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Musielak–Orlicz–Hardy Spaces Associated with Operators and Their Applications

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Abstract Let $\mathcal{X}$ be a metric space with doubling measure and L a nonnegative self-adjoint operator in $L^2(\mathcal{X})$ satisfying the Davies–Gaffney estimates. Let $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ be a function such that $\varphi(x, \cdot)$ is an Orlicz function, $\varphi(\cdot, t) \in \mathbb{A}_\infty(\mathcal{X})$ (the class of uniformly Muckenhoupt weights), its uniformly critical upper type index $I(\varphi) \in (0, 1]$, and it satisfies the uniformly reverse Hölder inequality of order $2/[2 - I(\varphi)]$. In this paper, the authors introduce a Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$, by the Lusin area function associated with the heat semigroup generated by L, and a Musielak–Orlicz BMO-type space $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$, which is further proved to be the dual space of $H_{\varphi,L}(\mathcal{X})$ and hence whose φ-Carleson measure characterization is deduced. Characterizations of $H_{\varphi,L}(\mathcal{X})$, including the atom, the molecule, and the Lusin area function associated with the Poisson semigroup of L, are presented. Using the atomic characterization, the authors characterize $H_{\varphi,L}(\mathcal{X})$ in terms of the Littlewood–Paley g_λ^*-function $g_{\lambda,L}^*$ and establish a Hörmander-type spectral multiplier theorem for L on $H_{\varphi,L}(\mathcal{X})$. Moreover, for the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ associated with the Schrödinger operator $L := -\Delta + V$, where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the authors obtain its several equivalent characterizations in terms of the non-tangential maximal function, the radial maximal function, the atom, and the molecule; finally, the authors show that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to the Musielak–Orlicz space $L^\varphi(\mathbb{R}^n)$ when $i(\varphi) \in (0, 1]$, and from $H_{\varphi,L}(\mathbb{R}^n)$ to the Musielak–Orlicz–Hardy space $H_\varphi(\mathbb{R}^n)$ when $i(\varphi) \in (\frac{n}{n+1}, 1]$, where $i(\varphi)$ denotes the uniformly critical lower type index of φ.

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1 Introduction

The real-variable theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$, initiated by Stein and Weiss [88], plays an important role in various fields of analysis (see, for example, [41, 72, 83, 87]). It is well known that the Hardy space $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$ is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$; for example, the classical Riesz transform is bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Moreover, the practicability of $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, as a substitute for $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, comes from its several equivalent real-variable characterizations, which were originally motivated by Fefferman and Stein in their seminal paper [41]. Among these characterizations, a very important and useful characterization of the Hardy spaces $H^p(\mathbb{R}^n)$ is their atomic characterizations, which were obtained by Coifman [22] when $n = 1$ and Latter [67] when $n > 1$. Moreover, a direct extension of the atomic characterization of the Hardy spaces is the molecular characterization established by Taibleson and Weiss [91].

On the other hand, as a natural generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum–Orlicz in [9] and Orlicz in [77], which has extensive applications in several branches of mathematics (see, for example, [5, 48, 55, 71, 79, 80] for more details). Moreover, the Orlicz–Hardy space, introduced and studied in [56, 89, 92], is also a suitable substitute of the Orlicz space in the study of the boundedness of operators (see, for example, [56–58, 60, 89, 92]). Furthermore, weighted local Orlicz–Hardy spaces and their dual spaces were also studied in [94]. All theories of these function spaces are intimately connected with the *Laplace operator* $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$.

Recall that the classical BMO space (the *space of functions with bounded mean oscillation*) was originally introduced by John and Nirenberg [61] to solve some problems in partial differential equations. Since Fefferman and Stein [41] proved that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$, the space $\mathrm{BMO}(\mathbb{R}^n)$ plays an important role in not only partial differential equations but also harmonic analysis (see, for example, [35, 41] for more details). Moreover, the generalized space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ was introduced and studied in [47, 56, 89, 92] and it was proved therein to be the dual space of the Orlicz–Hardy space $H_\Phi(\mathbb{R}^n)$, where Φ denotes the Orlicz function on $(0, \infty)$ and $\rho(t) := t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0, \infty)$. Here and in what follows, Φ^{-1} denotes the *inverse function* of Φ.

Recently, Ky [63] introduced a new Musielak–Orlicz–Hardy space, $H_\varphi(\mathbb{R}^n)$, via the grand maximal function, which contains both the Orlicz–Hardy space in [56, 89] and the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$ in [44, 90] as the special

cases. Here, $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a function such that $\varphi(x, \cdot)$ is an Orlicz function of uniformly upper type 1 and lower type p for some $p \in (0, 1]$ (see Sect. 2 below for the definitions of uniformly upper or lower types), and $\varphi(\cdot, t)$ is a Muckenhoupt weight, and $A_q(\mathbb{R}^n)$ with $q \in [1, \infty]$ denotes the *class of Muckenhoupt's weights* (see, for example, [43, 44, 46] for their definitions and properties). Moreover, the Musielak–Orlicz BMO-type space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ was also introduced and further proved to be the dual space of $H_\varphi(\mathbb{R}^n)$ in [63] by using the atomic characterization of $H_\varphi(\mathbb{R}^n)$ established in [63]. Furthermore, some interesting applications of the spaces $H_\varphi(\mathbb{R}^n)$ and $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ were given in [11, 13, 14, 63–66]. Moreover, the radial and the non-tangential maximal functions characterizations, the Littlewood–Paley function characterization and the molecular characterization of $H_\varphi(\mathbb{R}^n)$ were obtained in [54, 69]. As an application of the Lusin area function characterization of $H_\varphi(\mathbb{R}^n)$, the φ-Carleson measure characterization of the space $\mathrm{BMO}_\varphi(\mathbb{R}^n)$ was obtained in [54]. Furthermore, the local Musielak–Orlicz–Hardy space and its dual space were studied in [97]. It is worth pointing out that Musielak–Orlicz functions are the natural generalization of Orlicz functions (see, for example, [31, 32, 63, 73, 76]) and the motivation to study function spaces of Musielak–Orlicz type is attributed to their extensive applications to many branches of physics and mathematics (see, for example, [11–14, 31, 32, 63, 64, 68] for more details).

In recent years, the study of function spaces associated with different operators inspired great interests (see, for example, [6–8, 35–37, 51–53, 57–60, 86, 93] and their references). More precisely, Auscher, Duong, and McIntosh [6] initially studied the Hardy space $H^1_L(\mathbb{R}^n)$ associated with the operator L whose heat kernel satisfies a pointwise Poisson upper bound estimate. Based on this, Duong and Yan [36, 37] introduced the BMO-type space $\mathrm{BMO}_L(\mathbb{R}^n)$ associated with L and proved that the dual space of $H^1_L(\mathbb{R}^n)$ is just $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$. Moreover, Yan [93] further generalized these results to the Hardy space $H^p_L(\mathbb{R}^n)$ with $p \in (0, 1]$ close to 1 and its dual space. Also, the Orlicz–Hardy space and its dual space associated with such an L were studied in [60].

Moreover, Hofmann and Mayboroda [52] and Hofmann et al. [53] introduced the Hardy and Sobolev spaces associated with a second-order divergence form elliptic operator L on $\mathbb{R}^n$ with bounded measurable complex coefficients and these operators may not have the pointwise heat kernel bounds, and further established several equivalent characterizations for these spaces and studied their dual spaces. Meanwhile, the Orlicz–Hardy space and its dual space associated with L were independently studied in [58]. Furthermore, Orlicz–Hardy spaces associated with a second-order divergence form elliptic operator on the strongly Lipschitz domain of $\mathbb{R}^n$ were studied in [95, 96]. It is worth pointing out that the strongly Lipschitz domain of $\mathbb{R}^n$ is a special space of homogeneous type in the sense of Coifman and Weiss [25]. Recall that the Hardy spaces on strongly Lipschitz domains associated with the Laplace operator having some boundary conditions were originally and systematically studied by Chang et al. in [16–19] and Auscher et al. [8].

On the other hand, the Hardy space associated with the Schrödinger operator $-\Delta + V$ was studied in [39, 40], where the nonnegative potential V satisfies the reverse Hölder inequality (see, for example, [44, 46] for the definition of the reverse Hölder inequality). More generally, for nonnegative self-adjoint operators L satisfying the Davies–Gaffney estimates, Hofmann et al. [51] studied the Hardy space

$H^1_L(\mathcal{X})$ associated with L and its dual space on a metric measure space $\mathcal{X}$, which was extended to the Orlicz–Hardy space in [57]. As a special case of this setting, several equivalent characterizations and some applications of the Hardy space $H^1_L(\mathbb{R}^n)$ and the Orlicz–Hardy space $H_{\Phi,L}(\mathbb{R}^n)$ associated with the Schrödinger operator $L := -\Delta + V$ were, respectively, obtained in [51] and [57], where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, Song and Yan [86] studied the weighted Hardy space $H^1_{\omega,L}(\mathbb{R}^n)$ associated with the Schrödinger operator L, where $\omega \in A_1(\mathbb{R}^n)$. Very recently, some special Musielak–Orlicz–Hardy spaces associated with the Schrödinger operator $L := -\Delta + V$ on $\mathbb{R}^n$, where the nonnegative potential V satisfies the reverse Hölder inequality of order $n/2$, were studied by Ky [65, 66] and further applied to the study of commutators of singular integral operators associated with the operator L.

Let $\mathcal{X}$ be a metric measure space, L a nonnegative self-adjoint operator on $L^2(\mathcal{X})$ satisfying the Davies–Gaffney estimates, and $E(\lambda)$ the spectral resolution of L. For any bounded Borel function $m:\ [0,\infty) \to \mathbb{C}$, by using the spectral theorem, it is well known that the operator

$$m(L) := \int_0^\infty m(\lambda)\, dE(\lambda) \tag{1.1}$$

is well defined and bounded on $L^2(\mathcal{X})$. It is an interesting problem to find some sufficient conditions on m and L such that $m(L)$ in (1.1) is bounded on various function spaces on $\mathcal{X}$, which was extensively studied (see, for example, [2, 3, 10, 21, 30, 33, 34, 38, 50, 78] and their references). Specially, Duong and Yan [38] proved that $m(L)$ is bounded on the Hardy space $H^p_L(\mathcal{X})$, with $p \in (0,\infty)$, associated with L when $\mathcal{X}$ is a metric space with doubling measure and the function m satisfies a Hörmander-type condition.

Throughout the whole paper, let $\mathcal{X}$ be a metric space with doubling measure μ and L a nonnegative self-adjoint operator in $L^2(\mathcal{X})$ satisfying the Davies–Gaffney estimates. Let $\varphi : \mathcal{X} \times [0,\infty) \to [0,\infty)$ be a growth function as in Definition 2.4 below, which means that $\varphi(x,\cdot)$ is an Orlicz function (see Sect. 2.3 below), $\varphi(\cdot,t) \in \mathbb{A}_\infty(\mathcal{X})$ (the class of uniformly Muckenhoupt weights in Definition 2.3 below), and its uniformly critical upper type index $I(\varphi) \in (0,1]$ (see (2.10) below). Moreover, we *always assume* that $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ (see Definition 2.3 below). A typical example of such a φ is

$$\varphi(x,t) := \omega(x)\Phi(t) \tag{1.2}$$

for all $x \in \mathcal{X}$ and $t \in [0,\infty)$, where $\omega \in A_\infty(\mathcal{X})$ (the *class of Muckenhoupt weights*) and Φ is an Orlicz function on $[0,\infty)$ of upper type $p_1 \in (0,1]$ and lower type $p_2 \in (0,1]$ (see (2.9) below for the definition of types). Let $x_0 \in \mathcal{X}$. Another typical and useful example of the growth function φ is

$$\varphi(x,t) := \frac{t^\alpha}{[\ln(e+d(x,x_0))]^\beta + [\ln(e+t)]^\gamma} \tag{1.3}$$

for all $x \in \mathcal{X}$ and $t \in [0,\infty)$ with some $\alpha \in (0,1]$, $\beta \in [0,n)$ and $\gamma \in [0, 2\alpha(1+\ln 2)]$ (see Sect. 2.3 for more details). It is worth pointing out that such a function φ naturally

appears in the study of the pointwise multiplier characterization for the BMO-type space on the metric space with doubling measure (see [74]).

Motivated by [38, 51, 57, 63], in this paper, we study the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ and its dual space. More precisely, for all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$, define

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{d\mu(y)\,dt}{V(x,t)t} \right\}^{1/2}. \tag{1.4}$$

Here and in what follows, $\Gamma(x) := \{(y,t) \in \mathcal{X} \times (0,\infty) : \ d(x,y) < t\}$, d denotes the metric on $\mathcal{X}$, $B(x,t) := \{y \in \mathcal{X} : \ d(x,y) < t\}$, μ denotes the nonnegative Borel regular measure on $\mathcal{X}$ and $V(x,t) := \mu(B(x,t))$. The *Musielak–Orlicz–Hardy space* $H_{\varphi,L}(\mathcal{X})$ is then defined to be the completion of the set $\{f \in H^2(\mathcal{X}) : \ S_L(f) \in L^\varphi(\mathcal{X})\}$ with respect to the quasi-norm

$$\begin{aligned} \|f\|_{H_{\varphi,L}(\mathcal{X})} &:= \|S_L(f)\|_{L^\varphi(\mathcal{X})} \\ &:= \inf\left\{ \lambda \in (0,\infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) d\mu(x) \le 1 \right\}, \end{aligned}$$

where $H^2(\mathcal{X}) := \overline{R(L)}$ and $\overline{R(L)}$ denotes the *closure of the range of* L *in* $L^2(\mathcal{X})$.

In this paper, we first establish the atomic decomposition of $H_{\varphi,L}(\mathcal{X})$ and further obtain its molecular decomposition. Using the atomic and the molecular decompositions of $H_{\varphi,L}(\mathcal{X})$, we then prove that its dual space is the Musielak–Orlicz BMO-type space $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$, which is characterized by the φ-Carleson measure, and further establish the atomic and the molecular characterizations of $H_{\varphi,L}(\mathcal{X})$. We also obtain another characterization of $H_{\varphi,L}(\mathcal{X})$ via the Lusin area function associated with the Poisson semigroup of L. As applications, by using the atomic characterization, we prove that Littlewood–Paley functions g_L and $g^*_{\lambda,L}$ are bounded from $H_{\varphi,L}(\mathcal{X})$ to the Musielak–Orlicz space $L^\varphi(\mathcal{X})$; as a corollary, we characterize $H_{\varphi,L}(\mathcal{X})$ in terms of the Littlewood–Paley g^*_λ-function $g^*_{\lambda,L}$. We further establish a Hörmander-type spectral multiplier theorem for L on $H_{\varphi,L}(\mathcal{X})$ by using the atomic and the molecular characterizations of $H_{\varphi,L}(\mathcal{X})$. As further applications, we obtain several equivalent characterizations of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ associated with the Schrödinger operator $L := -\Delta + V$, where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, in terms of the Lusin-area function, the non-tangential maximal function, the radial maximal function, the atom, and the molecule. Finally, we show that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$ when $i(\varphi) \in (0,1]$ and from $H_{\varphi,L}(\mathbb{R}^n)$ to the Musielak–Orlicz–Hardy space $H_\varphi(\mathbb{R}^n)$ when $i(\varphi) \in (\frac{n}{n+1}, 1]$, where $i(\varphi)$ denotes the uniformly critical lower type index of φ (see (2.11) below).

The key step of the above approach is to establish the atomic (molecular) characterization of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$. To this end, we inherit a method used in [7, 57, 58]. We first establish the atomic decomposition of the Musielak–Orlicz tent space $T_\varphi(\mathcal{X} \times (0,\infty))$ associated with φ, whose proof implies that if $f \in T_\varphi(\mathcal{X} \times (0,\infty)) \cap T^2_2(\mathcal{X} \times (0,\infty))$, then the atomic decomposition of f holds true in both $T_\varphi(\mathcal{X} \times (0,\infty))$ and $T^2_2(\mathcal{X} \times (0,\infty))$. We point out that in this paper, by the assumptions on L, we only know that the Lusin area function S_L as

in (1.4) is bounded on $L^2(\mathcal{X})$ (see (2.7) below). To prove that the atomic decomposition of $f \in T_\varphi(\mathcal{X} \times (0,\infty)) \cap T_2^2(\mathcal{X} \times (0,\infty))$ holds true in $T_2^2(\mathcal{X} \times (0,\infty))$ (see Corollary 3.5 below), we need the *additional assumption* that $\varphi(\cdot, t)$ for all $t \in [0,\infty)$ belongs to the uniformly reverse Hölder class $\mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$. Then by the fact that the operator $\pi_{\Psi,L}$ in (4.2) below is bounded from $T_2^2(\mathcal{X} \times (0,\infty))$ to $L^2(\mathcal{X})$, we further obtain the $L^2(\mathcal{X})$-convergence of the corresponding atomic decomposition for functions in $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, since for all $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, $t^2 L e^{-t^2 L} f \in T_2^2(\mathcal{X} \times (0,\infty)) \cap T_\varphi(\mathcal{X} \times (0,\infty))$. This technique plays a fundamental role in the whole paper.

We remark that the method used to obtain the atomic characterization of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ in this paper is different from that in [86], but more close to the method in [15, 54, 57]. More precisely, in [86], the atomic characterization of the weighted Hardy space $H_L^1(\mathbb{R}^n)$, associated with the Schrödinger operator L, was established by using the Calderón reproducing formula associated with L and a subtle decomposition of all dyadic cubes in $\mathbb{R}^n$. However, in this paper, we establish the atomic characterization of $H_{\varphi,L}(\mathcal{X})$ by using the Calderón reproducing formula associated with L (see (4.14) below), the atomic decomposition of the Musielak–Orlicz tent space established in Theorem 3.1 below and some boundedness (see Proposition 4.6 below) of the operator $\pi_{\Psi,L}$ defined in (4.2) below. Moreover, we also point out that the notion of atoms in our atomic decomposition of the Musielak–Orlicz tent space is different from that in [15]. Since the weight also appears in the norm of atoms used by Bui and Duong [15] when establishing the atomic decomposition of elements in the weighted tent space, Bui and Duong [15] had to require the weight $\omega \in A_1(\mathcal{X}) \cap RH_{2/(2-p)}(\mathcal{X})$ in order to obtain the atomic decomposition of the weighted Hardy space $H^p_{\omega,L}(\mathcal{X})$ with $p \in (0,1]$ (see the proof of [15, Proposition 3.9] for the details). Instead of this, we do not use the weight in the norm of our $T_\varphi(\mathcal{X} \times (0,\infty))$-atoms. Due to this subtle choice, we are able to relax the requirements on the growth function into $\varphi \in \mathbb{A}_\infty(\mathcal{X}) \cap \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, which essentially improves the results of Bui and Duong [15] even when φ is as in (1.2).

Another important estimate, appeared in the approach of this paper, is that there exists a positive constant C such that, for any $\lambda \in \mathbb{C}$ and (φ, M)-atom α adapted to the ball B (or any (φ, M, ϵ)-molecule α adapted to the ball B),

$$\int_{\mathcal{X}} \varphi\big(x, S_L(\lambda\alpha)(x)\big)\, d\mu(x) \le C\varphi\big(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big); \tag{1.5}$$

see Definitions 4.3 and 4.4 below for the notions of (φ, M)-atoms and (φ, M, ϵ)-molecules. A main difficulty to prove (1.5) is how to take $S_L(\lambda\alpha)(x)$ out of the position of the time variable of φ. In [57, 58], to obtain (1.5) when φ is as in (1.2) with $\omega \equiv 1$, it was assumed that Φ is a concave Orlicz function on $(0,\infty)$. In this case, Jensen's inequality does the job. In the present setting, the spatial variable and the time variable of φ are combinative, so Jensen's inequality does not work even when φ is concave about the time variable. To overcome this difficulty, we subtly use the properties of φ which are the uniformly upper $p_1 \in (0,1]$ and lower type $p_2 \in (0,1]$ (see the proof of (4.5) below).

Precisely, this paper is organized as follows. In Sect. 2, we first recall some notions and notation on metric measure spaces and then describe some basic assumptions on

the operator L studied in this paper. We also recall some notation, some examples and some basic properties concerning growth functions considered in this paper.

In Sect. 3, we first recall some notions about tent spaces and then study the Musielak–Orlicz tent space $T_\varphi(\mathcal{X} \times (0,\infty))$ associated with growth function φ. The main target of this section is to establish the atomic characterization for $T_\varphi(\mathcal{X} \times (0,\infty))$ (see Theorem 3.1 below). Assume further that $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$. As a byproduct, we know that if $f \in T_\varphi(\mathcal{X} \times (0,\infty)) \cap T_2^2(\mathcal{X} \times (0,\infty))$, then the atomic decomposition of f holds true in both $T_\varphi(\mathcal{X} \times (0,\infty))$ and $T_2^2(\mathcal{X} \times (0,\infty))$, which plays an important role in the remainder of this paper (see Corollary 3.5 below). We point out that Theorem 3.1 and Corollary 3.5 completely cover [57, Theorem 3.1 and Corollary 3.1] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ concave.

In Sect. 4, we first introduce the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ and prove that the operator $\pi_{\Psi,L}$ in (4.2) below maps the Musielak–Orlicz tent space $T_\varphi(\mathcal{X} \times (0,\infty))$ continuously into $H_{\varphi,L}(\mathcal{X})$ (see Proposition 4.6 below). By this and the atomic decomposition of $T_\varphi(\mathcal{X} \times (0,\infty))$, we conclude that, for each $f \in H_{\varphi,L}(\mathcal{X})$, there exists an atomic decomposition of f holding true in $H_{\varphi,L}(\mathcal{X})$ (see Corollary 4.8 below). We should point out that to obtain the atomic decomposition of $H_{\varphi,L}(\mathcal{X})$, we borrow some ideas from [51, 57], and the estimate (1.5) is very important for this procedure. Via this atomic decomposition of $H_{\varphi,L}(\mathcal{X})$, we further prove that the dual space of $H_{\varphi,L}(\mathcal{X})$ is just the Musielak–Orlicz BMO-type space $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$ (see Theorem 4.16 below). As an application of this duality, we establish the φ-Carleson measure characterization of the space $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$ (see Theorem 4.19 below).

We remark that when φ is as in (1.2) with $\omega \equiv 1$ and Φ concave, the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ and the Musielak–Orlicz BMO-type space $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$ are the Orlicz–Hardy space $H_{\Phi,L}(\mathcal{X})$ and the BMO-type space $\mathrm{BMO}_{\rho,L}(\mathcal{X})$ introduced in [57] respectively.

In Sect. 5, by Proposition 4.9 and Theorem 4.16, we establish the equivalence between $H_{\varphi,L}(\mathcal{X})$ and the atomic (resp., molecular) Musielak–Orlicz–Hardy space $H^M_{\varphi,\mathrm{at}}(\mathcal{X})$ (resp., $H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$) (see Theorem 5.5 below). We notice that the series in $H^M_{\varphi,\mathrm{at}}(\mathcal{X})$ (resp., $H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$) is required to converge in the norm of $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$, where $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ denotes the dual space of $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$; while in Corollary 4.8 below, the atomic decomposition holds true in $H_{\varphi,L}(\mathcal{X})$. Applying its atomic characterization, we further characterize the Hardy space $H_{\varphi,L}(\mathcal{X})$ in terms of the Lusin area function associated with the Poisson semigroup of L (see Theorem 5.7 below). Observe that Theorems 5.5 and 5.7 completely cover [57, Theorems 5.1 and 5.2] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ concave.

In Sect. 6, we give some applications of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ to the boundedness of operators. More precisely, in Sect. 6.1, we prove that the Littlewood–Paley g-function g_L is bounded from $H_{\varphi,L}(\mathcal{X})$ to the Musielak–Orlicz space $L^\varphi(\mathcal{X})$ (see Theorem 6.3 below); in Sect. 6.2, we show that the g^*_λ-function $g^*_{\lambda,L}$ is bounded from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$ (see Theorem 6.7 below). As a corollary, we characterize $H_{\varphi,L}(\mathcal{X})$ in terms of the g^*_λ-function $g^*_{\lambda,L}$ (see Corollary 6.9 below). Observe that when $\mathcal{X} := \mathbb{R}^n$ and $L := -\Delta$, $g^*_{\lambda,L}$ is just the classical Littlewood–Paley g^*_λ-function. Moreover, the range of λ in Theorem 6.7 coincides

with the corresponding result on the classical Littlewood–Paley g_λ^*-function on $\mathbb{R}^n$ in the case that φ is as in (1.2) with that $\omega \in A_q(\mathbb{R}^n)$, $q \in [1, \infty)$, and $\Phi(t) := t^p$ for all $t \in [0, \infty)$, $p \in (0, 1]$ (see Remark 6.8 below). Thus, in some sense, the range of λ in Theorem 6.7 is sharp, which is attributed to the use of the unweighted norm in our definition of tent atoms, appearing in the atomic decomposition of the tent space $T_\varphi(\mathcal{X} \times (0, \infty))$. Finally, in Sect. 6.3, we establish a Hörmander-type spectral multiplier theorem for $m(L)$ as in (1.1) on $H_{\varphi,L}(\mathcal{X})$ (see Theorem 6.10 below). Let $p \in (0, 1]$. We remark that Theorem 6.10 covers [38, Theorem 1.1] in the case that $p \in (0, 1]$ by taking $\varphi(x, t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$. A typical example of the function m satisfying the condition of Theorem 6.10 is $m(\lambda) = \lambda^{i\gamma}$ for all $\lambda \in \mathbb{R}$ and some real value γ, where i denotes the *unit imaginary number* (see Corollary 6.13 below).

