : P^m \longrightarrow \mathbb{R}^n$, $m > 2$, having finite total curvature $\int_P \|B^P\|^m d\sigma < \infty$, its (extrinsic) volume growth, defined as the quotient $\frac{\mathrm{Vol}(\varphi(P)\cap B_t^{0,n})}{\omega_n t^n}$ is bounded from above by the number of ends of P, $\mathcal{E}(P)$, namely:

$$\lim_{t\to\infty} \frac{\mathrm{Vol}(\varphi(P) \cap B_t^{0,n})}{\omega_n t^n} \leq \mathcal{E}(P), \tag{1.1}$$

where $B_t^{b,n}$ denotes the metric t-ball in the real space form of constant curvature b, $\mathbb{K}^n(b)$, and $\|B^P\|$ denotes the Hilbert–Schmidt norm of the second fundamental form of P in $\mathbb{R}^n$. Moreover, if $\mathcal{E}(P) = 1$, the conclusion reached (using inequality (1.1)) is the Bernstein-type result outlined above, namely that P^m is an affine plane, i.e., totally geodesic in $\mathbb{R}^n$ (see Theorems 4.1, 5.1, and 5.2 in [1]).

In the paper [7] it was proven that inequality (1.1) is in fact an equality when the minimal submanifold in $\mathbb{R}^n$ exhibits an accurate decay of its extrinsic curvature $\|B^P\|$. Furthermore, in the paper [13] it was also proven that if the submanifold P has only one end and the decay of its extrinsic curvature $\|B^P\|$ is faster than linear (when the ambient space is $\mathbb{R}^n$) or than exponential (when the ambient space is $\mathbb{H}^n(b)$), then it is totally geodesic.

Within this study of the behavior at infinity of complete and minimal submanifolds with finite total curvature immersed in Euclidean space, it was also proven in [1] and in [20] that the immersion of a complete and minimal submanifold P in $\mathbb{R}^n$ or $\mathbb{H}^n(b)$ satisfying $\int_P \|B^P\|^m d\sigma < \infty$ is proper and that P is of finite topological type.

At this point we should mention the results reported in [2] and in [3], where new conditions have been stated on the decay of the extrinsic curvature for a completely immersed submanifold P in Euclidean space [2] and in a Cartan–Hadamard manifold [3], which guarantees the properness of the submanifold and the finiteness of its topology.

In view of these results, it seems natural to consider the following three issues:

(1) Can the properness/finiteness results in [2] and [3] be extended to submanifolds immersed in spaces which do not necessarily have non-positive curvature?
(2) Do we have an analogy to inequality (1.1) between the extrinsic volume growth and the number of ends when we consider a minimal submanifold (properly) immersed in hyperbolic space which exhibit an accurate extrinsic curvature decay?
(3) Moreover, is it possible to deduce from this inequality a Bernstein-type result in line with [1] and [13]?

In this paper we provide a (partial) answer to these questions, besides other lower bounds for the number of ends for (non-minimal) submanifolds in Euclidean and hyperbolic spaces and other gap results related to these estimates. As a preliminary view of our results, we have Theorems 1.1 and 1.2, which follow directly from our Theorem 3.5.

In Theorem 1.1 we have the answer to the last two questions, namely, we have obtained equation (1.1) but in the hyperbolic case and, moreover, we have also proven a Bernstein-type result for minimal submanifolds in hyperbolic space, along the lines studied by Kasue and Sugahara in [13] (see assertion (A-iv) of Theorem A).

Theorem 1.1 *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b)$ be a complete, proper, and minimal immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e., outside a compact):*

$$\|B_x^P\| \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}r(x)}},$$

where $r(x) = d_{\mathbb{H}^n(b)}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{H}^n(b)$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{H}^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$ and $\delta(r)$ is a smooth function such that $\delta(r) \to 0$ when $r \to \infty$. Then:

(1) *The finite number of ends $\mathcal{E}(P)$ is related to the volume growth by*

$$\operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t(o))}{\operatorname{Vol}(B_t^{b,m})} \leq \mathcal{E}(P),$$

where $D_t(o) = \{x \in P : r(x) < t\} = \{x \in P : \varphi(x) \in B_t^{b,n}(o)\}$ is the extrinsic ball of radius t in P (see Definition 2.1).

(2) *If P has only one end, P is totally geodesic in $\mathbb{H}^n(b)$.*

When the ambient manifold is $\mathbb{R}^n$, Theorem 1.2 encompasses a slightly less general version of the Bernstein-type result given in assertion (A-i) of Theorem A in [13].

Theorem 1.2 *Let* $\varphi : P^m \longrightarrow \mathbb{R}^n$ *be a complete non-compact, minimal, and proper immersion with* $m > 2$. *Let us suppose that for sufficiently large* R_0 *and for all points* $x \in P$ *such that* $r(x) > R_0$ (*i.e., outside the compact extrinsic ball* $D_{R_0}(o)$ *with* $\varphi^{-1}(o) \neq \emptyset$):

$$\|B_x^P\| \leq \frac{\epsilon(r(x))}{r(x)},$$

where $\epsilon(r)$ *is a smooth function such that* $\epsilon(r) \to 0$ *when* $r \to \infty$. *Then*:

(1) *The finite number of ends* $\mathcal{E}(P)$ *is related to the volume growth by*

$$\mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,m})} \leq \mathcal{E}(P).$$

(2) *If* P *has only one end*, P *is totally geodesic in* $\mathbb{R}^n$.

These results, which we shall prove in Sect. 8 (together with the corollaries in Sect. 4) follow from two main theorems, which are established in Sect. 3. In the first (Theorem 3.1) we show that a complete isometric immersion $\varphi : P^m \longrightarrow N^n$, $m > 2$, with controlled second fundamental form in a complete Riemannian manifold that possesses a pole and has controlled radial sectional curvatures is proper and has finite topology. In the second (Theorem 3.4) it is proven that a complete and proper isometric immersion $\varphi : P^m \longrightarrow M_w^n$, $m > 2$, with controlled second fundamental form in a radially symmetric space M_w^n with sectional curvatures bounded from below by a radial function, has its volume growth bounded from above by a quantity which involves its (finite) number of ends.

1.1 Outline

The structure of the paper can be outlined as follows. In Sect. 2 we present the definition of an extrinsic ball, together with the basic facts about the Hessian comparison theory of restricted distance function that we are going to use and an isoperimetric inequality for the extrinsic balls which plays an important role in the proof of Theorem 3.4. Section 3 is devoted to the statement of the main results (Theorems 3.1, 3.4, and 3.5). In Sect. 4 we will present two lists of results based on Theorems 3.1, 3.4, and 3.5: the first set of consequences is devoted to bounding the volume growth of a submanifold from above by the number of its ends, in several contexts, some Bernstein-type results also being obtained. In the second set of corollaries, some compactification theorems for submanifolds in $\mathbb{R}^n$, in $\mathbb{H}^n$, and in $\mathbb{H}^n \times \mathbb{R}^l$ are stated. Sections 5, 6, 7 are devoted to the proofs of the Theorems 3.1, 3.4, and 3.5, respectively. Theorems 1.1, 1.2, and the corollaries stated in Sect. 4 are proven in Sect. 8.