As applications, in Sect. 7, we study the Musielak–Orlicz–Hardy spaces $H_{\varphi,L}(\mathbb{R}^n)$ associated with the Schrödinger operator $L := -\Delta + V$, where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. As an application of Theorems 5.5 and 5.7, we characterize $H_{\varphi,L}(\mathbb{R}^n)$ in terms of the Lusin-area function associated with the Poisson semigroup of L, the atom, and the molecule (see Theorem 7.2 below). Moreover, characterizations of $H_{\varphi,L}(\mathbb{R}^n)$, in terms of the non-tangential maximal functions associated with the heat semigroup and the Poisson semigroup of L, the radial maximal functions associated with the heat semigroup and the Poisson semigroup of L, are also established (see Theorem 7.4 below). Observe that Theorem 7.4 completely covers [57, Theorem 6.4] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ satisfying that there exist $q_1, q_2 \in (0, \infty)$ such that $q_1 < 1 < q_2$ and $[\Phi(t^{q_2})]^{q_1}$ is a convex function on $(0, \infty)$. Finally, we show that the Riesz transform $\nabla L^{-1/2}$ associated with L is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$ when $i(\varphi) \in (0, 1]$, and from $H_{\varphi,L}(\mathbb{R}^n)$ to the Musielak–Orlicz–Hardy space $H_\varphi(\mathbb{R}^n)$ introduced by Ky [63] when $i(\varphi) \in (\frac{n}{n+1}, 1]$ (see Theorems 7.11 and 7.15 below). We remark that the boundedness of $\nabla L^{-1/2}$ from $H^1_L(\mathbb{R}^n)$ to the classical Hardy space $H^1(\mathbb{R}^n)$ was first established in [51, Theorem 8.6] and that Theorems 7.11 and 7.15 are respectively [57, Theorems 6.2 and 6.3] when φ is as in (1.2) with $\omega \equiv 1$ and Φ concave. We also point out that when $n = 1$ and $\varphi(x, t) := t$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, the Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ coincides with the Hardy space introduced by Czaja and Zienkiewicz [28]; if $L := -\Delta + V$ with V belonging to the reverse Hölder class $\mathrm{RH}_q(\mathbb{R}^n)$ for some $q \ge n/2$ and $n \ge 3$, and $\varphi(x, t) := t^p$ with $p \in (\frac{n}{n+1}, 1]$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, then the Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ coincides with the Hardy space introduced by Dziubański and Zienkiewicz [39, 40].

To prove Theorem 7.4 below, we borrow some ideas from the proof of [51, Theorem 8.2]. To this end, via invoking the Caccioppoli inequality associated with L, the special differential structure of L itself and the divergence theorem, we first establish a weighted "good-λ inequality" concerning the non-tangential maximal function $\mathcal{N}_P(f)$, associated with the Poisson semigroup of L, and the truncated variant of the Lusin area function $\widetilde{S}_P(f)$ in Lemma 7.8 below, which is a suitable substitute, in the present setting, of a distribution inequality concerning the non-tangential maximal function $\mathcal{N}_P(f)$ and the Lusin area function $\widetilde{S}_P(f)$, appeared in the proof of [51, Theorem 8.2] (see also [57, (6.5)]). We then use the Moser type local boundedness estimate from [51, Lemma 8.4] (see also Lemma 7.9 below), which is the substitute of the classical mean value property for harmonic functions in this setting. Moreover,

a more delicate estimate in (7.15) below than that used in the proof of [57, Theorem 6.4] is established, which leads us in Theorem 7.4 below to remove the additional assumption, appeared in [57, Theorem 6.4], that there exist $q_1, q_2 \in (0, \infty)$ such that $q_1 < 1 < q_2$ and $[\Phi(t^{q_2})]^{q_1}$ is a convex function on $(0, \infty)$ even when φ is as in (1.2) with $\omega \equiv 1$. The proof of Theorem 7.11 is a skillful application of the atomic characterization of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathbb{R}^n)$, a Davies–Gaffney type estimate (see [51, Lemma 8.5] or Lemma 7.10 below) and the $L^2(\mathbb{R}^n)$-boundedness of the Riesz transform $\nabla L^{-1/2}$. Furthermore, as an application of the atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$ obtained in Theorem 7.2 and the atomic characterization of the Musielak–Orlicz–Hardy space $H_\varphi(\mathbb{R}^n)$ established by Ky [63, Theorem 3.1] (see also Lemma 7.14 below), we obtain the boundedness of the Riesz transform $\nabla L^{-1/2}$ from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$ in Theorem 7.15 below. More precisely, for any given atom α as in Definition 4.3 below, we prove that

$$\nabla L^{-1/2}(\alpha) = \sum_j b_j$$

in $L^2(\mathbb{R}^n)$, where, for each j, b_j is a multiple of an atom introduced by Ky [63, Definition 2.4]. Observe that the atom in Definition 4.3 below is different from the atom in [63, Definition 2.4] in that the norm of the atom in Definition 4.3 is not weighted, but the atom introduced by Ky [63, Definition 2.4] is weighted and, moreover, that, in the present setting, $\nabla L^{-1/2}$ is known to be bounded on $L^p(\mathbb{R}^n)$ only with $p \in (1, 2]$. Thus, in order to prove that, for each j, b_j is a multiple of an atom as in [63, Definition 2.4], we need the assumption that $q(\varphi) < 2$ and $r(\varphi) > 2/[2 - q(\varphi)]$ (see (7.36) below for the details), where $q(\varphi)$ and $r(\varphi)$ are, respectively, as in (2.12) and (2.13) below.

We remark that there exist more applications of the results in this paper. For example, motivated by [64–66], in a forthcoming paper, we will apply the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ and the Musielak–Orlicz BMO-type space $\mathrm{BMO}_{\varphi,L}(\mathbb{R}^n)$ associated with the Schrödinger operator L, introduced in this paper, to the study of *pointwise multipliers* on BMO-type space associated with the Schrödinger operator L and *commutators* of singular integral operators associated with the operator L. This is reasonable, since φ in (1.3) naturally appears in the study of these problems in [74, 75]. Moreover, motivated by [8, 16–19], in another forthcoming paper, we will further establish various *maximal function characterizations* of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\Omega)$ on the strongly Lipschitz domain Ω of $\mathbb{R}^n$ associated with the Schrödinger operator L with some boundary conditions, which is a special case of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ introduced in this paper.

After the first version of this paper was put on arXiv, we learned from Dr. Bui that, in [15], Bui and Duong also introduced the weighted Hardy space $H^p_{L,\omega}(\mathcal{X})$, with $p \in (0, 1]$ and $\omega \in A_1(\mathcal{X})$ satisfying the reverse Hölder inequality of order $2/(2-p)$, by the Lusin area function associated with the heat semigroup generated by L. Moreover, Bui and Duong [15] established the atomic and the molecular characterizations of $H^p_{L,\omega}(\mathcal{X})$ and, as applications, obtained the boundedness on $H^p_{L,\omega}(\mathcal{X})$ of the generalized Riesz transforms associated with L and of the spectral multipliers of L. These

results are partially overlapped with the results of this paper when φ is as in (1.2) with $\Phi(t) := t^p$ for $p \in (0, 1]$ and $t \in [0, \infty)$. As have observed above, the atomic decomposition of the weighted tent space obtained in [15] and the Riesz transforms considered in [15] are different from these in this paper. We also point out that, it is motivated by [15], in the present version of this paper, we replace the assumption in the first version that the growth function φ is of uniformly upper type 1 by the assumption that φ is of uniformly upper type p_1 for some $p_1 \in (0, 1]$ and hence, in the main results of this paper, we improve the assumption in the first version that $\varphi \in \mathbb{RH}_2(\mathcal{X})$ into the weaker assumption that $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$.

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \ldots)$ to denote a *positive constant depending on the indicated parameters* γ, β, $\ldots$. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the *symbol* $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any measurable subset E of $\mathcal{X}$, we denote by $E^{\complement}$ the *set* $\mathcal{X} \setminus E$ and by χ_E its *characteristic function*. We also set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \cdots + \theta_n$. For any subsets $E, F \subset \mathcal{X}$, and $z \in \mathcal{X}$, let $\operatorname{dist}(E, F) := \inf_{x \in E, y \in F} d(x, y)$ and $\operatorname{dist}(z, E) := \inf_{x \in E} d(z, x)$.

2 Preliminaries

In Sect. 2.1, we first recall some notions on metric measure spaces and then, in Sect. 2.2, describe some basic assumptions on the operator L studied in this paper. In Sect. 2.3, we recall some notions concerning growth functions considered in this paper and also give some specific examples of growth functions satisfying the assumptions of this paper. Section 2.4 is devoted to recalling some properties of growth functions established in [63].

2.1 Metric Measure Spaces

Throughout the whole paper, we let $\mathcal{X}$ be a *set*, d a *metric* on $\mathcal{X}$, and μ a *nonnegative Borel regular measure* on $\mathcal{X}$. For all $x \in \mathcal{X}$ and $r \in (0, \infty)$, let

$$B(x, r) := \left\{ y \in \mathcal{X} : \ d(x, y) < r \right\}$$

and $V(x, r) := \mu(B(x, r))$. Moreover, we assume that there exists a constant $C_1 \in [1, \infty)$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$V(x, 2r) \le C_1 V(x, r) < \infty. \tag{2.1}$$

Observe that $(\mathcal{X}, d, \mu)$ is a *space of homogeneous type* in the sense of Coifman and Weiss [25]. Recall that in the definition of spaces of homogeneous type in [25, Chap. 3], d is assumed to be a quasi-metric. However, for simplicity, we always

assume that d is a metric. Notice that the doubling property (2.1) implies that the following strong homogeneity property that, for some positive constants C and n,

$$V(x, \lambda r) \le C\lambda^n V(x, r) \tag{2.2}$$

uniformly for all $\lambda \in [1, \infty)$, $x \in \mathcal{X}$, and $r \in (0, \infty)$. There also exist constants $C \in (0, \infty)$ and $N \in [0, n]$ such that, for all $x, y \in \mathcal{X}$ and $r \in (0, \infty)$,

$$V(x, r) \le C\left[1 + \frac{d(x, y)}{r}\right]^N V(y, r). \tag{2.3}$$

Indeed, the property (2.3) with $N = n$ is a simple corollary of the triangle inequality for the metric d and the strong homogeneity property (2.2). In the cases of Euclidean spaces and Lie groups of polynomial growth, N can be chosen to be 0.

In what follows, to simplify the notation, for each ball $B \subset \mathcal{X}$, set

$$U_0(B) := B \quad \text{and} \quad U_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for } j \in \mathbb{N}. \tag{2.4}$$

Furthermore, for $p \in (0, \infty]$, the *space of p-integrable functions on $\mathcal{X}$* is denoted by $L^p(\mathcal{X})$ and the *(quasi-)norm* of $f \in L^p(\mathcal{X})$ by $\|f\|_{L^p(\mathcal{X})}$.

2.2 Assumptions on Operators L

Throughout the whole paper, as in [51, 57], we always suppose that the considered operators L satisfy the following assumptions.

Assumption A L is a nonnegative self-adjoint operator in $L^2(\mathcal{X})$.

Assumption B The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies–Gaffney estimates, namely, there exist positive constants C_2 and C_3 such that, for all closed sets E and F in $\mathcal{X}$, $t \in (0, \infty)$, and $f \in L^2(E)$,

$$\left\|e^{-tL} f\right\|_{L^2(F)} \le C_2 \exp\left\{-\frac{[\operatorname{dist}(E, F)]^2}{C_3 t}\right\} \|f\|_{L^2(E)}. \tag{2.5}$$

Here and in what follows, $\operatorname{dist}(E, F) := \inf_{x \in E, y \in F} d(x, y)$ and $L^2(E)$ is the *set of all μ-measurable functions supported in E* such that $\|f\|_{L^2(E)} := \{\int_E |f(x)|^2 \, d\mu(x)\}^{1/2} < \infty$.

Examples of operators satisfying Assumptions A and B include second-order elliptic self-adjoint operators in divergence form on $\mathbb{R}^n$ with bounded measurable coefficients, (degenerate) Schrödinger operators with nonnegative potential or with magnetic field, and Laplace–Beltrami operators on all complete Riemannian manifolds (see, for example, [29, 42, 84, 85]).

By Assumptions A and B, we have the following results which were established in [51].

Lemma 2.1 *Let L satisfy Assumptions A and B. Then for every fixed $k \in \mathbb{N}$, the family of operators, $\{(t^2L)^k e^{-t^2L}\}_{t>0}$, satisfies the Davies–Gaffney estimates (2.5) with positive constants C_2 and C_3 only depending on n, N, and k.*

In what follows, for any operator T, let K_T denote its integral kernel. It is well known that if L satisfies Assumptions A and B, and $T := \cos(t\sqrt{L})$ with $t \in (0, \infty)$, then there exists a positive constant C_4 such that

$$\operatorname{supp} K_T \subset \mathcal{D}_t := \left\{(x, y) \in \mathcal{X} \times \mathcal{X} : \ d(x, y) \le C_4 t\right\} \tag{2.6}$$

(see, for example, [85, Theorem 2], [26, Theorem 3.14], and [51, Proposition 3.4]). This observation plays a key role in obtaining the atomic characterization of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ (see [51, 57] and Proposition 4.7 below).

Lemma 2.2 *Assume that L satisfies Assumptions A and B. Let $\psi \in C_c^\infty(\mathbb{R})$ be even and $\operatorname{supp}\psi \subset (-C_4^{-1}, C_4^{-1})$, where C_4 is as in (2.6). Let $\widetilde{\Phi}$ denote the Fourier transform of ψ. Then for every $\kappa \in \mathbb{N}$ and $t \in (0, \infty)$, the kernel $K_{(t^2L)^\kappa \widetilde{\Phi}(t\sqrt{L})}$ of $(t^2L)^\kappa \widetilde{\Phi}(t\sqrt{L})$ satisfies that $\operatorname{supp} K_{(t^2L)^\kappa \widetilde{\Phi}(t\sqrt{L})} \subset \{(x, y) \in \mathcal{X} \times \mathcal{X} : \ d(x, y) \le t\}$.*

For any given $\delta \in (0, \infty)$, let ϕ be a measurable function from $\mathbb{C}$ to $\mathbb{C}$ satisfying that there exists a positive constant $C(\delta)$ such that, for all $z \in \mathbb{C}$, $|\phi(z)| \le C(\delta)\frac{|z|^\delta}{1+|z|^{2\delta}}$. Then $\int_0^\infty |\phi(t)|^2 t^{-1}\,dt < \infty$. It was proved in [51, (3.14)] that, for all $f \in L^2(\mathcal{X})$,

$$\int_0^\infty \left\|\phi(t\sqrt{L})f\right\|_{L^2(\mathcal{X})}^2 \frac{dt}{t} \le \left\{\int_0^\infty \left|\phi(t)\right|^2 \frac{dt}{t}\right\} \|f\|_{L^2(\mathcal{X})}^2, \tag{2.7}$$

which is often used in what follows.

2.3 Growth Functions

We recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$ (see, for example, [73, 76, 79, 80]). The function Φ is said to be of *upper type p* (resp., *lower type p*) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$ (resp., $t \in [0, 1]$) and $s \in [0, \infty)$, $\Phi(st) \le Ct^p\Phi(s)$. If Φ is of both upper type p_1 and lower type p_2, then Φ is said to be of *type* (p_1, p_2). The function Φ is said to be of *strictly lower type p* if, for all $t \in [0, 1]$ and $s \in [0, \infty)$, $\Phi(st) \le t^p\Phi(s)$. Define

$$p_\Phi := \sup\left\{p \in [0, \infty) : \Phi(st) \le t^p\Phi(s) \text{ holds true for all } t \in [0, 1] \text{ and } s \in [0, \infty)\right\}. \tag{2.8}$$

It was proved in [58, Remark 2.1] that Φ is also of strictly lower type p_Φ; in other words, p_Φ is attainable.

For a given function $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ such that, for any $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of *uniformly upper type p* (resp., *uniformly lower*

type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $x \in \mathcal{X}$, $t \in [1, \infty)$ (resp., $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\varphi(x, st) \le C t^p \varphi(x, s); \tag{2.9}$$

φ is said to be of *positive uniformly upper type* (resp., *uniformly lower type*) if it is of uniformly upper type (resp., uniformly lower type) p for some $p \in (0, \infty)$. Moreover, let

$$I(\varphi) := \inf\bigl\{p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p\bigr\} \tag{2.10}$$

and

$$i(\varphi) := \sup\bigl\{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p\bigr\}. \tag{2.11}$$

In what follows, $I(\varphi)$ and $i(\varphi)$ are, respectively, called the *uniformly critical upper type index* and the *uniformly critical lower type index* of φ. Observe that $I(\varphi)$ and $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly upper type $I(\varphi)$ and uniformly lower type $i(\varphi)$ (see below for some examples).

Let $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ satisfy that $x \mapsto \varphi(x, t)$ is measurable for all $t \in [0, \infty)$. Following Ky [63], $\varphi(\cdot, t)$ is said to be *uniformly locally integrable* if, for all bounded subsets K of $\mathcal{X}$,

$$\int_K \sup_{t \in (0,\infty)} \left\{\varphi(x, t)\left[\int_K \varphi(y, t)\, d\mu(y)\right]^{-1}\right\} d\mu(x) < \infty.$$

Definition 2.3 Let $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ be uniformly locally integrable. The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$\mathbb{A}_q(\varphi) := \sup_{t \in (0,\infty)} \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \int_B \varphi(x, t)\, d\mu(x) \left\{\frac{1}{\mu(B)} \int_B \bigl[\varphi(y, t)\bigr]^{-q'/q}\, d\mu(y)\right\}^{q/q'} < \infty,$$

where $1/q + 1/q' = 1$, or

$$\mathbb{A}_1(\varphi) := \sup_{t \in (0,\infty)} \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \int_B \varphi(x, t)\, d\mu(x) \Bigl(\operatorname*{esssup}_{y \in B} \bigl[\varphi(y, t)\bigr]^{-1}\Bigr) < \infty.$$

Here the first supremums are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly reverse Hölder condition for some* $q \in (1, \infty]$, denoted by $\varphi \in \mathbb{RH}_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$\mathbb{RH}_q(\varphi) := \sup_{t \in (0,\infty)} \sup_{B \subset \mathcal{X}} \left\{\frac{1}{\mu(B)} \int_B \bigl[\varphi(x, t)\bigr]^q\, d\mu(x)\right\}^{1/q} \times \left\{\frac{1}{\mu(B)} \int_B \varphi(x, t)\, d\mu(x)\right\}^{-1} < \infty,$$

or

$$\mathbb{RH}_\infty(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \left\{\operatorname*{esssup}_{y\in B} \varphi(y,t)\right\} \left\{\frac{1}{\mu(B)} \int_B \varphi(x,t)\,d\mu(x)\right\}^{-1} < \infty.$$

Here the first supremums are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

Recall that in Definition 2.3, when $\mathcal{X} = \mathbb{R}^n$, $\mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$ was introduced by Ky [63].

Let $\mathbb{A}_\infty(\mathcal{X}) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathcal{X})$ and define the *critical indices* of φ as follows:

$$q(\varphi) := \inf\left\{q \in [1,\infty) : \varphi \in \mathbb{A}_q(\mathcal{X})\right\} \tag{2.12}$$

and

$$r(\varphi) := \sup\left\{q \in (1,\infty] : \varphi \in \mathbb{RH}_q(\mathcal{X})\right\}. \tag{2.13}$$

Observe that, if $q(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}(\mathcal{X})$, and there exists $\varphi \notin \mathbb{A}_1(\mathcal{X})$ such that $q(\varphi) = 1$ (see, for example, [62]). Similarly, if $r(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{RH}_{r(\varphi)}(\mathcal{X})$, and there exists $\varphi \notin \mathbb{RH}_\infty(\mathcal{X})$ such that $r(\varphi) = \infty$ (see, for example, [27]).

Now we introduce the notion of growth functions.

Definition 2.4 A function $\varphi:\ \mathcal{X} \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following hold true:

(i) φ is a *Musielak–Orlicz function*, namely,
  (i)$_1$ the function $\varphi(x,\cdot):\ [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathcal{X}$;
  (i)$_2$ the function $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.
(ii) $\varphi \in \mathbb{A}_\infty(\mathcal{X})$.
(iii) The function φ is of positive uniformly upper type p_1 for some $p_1 \in (0,1]$ and of uniformly lower type p_2 for some $p_2 \in (0,1]$.

Remark 2.5 By the definitions of the uniformly upper type and the uniformly lower type, we see that, if the growth function φ is of positive uniformly upper type p_1 and of positive uniformly lower type p_2, then $p_1 \geq p_2$.

Clearly, $\varphi(x,t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathcal{X})$ and Φ is an Orlicz function of upper type p_1 for some $p_1 \in (0,1]$ and of lower type p_2 for some $p_2 \in (0,1]$. It is known that, for $p \in (0,1]$, if $\Phi(t) := t^p$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of type (p,p); for $p \in [\frac{1}{2},1]$, if $\Phi(t) := t^p/\ln(e+t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type q for $q \in (0,p)$ and of upper type p; for $p \in (0,\frac{1}{2}]$, if $\Phi(t) := t^p \ln(e+t)$ for all $t \in [0,\infty)$, then Φ is an Orlicz function of lower type p and of upper type q for $q \in (p,1]$. Recall that if an Orlicz function is of upper type $p \in (0,1)$, then it is also of upper type 1.

Another typical and useful growth function is φ as in (1.3). It is easy to show that if φ is as in (1.3), then $\varphi \in \mathbb{A}_1(\mathcal{X})$, φ is of uniformly upper type α, $I(\varphi) = i(\varphi) = \alpha$, $i(\varphi)$ is not attainable, but $I(\varphi)$ is attainable. Moreover, it is worth to point

out that such function φ naturally appears in the study of the pointwise multiplier characterization for the BMO-type space on the metric space with doubling measure (see [74]). We also point out that when $\mathcal{X} = \mathbb{R}^n$, a similar example of such φ is given by Ky [63] replacing $d(x, x_0)$ by $|x|$, where $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^n$.

2.4 Some Basic Properties on Growth Functions

Throughout the whole paper, we *always assume that φ is a growth function* as in Definition 2.4. Let us now introduce the Musielak–Orlicz space.

The *Musielak–Orlicz space* $L^\varphi(\mathcal{X})$ is defined to be the set of all measurable functions f such that $\int_{\mathcal{X}} \varphi(x, |f(x)|)\, d\mu(x) < \infty$ with *Luxembourg norm*

$$\|f\|_{L^\varphi(\mathcal{X})} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}.$$

In what follows, for any measurable subset E of $\mathcal{X}$ and $t \in [0,\infty)$, let

$$\varphi(E,t) := \int_E \varphi(x,t)\, d\mu(x).$$

The following Lemmas 2.6 and 2.7 on the properties of growth functions are, respectively, [63, Lemmas 4.1 and 4.3].

Lemma 2.6 (i) *Let φ be a growth function. Then φ is uniformly σ-quasi-subadditive on $\mathcal{X} \times [0,\infty)$, namely, there exists a positive constant C such that, for all $(x, t_j) \in \mathcal{X} \times [0,\infty)$ with $j \in \mathbb{N}$, $\varphi(x, \sum_{j=1}^\infty t_j) \le C \sum_{j=1}^\infty \varphi(x, t_j)$.*

(ii) *Let φ be a growth function and $\widetilde{\varphi}(x,t) := \int_0^t \frac{\varphi(x,s)}{s}\, ds$ for all $(x,t) \in \mathcal{X} \times [0,\infty)$. Then $\widetilde{\varphi}$ is a growth function, which is equivalent to φ; moreover, $\widetilde{\varphi}(x,\cdot)$ is continuous and strictly increasing.*

Lemma 2.7 *Let c be a positive constant. Then there exists a positive constant C such that*

(i) $\int_{\mathcal{X}} \varphi(x, \frac{|f(x)|}{\lambda})\, d\mu(x) \le c$ *for some* $\lambda \in (0,\infty)$ *implies that* $\|f\|_{L^\varphi(\mathcal{X})} \le C\lambda$;
(ii) $\sum_j \varphi(B_j, \frac{t_j}{\lambda}) \le c$ *for some* $\lambda \in (0,\infty)$ *implies that*

$$\inf\left\{\alpha \in (0,\infty) : \sum_j \varphi\left(B_j, \frac{t_j}{\alpha}\right) \le 1\right\} \le C\lambda,$$

where $\{t_j\}_j$ is a sequence of positive numbers and $\{B_j\}_j$ a sequence of balls.

In what follows, for any given ball $B := B(x,t)$, with $x \in \mathcal{X}$ and $r \in (0,\infty)$, and $\lambda \in (0,\infty)$, we write λB for the λ-*dilated ball* of B, namely, $\lambda B := B(x, \lambda t)$.

We have the following properties for $\mathbb{A}_\infty(\mathcal{X})$, whose proofs are similar to those in [44–46], and we omit the details. In what follows, $\mathcal{M}$ denotes the *Hardy–Littlewood*

maximal function on $\mathcal{X}$, namely, for all $x \in \mathcal{X}$,

$$\mathcal{M}(f)(x) := \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls $B \ni x$.

Lemma 2.8 (i) $\mathbb{A}_1(\mathcal{X}) \subset \mathbb{A}_p(\mathcal{X}) \subset \mathbb{A}_q(\mathcal{X})$ *for* $1 \le p \le q < \infty$.

(ii) $\mathbb{RH}_\infty(\mathcal{X}) \subset \mathbb{RH}_p(\mathcal{X}) \subset \mathbb{RH}_q(\mathcal{X})$ *for* $1 < q \le p \le \infty$.

(iii) *If* $\varphi \in \mathbb{A}_p(\mathcal{X})$ *with* $p \in (1, \infty)$, *then there exists some* $q \in (1, p)$ *such that* $\varphi \in \mathbb{A}_q(\mathcal{X})$.

(iv) *If* $\varphi \in \mathbb{RH}_p(\mathcal{X})$ *with* $p \in (1, \infty)$, *then there exists some* $q \in (p, \infty)$ *such that* $\varphi \in \mathbb{RH}_q(\mathcal{X})$.

(v) $\mathbb{A}_\infty(\mathcal{X}) = \bigcup_{p \in [1,\infty)} \mathbb{A}_p(\mathcal{X}) \subset \bigcup_{q \in (1,\infty]} \mathbb{RH}_q(\mathcal{X})$.

(vi) *If* $p \in (1, \infty)$ *and* $\varphi \in \mathbb{A}_p(\mathcal{X})$, *then there exists a positive constant* C *such that, for all measurable functions* f *on* $\mathcal{X}$ *and* $t \in [0, \infty)$,

$$\int_{\mathcal{X}} \big[\mathcal{M}(f)(x)\big]^p \varphi(x,t)\, d\mu(x) \le C \int_{\mathcal{X}} \big|f(x)\big|^p \varphi(x,t)\, d\mu(x).$$

(vii) *If* $\varphi \in \mathbb{A}_p(\mathcal{X})$ *with* $p \in [1, \infty)$, *then there exists a positive constant* C *such that, for all balls* $B_1, B_2 \subset \mathcal{X}$ *with* $B_1 \subset B_2$ *and* $t \in [0, \infty)$, $\frac{\varphi(B_2,t)}{\varphi(B_1,t)} \le C[\frac{\mu(B_2)}{\mu(B_1)}]^p$.

(viii) *If* $\varphi \in \mathbb{RH}_q(\mathcal{X})$ *with* $q \in (1, \infty]$, *then there exists a positive constant* C *such that, for all balls* $B_1, B_2 \subset \mathcal{X}$ *with* $B_1 \subset B_2$ *and* $t \in [0, \infty)$, $\frac{\varphi(B_2,t)}{\varphi(B_1,t)} \ge C[\frac{\mu(B_2)}{\mu(B_1)}]^{(q-1)/q}$.

Remark 2.9 We remark that in the setting of the Euclidean space $\mathbb{R}^n$, Lemma 2.8(v) can be improved to $\mathbb{A}_\infty(\mathbb{R}^n) = \bigcup_{p \in [1,\infty)} \mathbb{A}_p(\mathbb{R}^n) = \bigcup_{q \in (1,\infty]} \mathbb{RH}_q(\mathbb{R}^n)$ (see, for example, [54, Lemma 2.4(iv)]). However, in the present setting, the inverse inclusion in Lemma 2.8(v) may not be true (see [90, p. 9] for a counterexample).

3 Musielak–Orlicz Tent Spaces $T_\varphi(\mathcal{X} \times (0, \infty))$

In this section, we study the Musielak–Orlicz tent space associated with the growth function. We first recall some notions as follows.

For any $\nu \in (0, \infty)$ and $x \in \mathcal{X}$, let $\Gamma_\nu(x) := \{(y,t) \in \mathcal{X} \times (0,\infty) : d(x,y) < \nu t\}$ be the *cone of aperture* ν *with vertex* $x \in \mathcal{X}$. For any closed subset F of $\mathcal{X}$, denote by $\mathcal{R}_\nu F$ the *union of all cones with vertices in* F, namely, $\mathcal{R}_\nu F := \bigcup_{x \in F} \Gamma_\nu(x)$ and, for any open subset O of $\mathcal{X}$, denote the *tent over* O by $T_\nu(O)$, which is defined as $T_\nu(O) := [\mathcal{R}_\nu(O^{\complement})]^{\complement}$. It is easy to see that $T_\nu(O) = \{(x,t) \in \mathcal{X} \times (0,\infty) : d(x, O^{\complement}) \ge \nu t\}$. In what follows, we denote $\Gamma_1(x)$ and $T_1(O)$ simply by $\Gamma(x)$ and $\widehat{O}$, respectively.

For all measurable functions g on $\mathcal{X} \times (0, \infty)$ and $x \in \mathcal{X}$, define

$$\mathcal{A}(g)(x) := \left\{ \int_{\Gamma(x)} |g(y,t)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right\}^{1/2}.$$

If $\mathcal{X} = \mathbb{R}^n$, Coifman, Meyer, and Stein [23] introduced the tent space $T_2^p(\mathbb{R}^{n+1}_+)$ for $p \in (0, \infty)$. Here and in what follows, $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. The tent space $T_2^p(\mathcal{X} \times (0, \infty))$ on spaces of homogenous type was introduced by Russ [82]. Recall that a measurable function g is said to belong to the *tent space* $T_2^p(\mathcal{X} \times (0, \infty))$ with $p \in (0, \infty)$, if $\|g\|_{T_2^p(\mathcal{X}\times(0,\infty))} := \|\mathcal{A}(g)\|_{L^p(\mathcal{X})} < \infty$. Moreover, Harboure, Salinas, and Viviani [47] and Jiang and Yang [57], respectively, introduced the Orlicz tent spaces $T_\Phi(\mathbb{R}^{n+1}_+)$ and $T_\Phi(\mathcal{X} \times (0, \infty))$.

Let φ be as in Definition 2.4. In what follows, we denote by $T_\varphi(\mathcal{X} \times (0, \infty))$ the *space of all measurable functions g on $\mathcal{X} \times (0, \infty)$ such that $\mathcal{A}(g) \in L^\varphi(\mathcal{X})$* and, for any $g \in T_\varphi(\mathcal{X} \times (0, \infty))$, define its *quasi-norm* by

$$\begin{aligned}\|g\|_{T_\varphi(\mathcal{X}\times(0,\infty))} &:= \|\mathcal{A}(g)\|_{L^\varphi(\mathcal{X})} \\ &= \inf\left\{\lambda \in (0, \infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{\mathcal{A}(g)(x)}{\lambda}\right) d\mu(x) \le 1\right\}.\end{aligned}$$

A function a on $\mathcal{X} \times (0, \infty)$ is called a $T_\varphi(\mathcal{X} \times (0, \infty))$-*atom* if

(i) there exists a ball $B \subset \mathcal{X}$ such that $\operatorname{supp} a \subset \widehat{B}$;

(ii) $\int_{\widehat{B}} |a(x, t)|^2 \frac{d\mu(x)\,dt}{t} \le \mu(B)\|\chi_B\|^{-2}_{L^\varphi(\mathcal{X})}$.

For functions in $T_\varphi(\mathcal{X} \times (0, \infty))$, we have the following atomic decomposition.

Theorem 3.1 *Let φ be as in Definition 2.4. Then for any $f \in T_\varphi(\mathcal{X} \times (0, \infty))$, there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $T_\varphi(\mathcal{X} \times (0, \infty))$-atoms such that, for almost every $(x, t) \in \mathcal{X} \times (0, \infty)$,*

$$f(x, t) = \sum_j \lambda_j a_j(x, t). \tag{3.1}$$

Moreover, there exists a positive constant C such that, for all $f \in T_\varphi(\mathcal{X} \times (0, \infty))$,

$$\begin{aligned}\Lambda\big(\{\lambda_j a_j\}_j\big) &:= \inf\left\{\lambda \in (0, \infty) : \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) \le 1\right\} \\ &\le C\|f\|_{T_\varphi(\mathcal{X}\times(0,\infty))},\end{aligned} \tag{3.2}$$

where, for each j, $\widehat{B_j}$ appears in the support of a_j.

We prove Theorem 3.1 by borrowing some ideas from the proof of [57, Theorem 3.1] (see also [23] and [82]). To this end, we first need some known facts as follows.

Let F be a closed subset of $\mathcal{X}$ and $O := F^{\complement}$. Assume that $\mu(O) < \infty$. For any fixed $\gamma \in (0, 1)$, we say that $x \in \mathcal{X}$ has the *global γ-density with respect to F* if, for all $r \in (0, \infty)$,

$$\frac{\mu(B(x, r) \cap F)}{\mu(B(x, r))} \ge \gamma.$$

Denote by F_γ^* the *set of all such* x. It is easy to prove that F_γ^* with $\gamma \in (0,1)$ is a closed subset of F. Let $\gamma \in (0,1)$ and $O_\gamma^* := (F_\gamma^*)^{\complement}$. Then it is easy to see that $O \subset O_\gamma^*$. Indeed, from the definition of O^*, we deduce that $O_\gamma^* = \{x \in \mathcal{X} : \widetilde{\mathcal{M}}(\chi_O)(x) > 1-\gamma\}$, where $\widetilde{\mathcal{M}}$ denotes the *centered Hardy–Littlewood maximal function on* $\mathcal{X}$, which, together with the fact that $\widetilde{\mathcal{M}}$ is of weak type $(1,1)$ (see [25]), further implies that there exists a positive constant $C(\gamma)$, depending on γ, such that $\mu(O_\gamma^*) \le C(\gamma)\mu(O)$. Recall that, for all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$\widetilde{\mathcal{M}}(f)(x) := \sup_{r\in(0,\infty)} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\,d\mu(y).$$

It is well known that there exists a positive constant C_5 such that, for all $x \in \mathcal{X}$ and $f \in L^1_{\mathrm{loc}}(\mathcal{X})$,

$$C_5^{-1}\widetilde{\mathcal{M}}(f)(x) \le \mathcal{M}(f)(x) \le C_5\widetilde{\mathcal{M}}(f)(x). \tag{3.3}$$

The following Lemma 3.2 was established in [82].

Lemma 3.2 *Let $\eta \in (0,1)$. Then there exist $\gamma_0 \in (0,1)$ and $C(\eta,\gamma_0) \in (0,\infty)$ such that, for any closed subset F of $\mathcal{X}$ whose complement has finite measure, $\gamma \in [\gamma_0, 1)$ and nonnegative measurable function H on $\mathcal{X} \times (0,\infty)$,*

$$\int_{\mathcal{R}_{1-\eta}(F_\gamma^*)} H(y,t)V(y,t)\,d\mu(y)\,dt \le C(\eta,\gamma_0)\int_F \left\{\int_{\Gamma(x)} H(y,t)\,d\mu(y)\,dt\right\} d\mu(x),$$

where F_γ^* denotes the set of points in $\mathcal{X}$ with the global γ-density with respect to F.

To prove Theorem 3.1, we need a covering lemma established in [24].

Lemma 3.3 *Let Ω be a proper open subset of finite measure of $\mathcal{X}$. For any $x \in \mathcal{X}$, define $r(x) := d(x, \Omega^{\complement})/10$. Then there exist a positive integer M and a sequence $\{x_j\}_j$ of points in $\mathcal{X}$ such that, if $r_j := r(x_j)$, then*

(i) $\Omega = \bigcup_j B(x_j, r_j)$;
(ii) $B(x_i, r_i/4) \cap B(x_j, r_j/4) = \emptyset$ *if* $i \ne j$;
(iii) *for each* j, $\sharp\{i : B(x_i, 5r_i) \cap B(x_j, 5r_j) \ne \emptyset\} \le M$, *where* $\sharp E$ *denotes the cardinality of the set* E;

Moreover, there exist nonnegative functions $\{\phi_j\}_j$ on $\mathcal{X}$ such that

(iv) *for all* j, $\operatorname{supp}\phi_j \subset B(x_j, 2r_j)$;
(v) *for all* j *and* $x \in B(x_j, r_j)$, $\phi_j(x) \ge 1/M$;
(vi) $\sum_j \phi_j = \chi_\Omega$.

Moreover, we also need the following Lemma 3.4, whose proof is similar to that of [63, Lemma 5.4]. We omit the details.

Lemma 3.4 *Let $f \in T_\varphi(\mathcal{X} \times (0,\infty))$ and $\Omega_k := \{x \in \mathcal{X} : \mathcal{A}(f)(x) > 2^k\}$ for all $k \in \mathbb{Z}$. Then there exists a positive constant C such that, for all $\lambda \in (0,\infty)$,*

$$\sum_{k\in\mathbb{Z}} \varphi\left(\Omega_k, \frac{2^k}{\lambda}\right) \le C \int_{\mathcal{X}} \varphi\left(x, \frac{\mathcal{A}(f)(x)}{\lambda}\right) d\mu(x).$$

Now we prove Theorem 3.1 by using Lemmas 3.3 and 3.4.

Proof of Theorem 3.1 Let $f \in T_\varphi(\mathcal{X} \times (0,\infty))$. For any $k \in \mathbb{Z}$, let $O_k := \{x \in \mathcal{X} : \mathcal{A}(f)(x) > 2^k\}$ and $F_k := O_k^{\complement}$. Since $f \in T_\varphi(\mathcal{X} \times (0,\infty))$, for each k, O_k is an open set of $\mathcal{X}$ with $\mu(O_k) < \infty$.

Let $\eta \in (0,1)$ and γ_0 be as in Lemma 3.2. Let $\gamma \in [\gamma_0, 1)$ such that $C_5(1-\gamma) \le 1/2$. In what follows, we denote $(F_{k,\gamma})^*$ and $(O_{k,\gamma})^*$ simply by F_k^* and O_k^*, respectively. We claim that $\operatorname{supp} f \subset \bigcup_{k\in\mathbb{Z}} T_{1-\eta}(O_k^*) \cup E$, where $E \subset \mathcal{X} \times (0,\infty)$ satisfies $\int_E \frac{d\mu(y)\,dt}{t} = 0$. Indeed, let (x,t) be the Lebesgue point of f and $(x,t) \notin \bigcup_{k\in\mathbb{Z}} T_{1-\eta}(O_k^*)$. Then there exists a sequence $\{y_k\}_{k\in\mathbb{Z}}$ of points such that $\{y_k\}_{k\in\mathbb{Z}} \subset B(x,(1-\eta)t)$ and for each k, $y_k \notin T_{1-\eta}(O_k^*)$, which implies that, for each $k \in \mathbb{Z}$, $\widetilde{\mathcal{M}}(\chi_{O_k})(y_k) \le 1-\gamma$. From this, (3.3) and $C_5(1-\gamma) \le 1/2$, we deduce that $\mu(B(x,t) \cap \{z \in \mathcal{X} : \mathcal{A}(f)(z) \le 2^k\}) \ge \mu(B(x,t))/2$. Letting $k \to -\infty$, we then see that $\mu(B(x,t) \cap \{z \in \mathcal{X} : \mathcal{A}(f)(z) = 0\}) \ge \mu(B(x,t))/2$. Therefore, there exists $y \in B(x,t)$ such that $f = 0$ almost everywhere in $\Gamma(y)$, which, together with Lebesgue's differentiation theorem (see [49, Theorem 1.8]), implies that $f(x,t) = 0$. By this, we know that the claim holds true.

If $O_k^* = \mathcal{X}$ for some $k \in \mathbb{Z}$, then $\mu(\mathcal{X}) < \infty$, which implies that $\mathcal{X}$ is a ball (see [74, Lemma 5.1]). In this case, set $I_k := \{1\}$, $B_{k,1} := \mathcal{X}$ and $\phi_{k,1} \equiv 1$. If O_k^* is a proper subset of $\mathcal{X}$, by Lemma 3.3 with $\Omega = O_k^*$, we obtain a set I_k of indices and balls $\{B_{k,j}\}_{j\in I_k} := \{B(x_{k,j}, 2r_{k,j})\}_{j\in I_k}$ and functions $\{\phi_{k,j}\}_{j\in I_k}$ satisfying that, for each $j \in I_k$, $\operatorname{supp}\phi_{k,j} \subset B(x_{x,j}, 2r_{k,j})$, and $\sum_{j\in I_k} \phi_{k,j} = \chi_{O_k^*}$. Thus, for each $(x,t) \in \mathcal{X} \times (0,\infty)$, we see that

$$(\chi_{T_{1-\eta}(O_k^*)} - \chi_{T_{1-\eta}(O_{k+1}^*)})(x,t) = \sum_{j\in I_k} \phi_{k,j}(x)(\chi_{T_{1-\eta}(O_k^*)} - \chi_{T_{1-\eta}(O_{k+1}^*)})(x,t).$$

From this, $\operatorname{supp} f \subset \{\bigcup_{k\in\mathbb{Z}} T_{1-\eta}(O_k^*) \cup E\}$, and $\int_E \frac{d\mu(y)\,dt}{t} = 0$, we infer that

$$f = \sum_{k\in\mathbb{Z}} f(\chi_{T_{1-\eta}(O_k^*)} - \chi_{T_{1-\eta}(O_{k+1}^*)}) = \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} f\phi_{k,j}(\chi_{T_{1-\eta}(O_k^*)} - \chi_{T_{1-\eta}(O_{k+1}^*)})$$

almost everywhere on $\mathcal{X} \times (0,\infty)$. For each k and j, let

$$a_{k,j} := 2^{-k} \|\chi_{B_{k,j}}\|_{L^\varphi(\mathcal{X})}^{-1} f\phi_{k,j}(\chi_{T_{1-\eta}(O_k^*)} - \chi_{T_{1-\eta}(O_{k+1}^*)})$$

and $\lambda_{kj} := 2^k \|\chi_{B_{k,j}}\|_{L^\varphi(\mathcal{X})}$. Then $f = \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} \lambda_{k,j} a_{k,j}$ almost everywhere. Similar to the proof of [82, (2.4)], we see that, for each k and j, $\operatorname{supp} a_{k,j} \subset \widehat{C(\eta)B_{k,j}}$, where $C(\eta) \in (1,\infty)$ is a positive constant independent of k and j. By Lemma 3.2,

$\operatorname{supp} a_{k,j} \subset (T_{1-\eta}(O^*_{k+1}))^{\complement} = \mathcal{R}_{1-\eta}(F^*_{k+1})$, and the definition of F_{k+1}, we know that, for each k and j,

$$\begin{aligned}
\|a_{k,j}\|^2_{T^2_2(\mathcal{X}\times(0,\infty))} &= \int_{\mathcal{X}\times(0,\infty)} |a_{k,j}(y,t)|^2 \frac{d\mu(y)\,dt}{t} \\
&\lesssim \int_{\mathcal{R}_{1-\eta}(F^*_{k+1})} |a_{k,j}(y,t)|^2 \frac{d\mu(y)\,dt}{t} \\
&\lesssim \int_{F_{k+1}} \left\{ \int_{\Gamma(x)} |a_{k,j}(y,t)|^2 \frac{d\mu(y)\,dt}{V(y,t)t} \right\} d\mu(x) \\
&\lesssim \int_{F_{k+1}\cap(C(\eta)B_{k,j})} \left[\mathcal{A}(a_{k,j})(x)\right]^2 d\mu(x) \\
&\lesssim 2^{-2k} \|\chi_{B_{k,j}}\|^{-2}_{L^\varphi(\mathcal{X})} \int_{F_{k+1}\cap(C(\eta)B_{k,j})} \left[\mathcal{A}(f)(x)\right]^2 d\mu(x) \\
&\lesssim V\big(C(\eta)B_{k,j}\big) \|\chi_{C(\eta)B_{k,j}}\|^{-2}_{L^\varphi(\mathcal{X})},
\end{aligned}$$

which implies that up to a harmless multiplicative constant, each $a_{k,j}$ is a $T_\varphi(\mathcal{X}\times(0,\infty))$-atom. Moreover, by (2.2), Lemma 2.7(i), and Lemma 3.4, we know that, for all $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_{j\in I_k} \varphi\left(C(\eta)B_{k,j}, \frac{|\lambda_{k,j}|}{\lambda \|\chi_{C(\eta)B_{k,j}}\|_{L^\varphi(\mathcal{X})}} \right) \\
&\quad \lesssim \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} \varphi\left(B_{k,j}, \frac{2^k}{\lambda} \right) \lesssim \sum_{k\in\mathbb{Z}} \varphi\left(O^*_k, \frac{2^k}{\lambda} \right) \\
&\quad \lesssim \int_{\mathcal{X}} \varphi\left(x, \frac{\mathcal{A}(f)(x)}{\lambda} \right) d\mu(x),
\end{aligned}$$

which implies that $\Lambda(\{\lambda_{k,j}a_{k,j}\}_{k\in\mathbb{Z},j}) \lesssim \|f\|_{T_\varphi(\mathcal{X}\times(0,\infty))}$. This finishes the proof of Theorem 3.1. □

Corollary 3.5 *Let φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, where $I(\varphi)$ is as in (2.10). If $f \in T_\varphi(\mathcal{X}\times(0,\infty)) \cap T^2_2(\mathcal{X}\times(0,\infty))$, then (3.1) in Theorem 3.1 holds true in both $T_\varphi(\mathcal{X}\times(0,\infty))$ and $T^2_2(\mathcal{X}\times(0,\infty))$.*

By the uniformly upper type p_1 property of φ with some $p_1 \in [I(\varphi), 1]$, Theorem 3.1 and its proof, similar to the proof of [54, Corollary 3.4], we can show Corollary 3.5 and omit the details here.

In what follows, let $T^b_\varphi(\mathcal{X}\times(0,\infty))$ and $T^{p,b}_2(\mathcal{X}\times(0,\infty))$ with $p \in (0,\infty)$ denote, respectively, the *set of all functions in $T_\varphi(\mathcal{X}\times(0,\infty))$ and $T^p_2(\mathcal{X}\times(0,\infty))$ with bounded support*. Here and in what follows, a function f on $\mathcal{X}\times(0,\infty)$ is said to have *bounded support* means that there exist a ball $B \subset \mathcal{X}$ and $0 < c_1 < c_2 < \infty$ such that $\operatorname{supp} f \subset B \times (c_1, c_2)$.

Proposition 3.6 *Let φ be as in Definition 2.4. Then $T^b_\varphi(\mathcal{X} \times (0,\infty)) \subset T^{2,b}_2(\mathcal{X} \times (0,\infty))$ as sets.*

The proof of Proposition 3.6 is an application of the uniformly lower type p_2 property of φ for some $p_2 \in (0, 1]$, which is similar to that of [54, Proposition 3.5]. We omit the details.

4 Musielak–Orlicz–Hardy Spaces $H_{\varphi,L}(\mathcal{X})$ and Their Duals

In this section, we always assume that the operator L satisfies Assumptions A and B, and the growth function φ is as in Definition 2.4. We introduce the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ associated with L via the Lusin-area function and give its dual space via the atomic and molecular decomposition of $H_{\varphi,L}(\mathcal{X})$. Let us begin with some notions.

In order to introduce the Musielak–Orlicz–Hardy space associated with L, we follow the ideas appeared in [6, 51] and first define the $L^2(\mathcal{X})$ adapted Hardy space

$$H^2(\mathcal{X}) := H^2_L(\mathcal{X}) := \overline{R(L)}, \tag{4.1}$$

where $\overline{R(L)}$ denotes the *closure of the range of L in $L^2(\mathcal{X})$*. Then $L^2(\mathcal{X})$ is the orthogonal sum of $H^2(\mathcal{X})$ and the null space $N(L)$, namely, $L^2(\mathcal{X}) = \overline{R(L)} \oplus N(L)$.

For all functions $f \in L^2(\mathcal{X})$, let the Lusin-area function $S_L(f)$ be as in (1.4). From (2.7), it follows that S_L is bounded on $L^2(\mathcal{X})$. Hofmann et al. [51] introduced the Hardy space $H^1_L(\mathcal{X})$ associated with L as the completion of $\{f \in H^2(\mathcal{X}) : S_L(f) \in L^1(\mathcal{X})\}$ with respect to the norm $\|f\|_{H^1_L(\mathcal{X})} := \|f\|_{L^1(\mathcal{X})}$. The Orlicz–Hardy space $H_{\Phi,L}(\mathcal{X})$ was introduced in [57] in a similar way.

Following [6, 51, 57], we now introduce the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ associated with L as follows.

Definition 4.1 Let L satisfy Assumptions A and B and φ be as in Definition 2.4. A function $f \in H^2(\mathcal{X})$ is said to be in $\widetilde{H}_{\varphi,L}(\mathcal{X})$ if $S_L(f) \in L^\varphi(\mathcal{X})$; moreover, define

$$\|f\|_{H_{\varphi,L}(\mathcal{X})} := \|S_L(f)\|_{L^\varphi(\mathcal{X})} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) d\mu(x) \le 1\right\}.$$

The *Musielak–Orlicz–Hardy space* $H_{\varphi,L}(\mathcal{X})$ is defined to be the completion of $\widetilde{H}_{\varphi,L}(\mathcal{X})$ in the quasi-norm $\|\cdot\|_{H_{\varphi,L}(\mathcal{X})}$.

Remark 4.2 (i) Notice that for $0 \neq f \in L^2(\mathcal{X})$, $\|S_L(f)\|_{L^\varphi(\mathcal{X})} = 0$ holds true if and only if $f \in N(L)$. Indeed, if $f \in N(L)$, then $t^2 L e^{-t^2 L} f = 0$ almost everywhere and hence $\|S_L(f)\|_{L^\varphi(\mathcal{X})} = 0$. Conversely, if $\|S_L(f)\|_{L^\varphi(\mathcal{X})} = 0$, then $t^2 L e^{-t^2 L} f = 0$ almost everywhere on $\mathcal{X} \times (0,\infty)$. Hence, for all $t \in (0,\infty)$, $(e^{-t^2 L} - I) f = \int_0^t -2s L e^{-s^2 L} f\, ds = 0$, which further implies that $Lf = L e^{-t^2 L} f = 0$ almost everywhere and $f \in N(L)$. Thus, in Definition 4.1, it is necessary to use $\overline{R(L)}$ rather

than $L^2(\mathcal{X})$ to guarantee $\|\cdot\|_{H_{\varphi,L}(\mathcal{X})}$ to be a quasi-norm (see also [51, Sect. 2] and [57, Remark 4.1(i)]).

Moreover, we know that, if the kernels of the semigroup $\{e^{-tL}\}_{t>0}$ satisfy the Gaussian upper bounded estimates, then $N(L)=\{0\}$ and hence $H^2(\mathcal{X})=L^2(\mathcal{X})$ (see, for example, [51, Sect. 2]).

(ii) It is easy to see that $\|\cdot\|_{H_{\varphi,L}(\mathcal{X})}$ is a quasi-norm.

(iii) From the Aoki–Rolewicz theorem in [4, 81], it follows that there exists a quasi-norm $|||\cdot|||$ on $\widetilde{H}_{\varphi,L}(\mathcal{X})$ and $\gamma\in(0,1]$ such that, for all $f\in\widetilde{H}_{\varphi,L}(\mathcal{X})$, $|||f|||\sim\|f\|_{H_{\varphi,L}(\mathcal{X})}$ and, for any sequence $\{f_j\}_j\subset\widetilde{H}_{\varphi,L}(\mathcal{X})$,

$$\left|\left|\left|\sum_j f_j\right|\right|\right|^{\gamma}\le\sum_j|||f_j|||^{\gamma}.$$

By the theorem of completion of Yosida [98, p. 56], it follows that $(\widetilde{H}_{\varphi,L}(\mathcal{X}),|||\cdot|||)$ has a completion space $(H_{\varphi,L}(\mathcal{X}),|||\cdot|||)$; namely, for any $f\in(H_{\varphi,L}(\mathcal{X}),|||\cdot|||)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^{\infty}\subset\widetilde{H}_{\varphi,L}(\mathcal{X})$ such that $\lim_{k\to\infty}|||f_k-f|||=0$. Moreover, if $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $(\widetilde{H}_{\varphi,L}(\mathcal{X}),|||\cdot|||)$, then there exists a unique $f\in H_{\varphi,L}(\mathcal{X})$ such that $\lim_{k\to\infty}|||f_k-f|||=0$. Furthermore, by the fact that $|||f|||\sim\|f\|_{H_{\varphi,L}(\mathcal{X})}$ for all $f\in\widetilde{H}_{\varphi,L}(\mathcal{X})$, we know that the spaces $(H_{\varphi,L}(\mathcal{X}),\|\cdot\|_{H_{\varphi,L}(\mathcal{X})})$ and $(H_{\varphi,L}(\mathcal{X}),|||\cdot|||)$ coincide with equivalent quasi-norms.

(iv) If $\varphi(x,t):=t$ for all $x\in\mathcal{X}$ and $t\in(0,\infty)$, the space $H_{\varphi,L}(\mathcal{X})$ is just the space $H^1_L(\mathcal{X})$ introduced by Hofmann et al. [51]. Moreover, if φ is as in (1.2) with $\omega\equiv1$ and Φ concave on $(0,\infty)$, the space $H_{\varphi,L}(\mathcal{X})$ is just the Orlicz–Hardy space $H_{\Phi,L}(\mathcal{X})$ introduced in [57].