2 Preliminaries

2.1 The Extrinsic Distance

Throughout the paper we assume that $\varphi : P^m \longrightarrow N^n$ is an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ (this is the precise meaning we shall give to the word *submanifold* throughout the text). Recall that a pole is a point o such that the exponential map

$$\exp_o \colon T_o N^n \to N^n$$

is a diffeomorphism. For every $x \in N^n - \{o\}$ we define $r(x) = r_o(x) = \mathrm{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from o*. We also denote by $r|_P$ or by r the composition $r \circ \varphi : P \to \mathbb{R}_+ \cup \{0\}$. This composition is called the *extrinsic distance function* from o in P^m. The gradients of r in N and $r|_P$ in P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Then we have the following basic relation, by virtue of the identification, given any point $x \in P$, between the tangent vector fields $X \in T_x P$ and $\varphi_{*_x}(X) \in T_{\varphi(x)} N$:

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \tag{2.1}$$

where $(\nabla^N r)^\perp(\varphi(x)) = \nabla^\perp r(\varphi(x))$ is perpendicular to $T_x P$ for all $x \in P$.

Definition 2.1 Given $\varphi : P^m \longrightarrow N^n$ an isometric immersion of a complete and connected Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$, we denote the *extrinsic metric balls* of radius $t > 0$ and center $o \in N$ by $D_t(o)$. They are defined as the subset of P:

$$D_t(o) = \{x \in P : r(\varphi(x)) < t\} = \{x \in P : \varphi(x) \in B_t^N(o)\},$$

where $B_t^N(o)$ denotes the open geodesic ball of radius t centered at the pole o in N^n. Note that the set $\varphi^{-1}(o)$ can be the empty set.

Remark 2.2 When the immersion φ is proper, the extrinsic domains $D_t(o)$ are precompact sets, with smooth boundaries $\partial D_t(o)$. The assumption on the smoothness of $\partial D_t(o)$ makes no restriction. Indeed, the distance function r is smooth in $N - \{o\}$ since N is assumed to possess a pole $o \in N$. Hence the composition $r|_P$ is smooth in P and consequently the radii t that produce smooth boundaries $\partial D_t(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

We now present the curvature restrictions which constitute the geometric framework of our study.

Definition 2.3 Let o be a point in a Riemannian manifold N and let $x \in N - \{o\}$. The sectional curvature $K_N(\sigma_x)$ of the two-plane $\sigma_x \in T_x N$ is then called an *o-radial sectional curvature* of N at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,N}(\sigma_x)$.

2.2 Model Spaces

Throughout this paper we shall assume that the ambient manifold N^n has its o-radial sectional curvatures $K_{o,N}(x)$ bounded from above by the expression $K_w(r(x)) = -w''(r(x))/w(r(x))$, which are precisely the radial sectional curvatures of the w-*model space* M_w^m we are going to define.

Definition 2.4 (See [21], [12], and [9]) A w-model M_w^m is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}_1^{m-1}$ (i.e., the unit $(m-1)$-sphere with standard metric), and warping function $w\colon [0, \Lambda[\to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $\Lambda = \infty$, then o_w is a pole of M_w^m.

Proposition 2.5 *The simply connected space forms* $\mathbb{K}^m(b)$ *of constant curvature* b *are* w-*models with warping functions*

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin(\sqrt{b}r) & \text{if } b > 0 \\ r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}r) & \text{if } b < 0 \end{cases}$$

Note that for $b > 0$ *the function* $w_b(r)$ *admits a smooth extension to* $r = \pi/\sqrt{b}$.

Proposition 2.6 (See Proposition 42 in Chap. 7 of [21]. See also [9] and [12]) *Let* M_w^m *be a* w-*model with warping function* $w(r)$ *and center* o_w. *The distance sphere* S_r^w *of radius* r *and center* o_w *in* M_w^m *is the fiber* $\pi^{-1}(r)$. *This distance sphere has the constant mean curvature* $\eta_w(r) = \frac{w'(r)}{w(r)}$.

On the other hand, the o_w-*radial sectional curvatures of* M_w^m *at every* $x \in \pi^{-1}(r)$ (*for* $r > 0$) *are all identical and determined by*

$$K_{o_w,M_w}(\sigma_x) = -\frac{w''(r)}{w(r)},$$

and the sectional curvatures of M_w^m *at every* $x \in \pi^{-1}(r)$ (*for* $r > 0$) *of the tangent planes to the fiber* S_r^w *are also all identical and determined by*

$$K(r) = K_{M_w}(\Pi_{S_r^w}) = \frac{1-(w'(r))^2}{w^2(r)}.$$

Remark 2.7 The w-model spaces are completely determined via w by the mean curvatures of the spherical fibers S_r^w:

$$\eta_w(r) = w'(r)/w(r),$$

by the volume of the fiber

$$\mathrm{Vol}(S_r^w) = V_0 w^{m-1}(r),$$

and by the volume of the corresponding ball, for which the fiber is the boundary

$$\mathrm{Vol}(B^w_r) = V_0 \int_0^r w^{m-1}(t)\,dt.$$

Here V_0 denotes the volume of the unit sphere $S^{0,m-1}_1$ (in general, the sphere of radius r in the real space form $\mathbb{K}^m(b)$ is denoted as $S^{b,m-1}_r$). The last two functions define the isoperimetric quotient function as follows:

$$q_w(r) = \mathrm{Vol}(B^w_r)/\,\mathrm{Vol}(S^w_r).$$

Besides the role of comparison controllers for the radial sectional curvatures of N^n, we shall also need two further purely intrinsic conditions on the model spaces:

Definition 2.8 A given w-model space M^m_w is called *balanced from below* and *balanced from above*, respectively, if the following weighted isoperimetric conditions are satisfied:

$$\text{Balanced from below:}\quad q_w(r)\eta_w(r) \geq 1/m \quad \text{for all } r \geq 0$$
$$\text{Balanced from above:}\quad q_w(r)\eta_w(r) \leq 1/(m-1) \quad \text{for all } r \geq 0$$

A model space is called *totally balanced* if it is balanced both from below and from above.

Remark 2.9 If $K_w(r) \geq -\eta^2_w(r)$, then M^m_w is balanced from above. If $K_w(r) \leq 0$, then M^m_w is balanced from below; see paper [15] for a detailed list of examples.

2.3 Hessian Comparison Analysis

The second-order analysis of the restricted distance function $r_{|P}$ defined on manifolds with a pole is governed by the Hessian comparison Theorem A in [9].

This comparison theorem can be stated as follows, when one of the spaces is a model space M^m_w (see [23]):

Theorem 2.10 (See [9], Theorem A) *Let $N = N^n$ be a manifold with a pole o, and let $M = M^m_w$ denote a w-model with center o_w. Suppose that every o-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the o_w-radial sectional curvatures in M^m_w as follows*:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \mathrm{dist}_N(o,x)$ from o in N. Then the Hessian of the distance function in N satisfies:

$$\begin{aligned}
\mathrm{Hess}^N(r(x))(X,X) &\geq \mathrm{Hess}^M(r(y))(Y,Y)\\
&= \eta_w(r)\left(\|X\|^2 - \langle \nabla^M r(y), Y\rangle^2_M\right)\\
&= \eta_w(r)\left(\|X\|^2 - \langle \nabla^N r(x), X\rangle^2_N\right) \qquad (2.2)
\end{aligned}$$

for every vector X in T_xN and for every vector Y in T_yM with $r(y) = r(x) = r$ and $\langle \nabla^M r(y), Y\rangle_M = \langle \nabla^N r(x), X\rangle_N$.

Remark 2.11 Note that inequality (2.2) is true along the geodesics emanating from o and o_w, which are free of conjugate points of o and o_w (see Remark 2.3 in [9]). Another relevant observation is that the bound given in inequality (2.2) does not depend on the dimension of the model space (see Remark 3.7 in [23]).