We now introduce (φ,M)-atoms and (φ,M,ϵ)-molecules as follows.

Definition 4.3 Let $M\in\mathbb{N}$. A function $\alpha\in L^2(\mathcal{X})$ is called a (φ,M)-*atom* associated with the operator L if there exist a function $b\in\mathcal{D}(L^M)$ and a ball $B\subset\mathcal{X}$ such that

(i) $\alpha=L^Mb$;
(ii) $\operatorname{supp}(L^kb)\subset B$, $k\in\{0,\dots,M\}$;
(iii) $\|(r_B^2L)^kb\|_{L^2(\mathcal{X})}\le r_B^{2M}[\mu(B)]^{1/2}\|\chi_B\|^{-1}_{L^{\varphi}(\mathcal{X})}$, $k\in\{0,\dots,M\}$.

Definition 4.4 Let $M\in\mathbb{N}$ and $\epsilon\in(0,\infty)$. A function $\beta\in L^2(\mathcal{X})$ is called a (φ,M,ϵ)-*molecule* associated with the operator L if there exist a function $b\in\mathcal{D}(L^M)$ and a ball $B\subset\mathcal{X}$ such that

(i) $\beta=L^Mb$;
(ii) for each $k\in\{0,\dots,M\}$ and $j\in\mathbb{Z}_+$, it holds true that

$$\|\left(r_B^2L\right)^kb\|_{L^2(U_j(B))}\le2^{-j\epsilon}r_B^{2M}\left[\mu(B)\right]^{1/2}\|\chi_B\|^{-1}_{L^{\varphi}(\mathcal{X})},$$

where $U_j(B)$ with $j\in\mathbb{Z}_+$ is as in (2.4).

Remark 4.5 Let Φ be a concave Orlicz function on $(0,\infty)$ with $p_\Phi\in(0,1]$. When $\varphi(x,t)=\Phi(t)$ for all $x\in\mathcal{X}$ and $t\in[0,\infty)$, the (φ,M)-atom is just the (Φ,M)-atom

introduced in [57]. However, the (φ, M, ϵ)-molecule is different from the (Φ, M, ϵ)-molecule in [57] even when $\varphi(x,t) = \Phi(t)$ for all $x \in \mathcal{X}$ and $t \in [0,\infty)$. More precisely, recall that β is called a (Φ, M, ϵ)-*molecule*, introduced in [57], if (ii) of Definition 4.4 is replaced by that, for each $k \in \{0,\dots,M\}$ and $j \in \mathbb{Z}_+$, it holds true that

$$\left\|\left(r_B^2 L\right)^k b\right\|_{L^2(U_j(B))} \le 2^{-j\epsilon} r_B^{2M}\left[\mu\left(2^j B\right)\right]^{-1/2}\left[\rho\left(\mu\left(2^j B\right)\right)\right]^{-1},$$

where $U_j(B)$ with $j \in \mathbb{Z}_+$ is as in (2.4) and ρ is given by $\rho(t) := t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$. Let p_2 be any lower type of Φ. Then for any $\epsilon \in (0,\infty)$, every (φ, M, ϵ)-molecule is a $(\Phi, M, \epsilon - n(1/p_2 - 1/2))$-molecule when $\varphi := \Phi$. Indeed, by [92, Proposition 2.1], we know that ρ is of upper type $1/p_2 - 1$, which, together with (2.2), implies that, for all $j \in \mathbb{N}$, $[\rho(\mu(2^j B))]^{-1} \gtrsim 2^{-jn(1/p_2-1)}[\rho(\mu(B))]^{-1}$. From this and (2.2), we further deduce that, for all $j \in \mathbb{N}$, $[\mu(2^j B)]^{-1/2}[\rho(\mu(2^j B))]^{-1} \gtrsim 2^{-jn(1/p_2-1/2)}[\mu(B)]^{-1/2}[\rho(\mu(B))]^{-1}$, which, together with the fact that $\|\chi_B\|_{L^\varphi(\mathcal{X})} = \mu(B)\rho(\mu(B))$, implies that the claim holds true. We point out that the notion of (φ, M, ϵ)-molecules is motivated by [70], which is convenient in applications (see, for example, [70] for more details).

4.1 Decompositions into Atoms and Molecules

Recall that a function f on $\mathcal{X} \times (0,\infty)$ is said to have *bounded support*, if there exist a ball $B \subset \mathcal{X}$ and $0 < c_1 < c_2 < \infty$ such that $\operatorname{supp} f \subset B \times (c_1, c_2)$. In what follows, let $L_b^2(\mathcal{X} \times (0,\infty))$ denote the *set of all functions* $f \in L^2(\mathcal{X} \times (0,\infty))$ *with bounded support*, $M \in \mathbb{N}$ and $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.2), (2.12), and (2.11). Let $\widetilde{\Phi}$ be as in Lemma 2.2 and $\Psi(t) := t^{2(M+1)}\widetilde{\Phi}(t)$ for all $t \in (0,\infty)$. For all $f \in L_b^2(\mathcal{X} \times (0,\infty))$ and $x \in \mathcal{X}$, define

$$\pi_{\Psi,L}(f)(x) := C_\Psi \int_0^\infty \Psi(t\sqrt{L})\big(f(\cdot,t)\big)(x)\frac{dt}{t}, \tag{4.2}$$

where C_Ψ is a positive constant such that

$$C_\Psi \int_0^\infty \Psi(t) t^2 e^{-t^2}\frac{dt}{t} = 1. \tag{4.3}$$

By (2.7) and Hölder's inequality, we easily see that, if $f \in L_b^2(\mathcal{X} \times (0,\infty))$, then $\pi_{\Psi,L}(f) \in L^2(\mathcal{X})$. Moreover, we have the following boundedness of $\pi_{\Psi,L}$.

Proposition 4.6 *Let L satisfy Assumptions A and B, $\pi_{\Psi,L}$ be as in (4.2), φ as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ being as in (2.10), and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then*

(i) *the operator $\pi_{\Psi,L}$, initially defined on the space $T_2^{2,b}(\mathcal{X} \times (0,\infty))$, extends to a bounded linear operator from $T_2^2(\mathcal{X} \times (0,\infty))$ to $L^2(\mathcal{X})$;*
(ii) *the operator $\pi_{\Psi,L}$, initially defined on the space $T_\varphi^b(\mathcal{X})$, extends to a bounded linear operator from $T_\varphi(\mathcal{X} \times (0,\infty))$ to $H_{\varphi,L}(\mathcal{X})$.*

Proof The conclusion (i) is just [57, Proposition 4.1(i)]. We only need to show (ii) of this proposition. Let $f \in T^b_\varphi(\mathcal{X} \times (0,\infty))$. Then by Proposition 3.6, Corollary 3.5 and (i), we know that

$$\pi_{\Psi,L}(f) = \sum_j \lambda_j \pi_{\Psi,L}(a_j) =: \sum_j \lambda_j \alpha_j$$

in $L^2(\mathcal{X})$, where $\{\lambda_j\}_j$ and $\{a_j\}_j$ satisfy (3.1) and (3.2), respectively. Recall that, for each j, $\operatorname{supp} a_j \subset \widehat{B_j}$ and B_j is a ball of $\mathcal{X}$. Moreover, from (2.7), we deduce that S_L is bounded on $L^2(\mathcal{X})$, which implies that, for all $x \in \mathcal{X}$, $S_L(\pi_{\Psi,L}(f))(x) \le \sum_j |\lambda_j| S_L(\alpha_j)(x)$. This, combined with Lemma 2.6(i), yields that

$$\int_{\mathcal{X}} \varphi\big(x, S_L\big(\pi_{\Psi,L}(f)\big)(x)\big)\, d\mu(x) \lesssim \sum_j \int_{\mathcal{X}} \varphi\big(x, |\lambda_j| S_L(\alpha_j)(x)\big)\, d\mu(x). \tag{4.4}$$

We now show that $\alpha_j = \pi_{\Psi,L}(a_j)$ is a multiple of a (φ, M)-atom for each j. Let

$$b_j := C_\Psi \int_0^\infty t^{2(M+1)} L\widetilde{\Phi}(t\sqrt{L})\big(a_j(\cdot,t)\big)\frac{dt}{t},$$

where C_Ψ is as in (4.3). Then for each j, from the definitions of α_j and b_j, it follows that $\alpha_j = L^M b_j$. Moreover, by Lemma 2.2, we know that, for each $k \in \{0,\dots,M\}$, $\operatorname{supp}(L^k b_j) \subset B_j$. Furthermore, for any $h \in L^2(B_j)$, from Hölder's inequality and (2.7), we infer that

$$\begin{aligned}
&\left|\int_{\mathcal{X}} \big(r_{B_j}^2 L\big)^k b_j(x) h(x)\, d\mu(x)\right| \\
&\quad = C_\Psi \left|\int_{\mathcal{X}} \int_0^\infty t^{2(M+1)} \big(r_{B_j}^2 L\big)^k L\widetilde{\Phi}(t\sqrt{L})\big(a_j(\cdot,t)\big)(x) h(x) \frac{d\mu(x)\, dt}{t}\right| \\
&\quad \lesssim r_{B_j}^{2M} \int_{\mathcal{X}} \int_0^{r_B} \big|a_j(y,t)\big(t^2 L\big)^{k+1} \widetilde{\Phi}(t\sqrt{L}) h(y)\big| \frac{d\mu(y)\, dt}{t} \\
&\quad \lesssim r_{B_j}^{2M} \|a_j\|_{T_2^2(\mathcal{X}\times(0,\infty))} \left\{\int_{\mathcal{X}} \int_0^\infty \big|\big(t^2 L\big)^{k+1} \widetilde{\Phi}(t\sqrt{L}) h(y)\big|^2 \frac{d\mu(y)\, dt}{t}\right\}^{1/2} \\
&\quad \lesssim r_{B_j}^{2M} \|a_j\|_{T_2^2(\mathcal{X}\times(0,\infty))} \|h\|_{L^2(\mathcal{X})} \lesssim r_{B_j}^{2M} \big[V(B_j)\big]^{1/2} \|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}^{-1} \|h\|_{L^2(\mathcal{X})},
\end{aligned}$$

which implies that $\|(r_{B_j}^2 L)^k b_j\|_{L^2(\mathcal{X})} \lesssim r_{B_j}^{2M} [V(B_j)]^{1/2} \|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}^{-1}$. Therefore, α_j is a (φ, M)-atom up to a harmless constant.

We claim that, for any $\lambda \in \mathbb{C}$ and (φ, M)-atom α supported in a ball $B \subset \mathcal{X}$,

$$\int_{\mathcal{X}} \varphi\big(x, S_L(\lambda\alpha)(x)\big)\, d\mu(x) \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right). \tag{4.5}$$

If (4.5) holds true, by (4.5), the facts that, for all $\lambda \in (0,\infty)$,

$$S_L\big(\pi_{\Psi,L}(f/\lambda)\big) = S_L\big(\pi_{\Psi,L}(f)\big)/\lambda$$

and $\pi_{\Psi,L}(f/\lambda) = \sum_j \lambda_j \alpha_j/\lambda$, and $S_L(\pi_{\Psi,L}(f)) \le \sum_j |\lambda_j| S_L(\alpha_j)$, we see that, for all $\lambda \in (0,\infty)$,

$$\int_{\mathcal{X}} \varphi\left(x, \frac{S_L(\pi_{\Psi,L}(f))(x)}{\lambda}\right) d\mu(x) \lesssim \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}}\right),$$

which, together with (3.2), implies that

$$\|\pi_{\Psi,L}(f)\|_{H_{\varphi,L}(\mathcal{X})} \lesssim \Lambda(\{\lambda_j \alpha_j\}_j) \lesssim \|f\|_{T_\varphi(\mathcal{X}\times(0,\infty))},$$

and hence completes the proof of (ii).

Now we prove (4.5). Write

$$\int_{\mathcal{X}} \varphi\big(x, S_L(\lambda\alpha)(x)\big)\, d\mu(x) = \sum_{k=0}^{\infty} \int_{U_k(B)} \varphi\big(x, |\lambda| S_L(\alpha)(x)\big)\, d\mu(x). \tag{4.6}$$

From the assumption $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, Lemma 2.8(iv) and the definition of $I(\varphi)$, we infer that, there exists $p_1 \in [I(\varphi), 1]$ such that φ is of uniformly upper type p_1 and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$. For $k \in \{0, \dots, 4\}$, by the uniformly upper type p_1 property of φ, Hölder's inequality, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$, the $L^2(\mathcal{X})$-boundedness of S_L and (2.2), we conclude that

$$\begin{aligned}
&\int_{U_k(B)} \varphi\big(x, |\lambda| S_L(\alpha)(x)\big)\, d\mu(x) \\
&\quad\lesssim \int_{U_k(B)} \varphi\big(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big\{1 + \big[S_L(\alpha)(x)\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_1}\big\}\, d\mu(x) \\
&\quad\lesssim \varphi\big(U_k(B), |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big) + \|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})} \\
&\qquad\times \left\{\int_{U_k(B)} \big[\varphi\big(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big]^{\frac{2}{2-p_1}}\, d\mu(x)\right\}^{\frac{2-p_1}{2}} \left\{\int_{U_k(B)} \big[S_L(\alpha)(x)\big]^2\, d\mu(x)\right\}^{\frac{p_1}{2}} \\
&\quad\lesssim \varphi\big(U_k(B), |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big) \lesssim \varphi\big(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big).
\end{aligned} \tag{4.7}$$

From the assumption that $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, it follows that, there exist $p_2 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2})$. Moreover, by the definitions of $i(\varphi)$ and $q(\varphi)$, we know that φ is of uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathcal{X})$. When $k \in \mathbb{N}$ with $k \ge 5$, from the uniformly upper type p_1 and lower type p_2 properties of φ, it follows that

$$\begin{aligned}
&\int_{U_k(B)} \varphi\big(x, |\lambda| S_L(\alpha)(x)\big)\, d\mu(x) \\
&\quad\lesssim \int_{U_k(B)} \varphi\big(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big[S_L(\alpha)(x)\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_1}\, d\mu(x) \\
&\qquad+ \int_{U_k(B)} \varphi\big(x, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big[S_L(\alpha)(x)\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_2}\, d\mu(x) \\
&\quad=: \mathrm{E}_k + \mathrm{F}_k.
\end{aligned} \tag{4.8}$$

To estimate E_k and F_k, we first estimate $\int_{U_k(B)}[S_L(\alpha)(x)]^2\,d\mu(x)$. Write

$$\begin{aligned}
&\int_{U_k(B)}\big[S_L(\alpha)(x)\big]^2\,d\mu(x)\\
&\quad=\int_{U_k(B)}\int_0^{\frac{d(x,x_B)}{4}}\int_{d(x,y)<t}\big|\big(t^2L\big)^{M+1}e^{-t^2L}b(y)\big|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t^{4M+1}}\,d\mu(x)\\
&\qquad+\int_{U_k(B)}\int_{\frac{d(x,x_B)}{4}}^{\infty}\int_{d(x,y)<t}\cdots\\
&\quad=:\mathrm{H}_k+\mathrm{I}_k.
\end{aligned}\tag{4.9}$$

We first estimate the term H_k. Let

$$G_k(B):=\big\{y\in\mathcal{X}:\ \text{there exists } x\in U_k(B) \text{ such that } d(x,y)<d(x,x_B)/4\big\}.$$

From $x\in U_k(B)$, it follows that $d(x,x_B)\in[2^{k-1}r_B,2^kr_B)$. Let $z\in B$ and $y\in G_k(B)$. Then $d(y,z)\ge d(x,x_B)-d(y,x)-d(z,x_B)\ge 3d(x,x_B)/4-r_B\ge 2^{k-2}r_B$, which implies that $\operatorname{dist}(G_k(B),B)\ge 2^{k-2}r_B$. By this, Fubini's theorem, (2.5), and (2.3), we know that

$$\begin{aligned}
\mathrm{H}_k&\lesssim\int_0^{2^{k+1}r_B}\int_{G_k(B)}\big|\big(t^2L\big)^{M+1}e^{-t^2L}b(y)\big|^2\,d\mu(y)\,\frac{dt}{t^{4M+1}}\\
&\lesssim\|b\|_{L^2(B)}^2\int_0^{2^{k+1}r_B}\exp\left\{-\frac{[\operatorname{dist}(G_k(B),B)]^2}{C_3t^2}\right\}\frac{dt}{t^{4M+1}}\\
&\lesssim r_B^{4M}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}\int_0^{2^{k+1}r_B}\left[\frac{t}{2^kr_B}\right]^{4M+1}\frac{dt}{t^{4M+1}}\\
&\lesssim 2^{-4kM}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}.
\end{aligned}\tag{4.10}$$

For I_k, from Lemma 2.1, it follows that

$$\begin{aligned}
\mathrm{I}_k&\lesssim\int_{2^{k-2}r_B}^{\infty}\int_{\mathcal{X}}\big|\big(t^2L\big)^{M+1}e^{-t^2L}b(y)\big|^2\,d\mu(y)\,\frac{dt}{t^{4M+1}}\\
&\lesssim\int_{2^{k-2}r_B}^{\infty}\|b\|_{L^2(B)}^2\frac{dt}{t^{4M+1}}\lesssim 2^{-4kM}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2},
\end{aligned}$$

which, together with (4.9) and (4.10), implies that, for all $k\in\mathbb{N}$ with $k\ge 5$,

$$\|S_L(\alpha)\|_{L^2(U_k(B))}\lesssim 2^{-2kM}\big[\mu(B)\big]^{1/2}\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}.\tag{4.11}$$

Now we estimate E_k. By Hölder's inequality, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$, (4.11), Lemma 2.8(vii), and (2.2), we conclude that

$$\begin{aligned}
\mathrm{E}_k &\le \left\{\int_{U_k(B)} \left[\varphi\left(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right)\right]^{\frac{2}{2-p_1}} d\mu(x)\right\}^{\frac{2-p_1}{2}} \\
&\quad \times \|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})} \left\{\int_{U_k(B)} \left[S_L(\alpha)(x)\right]^2 d\mu(x)\right\}^{\frac{p_1}{2}} \\
&\lesssim 2^{-2kMp_1} \frac{[\mu(B)]^{\frac{p_1}{2}}}{[\mu(2^{k+1}B)]^{\frac{p_1}{2}}} \varphi\left(2^{k+1}B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right) \\
&\lesssim 2^{-2kMp_1} \frac{[\mu(B)]^{\frac{p_1}{2}}}{[\mu(2^{k+1}B)]^{\frac{p_1}{2}}} \left[\frac{\mu(2^{k+1}B)}{\mu(B)}\right]^{q_0} \varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right) \\
&\lesssim 2^{-2kMp_1} \varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right)\left[\mu(B)\right]^{\frac{p_1}{2}-q_0}\left[\mu\left(2^{k+1}B\right)\right]^{q_0-\frac{p_1}{2}} \\
&\lesssim 2^{-k[2Mp_1-nq_0+\frac{n}{2}]} \varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right). \qquad (4.12)
\end{aligned}$$

Moreover, by Remark 2.5, we know that $p_1 \ge p_2$ and hence $2/(2-p_1) \ge 2/(2-p_2)$, which, together with $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and Lemma 2.8(ii), implies that $\varphi \in \mathbb{RH}_{2/(2-p_2)}(\mathcal{X})$. From this, Hölder's inequality, and (4.11), it follows that

$$\begin{aligned}
\mathrm{F}_k &\lesssim \left\{\int_{2^{k+1}B} \left[\varphi\left(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right)\right]^{\frac{2}{2-p_2}} d\mu(x)\right\}^{\frac{2-p_2}{2}} \\
&\quad \times \|\chi_B\|^{p_2}_{L^\varphi(\mathcal{X})} \left(2^{-2kM}\left[\mu(B)\right]^{\frac{1}{2}} \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right)^{p_2} \\
&\lesssim 2^{-2kMp_2} \|\chi_B\|^{-p_2}_{L^\varphi(\mathcal{X})} \left[\frac{\mu(B)}{\mu(2^{k+1}B)}\right]^{\frac{p_2}{2}} \varphi\left(2^{k+1}B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right) \\
&\lesssim 2^{-2kMp_2} \left[\mu(B)\right]^{\frac{p_2}{2}-q_0}\left[\mu\left(2^{k+1}B\right)\right]^{q_0-\frac{p_2}{2}} \varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right) \\
&\lesssim 2^{-k(2Mp_2+\frac{np_2}{2}-nq_0)} \varphi\left(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\right),
\end{aligned}$$

which, together with (4.6), (4.7), (4.8), (4.12), and $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2}) \ge \frac{n}{2}(\frac{q_0}{p_1} - \frac{1}{2})$, implies that (4.5) holds true. This finishes the proof of (ii) and hence Proposition 4.6. □

Proposition 4.7 *Let φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ being as in (2.10), and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then, for all $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that*

$$f = \sum_j \lambda_j \alpha_j \qquad (4.13)$$

in both $H_{\varphi,L}(\mathcal{X})$ and $L^2(\mathcal{X})$. Moreover, there exists a positive constant C such that, for all $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$,

$$\Lambda\big(\{\lambda_j\alpha_j\}_j\big) := \inf\left\{\lambda \in (0,\infty) : \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) \le 1\right\} \le C\|f\|_{H_{\varphi,L}(\mathcal{X})},$$

where for each j, $\operatorname{supp}\alpha_j \subset B_j$.

Proof Let $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$. Then by the H_∞-functional calculi for L and (4.3), we know that

$$f = C_\Psi \int_0^\infty \Psi(t\sqrt{L})t^2Le^{-t^2L}f\frac{dt}{t} = \pi_{\Psi,L}\big(t^2Le^{-t^2L}f\big) \tag{4.14}$$

in $L^2(\mathcal{X})$. Moreover, from Definition 4.1 and (2.7), we infer that $t^2Le^{-t^2L}f \in T_\varphi(\mathcal{X}\times(0,\infty)) \cap T_2^2(\mathcal{X}\times(0,\infty))$. Applying Theorem 3.1, Corollary 3.5 and Proposition 4.6 to $t^2Le^{-t^2L}f$, we conclude that

$$f = \pi_{\Psi,L}\big(t^2Le^{-t^2L}f\big) = \sum_j \lambda_j\pi_{\Psi,L}(a_j) =: \sum_j \lambda_j\alpha_j$$

in both $L^2(\mathcal{X})$ and $H_{\varphi,L}(\mathcal{X})$, and $\Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|t^2Le^{-t^2L}f\|_{T_\varphi(\mathcal{X}\times(0,\infty))} \sim \|f\|_{H_{\varphi,L}(\mathcal{X})}$. Furthermore, by the proof of Proposition 4.6, we know that, for each j, α_j is a (φ, M)-atom up to a harmless constant, which completes the proof of Proposition 4.7. □

Corollary 4.8 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ being as in (2.10), and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then for all $f \in H_{\varphi,L}(\mathcal{X})$, there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that $f = \sum_j \lambda_j\alpha_j$ in $H_{\varphi,L}(\mathcal{X})$. Moreover, there exists a positive constant C, independent of f, such that $\Lambda(\{\lambda_j\alpha_j\}_j) \le C\|f\|_{H_{\varphi,L}(\mathcal{X})}$.*

Proof If $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, then it follows, from Proposition 4.7, that all conclusions hold true.

If $f \in H_{\varphi,L}(\mathcal{X})$, since $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$, we then choose $\{f_k\}_{k\in\mathbb{Z}_+} \subset (H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X}))$ such that, for all $k \in \mathbb{Z}_+$, $\|f_k\|_{H_{\varphi,L}(\mathcal{X})} \le 2^{-k}\|f\|_{H_{\varphi,L}(\mathcal{X})}$ and $f = \sum_{k\in\mathbb{Z}_+} f_k$ in $H_{\varphi,L}(\mathcal{X})$. By Proposition 4.7, we see that, for all $k \in \mathbb{Z}_+$, there exist $\{\lambda_j^k\}_j \subset \mathbb{C}$ and (φ, M)-atoms $\{\alpha_j^k\}_j$ such that $f_k = \sum_j \lambda_j^k\alpha_j^k$ in $H_{\varphi,L}(\mathcal{X})$ and $\Lambda(\{\lambda_j^k\alpha_j^k\}_j) \lesssim \|f_k\|_{H_{\varphi,L}(\mathcal{X})}$. From this, we deduce that, for each $k \in \mathbb{Z}_+$,

$$\sum_j \varphi\left(B_j^k, \frac{|\lambda_j^k|}{\|f_k\|_{H_{\varphi,L}(\mathcal{X})}\|\chi_{B_j^k}\|_{L^\varphi(\mathcal{X})}}\right) \lesssim 1,$$

where, for each j, α_j^k is supported in the ball B_j^k, which, together with the uniformly lower type p_2 property of φ with $p_2 \in (0, i(\varphi))$, implies that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_+}\sum_j \varphi\left(B_j^k, \frac{|\lambda_j^k|}{\|f\|_{H_{\varphi,L}(\mathcal{X})}\|\chi_{B_j^k}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\quad\lesssim \sum_{k\in\mathbb{Z}_+}\sum_j \varphi\left(B_j^k, \frac{|\lambda_j^k|}{2^k\|f_k\|_{H_{\varphi,L}(\mathcal{X})}\|\chi_{B_j^k}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\quad\lesssim \sum_{k\in\mathbb{Z}_+} 2^{-kp_2}\\
&\quad\lesssim 1.
\end{aligned}$$

This further implies that $\Lambda(\{\lambda_j^k\alpha_j^k\}_{k\in\mathbb{Z}_+,j}) \lesssim \|f\|_{H_{\varphi,L}(\mathcal{X})}$ and hence finishes the proof of Corollary 4.8. □

Let $H^M_{\varphi,\mathrm{at,fin}}(\mathcal{X})$ and $H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X})$ denote the *sets of all finite combinations of* (φ, M)*-atoms and* (φ, M, ϵ)*-molecules*, respectively. Then we have the following dense conclusions.

Proposition 4.9 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ being as in (2.10), $\epsilon \in (n[q(\varphi)/i(\varphi) - 1/2], \infty)$ and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then the spaces $H^M_{\varphi,\mathrm{at,fin}}(\mathcal{X})$ and $H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X})$ are both dense in the space $H_{\varphi,L}(\mathcal{X})$.*

Proof From Corollary 4.8, it follows that $H^M_{\varphi,\mathrm{at,fin}}(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$.

To prove that $H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$, noticing that each (φ, M)-atom is a (φ, M, ϵ)-molecule, hence we know that $H^M_{\varphi,\mathrm{at,fin}}(\mathcal{X}) \subset H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X})$ and we only need to show that $H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X}) \subset H_{\varphi,L}(\mathcal{X})$. Let $\lambda \in \mathbb{C}$ and β be a (φ, M, ϵ)-molecule associated with a ball $B := B(x_B, r_B)$. Then there exists $b \in L^2(\mathcal{X})$ such that $\beta = L^M b$ and b satisfies Definition 4.4. Write

$$\begin{aligned}
&\int_{\mathcal{X}} \varphi\big(x, S_L(\lambda\beta)(x)\big)\,d\mu(x)\\
&\quad\lesssim \sum_{j=0}^{\infty}\int_{\mathcal{X}} \varphi\left(x, |\lambda|\left\{\int_0^{r_B}\int_{d(x,y)<t}\left|t^2Le^{-t^2L}(\chi_{U_j(B)}\beta)(y)\right|^2\frac{d\mu(y)\,dt}{V(x,t)t}\right\}^{1/2}\right)d\mu(x)\\
&\qquad+\sum_{j=0}^{\infty}\int_{\mathcal{X}} \varphi\left(x, |\lambda|\left\{\int_{r_B}^{\infty}\int_{d(x,y)<t}\cdots\right\}^{1/2}\right)d\mu(x) =: \sum_{j=0}^{\infty}\mathrm{E}_j + \sum_{j=0}^{\infty}\mathrm{F}_j. \qquad (4.15)
\end{aligned}$$

For each $j \in \mathbb{Z}_+$, let $B_j := 2^j B$. Then

$$\mathrm{E}_j = \sum_{k=0}^{\infty} \int_{U_k(B_j)} \varphi\left(x, |\lambda| \left\{ \int_0^{r_B} \int_{d(x,y)<t} \left| t^2 L e^{-t^2 L}(\chi_{U_j(B)}\beta)(y) \right|^2 \right.\right.$$
$$\left.\left. \times \frac{d\mu(y)\,dt}{V(x,t)t} \right\}^{1/2} \right) d\mu(x) =: \sum_{k=0}^{\infty} \mathrm{E}_{k,j}. \tag{4.16}$$

From the assumption $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, Lemma 2.8(iv) and the definition of $I(\varphi)$, we deduce that, there exists $p_1 \in [I(\varphi), 1]$ such that φ is of uniformly upper type p_1 and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$. Furthermore, by $\epsilon > n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$ and $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, we know that, there exist $p_2 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $\epsilon > n(\frac{q_0}{p_2} - \frac{1}{2})$ and $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2})$. Moreover, from the definitions of $i(\varphi)$ and $q(\varphi)$, we infer that φ is of uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathcal{X})$.

When $k \in \{0, \dots, 4\}$, by the uniformly upper type p_1 and lower type p_2 properties of φ, we see that

$$\mathrm{E}_{k,j} \lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1} \int_{U_k(B_j)} \varphi\big(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big) \big[S_L(\chi_{U_j(B)}\beta)(x)\big]^{p_1}\, d\mu(x)$$
$$+ \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2} \int_{U_k(B_j)} \varphi\big(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big) \big[S_L(\chi_{U_j(B)}\beta)(x)\big]^{p_2}\, d\mu(x)$$
$$=: \mathrm{G}_{k,j} + \mathrm{H}_{k,j}. \tag{4.17}$$

Now we estimate $\mathrm{G}_{k,j}$. By Hölder's inequality, the $L^2(\mathcal{X})$-boundedness of S_L, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and Lemma 2.8(vii), we conclude that

$$\mathrm{G}_{k,j} \lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1} \left\{ \int_{U_k(B_j)} \big[S_L(\chi_{U_j(B)}\beta)(x)\big]^2\, d\mu(x) \right\}^{\frac{p_1}{2}}$$
$$\times \left\{ \int_{U_k(B_j)} \big[\varphi\big(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big]^{\frac{2}{2-p_1}}\, d\mu(x) \right\}^{\frac{2-p_1}{2}}$$
$$\lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1} \|\beta\|_{L^2(\mathcal{X})}^{p_1} \big[\mu\big(2^{k+j}B\big)\big]^{-\frac{p_1}{2}} \varphi\big(2^{k+j}B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)$$
$$\lesssim 2^{-jp_1\epsilon} 2^{(k+j)n(q_0 - \frac{p_1}{2})} \varphi\big(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)$$
$$\sim 2^{-jp_1[\epsilon - n(\frac{q_0}{p_1} - \frac{1}{2})]} \varphi\big(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.18}$$

For $\mathrm{H}_{k,j}$, similarly, we see that $\mathrm{H}_{k,j} \lesssim 2^{-jp_2[\epsilon - n(q_0/p_2 - 1/2)]} \varphi(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1})$, which, together with (4.17), (4.18) and $p_1 \geq p_2$, implies that, for each $j \in \mathbb{Z}_+$ and $k \in \{0, \dots, 4\}$,

$$\mathrm{E}_{k,j} \lesssim 2^{-jp_2[\epsilon - n(\frac{q_0}{p_2} - \frac{1}{2})]} \varphi\big(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.19}$$

When $k \in \mathbb{N}$ with $k \geq 5$, to estimate $\mathrm{E}_{k,j}$, for $x \in \mathcal{X}$, let

$$S_{L,r_B}(x) := \left\{ \int_0^{r_B} \int_{d(x,y)<t} \left| t^2 L e^{-t^2 L}(\chi_{U_j(B)}\beta)(y) \right|^2 \frac{d\mu(y)\,dt}{V(x,t)t} \right\}^{1/2}.$$

Then from the uniformly upper type p_1 and lower type p_2 properties of φ, it follows that

$$\begin{aligned}
\mathrm{E}_{k,j} &\lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1} \int_{U_k(B_j)} \varphi\big(x, |\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big[S_{L,r_B}(x)\big]^{p_1}\,d\mu(x) \\
&\quad + \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2} \int_{U_k(B_j)} \varphi\big(x, |\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big[S_{L,r_B}(x)\big]^{p_2}\,d\mu(x) \\
&=: \mathrm{I}_{k,j} + \mathrm{K}_{k,j}. \qquad (4.20)
\end{aligned}$$

For each $k, j \in \mathbb{Z}_+$, let $\widetilde{U}_k(B_j) := \{y \in \mathcal{X} : 2^{j-2}2^k r_B \leq d(y, x_B) < 2^{j+1}2^k r_B\}$. It is easy to see that, when $k \geq 5$, $\mathrm{dist}(U_j(B), \widetilde{U}_k(B_j)) \gtrsim 2^{k+j} r_B$. Take $s \in (0, \infty)$ such that $s \in (n[\frac{q_0}{p_2} - \frac{1}{2}], 2M)$. Now we deal with the term $\mathrm{I}_{k,j}$. To this end, by (2.5), we see that

$$\begin{aligned}
&\int_{U_k(B_j)} \big[S_{L,r_B}(x)\big]^2\,d\mu(x) \\
&\quad = \int_{U_k(B_j)} \int_0^{r_B} \int_{d(x,y)<t} \left| t^2 L e^{-t^2 L}(\chi_{U_j(B)}\beta)(y) \right|^2 \frac{d\mu(y)\,dt}{V(x,t)t}\,d\mu(x) \\
&\quad \lesssim \int_0^{r_B} \int_{\widetilde{U}_k(B_j)} \left| t^2 L e^{-t^2 L}(\chi_{U_j(B)}\beta)(y) \right|^2 \frac{d\mu(y)\,dt}{t} \\
&\quad \lesssim \int_0^{r_B} \exp\left\{ -\frac{[\mathrm{dist}(U_j(B), \widetilde{U}_k(B_j))]^2}{C_3 t^2} \right\} \|\beta\|_{L^2(U_j(B))}^2 \frac{dt}{t} \\
&\quad \lesssim 2^{-2(k+j)s} \|\beta\|_{L^2(U_j(B))}^2, \qquad (4.21)
\end{aligned}$$

which, together with Hölder's inequality, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and Lemma 2.8(vii), implies that

$$\begin{aligned}
\mathrm{I}_{k,j} &\lesssim 2^{-(k+j)p_1 s} \|\beta\|_{L^2(U_j(B))}^{p_1} \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1} \big[\mu\big(2^{k+j}B\big)\big]^{-\frac{p_1}{2}} \varphi\big(2^{k+j}B, |\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big) \\
&\lesssim 2^{-jp_1[\epsilon+s-n(\frac{q_0}{p_1}-\frac{1}{2})]} 2^{-kp_1[s-n(\frac{q_0}{p_1}-\frac{1}{2})]} \varphi\big(B, |\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \qquad (4.22)
\end{aligned}$$

Now we estimate $\mathrm{K}_{k,j}$. From Hölder's inequality, (4.21), $\varphi \in \mathbb{RH}_{2/(2-p_2)}(\mathcal{X})$ and Lemma 2.8(vii), it follows that

$$\begin{aligned}
\mathrm{K}_{k,j} &\lesssim \left\{ \int_{U_k(B_j)} \big[\varphi\big(x, |\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big]^{\frac{2}{2-p_2}}\,d\mu(x) \right\}^{\frac{2-p_2}{2}} \\
&\quad \times \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2} \left\{ \int_{U_k(B_j)} \big[S_{L,r_B}(x)\big]^2\,d\mu(x) \right\}^{\frac{p_2}{2}}
\end{aligned}$$

$$\lesssim 2^{-(k+j)sp_2}\|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2}\|\beta\|_{L^2(\mathcal{X})}^{p_2}\big[\mu\big(2^{k+j}B\big)\big]^{-\frac{p_2}{2}}\varphi\big(2^{k+j}B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)$$
$$\lesssim 2^{-jp_2[\epsilon+s-n(\frac{q_0}{p_2}-\frac{1}{2})]}2^{-kp_2[s-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi\big(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.23}$$

By (4.20), (4.22), (4.23) and $p_1 \geq p_2$, we know that, when $k \in \mathbb{N}$ with $k \geq 5$ and $j \in \mathbb{Z}_+$,

$$\mathrm{E}_{k,j} \lesssim 2^{-jp_2[\epsilon+s-n(\frac{q_0}{p_2}-\frac{1}{2})]}2^{-kp_2[s-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi\big(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.24}$$

Now we deal with F_j. Write

$$\mathrm{F}_j = \sum_{k=0}^{\infty}\int_{U_k(B_j)}\varphi\Bigg(x,|\lambda|\Bigg\{\int_{r_B}^{\infty}\int_{d(x,y)<t}\big|t^2Le^{-t^2L}(\chi_{U_j(B)}\beta)(y)\big|^2$$
$$\times\frac{d\mu(y)\,dt}{V(x,t)t}\Bigg\}^{1/2}\Bigg)d\mu(x) =: \sum_{k=0}^{\infty}\mathrm{F}_{k,j}. \tag{4.25}$$

When $k \in \{0,\dots,4\}$, by the uniformly upper type p_1 and lower type p_2 properties of φ, Hölder's inequality, the $L^2(\mathcal{X})$-boundedness of S_L and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$, similar to the proof of (4.19), we see that

$$\mathrm{F}_{k,j} \lesssim 2^{-jp_2[\epsilon-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi\big(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.26}$$

When $k \in \mathbb{N}$ with $k \geq 5$, for any $x \in \mathcal{X}$, let

$$H_{L,r_B}(x) := \Bigg\{\int_{r_B}^{\infty}\int_{d(x,y)<t}\big|\big(t^2L\big)^{M+1}e^{-t^2L}(\chi_{U_j(B)}b)(y)\big|^2\frac{d\mu(y)\,dt}{V(x,t)t^{4M+1}}\Bigg\}^{1/2}.$$

Then from the uniformly upper type p_1 and lower type p_2 properties of φ, it follows that

$$\mathrm{F}_{k,j} \lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1}\int_{U_k(B_j)}\varphi\big(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big[H_{L,r_B}(x)\big]^{p_1}\,d\mu(x)$$
$$+\|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2}\int_{U_k(B_j)}\varphi\big(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big[H_{L,r_B}(x)\big]^{p_2}\,d\mu(x).$$

Similar to (4.21), we know that

$$\int_{U_k(B_j)}\big[H_{L,r_B}(x)\big]^2\,d\mu(x) \lesssim 2^{-2j\epsilon}2^{-2(k+j)s}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}.$$

Thus, similar to (4.24), we conclude that, when $k \in \mathbb{N}$ with $k \geq 5$ and $j \in \mathbb{Z}_+$,

$$\mathrm{F}_{k,j} \lesssim 2^{-jp_2[\epsilon+s-n(\frac{q_0}{p_2}-\frac{1}{2})]}2^{-kp_2[s-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi\big(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big). \tag{4.27}$$

Then from (4.15), (4.16), (4.19), (4.24), (4.25), (4.26) and (4.27), we infer that

$$\int_{\mathcal{X}} \varphi\big(x, |\lambda| S_L(\beta)(x)\big)\, d\mu(x) \lesssim \varphi\big(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big),$$

which implies that $\|\beta\|_{H_{\varphi,L}(\mathcal{X})} \lesssim 1$, and hence completes the proof of Proposition 4.9. □

4.2 Dual Spaces of $H_{\varphi,L}(\mathcal{X})$

In this subsection, we study the dual spaces of Musielak–Orlicz–Hardy spaces $H_{\varphi,L}(\mathcal{X})$. We begin with some notions.

Let $M \in \mathbb{N}$ and $\phi = L^M \nu$ be a function in $L^2(\mathcal{X})$, where $\nu \in \mathcal{D}(L^M)$. Following [51, 52, 57], for $\epsilon \in (0, \infty)$, $M \in \mathbb{N}$ and fixed $x_0 \in \mathcal{X}$, we introduce the space

$$\mathcal{M}_\varphi^{M,\epsilon}(L) := \big\{\phi = L^M \nu \in L^2(\mathcal{X}) : \|\phi\|_{\mathcal{M}_\varphi^{M,\epsilon}(L)} < \infty\big\},$$

where

$$\|\phi\|_{\mathcal{M}_\varphi^{M,\epsilon}(L)} := \sup_{j \in \mathbb{Z}_+} \left\{ 2^{j\epsilon} \big[V(x_0, 1)\big]^{-1/2} \|\chi_{B(x_0,1)}\|_{L^\varphi(\mathcal{X})} \sum_{k=0}^{M} \|L^k \nu\|_{L^2(U_j(B(x_0,1)))} \right\}.$$

Notice that, if $\phi \in \mathcal{M}_\varphi^{M,\epsilon}(L)$ with norm 1 and some $\epsilon \in (0, \infty)$, then ϕ is a (φ, M, ϵ)-molecule adapted to the ball $B(x_0, 1)$. Conversely, if β is a (φ, M, ϵ)-molecule adapted to any ball, then $\beta \in \mathcal{M}_\varphi^{M,\epsilon}(L)$.

Let A_t denote either $(I + t^2 L)^{-1}$ or $e^{-t^2 L}$ and f belong to the dual space of $\mathcal{M}_\varphi^{M,\epsilon}(L)$, $(\mathcal{M}_\varphi^{M,\epsilon}(L))^*$. We claim that $(I - A_t)^M f \in L^2_{\mathrm{loc}}(\mathcal{X})$ in the sense of distributions. Indeed, for any ball B, if $\psi \in L^2(B)$, then it follows, from the Davies–Gaffney estimates (2.5), that $(I - A_t)^M \psi \in \mathcal{M}_\varphi^{M,\epsilon}(L)$ for every $\epsilon \in (0, \infty)$. Thus, there exists a positive constant $C(t, r_B, \mathrm{dist}(B, x_0))$, depending on t, r_B and $\mathrm{dist}(B, x_0)$, such that

$$\begin{aligned} \big|\big\langle (I - A_t)^M f, \psi \big\rangle\big| &:= \big|\big\langle f, (I - A_t)^M \psi \big\rangle\big| \\ &\le C\big(t, r_B, \mathrm{dist}(B, x_0)\big) \|f\|_{(\mathcal{M}_\varphi^{M,\epsilon}(L))^*} \|\psi\|_{L^2(B)}, \end{aligned}$$

which implies that $(I - A_t)^M f \in L^2_{\mathrm{loc}}(\mathcal{X})$ in the sense of distributions.

Finally, for any $M \in \mathbb{N}$, define

$$\mathcal{M}_\varphi^M(\mathcal{X}) := \bigcap_{\epsilon > n[q(\varphi)/i(\varphi) - 1/2]} \big(\mathcal{M}_\varphi^{M,\epsilon}(L)\big)^*,$$

where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11).

Definition 4.10 Let φ be as in Definition 2.4, L satisfy Assumptions A and B, and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2),

(2.12) and (2.11). A functional $f \in \mathcal{M}_\varphi^M(\mathcal{X})$ is said to be in the *space* $\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})} := \sup_{B\subset\mathcal{X}} \frac{[\mu(B)]^{1/2}}{\|\chi_B\|_{L^\varphi(\mathcal{X})}} \left\{\int_B \left|\left(I - e^{-r_B^2 L}\right)^M f(x)\right|^2 d\mu(x)\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls B of $\mathcal{X}$.

By using Davies–Gaffney estimates (2.5) and the uniformly upper type and lower type properties of φ, similar to proofs of [52, Lemmas 8.1 and 8.3] or [57, Propositions 4.4 and 4.5], we obtain the following Propositions 4.11 and 4.12. Here, we omit the details.

Proposition 4.11 *Let φ, L and M be as in Definition 4.10. Then $f \in \mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$ if and only if $f \in \mathcal{M}_\varphi^M(\mathcal{X})$ and*

$$\sup_{B\subset\mathcal{X}} \frac{[\mu(B)]^{1/2}}{\|\chi_B\|_{L^\varphi(\mathcal{X})}} \left\{\int_B \left|\left(I - \left(I + r_B^2 L\right)^{-1}\right)^M f(x)\right|^2 d\mu(x)\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls B of $\mathcal{X}$. Moreover, the quantity appeared in the left-hand side of the above formula is equivalent to $\|f\|_{\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})}$.

Proposition 4.12 *Let φ, L and M be as in Definition 4.10. Then there exists a positive constant C such that, for all $f \in \mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$,*

$$\sup_{B\subset\mathcal{X}} \frac{[\mu(B)]^{1/2}}{\|\chi_B\|_{L^\varphi(\mathcal{X})}} \left\{\int_{\widehat{B}} \left|\left(t^2 L\right)^M e^{-t^2 L} f(x)\right|^2 \frac{d\mu(x)\,dt}{t}\right\}^{1/2} \le C\|f\|_{\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})},$$

where the supremum is taken over all balls B of $\mathcal{X}$.

The following Proposition 4.13 and Corollary 4.15 are a kind of Calderón reproducing formulae.

Proposition 4.13 *Let φ, L and M be as in Definition 4.10, $\epsilon \in (0,\infty)$ and $\widetilde{M} \in \mathbb{N}$ with $\widetilde{M} > M + \epsilon + \frac{N}{4} + \frac{nq(\varphi)}{2i(\varphi)}$, where N, n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.3), (2.2), (2.12) and (2.11). Fix $x_0 \in \mathcal{X}$. Assume that $f \in \mathcal{M}_\varphi^M(\mathcal{X})$ satisfies*

$$\int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{N+\epsilon+2nq_0/p_2}}\, d\mu(x) < \infty \tag{4.28}$$

for some $q_0 \in (q(\varphi),\infty)$ and $p_2 \in (0, i(\varphi))$. Then for all $(\varphi,\widetilde{M})$-atoms α,

$$\langle f,\alpha\rangle = \widetilde{C}_M \int_{\mathcal{X}\times(0,\infty)} \left(t^2 L\right)^M e^{-t^2 L} f(x) \overline{t^2 L e^{-t^2 L}\alpha(x)} \frac{d\mu(x)\,dt}{t},$$

where $\widetilde{C}_M$ is a positive constant satisfying $\widetilde{C}_M \int_0^\infty t^{2(M+1)} e^{-2t^2} \frac{dt}{t} = 1$.

The proof of Proposition 4.7 is a skillful application of the Davies–Gaffney estimates (2.5), the H_∞-functional calculi for L and the uniformly upper type and lower type properties of φ, which is similar to that of [57, Proposition 4.6]. We omit the details here.

To prove that Proposition 4.13 also holds true for all $f \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$, we need the following dyadic cubes on spaces of homogeneous type constructed by Christ [20, Theorem 11].

Lemma 4.14 *There exist a collection of open subsets, $\{Q^k_\alpha \subset \mathcal{X} : \ k \in \mathbb{Z}, \alpha \in I_k\}$, where I_k denotes some (possibly finite) index set depending on k, and constants $\delta \in (0,1)$, $a_0 \in (0,1)$ and $C_6 \in (0,\infty)$ such that*

(i) $\mu(\mathcal{X} \setminus \bigcup_\alpha Q^k_\alpha) = 0$ *for all* $k \in \mathbb{Z}$;
(ii) *if* $i \geq k$, *then either* $Q^i_\alpha \subset Q^k_\beta$ *or* $Q^i_\alpha \cap Q^k_\beta = \emptyset$;
(iii) *for each* (k,α) *and each* $i < k$, *there exists a unique* β *such that* $Q^k_\alpha \subset Q^i_\beta$;
(iv) *the diameter of* $Q^k_\alpha \leq C_6\delta^k$;
(v) *each* Q^k_α *contain some ball* $B(z^k_\alpha, a_0\delta^k)$.

From Proposition 4.13 and Lemma 4.14, we deduce the following weighted version of [57, Corollary 4.3].

Corollary 4.15 *Let φ, L and M be as in Definition 4.10, $\epsilon \in (0,\infty)$ and $\widetilde{M} \in \mathbb{N}$ with $\widetilde{M} > M + \epsilon + \frac{N}{4} + \frac{nq(\varphi)}{2i(\varphi)}$, where N, n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.3), (2.2), (2.12) and (2.11).*

Then for all $(\varphi, \widetilde{M})$-atoms α and $f \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$,

$$\langle f, \alpha\rangle = \widetilde{C}_M \int_{\mathcal{X}\times(0,\infty)} \left(t^2L\right)^M e^{-t^2L} f(x)\overline{t^2Le^{-t^2L}\alpha(x)}\frac{d\mu(x)\,dt}{t},$$

where $\widetilde{C}_M$ is as in Proposition 4.13.

Proof From $\widetilde{M} > M + \epsilon + \frac{N}{4} + \frac{nq(\varphi)}{2i(\varphi)}$, we deduce that there exist $q_0 \in (q(\varphi), \infty)$ and $p_2 \in (0, i(\varphi))$ such that $\widetilde{M} > M + \epsilon + \frac{N}{4} + \frac{nq_0}{2p_2}$. Let $\epsilon \in (0, \widetilde{M} - M - \frac{N}{4} - \frac{nq_0}{2p_2})$. By Proposition 4.13, we only need to show that (4.28) with such ϵ holds true for all $f \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$.

Let all the notation be the same as in Lemma 4.14. For each $j \in \mathbb{Z}$, choose $k_j \in \mathbb{Z}$ such that $C_6\delta^{k_j} \leq 2^j < C_6\delta^{k_j-1}$. Let $B := B(x_0, 1)$, where x_0 is as in (4.28), and

$$M_j := \left\{\beta \in I_{k_0} : Q^{k_0}_\beta \cap B\left(x_0, C_6\delta^{k_j-1}\right) \neq \emptyset\right\}.$$

Then for each $j \in \mathbb{Z}_+$,

$$U_j(B) \subset B\left(x_0, C_6\delta^{k_j-1}\right) \subset \bigcup_{\beta\in M_j} Q^{k_0}_\beta \subset B\left(x_0, 2C_6\delta^{k_j-1}\right). \tag{4.29}$$

From Lemma 4.14, it follows that the sets $\{Q_\beta^{k_0}\}_{\beta\in M_j}$ are disjoint. Moreover, by (iv) and (v) of Lemma 4.14, we know that there exists $z_\beta^{k_0}\in Q_\beta^{k_0}$ such that

$$B\big(z_\beta^{k_0}, a_0\delta^{k_0}\big)\subset Q_\beta^{k_0}\subset B\big(z_\beta^{k_0}, C_6\delta^{k_0}\big)\subset B\big(z_\beta^{k_0}, 1\big). \tag{4.30}$$

Then by Proposition 4.11, we know that

$$\begin{aligned}
\mathrm{H} &:= \left\{\int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{N+\epsilon+2nq_0/p_2}}\,d\mu(x)\right\}^{1/2}\\
&= \left\{\sum_{j\in\mathbb{Z}_+}\int_{U_j(B)} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{N+\epsilon+2nq_0/p_2}}\,d\mu(x)\right\}^{1/2}\\
&\le \sum_{j\in\mathbb{Z}_+} 2^{-j[(N+\epsilon)/2+nq_0/p_2]}\left\{\sum_{\beta\in M_j}\int_{Q_j^{k_0}}\big|\big[I-(I+L)^{-1}\big]^M f(x)\big|^2\,d\mu(x)\right\}^{1/2}\\
&\lesssim \sum_{j\in\mathbb{Z}_+} 2^{-j[(N+\epsilon)/2+nq_0/p_2]}\left\{\sum_{\beta\in M_j}\big[\mu\big(B\big(z_\beta^{k_0},1\big)\big)\big]^{-1}\right.\\
&\qquad\left.\times\|\chi_{B(z_\beta^{k_0},1)}\|_{L^\varphi(\mathcal{X})}^2\|f\|_{\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})}^2\right\}^{1/2}\\
&\lesssim \sum_{j\in\mathbb{Z}_+} 2^{-j[\epsilon/2+nq_0/p_2]}\big[\mu\big(B(x_0,1)\big)\big]^{-1/2}\sum_{\beta\in M_j}\|\chi_{B(z_\beta^{k_0},1)}\|_{L^\varphi(\mathcal{X})}\|f\|_{\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})}.
\end{aligned} \tag{4.31}$$

It follows, from the choice of k_0, that $\delta^{k_0}\sim 1$, which, together with the definition of φ, implies that $\|\chi_{B(z_\beta^{k_0},1)}\|_{L^\varphi(\mathcal{X})}\sim\|\chi_{B(z_\beta^{k_0},a_0\delta^{k_0})}\|_{L^\varphi(\mathcal{X})}$. By this and (4.30), we conclude that

$$\begin{aligned}
\sum_{\beta\in M_j}\|\chi_{B(z_\beta^{k_0},1)}\|_{L^\varphi(\mathcal{X})} &\sim \sum_{\beta\in M_j}\|\chi_{B(z_\beta^{k_0},a_0\delta^{k_0})}\|_{L^\varphi(\mathcal{X})}\\
&\lesssim \sum_{\beta\in M_j}\|\chi_{Q_\beta^{k_0}}\|_{L^\varphi(\mathcal{X})}\sim\|\chi_{\bigcup_{\beta\in M_j}Q_\beta^{k_0}}\|_{L^\varphi(\mathcal{X})}\\
&\lesssim \|\chi_{B(x_0,2C_6\delta^{k_j-1})}\|_{L^\varphi(\mathcal{X})}\lesssim\|\chi_{2^jB}\|_{L^\varphi(\mathcal{X})}.
\end{aligned} \tag{4.32}$$

Moreover, by $q_0\in(q(\varphi),\infty)$, the uniformly lower type p_2 property of φ and Lemma 2.8(vii), we conclude that, for all $j\in\mathbb{Z}_+$,

$$\begin{aligned}
&\int_{2^jB}\varphi\left(x,\frac{1}{2^{jnq_0/p_2}\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right)d\mu(x)\\
&\quad\lesssim 2^{-jnq_0}\varphi\big(2^jB,\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\lesssim 2^{-jnq_0}\left\{\frac{\mu(2^jB)}{\mu(B)}\right\}^{q_0}\varphi\big(B,\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\sim 1,
\end{aligned}$$

which implies that $\|\chi_{2^jB}\|_{L^\varphi(\mathcal{X})}\lesssim 2^{jnq_0/p_2}\|\chi_B\|_{L^\varphi(\mathcal{X})}$. From this, (4.31) and (4.32),

we deduce that

$$\mathrm{H} \lesssim \left[V\left(B(x_0,1)\right)\right]^{-1/2} \|\chi_B\|_{L^\varphi(\mathcal{X})} \|f\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})} < \infty,$$

which completes the proof of Corollary 4.15. □

Now we prove that $\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$ is just the dual space of $H_{\varphi,L}(\mathcal{X})$ by using Corollary 4.15.