We now present a technical result concerning the Hessian of a radial function, namely, a function which only depends on the distance function r. For the proof of this result, and the rest of the results in this subsection, we refer to the paper [23].

Proposition 2.12 *Let $N = N^n$ be a manifold with a pole o. Let $r = r(x) = \mathrm{dist}_N(o, x)$ be the distance from o to x in N. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. Then, given $q \in N$ and $X, Y \in T_q N$,*

$$\begin{aligned}\mathrm{Hess}^N F \circ r|_q(X, Y) = F''(r)(\nabla^N r \otimes \nabla^N r)(X, Y)\\ + F'(r)\,\mathrm{Hess}^N r|_q(X, Y).\end{aligned} \tag{2.3}$$

Now, let us consider a complete isometric immersion $\varphi : P^m \longrightarrow N$ in a Riemannian ambient manifold N^n with pole o, and with distance function to the pole r. We are going to see how the Hessians (in P and in N) of a radial function defined in the submanifold are related via the second fundamental form B^P of the submanifold P in N. As before, given any $q \in P$, we identify the tangent vectors $X \in T_q P$ with $\varphi_{*_q} X \in T\varphi(q)N$ throughout the following results.

Proposition 2.13 *Let N^n be a manifold with a pole o, and let us consider an isometric immersion $\varphi : P^m \longrightarrow N$. If $r|_P$ is the extrinsic distance function, then, given $q \in P$ and $X, Y \in T_q P$,*

$$\mathrm{Hess}^P r|_q(X, Y) = \mathrm{Hess}^N r|_{\varphi(q)}(X, Y) + \langle B^P_q(X, Y), \nabla^N r|_q\rangle, \tag{2.4}$$

where B^P_q is the second fundamental form of P in N at the point $q \in P$.

Now, we apply Proposition 2.12 to $F \circ r|_P = F \circ r \circ \varphi$ (considering P as the Riemannian manifold where the function is defined) to obtain an expression for $\mathrm{Hess}^P F \circ r|_P(X, Y)$. Then, let us apply Proposition 2.13 to $\mathrm{Hess}^P r|_P(X, Y)$, and finally we get:

Proposition 2.14 *Let $N = N^n$ be a manifold with a pole o, and let P^m denote an immersed submanifold in N. Let $r|_P$ be the extrinsic distance function. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. Then, given $q \in P$ and $X, Y \in T_q P$:*

$$\begin{aligned}\mathrm{Hess}^P F \circ r|_q(X, Y) = F''(r(q))\langle \nabla^N r|_q, X\rangle\langle \nabla^N r|_q, Y\rangle\\ + F'(r(q))\{\mathrm{Hess}^N r|_q(X, Y) + \langle \nabla^N r|_q, B^P_q(X, Y)\rangle\}.\end{aligned} \tag{2.5}$$

2.4 Comparison Constellations and Isoperimetric Inequalities

The isoperimetric inequalities satisfied by the extrinsic balls in minimal submanifolds are at the base of the monotonicity of the volume growth function $f(r) = \frac{\mathrm{Vol}(D_r)}{\mathrm{Vol}(B_r^w)}$, a key result to prove Theorem 1.1. We have the following theorem.

Theorem 2.15 (See [15–18], and [22]) *Let $\varphi : P^m \longrightarrow N^n$ be a complete, proper, and minimal immersion in an ambient Riemannian manifold N^n which possesses at least one pole $o \in N$. Let us suppose that the o-radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of the w-model space M_w^m:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \quad \forall x \in N,$$

and let us also assume that M_w^m is balanced from below. Let D_r be an extrinsic r-ball in P^m, with center at a pole $o \in N$ in the ambient space N. Then:

$$\frac{\mathrm{Vol}(\partial D_r)}{\mathrm{Vol}(D_r)} \geq \frac{\mathrm{Vol}(S_r^w)}{\mathrm{Vol}(B_r^w)} \quad \textit{for all } r > 0. \tag{2.6}$$

Furthermore, if $\varphi^{-1}(o) \neq \emptyset$,

$$\mathrm{Vol}(D_r) \geq \mathrm{Vol}(B_r^w) \quad \textit{for all } r > 0. \tag{2.7}$$

Moreover, if equality in inequalities (2.6) or (2.7) holds for some fixed radius R and if the balance of M_w^m from below is sharp $q_w(r)\eta_w(r) > 1/m$ for all r, then D_R is a minimal cone in the ambient space N^n, so if N^n is the hyperbolic space $\mathbb{H}^n(b)$, $b < 0$, then P^m is totally geodesic in $\mathbb{H}^n(b)$.

If, on the other hand, the ambient space is $\mathbb{R}^n$ and equality in inequalities (2.6) or (2.7) holds for all radii $r > 0$, then P^m is totally geodesic in $\mathbb{R}^n$.

On the other hand, and also as a consequence of inequality (2.6), the volume growth function $f(r) = \frac{\mathrm{Vol}(D_r)}{\mathrm{Vol}(B_r^w)}$ is a non-decreasing function of r.

3 Main Results

In this section we prove our main results. In doing so, we establish a set of conditions that ensure our submanifolds are properly immersed and have finite topology, as well as (under certain conditions) bounding from below the number of their ends.

To obtain the finiteness of the topology in Theorem 3.1, we show that the restricted (to the submanifold) extrinsic distance to a fixed pole (in the ambient manifold) has no critical points outside a compact, and then we apply classical Morse theory. To show that the extrinsic distance function has no critical points we compute its Hessian as it appears in [15] and [23]. These results are, in turn, based on the Jacobi-Index analysis for the Hessian of the distance function given in [9], in particular, its Theorem A (see Sect. 2.3). This comparison theorem is different from the Hessian comparison Theorem 2.1 used in [3]. While in this last theorem, the space used as a model for comparison purposes is the real space form with constant sectional curvature equal to the bound on the sectional curvatures of the given Riemannian manifold, in our adaptation of Theorem A in [9] (see Theorem 2.10), only the sectional curvatures of the planes containing radial directions from the pole are bounded by the corresponding radial sectional curvatures in a radially symmetric space used as a model.

Here, we also note that although we use the definition of pole given by Greene and Wu in [9] (namely, the exponential must be a diffeomorphism at a pole), in fact, the comparison of the Hessians in Theorem A in [9] (and in our Theorem 2.10) holds along radial geodesics from the poles defined as those points which have no conjugate points, as in [3].

Theorem 3.1 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ and satisfying $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that*:

(1) *The o-radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of the w-model space M_w^m*:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \quad \forall x \in N.$$

(2) *The second fundamental form B_x^P in $x \in P$ satisfies that, for sufficiently large radius R_0, and for some constant $c \in]0, 1[$*:

$$\|B_x^P\| \leq c\eta_w(\rho^P(x)) \quad \forall x \in P - B_{R_0}^P(x_o),$$

where $\rho^P(x)$ denotes the intrinsic distance in P from some fixed $x_o \in \varphi^{-1}(o)$ to x.

(3) *For any $r > 0$, $w'(r) \geq d > 0$ and $(\eta_w(r))' \leq 0$.*

Then P is properly immersed in N and it is C^∞-diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

Remark 3.2 To show that φ is proper, we shall use Theorem 2.10. Hence, as was pointed out before, it is enough to assume that o is a pole in the sense that there are no conjugate points along any geodesic emanating from o (see [8] and [24]). Therefore, our statement about the properness of the immersion includes ambient manifolds N that admit non-negative sectional curvatures, unlike the ambient manifold in Theorem 1.2 in [3].