Theorem 4.16 *Let L satisfy Assumptions A and B, φ be as in (2.4) with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ being as in (2.10), $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$ and $\widetilde{M} \in \mathbb{N}$ with $\widetilde{M} > M + \frac{N}{4} + \frac{nq(\varphi)}{2i(\varphi)}$, where n, N, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3), (2.12) and (2.11). Then the dual space of $H_{\varphi,L}(\mathcal{X})$, $(H_{\varphi,L}(\mathcal{X}))^*$, coincides with the space $\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$ in the following sense*:

(i) *Let $g \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$. Then the linear functional ℓ, which is initially defined on $H^{\widetilde{M}}_{\varphi,\mathrm{at,fin}}(\mathcal{X})$ by*

$$\ell(f) := \langle g, f\rangle, \tag{4.33}$$

has a unique extension to $H_{\varphi,L}(\mathcal{X})$ with $\|\ell\|_{(H_{\varphi,L}(\mathcal{X}))^*} \le C\|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})}$, where C is a positive constant independent of g.

(ii) *Conversely, let $\epsilon \in (n[q(\varphi)/i(\varphi) - 1/2], \infty)$. Then for any $\ell \in (H_{\varphi,L}(\mathcal{X}))^*$, there exists $g \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$ such that (4.33) holds true for all $f \in H^{M,\epsilon}_{\varphi,\mathrm{mol,fin}}(\mathcal{X})$ and $\|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})} \le C\|\ell\|_{(H_{\varphi,L}(\mathcal{X}))^*}$, where C is a positive constant independent of ℓ.*

Proof Let $g \in \mathrm{BMO}^M_{\varphi,L}(\mathcal{X})$. For any $f \in H^{\widetilde{M}}_{\varphi,\mathrm{at,fin}}(\mathcal{X})$, by Proposition 4.9, we know that $t^2Le^{-t^2L}f \in T_\varphi(\mathcal{X}\times(0,\infty))$. From this and Theorem 3.1, it follows that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and $T_\varphi(\mathcal{X}\times(0,\infty))$-atoms $\{a_j\}_j$ supported in $\{\widehat{B}_j\}_j$ such that (3.2) holds true. Moreover, by the uniformly upper type p_1 property of φ, we know that $\sum_j |\lambda_j| \lesssim \Lambda(\{\lambda_j a_j\}_j)$, where $\Lambda(\{\lambda_j a_j\}_j)$ is as in (3.2). This, together with Corollary 4.15, Hölder's inequality, and Proposition 4.12, yields that

$$\begin{aligned}
|\langle g, f\rangle| &= \left|\widetilde{C}_M \int_0^\infty \int_{\mathcal{X}} \left(t^2L\right)^M e^{-t^2L} g(x) \overline{t^2Le^{-t^2L}f(x)} \frac{d\mu(x)\,dt}{t}\right| \\
&\lesssim \sum_j |\lambda_j| \int_0^\infty \int_{\mathcal{X}} \left|\left(t^2L\right)^M e^{-t^2L} g(x) a_j(x,t)\right| \frac{d\mu(x)\,dt}{t} \\
&\lesssim \sum_j |\lambda_j| \|a_j\|_{T_2^2(\mathcal{X}\times(0,\infty))} \left\{\int_{\widehat{B}_j} \left|\left(t^2L\right)^M e^{-t^2L} g(x)\right|^2 \frac{d\mu(x)\,dt}{t}\right\}^{1/2} \\
&\lesssim \sum_j |\lambda_j| \|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})} \lesssim \Lambda\left(\{\lambda_j\alpha_j\}_j\right) \|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})} \\
&\lesssim \left\|t^2Le^{-t^2L}f\right\|_{T_\varphi(\mathcal{X}\times(0,\infty))} \|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})} \lesssim \|f\|_{H_{\varphi,L}(\mathcal{X})} \|g\|_{\mathrm{BMO}^M_{\varphi,L}(\mathcal{X})},
\end{aligned}$$

which, together with Proposition 4.9, implies that (i) holds true.

Conversely, let $\ell \in (H_{\varphi,L}(\mathcal{X}))^*$. If $g \in \mathcal{M}_\varphi^{M,\epsilon}(L)$, then g is a multiple of a (φ, M, ϵ)-molecule. Moreover, if $\epsilon > n[q(\varphi)/i(\varphi) - 1/2]$, then by Proposition 4.9, we see that $g \in H_{\varphi,L}(\mathcal{X})$, and hence $\mathcal{M}_\varphi^{M,\epsilon}(L) \subset H_{\varphi,L}(\mathcal{X})$. Therefore, $\ell \in \mathcal{M}_\varphi^M(\mathcal{X})$.

Moreover, for any ball $B \subset \mathcal{X}$, let $\phi \in L^2(B)$ with

$$\|\phi\|_{L^2(B)} \le \big[\mu(B)\big]^{1/2} \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}$$

and $\widetilde{\beta} := (I - (I + r_B^2 L)^{-1})^M \phi$. Obviously, $\widetilde{\beta} = (r_B^2 L)^M (I + r_B^2 L)^{-M} \phi =: L^M \widetilde{b}$. Then from the fact that $(I + r_B^2 L)^{-1}$ satisfies the Davies–Gaffney estimates (2.5) with $[\mathrm{dist}(E, F)]^2$ and t^2, respectively, replaced by $\mathrm{dist}(E, F)$ and t, we infer that, for each $j \in \mathbb{Z}_+$ and $k \in \{0, \dots, M\}$,

$$\begin{aligned}
\big\|\big(r_B^2 L\big)^k \widetilde{b}\big\|_{L^2(U_j(B))} &= r_B^{2M} \big\|\big(I - \big(I + r_B^2 L\big)^{-1}\big)^k \big(I + r_B^2 L\big)^{-(M-k)} \phi\big\|_{L^2(U_j(B))} \\
&\lesssim r_B^{2M} \exp\left\{-\frac{\mathrm{dist}(B, U_j(B))}{C_3 r_B}\right\} \|\phi\|_{L^2(B)} \\
&\lesssim 2^{-j\epsilon} r_B^{2M} \big[\mu(B)\big]^{1/2} \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1},
\end{aligned}$$

where $M \in \mathbb{N}$ and $2M > n[q(\varphi)/i(\varphi) - 1/2]$. Thus, $\widetilde{\beta}$ is a multiple of a (φ, M, ϵ)-molecule. Since $(I - (I + r_B^2 L)^{-1})^M \ell$ is well defined and belongs to $L^2_{\mathrm{loc}}(\mathcal{X})$ for every $t \in (0, \infty)$, by $\|\widetilde{\beta}\|_{H_{\varphi,L}(\mathcal{X})} \lesssim 1$, we know that

$$\big|\big\langle\big(I - \big(I + r_B^2 L\big)^{-1}\big)^M \ell, \phi\big\rangle\big| = \big|\big\langle \ell, \big(I - \big(I + r_B^2 L\big)^{-1}\big)^M \phi\big\rangle\big| = \big|\langle \ell, \widetilde{\beta}\rangle\big| \lesssim \|\ell\|_{(H_{\varphi,L}(\mathcal{X}))^*},$$

which further implies that

$$\begin{aligned}
&\frac{[\mu(B)]^{1/2}}{\|\chi_B\|_{L^\varphi(\mathcal{X})}} \left\{\int_B \big|\big(I - \big(I + r_B^2 L\big)^{-1}\big)^M \ell(x)\big|^2 \, d\mu(x)\right\}^{1/2} \\
&\quad \lesssim \sup_{\|\phi\|_{L^2(B)} \le 1} \left|\left\langle \ell, \big(I - \big(I + r_B^2 L\big)^{-1}\big)^M \frac{[\mu(B)]^{1/2} \phi}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right\rangle\right| \lesssim \|\ell\|_{(H_{\varphi,L}(\mathcal{X}))^*}.
\end{aligned}$$

From this and Proposition 4.11, it follows that $\ell \in \mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$, which completes the proof of Theorem 4.16. □

Remark 4.17 By Theorem 4.16, we know that, for all $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, the spaces $\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$ coincide with equivalent norms; thus, in what follows, we denote $\mathrm{BMO}_{\varphi,L}^M(\mathcal{X})$ simply by $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$.

Definition 4.18 A measure $d\mu$ on $\mathcal{X} \times (0, \infty)$ is called a *φ-Carleson measure* if

$$\|d\mu\|_\varphi := \sup_{B \subset \mathcal{X}} \frac{[\mu(B)]^{1/2}}{\|\chi_B\|_{L^\varphi(\mathcal{X})}} \left\{\int_{\widehat{B}} \big|d\mu(x, t)\big|\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ and $\widehat{B}$ denotes the tent over B.

Using Theorem 4.16 and Proposition 4.12, we obtain the following φ-Carleson measure characterization of $\mathrm{BMO}_{\varphi,L}(\mathcal{X})$, whose proof is similar to that of [57, Theorem 4.2]. We omit the details.

Theorem 4.19 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ as in (2.10), and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then the following conditions are equivalent*:

(i) $f \in \mathrm{BMO}_{\varphi,L}(\mathcal{X})$;
(ii) $f \in \mathcal{M}_{\varphi}^{M}(\mathcal{X})$ *satisfies* (4.28) *for some* $q_0 \in (q(\varphi), \infty)$, $p_2 \in (0, i(\varphi))$ *and* $\epsilon \in (0, \infty)$, *and* $d\mu_f$ *is a φ-Carleson measure, where* $d\mu_f$ *is defined by*

$$d\mu_f := \left|\left(t^2 L\right)^M e^{-t^2 L} f(x)\right|^2 \frac{d\mu(x)\,dt}{t}.$$

Moreover, $\|f\|_{\mathrm{BMO}_{\varphi,L}(\mathcal{X})}$ *and* $\|d\mu_f\|_{\varphi}$ *are comparable.*

5 Equivalent Characterizations of $H_{\varphi,L}(\mathcal{X})$

In this section, we establish several equivalent characterizations of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$ in terms of the atom, the molecule and the Lusin-area function associated with the Poisson semigroup generated by L. We begin with some notions.

Definition 5.1 Let L satisfy Assumptions A and B, φ be as in Definition 2.4 and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). A distribution $f \in (\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ is said to be in the *space* $H_{\varphi,\mathrm{at}}^{M}(\mathcal{X})$ if there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that $f = \sum_j \lambda_j \alpha_j$ in $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ and

$$\sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{\varphi}(\mathcal{X})}}\right) < \infty,$$

where, for each j, $\operatorname{supp}\alpha_j \subset B_j$. Moreover, for any $f \in H_{\varphi,\mathrm{at}}^{M}(\mathcal{X})$, its *quasi-norm* is defined by $\|f\|_{H_{\varphi,\mathrm{at}}^{M}(\mathcal{X})} := \inf\{\Lambda(\{\lambda_j\alpha_j\}_j)\}$, where $\Lambda(\{\lambda_j\alpha_j\}_j)$ is the same as in Proposition 4.7 and the infimum is taken over all possible decompositions of f as above.

Definition 5.2 Let L satisfy Assumptions A and B, φ be as in Definition 2.4, $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$ and $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). A distribution $f \in (\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ is said to be in the *space* $H_{\varphi,\mathrm{mol}}^{M,\epsilon}(\mathcal{X})$ if there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\beta_j\}_j$ of (φ, M, ϵ)-molecules such that $f = \sum_j \lambda_j \beta_j$ in $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ and

$$\sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{\varphi}(\mathcal{X})}}\right) < \infty,$$

where, for each j, β_j is associated with the ball B_j. Moreover, for any $f \in H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$, its *quasi-norm* is defined by $\|f\|_{H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})} := \inf\{\Lambda(\{\lambda_j\beta_j\}_j)\}$, where $\Lambda(\{\lambda_j\beta_j\}_j)$ is the same as in Proposition 4.7 and the infimum is taken over all possible decompositions of f as above.

For all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$, define the Lusin area function associated with the Poisson semigroup of L by

$$S_P f(x) := \left\{\int_{\Gamma(x)} \left|t\sqrt{L}e^{-t\sqrt{L}} f(y)\right|^2 \frac{d\mu(y)\,dt}{V(x,t)t}\right\}^{1/2}. \tag{5.1}$$

Similar to Definition 4.1, we introduce the space $H_{\varphi,S_P}(\mathcal{X})$ as follows.

Definition 5.3 Let L satisfy Assumptions A and B and φ be as in Definition 2.4. A function $f \in H^2(\mathcal{X})$ is said to be in $\widetilde{H}_{\varphi,S_P}(\mathcal{X})$ if $S_P(f) \in L^\varphi(\mathcal{X})$; moreover, define

$$\begin{aligned}\|f\|_{H_{\varphi,S_P}(\mathcal{X})} &:= \|S_P(f)\|_{L^\varphi(\mathcal{X})}\\ &:= \inf\left\{\lambda \in (0,\infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{S_P(f)(x)}{\lambda}\right) d\mu(x) \le 1\right\}.\end{aligned}$$

The *Musielak–Orlicz–Hardy space* $H_{\varphi,S_P}(\mathcal{X})$ is defined to be the completion of $\widetilde{H}_{\varphi,S_P}(\mathcal{X})$ in the quasi-norm $\|\cdot\|_{H_{\varphi,S_P}(\mathcal{X})}$.

We now show that the spaces $H_{\varphi,L}(\mathcal{X})$, $H^M_{\varphi,\mathrm{at}}(\mathcal{X})$, $H^{M\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$ and $H_{\varphi,S_P}(\mathcal{X})$ coincide with equivalent quasi-norms.

5.1 Atomic and Molecular Characterizations

In this subsection, we establish the atomic and the molecular characterizations of the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$. First we need the following Proposition 5.4 whose proof is similar to that of [57, Proposition 5.1]. We omit the details.

Lemma 5.4 *Let L satisfy Assumptions A and B and φ be as in Definition 2.4. Fix $t \in (0,\infty)$ and $\widetilde{B} := B(x_0, R)$. Then there exists a positive constant $C(t, R, \widetilde{B})$, depending on t, R and $\widetilde{B}$, such that, for all $\phi \in L^2(\widetilde{B})$, $t^2Le^{-t^2L}\phi \in \mathrm{BMO}_{\varphi,L}(\mathcal{X})$ and*

$$\left\|t^2Le^{-t^2L}\phi\right\|_{\mathrm{BMO}_{\varphi,L}(\mathcal{X})} \le C(t,R,\widetilde{B})\|\phi\|_{L^2(\widetilde{B})}.$$

From Lemma 5.4, it follows that, for each $f \in (\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$, $t^2Le^{-t^2L}f$ is well defined. Indeed, for any ball $B := B(x_B, r_B)$ and $\phi \in L^2(B)$, by Lemma 5.4, we know that there exists a positive constant $C(t,B)$, depending on t and B, such that

$$\left|\left\langle t^2Le^{-t^2L}f, \phi\right\rangle\right| := \left|\left\langle f, t^2Le^{-t^2L}\phi\right\rangle\right| \le C(t,B)\|\phi\|_{L^2(B)}\|f\|_{(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*},$$

which implies that $t^2Le^{-t^2L}f \in L^2_{\mathrm{loc}}(\mathcal{X})$ in the sense of distributions.

Theorem 5.5 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ as in (2.10), $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$ and $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then the spaces $H_{\varphi,L}(\mathcal{X})$, $H^M_{\varphi,\mathrm{at}}(\mathcal{X})$ and $H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$ coincide with equivalent quasi-norms.*

Proof By Theorem 4.16, we know that $(H_{\varphi,L}(\mathcal{X}))^* = \mathrm{BMO}_{\varphi,L}(\mathcal{X})$, which, together with Corollary 4.8, further implies that, for any $f \in H_{\varphi,L}(\mathcal{X})$, its atomic decomposition (4.13) also holds true in $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$. Thus, $H_{\varphi,L}(\mathcal{X}) \subset H^M_{\varphi,\mathrm{at}}(\mathcal{X})$. Moreover, since every (φ, M)-atom is a (φ, M, ϵ)-molecule for all $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$, the inclusion $H^M_{\varphi,\mathrm{at}}(\mathcal{X}) \subset H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$ is obvious.

Let us finally prove that $H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X}) \subset H_{\varphi,L}(\mathcal{X})$. Suppose that $f \in H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})$. Then there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\beta_j\}_j$ of (φ, M, ϵ)-molecules such that $f = \sum_j \lambda_j \beta_j$ in $(\mathrm{BMO}_{\varphi,L}(\mathcal{X}))^*$ and $\Lambda(\{\lambda_j\beta_j\}_j) < \infty$.

For all $x \in \mathcal{X}$, from Lemma 5.4, it follows that

$$\begin{aligned}
S_L(f)(x) &= \left\{\int_0^\infty \left\|t^2 L e^{-t^2 L} f\right\|^2_{L^2(B(x,t))} \frac{dt}{V(x,t)t}\right\}^{1/2} \\
&= \left\{\int_0^\infty \left(\sup_{\|\phi\|_{L^2(B(x,t))}\le 1} \left|\left\langle \sum_j \lambda_j \beta_j, t^2 L e^{-t^2 L}\phi\right\rangle\right|\right)^2 \frac{dt}{V(x,t)t}\right\}^{1/2} \\
&\le \sum_j \left\{\int_0^\infty \left(\sup_{\|\phi\|_{L^2(B(x,t))}\le 1} \left|\left\langle t^2 L e^{-t^2 L}(\lambda_j\beta_j), \phi\right\rangle\right|\right)^2 \frac{dt}{V(x,t)t}\right\}^{1/2} \\
&\le \sum_j S_L(\lambda_j\beta_j)(x).
\end{aligned}$$

By this, the proof of Proposition 4.9 and Lemma 2.2(i), we conclude that, for $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$,

$$\begin{aligned}
\int_{\mathcal{X}} \varphi\big(x, S_L(f)(x)\big)\, d\mu(x) &\lesssim \sum_j \int_{\mathcal{X}} \varphi\big(x, S_L(\lambda_j\beta_j)(x)\big)\, d\mu(x) \\
&\lesssim \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}}\right),
\end{aligned}$$

where, for each j, β_j is associated with the ball B_j, which further implies that $\|f\|_{H_{\varphi,L}(\mathcal{X})} \lesssim \Lambda(\{\lambda_j\beta_j\}_j)$. Then by taking the infimum over all decompositions of f as above, we see that

$$\|f\|_{H_{\varphi,L}(\mathcal{X})} \lesssim \|f\|_{H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})},$$

which completes the proof of Theorem 5.5. □

5.2 The Lusin Area Function Characterization

In this subsection, we characterize the space $H_{\varphi,L}(\mathcal{X})$ by the Lusin area function S_P as in (5.1). First, by using the subadditivity and continuity of φ, and the uniformly upper type p_1 property of φ for some $p_1 \in (0,1]$, similar to the proof of [57, Lemma 5.2], we obtain the following auxiliary conclusion. We omit the details here.

Recall that a *nonnegative sublinear operator* T means that T is sublinear and $T(f) \geq 0$ for all f in the domain of T.

Lemma 5.6 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Suppose that T is a linear (resp., nonnegative sublinear) operator which maps $L^2(\mathcal{X})$ continually into weak-$L^2(\mathcal{X})$. If there exists a positive constant C such that, for all $\lambda \in \mathbb{C}$ and (φ, M)-atoms α,*

$$\int_{\mathcal{X}} \varphi\big(x, T(\lambda\alpha)(x)\big)\, d\mu(x) \leq C\varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right), \tag{5.2}$$

then T extends to a bounded linear (resp., sublinear) operator from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$; moreover, there exists a positive constant C such that, for all $f \in H_{\varphi,L}(\mathcal{X})$, $\|Tf\|_{L^\varphi(\mathcal{X})} \leq C\|f\|_{H_{\varphi,L}(\mathcal{X})}$.

Theorem 5.7 *Let L satisfy Assumptions A and B, and φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ as in (2.10). Then the spaces $H_{\varphi,L}(\mathcal{X})$ and $H_{\varphi,S_P}(\mathcal{X})$ coincide with equivalent quasi-norms.*

Proof We first prove $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X}) \subset H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X})$. From (2.7), it follows that S_P is bounded on $L^2(\mathcal{X})$. Thus, by Lemma 5.6, to prove that $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X}) \subset H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X})$, we only need to show that (5.2) holds true with $T := S_P$, where $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$. From (2.5), the subordination formulae associated with L (see, for example, [57, (5.3)]) and the uniformly upper type $p_1 \in [I(\varphi), 1]$ and lower type $p_2 \in (0, i(\varphi))$ properties of φ, similar to the proof of (4.5), we can show (5.2) holds true with $T := S_P$. We omit the details.

Conversely, we show that $H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X}) \subset H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$. Let $f \in H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X})$. Then $t\sqrt{L}e^{-t\sqrt{L}} f \in T_\varphi(\mathcal{X} \times (0,\infty))$, which, together with Proposition 4.6(ii), implies that $\pi_{\Psi,L}(t\sqrt{L}e^{-t\sqrt{L}} f) \in H_{\varphi,L}(\mathcal{X})$. Furthermore, from the H_∞ functional calculi, we infer that

$$f = \frac{\widetilde{C}_\Psi}{C_\Psi} \pi_{\Psi,L}\big(t\sqrt{L}e^{-t\sqrt{L}} f\big)$$

in $L^2(\mathcal{X})$, where $\widetilde{C}_\Psi$ is a positive constant such that $\widetilde{C}_\Psi \int_0^\infty \Psi(t) t e^{-t} \frac{dt}{t} = 1$ and C_Ψ is as in (4.2). This, combined with $\pi_{\Psi,L}(t\sqrt{L}e^{-t\sqrt{L}} f) \in H_{\varphi,L}(\mathcal{X})$, implies that $f \in H_{\varphi,L}(\mathcal{X})$. Therefore, we know that $H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X}) \subset H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$.

From the above argument, it follows that $H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X}) = H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$ with equivalent norms, which, together with the fact that $H_{\varphi,S_P}(\mathcal{X}) \cap H^2(\mathcal{X})$

and $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$ are, respectively, dense in $H_{\varphi,S_P}(\mathcal{X})$ and $H_{\varphi,L}(\mathcal{X})$, and a density argument, implies that the spaces $H_{\varphi,S_P}(\mathcal{X})$ and $H_{\varphi,L}(\mathcal{X})$ coincide with equivalent norms. This finishes the proof of Theorem 5.7. □

6 Applications

In this section, we give some applications of the Musielak–Orlicz–Hardy space to the boundedness of operators. More precisely, in Sect. 6.1, we prove that the Littlewood–Paley g-function g_L is bounded from $H_{\varphi,L}(\mathcal{X})$ to the Musielak–Orlicz space $L^\varphi(\mathcal{X})$; in Sect. 6.2, we show that the Littlewood–Paley g_λ^*-function $g_{\lambda,L}^*$ is bounded from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$; in Sect. 6.3, we prove that the spectral multipliers associated with L is bounded on $H_{\varphi,L}(\mathcal{X})$.

6.1 Boundedness of Littlewood–Paley g-Functions g_L

We begin with the definition of the Littlewood–Paley g-function g_L associated with L.

Definition 6.1 For all functions $f \in L^2(\mathcal{X})$, the *g-function* $g_L(f)$ is defined by setting, for all $x \in \mathcal{X}$,

$$g_L(f)(x) := \left\{ \int_0^\infty \left| t^2 L e^{-t^2 L} f(x) \right|^2 \frac{dt}{t} \right\}^{1/2}.$$

To establish the main result of this subsection, we need the following Lemma 6.2, which is a simple corollary of (2.7).

Lemma 6.2 *Let L satisfy Assumptions A and B and g_L be as in Definition 6.1. Then g_L is bounded on $L^2(\mathcal{X})$.*

The main result of this subsection is as follows.

Theorem 6.3 *Let L satisfy Assumptions A and B and φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ as in (2.10). Then g_L is bounded from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$.*

Proof Let $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then there exist $q_0 \in (q(\varphi), \infty)$ and $p_2 \in (0, i(\varphi))$ such that $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2})$, φ is of uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathcal{X})$. We first assume that $f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$. To show Theorem 6.3, it suffices to show that, for any $\lambda \in \mathbb{C}$ and (φ, M)-atom α supported in the ball $B := B(x_B, r_B)$,

$$\int_{\mathcal{X}} \varphi\big(x, g_L(\lambda\alpha)(x)\big)\, d\mu(x) \lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right). \tag{6.1}$$

Indeed, if (6.1) holds true, it follows, from Proposition 4.7, that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that $f = \sum_j \lambda_j \alpha_j$ in $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|f\|_{H_{\varphi,L}(\mathcal{X})}$, which, together with Lemmas 6.2 and 2.6(i), and (6.1), implies that, for all $\lambda \in (0,\infty)$,

$$\begin{aligned}\int_{\mathcal{X}} \varphi\left(x, \frac{g_L(f)(x)}{\lambda}\right) d\mu(x) &\lesssim \sum_j \int_{\mathcal{X}} \varphi\left(x, \frac{g_L(\lambda_j\alpha_j)(x)}{\lambda}\right) d\mu(x)\\ &\lesssim \sum_j \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathcal{X})}}\right),\end{aligned}$$

where, for each j, $\operatorname{supp}\alpha_j \subset B_j$. By this, we see that $\|g_L(f)\|_{L^\varphi(\mathcal{X})} \lesssim \Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|f\|_{H_{\varphi,L}(\mathcal{X})}$. Since $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$, a density argument then gives the desired conclusion.