On the other hand, to prove the finiteness of the topology of P we need to assume that the ambient manifold N possesses a pole as defined in [9], namely, a point $p \in N$ where $\exp_p$ is a C^∞ diffeomorphism. However, although our ambient manifold must be diffeomorphic to $\mathbb{R}^n$ in this case (as in Theorem 1.2 in [3], where the ambient space must be a Cartan–Hadamard manifold), it also admits non-negative sectional curvatures.

To complete the benchmarking with the hypotheses in [2] and [3], we are going to compare assumptions (2) and (3) in Theorem 3.1 with the notion of "submanifold with *tamed* second fundamental form" introduced in [2]. It is straightforward to check that if $\varphi : P^m \longrightarrow N^n$ is an immersion of a complete Riemannian m-manifold P^m into a complete Riemannian manifold N^n with sectional curvatures $K_N \leq b \leq 0$, and P has tamed second fundamental form, in the sense of Definition 1.1 in [3], then there exists $R_0 > 0$ such that for all $r \geq R_0$, the quantity

$$a_r := \operatorname{Sup}\left\{ \frac{w_b}{w_b'}(\rho^P(x)) \| B_x^P \| : x \in P - B_r^P \right\}$$

satisfies $a_r < 1$.

Hence, taking $r = R_0$, we have that for all $x \in P - B_{R_0}^P$, and some $c \in (0, 1)$,

$$\| B_x^P \| \leq c \eta_{w_b}(\rho^P(x)).$$

On the other hand, when $b \leq 0$, then $w_b'(r) \geq 1 > 0$ $\forall r > 0$ and $(\eta_{w_b}(r))' \leq 0$ $\forall r > 0$.

If the o-radial sectional curvatures of the ambient space N are bounded from above by $-G(r(x)) = -\frac{w''(r(x))}{w(r(x))}$, then the condition $w'(r) \geq d > 0$ can be achieved provided the criteria $t \int_t^\infty G_-(s) ds \leq \frac{1}{4}$ is fulfilled, where $G_- = \min\{-G, 0\}$ (see [4] and [5]).

All these observations lead us to consider our Theorem 3.1 as a natural and slight generalization of assertions (b) and (c) of Theorem 1.2 in [3].

Observe that if we assume the properness of the immersion, we obtain the following version of Theorem 3.1, where we can remove the hypothesis about the decrease in the function $\eta_w(r)$ because the norm of the second fundamental form $\| B_x^P \|$ is bounded by the value of η_w at $r(x)$ instead of $\rho^P(x)$:

Theorem 3.3 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric and proper immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ and satisfying $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that, as in Theorem 3.1, the o-radial sectional curvatures of N are bounded from above as*:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \quad \forall x \in N$$

and for any $r > 0$, $w'(r) \geq d > 0$. Let us assume, moreover, that the second fundamental form B_x^P in $x \in P$ satisfies that, for sufficiently large radius R_0:

$$\| B_x^P \| \leq c \eta_w(r(x)) \quad \forall x \in P - D_{R_0}(o),$$

where c is a positive constant such that $c < 1$.

Then P is C^∞-diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

We are now going to see how to estimate the area growth function of P, defined as $g(r) = \frac{\operatorname{Vol}(\partial D_r)}{\operatorname{Vol}(S_r^w)}$, by the number of ends of the immersion P, $\mathcal{E}(P)$, when the ambient space N is a radially symmetric space.

Theorem 3.4 *Let $\varphi : P^m \longrightarrow M^n_w$ be an isometric and proper immersion of a complete non-compact Riemannian m-manifold P^m into a model space M^n_w with pole o_w. Suppose that $\varphi^{-1}(o_w) \neq \emptyset$, $m > 2$, and moreover*:

(1) *The norm of second fundamental form B^P_x in $x \in P$ is bounded from above outside a (compact) extrinsic ball $D_{R_0}(o) \subseteq P$ with sufficiently large radius R_0 by*:

$$\|B^P_x\| \leq \frac{\epsilon(r(x))}{(w'(r(x)))^2}\eta_w(r(x)) \quad \forall x \in P - D_{R_0},$$

where ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$.

(2) *For sufficiently large r, $w'(r) \geq d > 0$.*

Then, for sufficiently large r, we have:

$$\frac{\mathrm{Vol}(\partial D_r)}{\mathrm{Vol}(S^w_r)} \leq \frac{\mathcal{E}(P)}{(1 - 4\epsilon(r))^{\frac{(m-1)}{2}}}, \tag{3.1}$$

where $\mathcal{E}(P)$ is the (finite) number of ends of P.

When we consider minimal immersions in model spaces, we have the following result, which is an immediate corollary from the theorem above, and Theorem 2.15 in Sect. 2.

Theorem 3.5 *Let $\varphi : P^m \longrightarrow M^n_w$ be a complete non-compact, proper, and minimal immersion into a balanced from below model space M^n_w with pole o_w. Suppose that $\varphi^{-1}(o_w) \neq \emptyset$ and $m > 2$. Let us assume moreover hypotheses* (1) *and* (2) *in Theorem 3.4.*

Then

(1) *The (finite) number of ends $\mathcal{E}(P)$ is related to the (finite) volume growth by*

$$1 \leq \lim_{r\to\infty} \frac{\mathrm{Vol}(D_r)}{\mathrm{Vol}(B^w_r)} \leq \mathcal{E}(P). \tag{3.2}$$

(2) *If P has only one end, P is a minimal cone in M^n_w.*

4 Corollaries

As we said in the Introduction, we have divided the list of results based on Theorem 3.1 and on Theorem 3.4 into two series of corollaries. The first set of consequences follows along the lines of Theorems 1.1 and 1.2 (which are in fact the main representatives of these results), presenting upper bounds for the volume and area growth of a complete and proper immersion in the real space form $\mathbb{K}^n(b)$, ($b \leq 0$), in terms of the number of its ends. The second set of corollaries includes the statement of compactification theorems for complete and proper immersions in $\mathbb{R}^n$, $\mathbb{H}^n(b)$, and $\mathbb{H}^n(b) \times \mathbb{R}^l$.

The first of these corollaries constitutes a non-minimal version of Theorem 1.1:

Corollary 4.1 *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b)$ be a complete non-compact and proper immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e., outside the compact extrinsic ball $D_{R_0}(o)$ with $\varphi^{-1}(o) \neq \emptyset$),*

$$\|B_x^P\| \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}r(x)}},$$

where $r(x) = d_{\mathbb{H}^n(b)}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{H}^n(b)$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{H}^n(b)$, and $\delta(r)$ is a smooth function such that $\delta(r) \to 0$ when $r \to \infty$. Let $\{t_i\}_{i=1}^{\infty}$ be any non-decreasing sequence such that $t_i \to \infty$ when $i \to \infty$. Then the finite number of ends $\mathcal{E}(P)$ is related to the area growth of P by:

$$\liminf_{i\to\infty} \frac{\operatorname{Vol}(\partial D_{t_i})}{\operatorname{Vol}(S_{t_i}^{b,m-1})} \leq \mathcal{E}(P).$$

The corresponding non-minimal statement of Theorem 1.2 is:

Corollary 4.2 *Let $\varphi : P^m \longrightarrow \mathbb{R}^n$ be a complete non-compact and proper immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$ (i.e., outside the compact extrinsic ball $D_{R_0}(o)$ with $\varphi^{-1}(o) \neq \emptyset$),*

$$\|B_x^P\| \leq \frac{\epsilon(r(x))}{r(x)},$$

where $r(x) = d_{\mathbb{R}^n}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{R}^n$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{R}^n$ and $\epsilon(r)$ is a smooth function such that $\epsilon(r) \to 0$ when $r \to \infty$. Let $\{t_i\}_{i=1}^{\infty}$ be any non-decreasing sequence such that $t_i \to \infty$ when $i \to \infty$. Then the finite number of ends $\mathcal{E}(P)$ is related to the area growth by:

$$\liminf_{i\to\infty} \frac{\operatorname{Vol}(\partial D_{t_i})}{\operatorname{Vol}(S_{t_i}^{0,m-1})} \leq \mathcal{E}(P).$$

Concerning compactification, we have the following result given by Bessa, Jorge, and Montenegro in [2], and by Bessa and Costa in [3]:

Corollary 4.3 *Let $\varphi : P^m \longrightarrow \mathbb{K}^n(b)$ be a complete non-compact immersion in the real space form $\mathbb{K}^n(b)$, $(b \leq 0)$. Let us suppose that for all points $x \in P \setminus B_{R_0}^P(o)$ (for sufficiently large R_0, where o is a pole in $\mathbb{K}^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$):*

$$\|B_x^P\| \leq c h_b(\rho^P(x)),$$

where $\rho^P(x)$ is the (intrinsic) distance to a fixed $x_o \in \varphi^{-1}(o)$ and c is a positive constant such that $c < 1$, and

$$h_b(r) = \eta_{w_b}(r) = \begin{cases} 1/r & \text{if } b = 0 \\ \sqrt{-b}\coth(\sqrt{-b}r) & \text{if } b < 0 \end{cases}$$

is the mean curvature of the geodesic spheres in $\mathbb{K}^n(b)$. Then P is properly immersed in $\mathbb{K}^n(b)$ and it is diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

Our last result concerns isometric immersions in $\mathbb{H}^n(b) \times \mathbb{R}^l$:

Corollary 4.4 *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b) \times \mathbb{R}^l$ be a complete non-compact immersion. Let us consider a pole $o \in \mathbb{H}^n(b) \times \mathbb{R}^l$ such that $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that for all points $x \in P \setminus B^P_{R_0}(x_o)$, where $x_o \in \varphi^{-1}(o)$ and for sufficiently large R_0:*

$$\|B_x\| \leq \frac{c}{\rho^P(x)}.$$

Here, $\rho^P(x)$ denotes the intrinsic distance in P from the fixed $x_o \in \varphi^{-1}(o)$ to x and c is a positive constant such that $c < 1$. Then P is properly immersed in $\mathbb{H}^n(b) \times \mathbb{R}^l$ and it is diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

5 Proof of Theorem 3.1

5.1 P Is Properly Immersed

Let us define the following function:

$$F(r) := \int_0^r w(t)dt. \tag{5.1}$$

Observe that F is injective, because $F'(r) = w(r) > 0\ \forall r > 0$, and $F(r) \to \infty$ when $r \to \infty$. Applying Theorem 2.10 and Proposition 2.14, we obtain, for all $x \in P$ and given $X \in T_xP$:

$$\begin{aligned}\mathrm{Hess}^P_x F(r)(X,X) &\geq w'(r(x))\|X\|^2 + w(r(x))\langle B^P_x(X,X), \nabla^N r\rangle \\ &\geq w'(r(x))\|X\|^2 - w(r(x))\|B^P_x\|\|X\|^2.\end{aligned} \tag{5.2}$$

By hypothesis there exists a geodesic ball $B^P_{r_1}(x_0)$ in P, with $r_1 \geq R_0$, such that for any $x \in P \setminus B^P_{r_1}(x_0)$, $\|B^P_x\| \leq c\eta_w(\rho^P(x))$. On the other hand, as $\eta_w(r)$ is non-increasing and $r(x) \leq \rho^P(x)$ because φ is isometric, we have $c\eta_w(\rho^P(x)) \leq c\eta_w(r(x))$, so if $x \in P \setminus B^P_{r_1}$:

$$\begin{aligned}\mathrm{Hess}^P_x F(r)(X,X) &\geq w'(r(x))\|X\|^2 - w(r)c\eta_w(\rho^P(x))\|X\|^2 \\ &\geq w'(r(x))\|X\|^2(1-c) \geq d(1-c) > 0.\end{aligned} \tag{5.3}$$

The above result implies that there exists $r_1 \geq R_0$ such that $F \circ r$ is a strictly convex function outside the geodesic ball in P centered at x_0, $B^P_{r_1}(x_0)$. And hence, as $r(x) \leq \rho^P(x)$ for all $x \in P$ (and therefore $B^P_{r_1}(x_0) \subseteq D_{r_1}$), $F \circ r$ is a strictly convex function outside the extrinsic disc D_{r_1}.

Let $\sigma : [0, \rho^P(x)] \to P^m$ be a minimizing geodesic from x_0 to x.
If we denote as $f = F \circ r$, let us define $h : \mathbb{R} \to \mathbb{R}$ as

$$h(s) = F(r(\sigma(s))) = f(\sigma(s)).$$

Then,

$$(f \circ \sigma)'(s) = h'(s) = \sigma'(s)(f) = \langle \nabla^P f(\sigma(s)), \sigma'(s) \rangle \tag{5.4}$$

and hence,

$$\begin{aligned}(f \circ \sigma)''(s) = h''(s) &= \sigma'(s)(\langle \nabla^P f(\sigma(s)), \sigma'(s) \rangle) \\ &= \langle \nabla^P_{\sigma'(s)} \nabla^P f(\sigma(s)), \sigma'(s) \rangle + \langle \nabla^P f(\sigma(s)), \nabla^P_{\sigma'(s)} \sigma'(s) \rangle \\ &= \mathrm{Hess}^P_{\sigma(s)} f(\sigma(s))(\sigma'(s), \sigma'(s)). \end{aligned} \tag{5.5}$$

From (5.3) we have that $(f \circ \sigma)''(\tau) = \mathrm{Hess}^P f(\sigma(\tau))(\sigma', \sigma') \geq d(1-c)$ for all $\tau \geq r_1$. And for $\tau < r_1$, $(f \circ \sigma)''(\tau) \geq a = \inf_{x \in B^P_{r_1}} \{\mathrm{Hess}^P f(x)(\nu, \nu), |\nu| = 1\}$. Then

$$\begin{aligned}(f \circ \sigma)'(s) &= (f \circ \sigma)'(0) + \int_0^s (f \circ \sigma)''(\tau) d\tau \\ &\geq (f \circ \sigma)'(0) + \int_0^{r_1} a\, d\tau + d \int_{r_1}^s (1-c) d\tau \\ &\geq (f \circ \sigma)'(0) + a r_1 + d(1-c)(s - r_1). \end{aligned} \tag{5.6}$$

On the other hand, as

$$\nabla^P f(\sigma(s)) = \nabla^P F(r(\sigma(s))) = F'(r(\sigma(s))) \nabla^P r|_{\sigma(s)} = w(r(\sigma(s))) \nabla^P r|_{\sigma(s)} \tag{5.7}$$

then

$$\nabla^P f(\sigma(0)) = w(r(\sigma(0))) \nabla^P r|_{\sigma(0)} = w(0) \nabla^P r|_{\sigma(0)} = 0,$$

so we have that

$$(f \circ \sigma)'(0) = \langle \nabla^P f(\sigma(0)), \sigma'(0) \rangle = 0. \tag{5.8}$$

We also have that $(f \circ \sigma)(0) = F(r(\sigma(0))) = F(0) = 0$. Hence, by applying inequality (5.6):

$$f(\sigma(s)) = (f \circ \sigma)(0) + \int_0^s (f \circ \sigma)'(\tau) d\tau \geq a r_1 s + d(1-c) \left\{ \frac{1}{2} s^2 - r_1 s \right\}. \tag{5.9}$$

Therefore,

$$F(r(x)) = f(x) = f(\sigma(\rho^P(x))) = \int_0^{\rho^P(x)} (f \circ \sigma)'(s)\, ds$$

$$\geq \int_0^{\rho^P(x)} ar_1 + d(1-c)(s-r_1)\,ds$$
$$= ar_1\rho^P(x)\ + d(1-c)\left(\frac{\rho^P(x)^2}{2} - r_1\rho^P(x)\right). \tag{5.10}$$

Hence, if $\rho^P \to \infty$, then $F(r(x)) \to \infty$ and then, as F is strictly increasing, $r \to \infty$, so the immersion is proper.

5.2 *P* Has Finite Topology

We are going to see that $\nabla^P r$ never vanishes on $P \setminus D_{r_1}$. To show this, we consider, as in the previous subsection, any geodesic in P emanating from the pole o, $\sigma(s)$. Using inequality (5.6), we have that

$$\langle \nabla^P f(\sigma(s)), \sigma'(s)\rangle = (f\circ\sigma)'(s) \geq ar_1 + d(1-c)(s-r_1) > 0, \quad \forall s > r_1. \tag{5.11}$$

Hence, as $\|\sigma'(s)\| = 1\ \forall s$, then $\|\nabla^P f(\sigma(s))\| > 0$ for all $s > r_1$. But we have computed $\nabla^P f(\sigma(s)) = w(r(\sigma(s)))\nabla^P r|_{\sigma(s)}$, so as $w(r) > 0\ \forall r > 0$, then $\|\nabla^P r|_{\sigma(s)}\| > 0\ \forall s > r_1$ and hence $\nabla^P r|_{\sigma(s)} \neq 0\ \forall s > r_1$. We have proven that $\nabla^P r$ never vanishes on $P \setminus B^P_{r_1}$, so we also have that $\nabla^P r$ never vanishes on $P \setminus D_{r_1}$. Let

$$\phi : \partial D_{r_1} \times [r_1, +\infty) \to P \setminus D_{r_1}$$

be the integral flow of a vector field $\frac{\nabla^P r}{\|\nabla^P r\|^2}$ with

$$\phi(p, r_1) = p \in \partial D_{r_1}.$$

It is obvious that $r(\phi(p,t)) = t$ and

$$\phi(\cdot, t) : \partial D_{r_1} \to \partial D_t$$

is a diffeomorphism. So P has finitely many ends, and each of its ends is of finite topological type.

In fact, by applying Theorem 3.1 in [19], we conclude that, as the extrinsic annuli $A_{r_1,R}(o) = D_R(o) \setminus D_{r_1}(o)$ contain no critical points of the extrinsic distance function $r : P \longrightarrow \mathbb{R}^+$, then $D_R(o)$ is diffeomorphic to $D_{r_1}(o)$ for all $R \geq r_1$ and hence the annuli $A_{r_1,R}(o)$ are diffeomorphic to $\partial D_{r_1} \times [r_1, R]$.

Remark 5.1 To demonstrate Theorem 3.3, we use the same argument as that shown at the beginning of the proof of Theorem 3.1: with the same function $F(r)$ we obtain inequality (5.2). But now, as our hypothesis, we have that $\|B^P_x\| \leq c\eta_w(r(x))$, so inequality $\eta'_w(r) \leq 0$ is not necessary to obtain inequality (5.3).

6 Proof of Theorem 3.4

Initially, we are going to see that P has finite topology. As P is properly immersed, we shall apply Theorem 3.3, and to do so, it must be checked that the hypotheses in that theorem are fulfilled. First, we have hypothesis (1) in Theorem 3.3 because $N = M^n_w$. On the other hand, as $w'(r) \geq d > 0\ \forall r > 0$ and, for some R_0, we have that $\|B^P_x\| \leq \frac{\epsilon(r(x))}{(w'(r(x)))^2}\eta_w(r(x))\ \forall x \in P - D_{R_0}$, where ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$, hence $0 \leq \lim_{r\to\infty} \frac{\epsilon(r)}{(w'(r))^2} \leq \lim_{r\to\infty} \frac{\epsilon(r)}{d^2} = 0$. Therefore, for a constant $c < 1$, there exists R_0 such that $\|B^P_x\| \leq c\eta_w(r(x))\ \forall x \in P - D_{R_0}$. Therefore, as $\varphi : P \longrightarrow M^n_w$ is a proper immersion, by Theorem 3.3 we have that P has finite topological type and thus P has finitely many ends, each of finite topological type. Hence we have, in a way analogous to [1], and for $r_1 \geq R_0$ as in Sect. 5:

$$P - D_{r_1} = \bigcup_{k=1}^{\mathcal{E}(P)} V_k; \tag{6.1}$$

where V_k are disjoint, smooth domains in P. Throughout the rest of the proof, we will work on each end V_k separately. Let V denote one element of the family $\{V_k\}_{k=1}^{\mathcal{E}(P)}$ and, given a fixed radius $t > r_1$, let $\partial V(t)$ denote the set $\partial V(t) = V \cap \partial D_t = V \cap S^w_t$, where S^w_t is the geodesic t-sphere in M^n_w. This set is a hypersurface in P^m, with normal vector $\frac{\nabla^P r}{\|\nabla^P r\|}$, and we are going to estimate its sectional curvatures when $t \to \infty$.

Suppose that e_i, e_j are two orthonormal vectors of $T_p\partial V(t)$ on the point $p \in \partial V(t)$. Then the sectional curvature of the plane expanded by e_i, e_j, using Gauss's formula, is:

$$\begin{aligned}
K_{\partial V(t)}(e_i, e_j) &= K_P(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle - \|B^{\partial V-P}(e_i, e_j)\|^2 \\
&= K_N(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle \\
&\quad - \|B^{\partial V-P}(e_i, e_j)\|^2 + \langle B^P(e_i, e_i), B^P(e_j, e_j)\rangle - \|B^P(e_i, e_j)\|^2 \\
&\geq K_N(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle \\
&\quad - \|B^{\partial V-P}(e_i, e_j)\|^2 - 2\|B^P\|^2,
\end{aligned} \tag{6.2}$$

where $B^{\partial V-P}$ is the second fundamental form of $\partial V(t)$ in P. But this second fundamental form is for two vector fields X, Y in $T\partial V(t)$:

$$\begin{aligned}
B^{\partial V-P}(X, Y) &= \left\langle \nabla^P_X Y, \frac{\nabla^P r}{||\nabla^P r||}\right\rangle \frac{\nabla^P r}{||\nabla^P r||} = \langle \nabla^P_X Y, \nabla^P r\rangle \frac{\nabla^P r}{||\nabla^P r||^2} \\
&= X(\langle Y, \nabla^P r\rangle)\frac{\nabla^P r}{||\nabla^P r||^2} - \langle Y, \nabla^P_X \nabla^P r\rangle \frac{\nabla^P r}{||\nabla^P r||^2} \\
&= -\operatorname{Hess}^P r(X, Y)\frac{\nabla^P r}{||\nabla^P r||^2}.
\end{aligned} \tag{6.3}$$