Now we prove (6.1). First we see that

$$\int_{\mathcal{X}} \varphi\big(x, g_L(\lambda\alpha)(x)\big)\, d\mu(x) = \sum_{j\in\mathbb{Z}_+} \int_{U_j(B)} \varphi\big(x, |\lambda| g_L(\alpha)(x)\big)\, d\mu(x) =: \sum_{j\in\mathbb{Z}_+} \mathrm{H}_j. \tag{6.2}$$

From the assumption $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, Lemma 2.8(iv) and the definition of $I(\varphi)$, we infer that, there exists $p_1 \in [I(\varphi), 1]$ such that φ is of uniformly upper type p_1 and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$. When $j \in \{0,\dots,4\}$, by the uniformly upper type p_1 property of φ, Hölder's inequality, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and Lemmas 6.2 and 2.8(vi), we know that

$$\begin{aligned}\mathrm{H}_j &\le \int_{U_j(B)} \varphi\left(x, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right)\Big(1 + \big[g_L(\alpha)(x)\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_1}\Big)\, d\mu(x)\\ &\lesssim \varphi\left(2^j B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right) + \|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})}\|g_L(\alpha)\|^{p_1}_{L^2(\mathcal{X})}\\ &\quad\times\left\{\int_{2^jB}\Big[\varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\Big]^{\frac{2}{2-p_1}} d\mu(x)\right\}^{\frac{2-p_1}{2}}\\ &\lesssim \varphi\left(2^j B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right)\\ &\lesssim \varphi\left(B, \frac{|\lambda|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right).\end{aligned} \tag{6.3}$$

When $j \in \mathbb{N}$ with $j \ge 5$, from the uniformly upper type p_1 and lower type p_2 properties of φ, it follows that

$$\begin{aligned}\mathrm{H}_j &\lesssim \|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})} \int_{U_j(B)} \varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big[g_L(\alpha)(x)\big]^{p_1} d\mu(x)\\ &\quad + \|\chi_B\|^{p_2}_{L^\varphi(\mathcal{X})} \int_{U_j(B)} \varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big[g_L(\alpha)(x)\big]^{p_2} d\mu(x)\\ &=: \mathrm{E}_j + \mathrm{F}_j.\end{aligned} \tag{6.4}$$

To deal with E_j and F_j, we first estimate $\int_{U_j(B)}[g_L(\alpha)(x)]^2\,d\mu(x)$. By the definition of g_L, we see that

$$\int_{U_j(B)}\left[g_L(\alpha)(x)\right]^2 d\mu(x)=\int_0^{r_B}\int_{U_j(B)}\left|t^2Le^{-t^2L}\alpha(x)\right|^2 d\mu(x)\,\frac{dt}{t}+\int_{r_B}^{\infty}\cdots. \tag{6.5}$$

Take $s_0\in(0,\infty)$ such that $s_0\in(n[\frac{q_0}{p_2}-\frac{1}{2}],2M)$. From (2.5), we infer that

$$\begin{aligned}
&\int_0^{r_B}\int_{U_j(B)}\left|t^2Le^{-t^2L}\alpha(x)\right|^2 d\mu(x)\,\frac{dt}{t}\\
&\lesssim\int_0^{r_B}\exp\left\{-\frac{(2^jr_B)^2}{C_3t^2}\right\}\|\alpha\|_{L^2(B)}^2\,\frac{dt}{t}\\
&\lesssim\left\{\int_0^{r_B}\frac{t^{2s_0}}{(2^jr_B)^{2s_0}}\,\frac{dt}{t}\right\}\|\alpha\|_{L^2(B)}^2\sim 2^{-2js_0}\|\alpha\|_{L^2(B)}^2\\
&\lesssim 2^{-2js_0}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}.
\end{aligned}\tag{6.6}$$

Moreover, by the definition of α, we know that there exists $b\in L^2(B)$ such that $\alpha=L^Mb$ and $\|b\|_{L^2(B)}\le r_B^{2M}[\mu(B)]^{1/2}\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}$. From this and (2.5), it follows that

$$\begin{aligned}
&\int_{r_B}^{\infty}\int_{U_j(B)}\left|t^2Le^{-t^2L}\alpha(x)\right|^2 d\mu(x)\,\frac{dt}{t}\\
&=\int_{r_B}^{\infty}\int_{U_j(B)}\left|\left(t^2L\right)^{M+1}e^{-t^2L}b(x)\right|^2 d\mu(x)\,\frac{dt}{t^{4M+1}}\\
&\lesssim\int_{r_B}^{\infty}\exp\left\{-\frac{(2^jr_B)^2}{C_3t^2}\right\}\|b\|_{L^2(B)}^2\,\frac{dt}{t^{4M+1}}\\
&\lesssim\left\{\int_{r_B}^{\infty}\frac{t^{2s_0}}{(2^jr_B)^{2s_0}}\,\frac{dt}{t^{4M+1}}\right\}\|b\|_{L^2(B)}^2\\
&\lesssim 2^{-2js_0}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2},
\end{aligned}$$

which, together with (6.5) and (6.6), implies that

$$\int_{U_j(B)}\left[g_L(\alpha)(x)\right]^2 d\mu(x)\lesssim 2^{-2js_0}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}. \tag{6.7}$$

Thus, by Hölder's inequality, (6.7), $\varphi\in\mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and (2.2), we conclude that, for all $j\in\mathbb{N}$ with $j\ge 5$,

$$\begin{aligned}
\mathrm{E}_j &\lesssim \|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_1}\left\{\int_{2^jB}\left[\varphi\left(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)\right]^{\frac{2}{2-p_1}}d\mu(x)\right\}^{\frac{2-p_1}{2}}\\
&\quad\times\left\{\int_{U_j(B)}\left[g_L(\alpha)(x)\right]^2d\mu(x)\right\}^{\frac{p_1}{2}}\\
&\lesssim 2^{-jp_1s_0}\left[\mu(B)\right]^{\frac{1}{2}-q_0}\left[\mu\left(2^jB\right)\right]^{q_0-\frac{1}{2}}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)\\
&\lesssim 2^{-jp_1[s_0-n(\frac{q_0}{p_1}-\frac{1}{2})]}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right). \qquad (6.8)
\end{aligned}$$

Similarly, by using Hölder's inequality, (6.7), $\varphi\in\mathbb{RH}_{2/(2-p_1)}(\mathcal{X})\subset\mathbb{RH}_{2/(2-p_2)}(\mathcal{X})$ and Lemma 2.8(vii), we see that $\mathrm{F}_j\lesssim 2^{-jp_2[s_0-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1})$, which, together with (6.8), (6.4) and $p_1\geq p_2$, implies that, for each $j\in\mathbb{N}$ with $j\geq 5$,

$$\mathrm{H}_j\lesssim 2^{-jp_2[s_0-n(\frac{q_0}{p_2}-\frac{1}{2})]}\varphi\left(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right).$$

From this, $s_0>n(\frac{q_0}{p_2}-\frac{1}{2})$, (6.2) and (6.3), we infer that (6.1) holds true, which completes the proof of Theorem 6.3. □

Remark 6.4 When $\mathcal{X}:=\mathbb{R}^n$, L is a nonnegative self-adjoint elliptic operator in $L^2(\mathbb{R}^n)$ and φ as in (1.2) with $\omega\equiv 1$ and Φ concave, Theorem 6.3 was obtained in [58, Theorem 7.1].

6.2 Boundedness of Littlewood–Paley g_λ^*-Functions $g_{\lambda,L}^*$

In this subsection, we establish the boundedness of the Littlewood–Paley g_λ^*-function $g_{\lambda,L}^*$ associated with L from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$. We begin with the definition of the Littlewood–Paley g_λ^*-function $g_{\lambda,L}^*$.

Definition 6.5 Let $\lambda\in(0,\infty)$ and L satisfy Assumptions A and B. For all $f\in L^2(\mathcal{X})$, the *g_λ^*-function associated with L*, $g_{\lambda,L}^*(f)$, is defined by setting, for all $x\in\mathcal{X}$,

$$g_{\lambda,L}^*(f)(x):=\left\{\int_0^\infty\int_{\mathcal{X}}\left[\frac{t}{t+d(x,y)}\right]^\lambda\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(y)\,dt}{V(x,t)t}\right\}^{1/2}.$$

To prove the boundedness of $g_{\lambda,L}^*$ from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$, we need the following auxiliary conclusion.

Lemma 6.6 *Let $\alpha\in(0,\infty)$ and*

$$S_L^\alpha(f)(x):=\left\{\int_0^\infty\int_{B(x,\alpha t)}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(y)\,dt}{V(x,t)t}\right\}^{1/2}$$

for all $f\in L^2(\mathcal{X})$ and $x\in\mathcal{X}$. Then there exists a positive constant C such that, for all $f\in L^2(\mathcal{X})$, $\|S_L^\alpha(f)\|_{L^2(\mathcal{X})}\leq C\alpha^{n/2}(1+\alpha)^{N/2}\|f\|_{L^2(\mathcal{X})}$, where n and N are, respectively, as in (2.2) and (2.3).

Proof By the definition of S_L^α, Fubini's theorem, (2.2), (2.3) and (2.7), we see that

$$\begin{aligned}\|S_L^\alpha(f)\|_{L^2(\mathcal{X})}^2 &= \int_{\mathcal{X}}\int_0^\infty\int_{B(x,\alpha t)}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(y)\,dt}{V(x,t)t}\,d\mu(x)\\ &\le (1+\alpha)^N\int_{\mathcal{X}}\int_0^\infty\int_{B(y,\alpha t)}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(x)}{V(y,t)}\frac{d\mu(y)\,dt}{t}\\ &\lesssim \alpha^n(1+\alpha)^N\int_0^\infty\int_{\mathcal{X}}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(y)\,dt}{t}\\ &\lesssim \alpha^n(1+\alpha)^N\|f\|_{L^2(\mathcal{X})}^2,\end{aligned}$$

which is desired, and hence completes the proof of Lemma 6.6. □

Now we give the main result of this subsection.

Theorem 6.7 *Let L satisfy Assumptions A and B, φ be as in Definition 2.4 with $\varphi\in\mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$ and $I(\varphi)$ as in (2.10), and $\lambda\in([2nq(\varphi)+NI(\varphi)]/i(\varphi),\infty)$, where n, N, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3), (2.12) and (2.11). Then the operator $g^*_{\lambda,L}$ is bounded from $H_{\varphi,L}(\mathcal{X})$ to $L^\varphi(\mathcal{X})$.*

Proof Let $M\in\mathbb{N}$ with $M>\frac{n}{2}[\frac{q(\varphi)}{i(\varphi)}-\frac{1}{2}]$ and $\lambda\in([2nq(\varphi)+NI(\varphi)]/i(\varphi),\infty)$, where n, N, $q(\varphi)$, $I(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3), (2.12), (2.10) and (2.11). Then by the assumption $\varphi\in\mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, Lemma 2.8(iv) and the definitions of $q(\varphi)$, $I(\varphi)$ and $i(\varphi)$, we know that, there exist $q_0\in(q(\varphi),\infty)$, $p_1\in[I(\varphi),1]$ and $p_2\in(0,i(\varphi))$ such that $M>\frac{n}{2}(\frac{q_0}{p_2}-\frac{1}{2})$, $\lambda>(2nq_0+Np_1)/p_2$, φ is of uniformly upper type p_1 and uniformly lower type p_2, and $\varphi\in\mathbb{RH}_{2/(2-p_1)}(\mathcal{X})\cap\mathbb{A}_{q_0}(\mathcal{X})$. To show Theorem 6.7, similar to the proof of Theorem 6.3, it suffices to show that, for all $\gamma\in\mathbb{C}$ and (φ,M)-atoms α supported in the ball $B:=B(x_B,r_B)$,

$$\int_{\mathcal{X}}\varphi\big(x,g^*_{\lambda,L}(\gamma\alpha)(x)\big)\,d\mu(x)\lesssim\varphi\left(B,\frac{|\gamma|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right).\tag{6.9}$$

In order to prove (6.9), it suffices to show that, for all $k\in\mathbb{Z}_+$,

$$\int_{\mathcal{X}}\varphi\big(x,2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x)\big)\,d\mu(x)\lesssim 2^{-\frac{kp_2}{2}(\lambda-\frac{2nq_0+Np_1}{p_2})}\varphi\left(B,\frac{|\gamma|}{\|\chi_B\|_{L^\varphi(\mathcal{X})}}\right).\tag{6.10}$$

Indeed, if (6.10) holds true, from the definition of $g^*_{\lambda,L}$, it follows that, for all $x\in\mathcal{X}$,

$$\begin{aligned}&g_L^*(\gamma\alpha)(x)\\ &\quad\lesssim\left\{\int_0^\infty\int_{B(x,t)}\left|t^2Le^{-t^2L}(\gamma\alpha)(y)\right|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}+\sum_{k=1}^\infty 2^{-k\lambda}\int_0^\infty\int_{B(x,2^kt)}\cdots\right\}^{1/2}\\ &\quad\lesssim\sum_{k=0}^\infty 2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x),\end{aligned}$$

which, together with (6.10), Lemma 2.6(i) and $\lambda > (2nq_0 + Np_1)/p_2$, implies that

$$\begin{aligned}\int_{\mathcal{X}} \varphi\big(x, g^*_{\lambda,L}(\gamma\alpha)(x)\big)\,d\mu(x) &\lesssim \sum_{k=0}^{\infty}\int_{\mathcal{X}} \varphi\big(x, 2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x)\big)\,d\mu(x)\\ &\lesssim \sum_{k=0}^{\infty} 2^{-\frac{kp_2}{2}(\lambda-\frac{2nq_0+Np_1}{p_2})}\varphi\big(B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\\ &\lesssim \varphi\big(B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big).\end{aligned}$$

Thus, (6.9) holds true.

Now we prove (6.10). For each $k \in \mathbb{Z}_+$, let $B_k := 2^k B$. Then

$$\int_{\mathcal{X}} \varphi\big(x, 2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x)\big)\,d\mu(x) = \sum_{j=0}^{\infty}\int_{U_j(B_k)} \cdots. \tag{6.11}$$

For $j \in \{0, \dots, 4\}$, then by the uniformly upper type p_1 and lower type p_2 properties of φ, Hölder's inequality, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$, Lemmas 6.6 and 2.8(vi), we know that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}&\int_{U_j(B_k)} \varphi\big(x, 2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x)\big)\,d\mu(x)\\ &\quad\lesssim \int_{U_j(B_k)} \varphi\big(x, 2^{-k\lambda/2}|\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big(1 + \big[S_L^{2^k}(\alpha)(x)\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_1}\big)\,d\mu(x)\\ &\quad\lesssim 2^{-k\lambda p_2/2}\varphi\big(2^{j+k}B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\\ &\qquad + 2^{-k\lambda p_2/2}\|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})}\left\{\int_{U_j(B_k)}\big[S_L^{2^k}(\alpha)(x)\big]^2\,d\mu(x)\right\}^{\frac{p_1}{2}}\\ &\qquad \times\left\{\int_{U_j(B_k)}\big[\varphi\big(x, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big]^{\frac{2}{2-p_1}}\,d\mu(x)\right\}^{\frac{2-p_1}{2}}\\ &\quad\lesssim 2^{-k(\frac{\lambda p_2}{2}-nq_0)}\varphi\big(B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big) + 2^{-\frac{k\lambda p_2}{2}}2^{\frac{k(n+N)p_1}{2}}\|\alpha\|^{p_1}_{L^2(\mathcal{X})}\|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})}\\ &\qquad \times\big[\mu\big(2^{j+k}B\big)\big]^{q_0-\frac{p_1}{2}}\big[\mu(B)\big]^{-q_0}\varphi\big(B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\\ &\quad\lesssim 2^{-k(\frac{\lambda p_2}{2}-nq_0-\frac{Np_1}{2})}\varphi\big(B, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big).\end{aligned} \tag{6.12}$$

When $j \in \mathbb{N}$ with $j \geq 5$, from the uniformly upper type p_1 and lower type p_2 properties of φ, we deduce that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}&\int_{U_j(B_k)} \varphi\big(x, 2^{-k\lambda/2}S_L^{2^k}(\gamma\alpha)(x)\big)\,d\mu(x)\\ &\quad\lesssim 2^{-k\lambda p_2/2}\|\chi_B\|^{p_1}_{L^\varphi(\mathcal{X})}\int_{U_j(B_k)} \varphi\big(x, |\gamma|\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big)\big[S_L^{2^k}(\alpha)(x)\big]^{p_1}\,d\mu(x)\end{aligned}$$

$$+2^{-k\lambda p_2/2}\|\chi_B\|_{L^\varphi(\mathcal{X})}^{p_2}\int_{U_j(B_k)}\varphi\big(x,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big)\big[S_L^{2^k}(\alpha)(x)\big]^{p_2}\,d\mu(x)$$
$$=:\mathrm{H}_{j,k}+\mathrm{I}_{j,k}. \tag{6.13}$$

To estimate $\mathrm{H}_{j,k}$ and $\mathrm{I}_{j,k}$, we need to estimate $\int_{U_j(B_k)}|S_L^{2^k}(\alpha)(x)|^2\,d\mu(x)$. We first see that

$$\int_{U_j(B_k)}\big[S_L^{2^k}(\alpha)(x)\big]^2\,d\mu(x)$$
$$=\int_{U_j(B_k)}\int_0^{r_B}\int_{B(x,2^kt)}\big|t^2Le^{-t^2L}(\alpha)(y)\big|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x)+\int_{U_j(B_k)}\int_{r_B}^{\infty}\cdots$$
$$=:\mathrm{J}_{j,k}+\mathrm{K}_{j,k}. \tag{6.14}$$

Take $s\in(0,\infty)$ such that $s\in(n[\frac{q_0}{p_2}-\frac{1}{2}],2M)$. Moreover, for each $j\in\mathbb{N}$ with $j\geq 5$ and $k\in\mathbb{Z}_+$, let $\widetilde{U_j}(B_k):=\{z\in\mathcal{X}:2^{j-2}2^kr_B\leq d(z,x_B)<2^{j+1}2^kr_B\}$. Then for any $x\in U_j(B_k)$, $t\in(0,r_B)$ and $y\in\mathcal{X}$ with $d(x,y)<2^kt$, we see that $y\in\widetilde{U_j}(B_k)$. From this, (2.3), Fubini's theorem and (2.5), it follows that

$$\mathrm{J}_{j,k}\lesssim 2^{k(N+n)}\int_0^{r_B}\int_{\widetilde{U_j}(B_k)}\big|t^2Le^{-t^2L}(\alpha)(y)\big|^2\frac{d\mu(y)\,dt}{t}$$
$$\lesssim 2^{k(N+n)}\|\alpha\|_{L^2(B)}^2\int_0^{r_B}e^{-\frac{[2^{j+k}r_B]^2}{c_3t^2}}\frac{dt}{t}\lesssim 2^{-2js}2^{-k(2s-N-n)}\|\alpha\|_{L^2(B)}^2. \tag{6.15}$$

Furthermore, by the definition of α, we know that there exists $b\in L^2(B)$ such that $\alpha=L^Mb$ and $\|b\|_{L^2(\mathcal{X})}\leq r_B^{2M}[\mu(B)]^{1/2}\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}$. From this, we deduce that

$$\mathrm{K}_{j,k}\lesssim\int_{U_j(B_k)}\int_{r_B}^{2^{j-3}r_B}\int_{B(x,2^kt)}\big|\big(t^2L\big)^{M+1}e^{-t^2L}(b)(y)\big|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t^{4M+1}}\,d\mu(x)$$
$$+\int_{U_j(B_k)}\int_{2^{j-3}r_B}^{\infty}\cdots=:\mathrm{K}_{j,k,1}+\mathrm{K}_{j,k,2}. \tag{6.16}$$

We first estimate $\mathrm{K}_{j,k,1}$. Let $x\in U_j(B_k)$, $t\in[r_B,2^{j-3}r_B)$ and $y\in\mathcal{X}$ with $d(x,y)<2^kt$. Then

$$d(y,x_B)\leq d(x,y)+d(x,x_B)\leq 2^kt+2^j2^kr_B\leq 2^{j+1}2^kr_B$$

and

$$d(y,x_B)\geq d(x,x_B)-d(x,y)\geq 2^{j-1}2^kr_B-2^{j-3}2^kr_B\geq 2^{j-3}2^kr_B.$$

From this, (2.3), Fubini's theorem and (2.5), we infer that

$$\mathrm{K}_{j,k,1}\lesssim 2^{k(N+n)}\int_{r_B}^{2^{j-3}r_B}\int_{\widetilde{U_j}(B_k)}\big|\big(t^2L\big)^{M+1}e^{-t^2L}(b)(y)\big|^2\frac{d\mu(y)\,dt}{t^{4M+1}}$$

$$\lesssim 2^{k(N+n)}\|b\|_{L^2(B)}^2\int_{r_B}^{2^{j-3}r_B} e^{-\frac{[2^{j+k-3}r_B]^2}{C_3t^2}}\frac{dt}{t^{4M+1}}$$
$$\lesssim 2^{-2js}2^{-k(2s-N-n)}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}. \tag{6.17}$$

For $\mathrm{K}_{j,k,2}$, by (2.3), Fubini's theorem and (2.5), we see that

$$\mathrm{K}_{j,k,2}\lesssim 2^{k(N+n)}\|b\|_{L^2(B)}^2\int_{2^{j-3}r_B}^{\infty}\frac{dt}{t^{4M+1}}\lesssim 2^{-2js}2^{k(N+n)}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2},$$

which, together with (6.17) and (6.16), implies that, for all $j\in\mathbb{N}$ with $j\geq 5$ and $k\in\mathbb{Z}_+$,

$$\mathrm{K}_{j,k}\lesssim 2^{-2js}2^{k(N+n)}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}.$$

From this, (6.14) and (6.15), it follows that, for all $j\in\mathbb{N}$ with $j\geq 5$ and $k\in\mathbb{Z}_+$,

$$\int_{U_j(B_k)}\left[S_L^{2^k}(\alpha)(x)\right]^2d\mu(x)\lesssim 2^{-2js}2^{k(N+n)}\mu(B)\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-2}. \tag{6.18}$$

By (6.18), Hölder's inequality, $\varphi\in\mathbb{RH}_{2/(2-p_1)}(\mathcal{X})$ and Lemma 2.8(vii), we conclude that

$$\mathrm{H}_{j,k}\lesssim 2^{-\frac{k\lambda p_2}{2}}\|\chi_B\|_{L^\varphi(B)}^{p_1}\left\{\int_{U_j(B_k)}\left[\varphi\left(x,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)\right]^{\frac{2}{2-p_1}}d\mu(x)\right\}^{\frac{2-p_1}{2}}$$
$$\times\left\{\int_{U_j(B_k)}\left[S_L^{2^k}(\alpha)(x)\right]^2d\mu(x)\right\}^{\frac{p_1}{2}}$$
$$\lesssim 2^{-\frac{k\lambda p_2}{2}}2^{-jsp_1}2^{\frac{k(N+n)p_1}{2}}\left[\mu\left(2^{j+k}B\right)\right]^{q_0-\frac{p_1}{2}}\left[\mu(B)\right]^{\frac{p_1}{2}-q_0}\varphi\left(B,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$
$$\lesssim 2^{-jp_1[s-n(\frac{q_0}{p_1}-\frac{1}{2})]}2^{-kp_2(\frac{\lambda}{2}-\frac{nq_0}{p_2}-\frac{Np_1}{2p_2})}\varphi\left(B,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right). \tag{6.19}$$

For $\mathrm{I}_{j,k}$, similar to (6.19), we see that

$$\mathrm{I}_{j,k}\lesssim 2^{-\frac{k\lambda p_2}{2}}\|\chi_B\|_{L^\varphi(B)}^{p_2}\left\{\int_{U_j(B_k)}\left[\varphi\left(x,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)\right]^{\frac{2}{2-p_2}}d\mu(x)\right\}^{\frac{2-p_2}{2}}$$
$$\times\left\{\int_{U_j(B_k)}\left[S_L^{2^k}(\gamma\alpha)(x)\right]^2d\mu(x)\right\}^{\frac{p_2}{2}}$$
$$\lesssim 2^{-\frac{k\lambda p_2}{2}}2^{-jsp_2}2^{\frac{k(N+n)p_2}{2}}\left[\mu\left(2^{j+k}B\right)\right]^{q_0-\frac{p_2}{2}}\left[\mu(B)\right]^{\frac{p_2}{2}-q_0}\varphi\left(B,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right)$$
$$\lesssim 2^{-jp_2[s-n(\frac{q_0}{p_2}-\frac{1}{2})]}2^{-kp_2(\frac{\lambda}{2}-\frac{nq_0}{p_2}-\frac{N}{2})}\varphi\left(B,|\gamma|\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\right),$$

which, together with (6.11), (6.12), (6.13), (6.19), $p_1 \geq p_2$ and $s > n(\frac{q_0}{p_2} - \frac{1}{2})$, implies that

$$\int_{\mathcal{X}} \varphi\big(x, 2^{-k\lambda/2} S_L^{2^k}(\gamma\alpha)(x)\big)\, d\mu(x) \lesssim 2^{-\frac{kp_2}{2}(\lambda - \frac{2nq_0+Np_1}{p_2})} \varphi\big(B, |\gamma| \|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1}\big).$$

From this, we deduce that (6.10) holds true, which completes the proof of Theorem 6.7. □

Remark 6.8 We remark that when $\mathcal{X} := \mathbb{R}^n$ and $L := -\Delta$, $g^*_{\lambda,L}$ is just the classical Littlewood–Paley g^*_λ-function.

Let $p \in (0, 1]$, $\omega \in A_q(\mathbb{R}^n)$ with $q \in [1, \infty)$ and $\varphi(x, t) := \omega(x)t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$. We point out that, in this case, the range of λ in Theorem 6.7 coincides with the result on the classical Littlewood–Paley g^*_λ-function on $\mathbb{R}^n$ (see, for example, [1, Theorem 2]).

From Theorem 6.7 and the fact that $S_L(f) \leq g^*_{\lambda,L}(f)$ pointwise for all $f \in L^2(\mathcal{X})$, we immediately deduce the following Littlewood–Paley g^*_λ-function $g^*_{\lambda,L}$ characterization of $H_{\varphi,L}(\mathcal{X})$.