Then, since for all $X, Y \in T_p M^n_w$,

$$\operatorname{Hess}^{M^n_w} r(X, Y) = \eta_w(r)\langle X, Y\rangle - \langle X, \nabla^{M^n_w} r\rangle\langle Y, \nabla^{M^n_w} r\rangle, \tag{6.4}$$

we have (using the fact that e_i are tangent to the fiber S_t^w and Proposition 2.6) that:

$$K_{M_w^n}(e_i, e_j) = K(t) = \frac{1}{w^2(t)} - \eta_w^2(t). \tag{6.5}$$

So for any $p \in \partial V(t)$ such that $t = r(p)$ is sufficiently large:

$$\begin{aligned}
K_{\partial V(t)}(e_i, e_j) &\geq K_{M_w^n}(e_i, e_j) + \frac{\operatorname{Hess}_p^P r(e_i, e_i)\operatorname{Hess}_p^P r(e_j, e_j)}{||\nabla^P r||^2} \\
&\quad - \frac{\operatorname{Hess}_p^P r(e_i, e_j)^2}{||\nabla^P r||^2} - 2\|B^P\|^2 \\
&\geq K(t) + \frac{\left(\eta_w(t) - \|B^P\|\right)^2 - \|B^P\|^2}{||\nabla^P r||^2} - 2\|B^P\|^2 \\
&\geq \eta_w^2(t)\left(1 - 2\frac{\|B^P\|}{\eta_w(t)} - 2\left(\frac{\|B^P\|}{\eta_w(t)}\right)^2 + \frac{K(t)}{\eta_w^2(t)}\right) \\
&\geq \eta_w^2(t)\left(1 - 4\frac{\|B^P\|}{\eta_w(t)} + \frac{K(t)}{\eta_w^2(t)}\right) \\
&= \eta_w^2(t)\left(1 + \frac{K(t)}{\eta_w^2(t)}\right)\left(1 - 4\frac{\frac{\|B^P\|}{\eta_w(t)}}{1 + \frac{K(t)}{\eta_w^2(t)}}\right) \\
&\geq \frac{1}{w^2(t)}\left(1 - 4\|B^P\|w'(t)w(t)\right) \geq \frac{1}{w^2(t)}\left(1 - 4\epsilon(t)\right),
\end{aligned} \tag{6.6}$$

where we recall that, by hypothesis, $\|B^P\| \leq \frac{\epsilon(t)}{(w'(t))^2}\eta_w(t)$ for all $t = r(x) > R_0$, and ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$.

If we denote as $\delta(t) = \frac{1}{w^2(t)}(1 - 4\epsilon(t))$, for each sufficiently large t we have that $K_{\partial V(t)}(e_i, e_j) \geq \delta(t)$ holds everywhere on $\partial V(t)$ and $\delta(t)$ is a positive constant. Then, the Ricci curvature of $\partial V(t)$ is bounded from below, for these sufficiently large radius t, as

$$\operatorname{Ricc}_{\partial V(t)}(\xi, \xi) \geq \delta(t)(m-2)\|\xi\|^2 > 0 \quad \forall \xi \in T\partial V(t)$$

So, by applying Myers's Theorem, $\partial V(t)$ is compact and has diameter $d(\partial V(t)) \leq \frac{\pi}{\sqrt{\delta(t)}}$ (see [24]). On the other hand, by applying Bishop's Theorem (see Theorem 6 in [6]), we obtain:

$$\operatorname{Vol}(\partial V(t)) \leq \frac{\operatorname{Vol}(S^{0,m-1}(1))}{\sqrt{\delta(t)^{m-1}}} \tag{6.7}$$

and hence

$$\begin{aligned}
\frac{\operatorname{Vol}(\partial V(t))}{\operatorname{Vol}(S_t^w)} &\leq \frac{1}{w(t)^{m-1}\sqrt{\delta(t)^{m-1}}} \\
&= \frac{1}{(1 - 4\epsilon(t))^{(m-1)/2}}.
\end{aligned} \tag{6.8}$$

Therefore, since for t large enough $\mathrm{Vol}(\partial D_t(o)) \leq \sum_{i=1}^{\mathcal{E}(P)} \mathrm{Vol}(\partial V_i(t))$, where V_i denotes each end of P then:

$$\frac{\mathrm{Vol}(\partial D_t(o))}{\mathrm{Vol}(S_t^w)} \leq \frac{\mathcal{E}(P)}{(1-4\epsilon(t))^{(m-1)/2}}. \tag{6.9}$$

7 Proof of Theorem 3.5

To show assertion (1) we apply Theorem 2.15 and inequality (3.1) in Theorem 3.4 to obtain, for sufficiently large r (we suppose that $\varphi^{-1}(o_w) \neq \emptyset$, and take $o \in \varphi^{-1}(o_w)$ in order to have that $\mathrm{Vol}(D_r(o)) \geq \mathrm{Vol}(B_r^w)$ for all $r > 0$):

$$1 \leq \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq \frac{\mathrm{Vol}(\partial D_r(o))}{\mathrm{Vol}(S_r^w)} \leq \frac{\mathcal{E}(P)}{(1-4\epsilon(r))^{(m-1)/2}}. \tag{7.1}$$

Moreover, we know (again using Theorem 2.15) that the volume growth function is non-decreasing.

Therefore, taking limits in (7.1) when r goes to ∞, we obtain:

$$1 \leq \lim_{r\to\infty} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} = \mathrm{Sup}_{r>0} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq \mathcal{E}(P). \tag{7.2}$$

Now, to prove assertion (2), if P has one end, we have that:

$$1 \leq \mathrm{Sup}_{r>0} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq 1. \tag{7.3}$$

Hence, as $f(r) = \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)}$ is non-decreasing, then $f(r) = 1\ \forall r > 0$, so we have equality in inequality (2.6) for all $r > 0$, and P is a minimal cone (see [16] for details).

8 Proof of Theorems 1.1 and 1.2 and the Corollaries

8.1 Proof of Theorem 1.1

We are going to apply Theorem 3.5. To do this, we must check hypotheses (1) and (2) in Theorem 3.4.

In this case we have that the ambient manifold is the hyperbolic space $\mathbb{H}^n(b)$. Therefore all of its points are poles, so there exist at least $o \in \mathbb{H}^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$. As is known, hyperbolic space $\mathbb{H}^n(b)$ is a model space with $w(r) = w_b(r) = \frac{1}{\sqrt{-b}} \sinh \sqrt{-b}r$ so $w_b'(r) = \cosh \sqrt{-b}r \geq 1\ \forall r > 0$.

Therefore, hypothesis (2) in Theorem 3.4 is fulfilled in this context. Concerning hypothesis (1), it is straightforward that

$$\|B_x^P\| \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}r(x)}} \leq \frac{\epsilon(r)\sqrt{-b}}{\sinh\sqrt{-b}r \cosh\sqrt{-b}r}$$

$$= \frac{\epsilon(r)}{\cosh^2 \sqrt{-b}r} \sqrt{-b} \coth \sqrt{-b}r = \frac{\epsilon(r)}{(w_b'(r))^2} \eta_{w_b}(r), \tag{8.1}$$

where $\epsilon(r) = \frac{\delta(r(x))}{4\sqrt{-b}}$ goes to 0 when r goes to ∞.

Hence, hypothesis (1) in Theorem 3.4 is also fulfilled, and so by applying inequality (3.2) in Theorem 3.5 (because P is minimal):

$$1 \leq \lim_{r\to\infty} \frac{\text{Vol}(D_r)}{\text{Vol}(B_r^{w_b})} \leq \mathcal{E}(P). \tag{8.2}$$

Finally, when P has one end, then $\lim_{r\to\infty} \frac{\text{Vol}(D_r)}{\text{Vol}(B_r^{w_b})} = 1$. Since P is minimal, by Theorem 2.15, $f(r) = \frac{\text{Vol}(D_r)}{\text{Vol}(B_r^{w_b})}$ is a monotone non-decreasing function and, on the other hand, $f(r) \geq 1\ \forall r > 0$ because of inequality (2.7). Hence $f(r) = 1\ \forall r > 0$, so $f'(r) = 0\ \forall r > 0$. This last equality implies equality in inequality (2.6) for all $r > 0$ (see [16] or [17] for details), and we apply the equality assertion in Theorem 2.15 to conclude that P is totally geodesic in $\mathbb{H}^n(b)$.

8.2 Proof of Theorem 1.2

In this case, we apply Theorem 3.5, with $M_w^n = \mathbb{R}^n$, i.e., with $w(r) = w_0(r) = r$, $(b = 0)$. Hence, $w_0'(r) = 1 > 0\ \forall r > 0$ and $\eta_0(r) = \frac{1}{r}$ and hypotheses (1) and (2) in this theorem are trivially satisfied.

When P has only one end we conclude, as before, that the volume growth function is constant, so we conclude equality in (2.6) for all radii $r > 0$. Hence on applying the corresponding equality assertion in Theorem 2.15, P is totally geodesic in $\mathbb{R}^n$.

8.3 Proof of Corollary 4.1

We are now considering a complete and proper immersion in $\mathbb{H}^n(b)$, as in Theorem 1.1, but P is not necessarily minimal. In this setting hypotheses (1) and (2) in Theorem 3.4 are fulfilled (as we have verified in the proof above, without using minimality). Hence, by taking limits in (3.1) when we consider an increasing sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ when $i \to \infty$, we have:

$$\liminf_{i\to\infty} \frac{\text{Vol}(\partial D_{t_i})}{\text{Vol}(S_{t_i}^{b,m-1})} \leq \mathcal{E}(P).$$

8.4 Proof of Corollary 4.2

Hypotheses (1) and (2) in Theorem 3.4 are trivially satisfied, and the result is obtained by arguing as in the proof of Corollary 4.1.

8.5 Proof of Corollary 4.3

We apply Theorem 3.1. Our ambient manifold is $\mathbb{K}^n(b)$, $(b \leq 0)$, so hypothesis (1) about the bounds for the radial sectional curvature holds, and as $w(r) = w_b(r)$, hence

$w_b'(r) \geq 1 > 0\ \forall r > 0$ and $\eta_{w_b}'(r) \leq 0\ \forall r > 0$. This means that hypothesis (3) is fulfilled. Hypothesis (2) in Theorem 3.1 holds because

$$\|B_x^P\| \leq ch_b(\rho^P(x)),$$

where $\rho^P(x)$ is the (intrinsic) distance to a fixed $x_o \in \varphi^{-1}(o)$ and c is a positive constant such that $c < 1$.

8.6 Proof of Corollary 4.4

We again apply Theorem 3.1, taking into account that the ambient space is the Cartan–Hadamard manifold $\mathbb{H}^n(b) \times \mathbb{R}^l$, and the model space used for comparison is $\mathbb{R}^m$, with $w(r) = w_0(r) = r$.

Acknowledgements The authors wish to thank the referee for his/her useful suggestions.

References

1. Anderson, M.: The compactification of a minimal submanifold by the Gauss map. Preprint IHES (1984)
2. Bessa, G.P., Jorge, L., Montenegro, J.F.: Complete submanifolds of $\mathbb{R}^n$ with finite topology. Commun. Anal. Geom. **15**(4), 725–732 (2007)
3. Bessa, G.P., Costa, M.S.: On submanifolds with Tamed second fundamental form. Glasg. Math. J. **51**, 669–680 (2009)
4. Bianchini, B., Mari, L., Rigoli, M.: Some generalizations of Calabi compactness theorem. Preprint (2011). arXiv:1112.3703v1
5. Bianchini, B., Mari, L., Rigoli, M.: On some aspects of oscillation theory and geometry. To appear in Mem. Amer. Math. Soc. Preprint
6. Chavel, I.: Eigenvalues in Riemannian geometry. Pure and Applied Mathematics, vol. 115. Academic Press, Orlando (1984). Including a chapter by Burton Randol, with an appendix by Jozef Dodziuk
7. Chen, Q.: On the volume growth and the topology of complete minimal submanifolds of a Euclidean space. J. Math. Sci. Univ. Tokyo **2**, 657–669 (1995)
8. do Carmo, M.P.: do Carmo Riemannian Geometry. Birkhäuser, Boston (1992)
9. Greene, R., Wu, H.: Function theory on manifolds which possess a pole. Lecture Notes in Math., vol. 699. Springer, Berlin (1979)
10. Greene, R., Wu, H.: Gap theorems for noncompact Riemannian manifolds. Duke Math. J. **49**, 731–756 (1982)
11. Greene, R., Wu, H.: On a new gap phenomenon in Riemannian geometry. Proc. Nat. Acad. Sci. USA **79**, 714–715 (1982)
12. Grigor'yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Am. Math. Soc. **36**, 135–249 (1999)
13. Kasue, A., Sugahara, K.: Gap theorems for certain submanifolds of Euclidean spaces and hyperbolic space forms. Osaka J. Math. **24**, 679–704 (1987)
14. Kasue, A.: Gap theorems for minimal submanifolds of Euclidean space. J. Math. Soc. Jpn. **38**(3), 473–492 (1986)
15. Markvorsen, S., Palmer, V.: Torsional rigidity of minimal submanifolds. Proc. Lond. Math. Soc. (3) **93**, 253–272 (2006)
16. Markvorsen, S., Palmer, V.: The relative volume growth of minimal submanifolds. Arch. Math. **79**, 507–514 (2002)
17. Markvorsen, S., Palmer, V.: Generalized isoperimetric inequalities for extrinsic balls in minimal submanifolds. J. Reine Angew. Math. **551**, 101–121 (2002)
18. Markvorsen, S., Palmer, V.: On the isoperimetric rigidity of extrinsic minimal balls. Differ. Geom. Appl. **18**, 47–54 (2003)

19. Milnor, J.: Morse Theory. Lecture Notes in Math., vol. 699. Springer, Berlin (1979)
20. De Oliveira, G.: Compactification of minimal submanifolds of hyperbolic space. Commun. Anal. Geom. **1**, 1–29 (1993)
21. O'Neill, B.: Semi-Riemannian Geometry; With Applications to Relativity. Academic Press, New York (1983)
22. Palmer, V.: Isoperimetric inequalities for extrinsic balls in minimal submanifolds and their applications. J. Lond. Math. Soc. (2) **60**(2), 607–616 (1999)
23. Palmer, V.: On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature. In: Simon Stevin Transactions on Geometry, vol. 1, pp. 131–159. Simon Stevin Institute for Geometry, Tilburg (2010)
24. Sakai, T.: Riemannian Geometry. Translations of Mathematical Monographs, vol. 149. AMS, Reading (1996). Addison-Wesley, Reading (1990)
25. Schoen, R.: Uniqueness, symmetry and embeddedness of minimal surfaces. J. Differ. Geom. **18**, 791–809 (1983)