Corollary 6.9 *Let L satisfy Assumptions A and B, $g^*_{\lambda,L}$ be as in Definition 6.5 and φ as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, where $I(\varphi)$ is as in (2.10). Assume further that $\lambda \in ([2nq(\varphi) + NI(\varphi)]/i(\varphi), \infty)$, where n, N, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3), (2.12) and (2.11). Then $f \in H_{\varphi,L}(\mathcal{X})$ if and only if $g^*_{\lambda,L}(f) \in L^\varphi(\mathcal{X})$; moreover, $\|f\|_{H_{\varphi,L}(\mathcal{X})} \sim \|g^*_{\lambda,L}(f)\|_{L^\varphi(\mathcal{X})}$ with the implicit constants independent of f.*

6.3 Boundedness of Spectral Multipliers

In this subsection, we prove a Hörmander-type spectral multiplier theorem for L on the Musielak–Orlicz–Hardy space $H_{\varphi,L}(\mathcal{X})$. We begin with some notions.

Let L satisfy Assumptions A and B, and $m(L)$ be as in (1.1). Let ϕ be a nonnegative C_c^∞ function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi \subset (1/4, 1) \quad \text{and} \quad \sum_{\ell\in\mathbb{Z}} \phi\big(2^{-\ell}\lambda\big) = 1 \quad \text{for all } \lambda \in (0, \infty). \tag{6.20}$$

Let $s \in [0, \infty)$. Recall that $C^s(\mathbb{R})$ is the *space of all functions m* on $\mathbb{R}$ for which

$$\|m\|_{C^s(\mathbb{R})} := \begin{cases} \sum_{k=0}^s \sup_{\lambda\in\mathbb{R}} |m^{(k)}(\lambda)|, & s \in \mathbb{Z}_+, \\ \|m^{(\lfloor s\rfloor)}\|_{\mathrm{Lip}(s-\lfloor s\rfloor)} + \sum_{k=0}^{\lfloor s\rfloor} \sup_{\lambda\in\mathbb{R}} |m^{(k)}(\lambda)|, & s \notin \mathbb{Z}_+ \end{cases}$$

is finite, where $m^{(k)}$, $k \in \mathbb{N}$, denotes the *k-order derivative of m*, and $\|m^{(\lfloor s\rfloor)}\|_{\mathrm{Lip}(s-\lfloor s\rfloor)}$ is defined by

$$\big\|m^{(\lfloor s\rfloor)}\big\|_{\mathrm{Lip}(s-\lfloor s\rfloor)} := \sup_{x,y\in\mathbb{R}, x\neq y} \frac{|m^{(\lfloor s\rfloor)}(x) - m^{(\lfloor s\rfloor)}(y)|}{|x-y|^{s-\lfloor s\rfloor}}.$$

Now we state the main result of this subsection as follows.

Theorem 6.10 *Let L satisfy Assumptions A and B and φ be as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathcal{X})$, where $I(\varphi)$ is as in (2.10). Assume that ϕ is a nonnegative $C_c^\infty(\mathbb{R})$ function satisfying (6.20). If the bounded Borel function $m : [0,\infty) \to \mathbb{C}$ satisfies that, for some $s \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11),*

$$C(\phi, s) := \sup_{t\in(0,\infty)} \|\phi(\cdot)m(t\cdot)\|_{C^s(\mathbb{R})} + |m(0)| < \infty, \tag{6.21}$$

then $m(L)$ is bounded on $H_{\varphi,L}(\mathcal{X})$ and there exists a positive constant C such that, for all $f \in H_{\varphi,L}(\mathcal{X})$,

$$\|m(L)f\|_{H_{\varphi,L}(\mathcal{X})} \le C\|f\|_{H_{\varphi,L}(\mathcal{X})}.$$

Remark 6.11 (i) A typical example of the function m satisfying the condition of Theorem 6.10 is $m(\lambda) = \lambda^{i\gamma}$ for all $\lambda \in \mathbb{R}$ and some real-valued γ, where i denotes the *unit imaginary number* (see Corollary 6.13 below).

(ii) Theorem 6.10 covers the results of [38, Theorem 1.1] in the case when $p \in (0, 1]$, by taking $\varphi(x, t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$.

To prove Theorem 6.10, we need the following Lemma 6.12.

Lemma 6.12 *Let φ and L be as in Theorem 6.10, and m a bounded Borel function and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Assume that there exist $D \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$ and $C \in (0, \infty)$ such that, for every $j \in \{2, 3, \ldots\}$, any ball $B := B(x_B, r_B)$ and $f \in L^2(\mathcal{X})$ with* $\operatorname{supp} f \subset B$,

$$\left\|m(L)\left(I - e^{-r_B^2 L}\right)^M f\right\|_{L^2(U_j(B))} \le C2^{-jD}\|f\|_{L^2(B)}. \tag{6.22}$$

Then $m(L)$ can extend to a bounded linear operator on $H_{\varphi,L}(\mathcal{X})$. More precisely, there exists a positive constant C such that, for all $f \in H_{\varphi,L}(\mathcal{X})$, $\|m(L)f\|_{H_{\varphi,L}(\mathcal{X})} \le C\|f\|_{H_{\varphi,L}(\mathcal{X})}$.

Proof We borrow some ideas from [38]. Notice that since $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$, we can define $m(L)$ on $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$. Once we prove that $m(L)$ is bounded from $H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$ to $H_{\varphi,L}(\mathcal{X})$, by a density argument, we then see that the operator $m(L)$ can be extended to $H_{\varphi,L}(\mathcal{X})$.

Let $f \in H_{\varphi,L}(\mathcal{X}) \cap H^2(\mathcal{X})$ and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$. To prove the desired conclusion, it suffices to prove that, for any $(\varphi, 2M)$-atom α, $m(L)\alpha$ is a constant multiple of a (φ, M, ϵ)-molecule with $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$. Indeed, if this holds true, by Proposition 4.7, we know that there exist $\{\lambda_j\} \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi, 2M)$-atoms such that $f = \sum_j \lambda_j\alpha_j$ in $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ and

$\Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|f\|_{H_{\varphi,L}(\mathcal{X})}$. From this and the $L^2(\mathcal{X})$-boundedness of $m(L)$, we infer that $m(L)f = \sum_j \lambda_j(m(L)\alpha_j)$ is a molecular decomposition of $m(L)f$ and

$$\|m(L)f\|_{H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathcal{X})} \lesssim \Lambda\big(\big\{\lambda_j\big(m(L)\alpha_j\big)\big\}_j\big) \lesssim \Lambda\big(\{\lambda_j\alpha_j\}_j\big) \sim \|f\|_{H_{\varphi,L}(\mathcal{X})}.$$

Let α be a $(\varphi, 2M)$-atom. Then there exists a function $b \in \mathcal{D}(L^{2M})$ such that $\alpha = L^{2M}b$ satisfies (ii) and (iii) of Definition 4.3. From the spectral theory, it follows that $m(L)\alpha = L^M(m(L)L^M b)$. Furthermore, by the definition of (φ, M, ϵ)-molecules, it remains to prove that, for all $k \in \{0, \dots, M\}$ and $j \in \mathbb{Z}_+$,

$$\big\|\big(r_B^2 L\big)^k m(L)L^M b\big\|_{L^2(U_j(B))} \lesssim 2^{-j\epsilon} r_B^{2M}\big[\mu(B)\big]^{1/2}\|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}. \tag{6.23}$$

From the $L^2(\mathcal{X})$-boundedness of $m(L)$, the H_∞-functional calculi for L and (2.5), similar to the proof of [38, (3.4)], it follows that (6.23) holds true. We omit the details and hence complete the proof of Lemma 6.12. □

Now we give the proof of Theorem 6.10 by using Lemma 6.12.

Proof of Theorem 6.10 We borrow some ideas from [33, 38]. Since that m satisfies (6.21) if and only if the function $\lambda \to m(\lambda^2)$ satisfies the same property, similar to the proof of [38, Theorem 1.1], we may consider $m(\sqrt{L})$ instead of $m(L)$. By $m(\lambda) = m(\lambda) - m(0) + m(0)$, we know that $m(\sqrt{L}) = (m(\cdot) - m(0))(\sqrt{L}) + m(0)I$. Replacing m by $m - m(0)$, without loss of generality, we may assume, in the following, that $m(0) = 0$. Let ϕ be a function as in (6.20). Then for all $\lambda \in (0, \infty)$,

$$m(\lambda) = \sum_{\ell\in\mathbb{Z}} \phi\big(2^{-\ell}\lambda\big)m(\lambda) =: \sum_{\ell\in\mathbb{Z}} m_\ell(\lambda).$$

Moreover, from (1.1), it follows that the sequence $\sum_{\ell=-N}^{N} m_\ell(\sqrt{L})$ converges strongly in $L^2(\mathcal{X})$ to $m(\sqrt{L})$. We shall prove that $\sum_{\ell=-N}^{N} m_\ell(\sqrt{L})$ is bounded on $H_{\varphi,L}(\mathcal{X})$ with its bound independent of N. This, together with the strong convergence of (1.1) in $L^2(\mathcal{X})$, the fact that $H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $H_{\varphi,L}(\mathcal{X})$ and a density argument, then gives the desired conclusion.

Now fix $s \in \mathbb{R}$ with $s > n[q(\varphi)/i(\varphi) - 1/2]$. Let $M \in \mathbb{N}$ with $M > s/2$. For any $\ell \in \mathbb{Z}$, $r, \lambda \in (0, \infty)$, we set $F_{r,M}(\lambda) := m(\lambda)(1 - e^{-(r\lambda)^2})^M$ and $F^\ell_{r,M}(\lambda) := m_\ell(\lambda)(1 - e^{-(r\lambda)^2})^M$. Then we see that

$$m(\sqrt{L})\big(I - e^{-r^2L}\big)^M = F_{r,M}(\sqrt{L}) = \lim_{N\to\infty} \sum_{\ell=-N}^{N} F^\ell_{r,M}(\sqrt{L}) \tag{6.24}$$

in $L^2(\mathcal{X})$. Fix a ball B. For all $b \in L^2(\mathcal{X})$ with $\operatorname{supp} b \subset B$, by using the $L^2(\mathcal{X})$-boundedness of $m(L)$ and (6.20), similar to the proof of [38, (4.8)], we know that, for all $\ell \in \mathbb{Z}$ and $j \in \mathbb{N}$ with $j \geq 3$,

$$\big\|F^\ell_{r_B,M}(\sqrt{L})b\big\|_{L^2(U_j(B))} \lesssim C(\phi, s)2^{-sj}\big(2^\ell r_B\big)^{-s}\min\big\{1, \big(2^\ell r_B\big)^{2M}\big\}\|b\|_{L^2(B)}, \tag{6.25}$$

which, together with (6.24), $s > n[q(\varphi)/i(\varphi) - 1/2]$ and $M > s/2$, implies that, for all $j \in \mathbb{N}$ with $j \geq 3$,

$$\begin{aligned}
&\left\|m(\sqrt{L})\left(I - e^{-r_B^2 L}\right)^M b\right\|_{L^2(U_j(B))} \\
&\lesssim 2^{-js} \lim_{N\to\infty} \sum_{\ell=-N}^{N} \left(2^\ell r_B\right)^{-s} \min\left\{1, \left(2^\ell r_B\right)^{2M}\right\} \|b\|_{L^2(B)} \\
&\lesssim 2^{-js} \left[\sum_{\{\ell\in\mathbb{Z}:2^\ell r_B>1\}} \left(2^\ell r_B\right)^{-s} + \sum_{\{\ell\in\mathbb{Z}:2^\ell r_B\leq 1\}} \left(2^\ell r_B\right)^{2M-s}\right] \|b\|_{L^2(B)} \\
&\lesssim 2^{-js} \|b\|_{L^2(B)}.
\end{aligned}$$

By this, we know that the assumptions of Lemma 6.12 are satisfied, and hence the desired conclusion of Theorem 6.10 holds true, which completes the proof of Theorem 6.10. □

In the following corollary, we obtain the boundedness of imaginary powers of self-adjoint operators on Musielak–Orlicz–Hardy spaces $H_{\varphi,L}(\mathcal{X})$.

Corollary 6.13 *Let φ and L be as in Theorem 6.10. Then for any $\gamma \in \mathbb{R}$, the operator $L^{i\gamma}$ is bounded on $H_{\varphi,L}(\mathcal{X})$. Moreover, for any $\epsilon \in (0,\infty)$, there exists a positive constant $C(\epsilon)$, depending on ϵ, such that, for all $f \in H_{\varphi,L}(\mathcal{X})$,*

$$\|L^{i\gamma} f\|_{H_{\varphi,L}(\mathcal{X})} \leq C(\epsilon)(1+|\gamma|)^{n[\frac{q(\varphi)}{i(\varphi)}-\frac{1}{2}]+\epsilon} \|f\|_{H_{\varphi,L}(\mathcal{X})},$$

where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11).

Proof We apply Theorem 6.10 with $m(\lambda) := \lambda^{i\gamma}$ for all $\lambda \in (0,\infty)$. In this case it is easy to show that, for $s > n[q(\varphi)/i(\varphi) - 1/2]$, $C(\phi, s) \lesssim (1+|\gamma|)^s$, where $C(\phi, s)$ is as in (6.21) (see, for example, [38, Corollary 4.3]). From this, (6.25) and the proof of Theorem 6.10, we deduce that, for all $\epsilon \in (0,\infty)$, there exists a positive constant $C(\epsilon)$, depending on ϵ, such that, for all $f \in H_{\varphi,L}(\mathcal{X})$,

$$\|L^{i\gamma} f\|_{H_{\varphi,L}(\mathcal{X})} \leq C(\epsilon)(1+|\gamma|)^{n[\frac{q(\varphi)}{i(\varphi)}-\frac{1}{2}]+\epsilon} \|f\|_{H_{\varphi,L}(\mathcal{X})},$$

which completes the proof of Corollary 6.13. □

7 Applications to Schrödinger Operators

In this section, let $\mathcal{X} := \mathbb{R}^n$ and

$$L := -\Delta + V \tag{7.1}$$

be a Schrödinger operator, where $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We establish several equivalent characterizations of the corresponding Musielak–Orlicz–Hardy spaces $H_{\varphi,L}(\mathcal{X})$, in

terms of the atom, the molecular, the Lusin-area function associated with the Poisson semigroup of L, the non-tangential and the radial maximal functions associated with the heat semigroup generated by L, and the non-tangential and the radial maximal functions associated with the Poisson semigroup generated by L. Moreover, we prove that the Riesz transform $\nabla L^{-1/2}$ associated with L is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^{\varphi}(\mathbb{R}^n)$ when $i(\varphi) \in (0, 1]$, and from $H_{\varphi,L}(\mathbb{R}^n)$ to the Musielak–Orlicz–Hardy space $H_{\varphi}(\mathbb{R}^n)$ introduced by Ky [63] when $i(\varphi) \in (\frac{n}{n+1}, 1]$.

Since V is a nonnegative function, from the Feynman–Kac formula, we deduce that the kernel of the semigroup e^{-tL}, h_t, satisfies that, for all $x, y \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$0 \le h_t(x, y) \le (4\pi t)^{-n/2} \exp\left\{-\frac{|x-y|^2}{4t}\right\}.$$

Remark 7.1 (i) By Remark 4.2(i), we know that, in this case, $H^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

(ii) In this section, for the sake of convenience, we choose the norm on $\mathbb{R}^n$ to be the *supremum norm*; namely, for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $|x| := \max\{|x_1|, \dots, |x_n|\}$, for which balls determined by this norm are cubes associated with the usual Euclidean norm with sides parallel to the axes.

It is easy to see that L satisfies Assumptions A and B, which, combined with Theorems 5.5 and 5.7, immediately implies the following conclusions. We omit the details.

Theorem 7.2 *Let L be as in (7.1) and φ as in Definition 2.4 with $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathbb{R}^n)$, where $I(\varphi)$ is as in (2.10). Assume further that $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$ and $\epsilon \in (n[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}], \infty)$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$, $H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n)$, $H^{M,\epsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ and $H_{\varphi,S_P}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

For any $\beta \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\mathcal{N}_h^{\beta}(f)(x) := \sup_{y \in B(x,\beta t), t \in (0,\infty)} \left|e^{-t^2 L}(f)(y)\right|,$$

$$\mathcal{N}_P^{\beta}(f)(x) := \sup_{y \in B(x,\beta t), t \in (0,\infty)} \left|e^{-t\sqrt{L}}(f)(y)\right|,$$

$\mathcal{R}_h(f)(x) := \sup_{t\in(0,\infty)} |e^{-t^2 L}(f)(x)|$ and $\mathcal{R}_P(f)(x) := \sup_{t\in(0,\infty)} |e^{-t\sqrt{L}}(f)(x)|$. We *denote $\mathcal{N}_h^1(f)$ and $\mathcal{N}_P^1(f)$ simply by $\mathcal{N}_h(f)$ and $\mathcal{N}_P(f)$*, respectively.

Definition 7.3 Let L be as in (7.1) and φ as in Definition 2.4. A function $f \in H^2(\mathbb{R}^n)$ is said to be in $\widetilde{H}_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$ if $\mathcal{N}_h(f) \in L^{\varphi}(\mathbb{R}^n)$; moreover, let $\|f\|_{H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)} := \|\mathcal{N}_h(f)\|_{L^{\varphi}(\mathbb{R}^n)}$. The *Musielak–Orlicz–Hardy space* $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$ is defined to be the completion of $\widetilde{H}_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)}$.

The *spaces* $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)$ and $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)$ are defined in a similar way.

Then we give the following several equivalent characterizations of $H_{\varphi,L}(\mathbb{R}^n)$ in terms of maximal functions associated with L.

Theorem 7.4 *Assume that φ and L are as in Theorem 7.2. Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)$ and $H_{\varphi,S_P}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Remark 7.5 Theorem 7.2 completely covers [57, Theorem 6.1] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ concave. Theorem 7.4 completely covers [57, Theorem 6.4] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ satisfying that Φ is concave on $(0,\infty)$ and there exist $q_1, q_2 \in (0,\infty)$ such that $q_1 < 1 < q_2$ and $[\Phi(t^{q_2})]^{q_1}$ is a convex function on $(0,\infty)$.

To prove Theorem 7.4, we first establish the following Proposition 7.6.

Proposition 7.6 *Let φ and L be as in Theorem 7.2. Then $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for all $f \in H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\|f\|_{H_{\varphi,S_P}(\mathbb{R}^n)} \le C\|f\|_{H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)}$.*

To prove Proposition 7.6, we first introduce some notions. Let $\alpha \in (0,\infty)$ and $\epsilon, R \in (0,\infty)$ with $\epsilon < R$. For $f \in L^2(\mathbb{R}^n)$, define the *truncated Lusin-area function* $S_P^{\epsilon,R,\alpha}(f)(x)$ for all $x \in \mathbb{R}^n$, by setting,

$$S_P^{\epsilon,R,\alpha}(f)(x) := \left\{ \int_{\Gamma_\alpha^{\epsilon,R}(x)} \left| t\sqrt{L}e^{-t\sqrt{L}}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2},$$

where

$$\Gamma_\alpha^{\epsilon,R}(x) := \left\{ (y,t) \in \mathbb{R}^n \times (\epsilon, R) : |x-y| < \alpha t \right\}. \tag{7.2}$$

Then we have the following conclusion about the truncated Lusin-area function.

Lemma 7.7 *Let φ be as in Definition 2.4 and $\alpha, \beta \in (0,\infty)$. Then for all $0 \le \epsilon < R < \infty$ and $f \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi\big(x, S_P^{\epsilon,R,\alpha}(f)(x)\big)\,dx \sim \int_{\mathbb{R}^n} \varphi\big(x, S_P^{\epsilon,R,\beta}(f)(x)\big)\,dx,$$

where the implicit constants are independent of ϵ, R and f.

Proof First we recall two useful conclusions established in [23]. Let $\alpha, \beta \in (0,\infty)$, $\epsilon, R \in (0,\infty)$ with $\epsilon < R$. Then for any closed subset F of $\mathbb{R}^n$ whose complement has finite measure and any nonnegative measurable function H on $\mathbb{R}^n \times (0,\infty)$,

$$\int_F \left\{ \int_{\Gamma_\alpha^{\epsilon,R}(x)} H(y,t)\,dy\,dt \right\} dx \lesssim \int_{\mathcal{R}_\alpha^{\epsilon,R}(F)} H(y,t)t^n\,dy\,dt, \tag{7.3}$$

where $\Gamma_\alpha^{\epsilon,R}(x)$ is as in (7.2), $\mathcal{R}_\alpha^{\epsilon,R}(F) := \bigcup_{x\in F}\Gamma_\alpha^{\epsilon,R}(x)$ and the implicit constants are independent of F, ϵ, R and H. Let $\gamma \in (0,1)$ and F_γ^* be as in Sect. 3. Then

$$\int_{\mathcal{R}_\alpha^{\epsilon,R}(F_\gamma^*)} H(y,t)t^n\,dy\,dt \lesssim \int_F \left\{\int_{\Gamma_\beta^{\epsilon,R}(x)} H(y,t)\,dy\,dt\right\}dx. \tag{7.4}$$

Let $\alpha,\beta \in (0,\infty)$. Without loss of generality, we may assume that $\alpha > \beta$. Let $\epsilon, R \in (0,\infty)$ with $\epsilon < R$ and $f \in L^2(\mathbb{R}^n)$. Fix $\lambda \in (0,\infty)$. Let $\gamma \in (0,1)$, $F := \{x \in \mathbb{R}^n : S_P^{\epsilon,R,\beta}(f)(x) \le \lambda\}$ and $O := \mathbb{R}^n \setminus F$. Assume that F_γ^* and O_γ^* are as in Sect. 3. Then by (7.3) with $F := F_\gamma^*$ and $H(y,t) := |t\sqrt{L}e^{-t\sqrt{L}}(f)(y)|^2 t^{-(n+1)}$, we know that

$$\int_{F_\gamma^*}\left[S_P^{\epsilon,R,\alpha}(f)(x)\right]^2 dx \lesssim \int_{\mathcal{R}_\alpha^{\epsilon,R}(F_\gamma^*)}\left|t\sqrt{L}e^{-t\sqrt{L}}(f)(y)\right|^2 t^{-1}\,dy\,dt.$$

This, combined with (7.4) by choosing $H(y,t) := |t\sqrt{L}e^{-t\sqrt{L}}(f)(y)|^2 t^{-(n+1)}$, yields that

$$\int_{F_\gamma^*}\left[S_P^{\epsilon,R,\alpha}(f)(x)\right]^2 dx \lesssim \int_F\left[S_P^{\epsilon,R,\beta}(f)(x)\right]^2 dx. \tag{7.5}$$

Let $q \in (q(\varphi),\infty)$. Then $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, which, together with (7.5) and Lemma 2.8(vi), implies that, for all $t \in (0,\infty)$,

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^n : S_P^{\epsilon,R,\alpha}(f)(x)>\lambda\}} \varphi(x,t)\,dx \\
&\quad\le \int_{O_\gamma^*}\varphi(x,t)\,dx + \int_{\{x\in F_\gamma^* : S_P^{\epsilon,R,\alpha}(f)(x)>\lambda\}}\cdots \\
&\quad\lesssim \int_{\{x\in\mathbb{R}^n : \mathcal{M}(\chi_O)(x)>1-\gamma\}}\varphi(x,t)\,dx + \int_{\{x\in F_\gamma^* : S_P^{\epsilon,R,\alpha}(f)(x)>\lambda\}}\cdots \\
&\quad\lesssim \int_{\mathbb{R}^n}|\chi_O(x)|^q\varphi(x,t)\,dx + \frac{1}{\lambda^2}\int_F\left[S_P^{\epsilon,R,\beta}(f)(x)\right]^2\varphi(x,t)\,dx \\
&\quad\sim \int_{\{x\in\mathbb{R}^n : S_P^{\epsilon,R,\beta}(f)(x)>\lambda\}}\varphi(x,t)\,dx + \frac{1}{\lambda^2}\int_F\left[S_P^{\epsilon,R,\beta}(f)(x)\right]^2\varphi(x,t)\,dx.
\end{aligned}$$

From this, the fact that $\varphi(x,t) \sim \int_0^t \frac{\varphi(x,s)}{s}\,ds$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, Fubini's theorem and the uniformly upper type p_1 property of φ with $p_1 \in (0,1]$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\varphi\big(x, S_P^{\epsilon,R,\alpha}(f)(x)\big)\,dx \\
&\quad\sim \int_{\mathbb{R}^n}\left\{\int_0^{S_P^{\epsilon,R,\alpha}(f)(x)}\frac{\varphi(x,t)}{t}\,dt\right\}dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^\infty \int_{\{x\in\mathbb{R}^n: S_P^{\epsilon,R,\alpha}(f)(x)>t\}} \frac{\varphi(x,t)}{t}\,dx\,dt \\
&\lesssim \int_0^\infty \frac{1}{t}\int_{\{x\in\mathbb{R}^n: S_P^{\epsilon,R,\beta}(f)(x)>t\}} \varphi(x,t)\,dx\,dt \\
&\quad + \int_0^\infty \frac{1}{t^3}\int_F \big[S_P^{\epsilon,R,\beta}(f)(x)\big]^2\varphi(x,t)\,dx\,dt \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, S_P^{\epsilon,R,\beta}(f)(x)\big)\,dx + \int_{\mathbb{R}^n}\left\{\int_{S_P^{\epsilon,R,\beta}(f)(x)}^\infty \frac{\varphi(x,t)}{t^3}\,dt\right\}\big[S_P^{\epsilon,R,\beta}(f)(x)\big]^2\,dx \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, S_P^{\epsilon,R,\beta}(f)(x)\big)\,dx + \int_{\mathbb{R}^n}\big[S_P^{\epsilon,R,\beta}(f)(x)\big]^{2-p_1}\varphi\big(x, S_P^{\epsilon,R,\beta}(f)(x)\big) \\
&\quad \times \left\{\int_{S_P^{\epsilon,R,\beta}(f)(x)}^\infty \frac{1}{t^{3-p_1}}\,dt\right\}dx \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, S_P^{\epsilon,R,\beta}(f)(x)\big)\,dx,
\end{aligned}
$$

which completes the proof of Lemma 7.7. □

Let $\alpha \in (0,\infty)$ and $\epsilon, R \in (0,\infty)$ with $\epsilon < R$. For $f \in L^2(\mathbb{R}^n)$, define the *truncated Lusin-area function* $\widetilde{S}_P^{\epsilon,R,\alpha}(f)(x)$ for all $x \in \mathbb{R}^n$, by setting,

$$
\widetilde{S}_P^{\epsilon,R,\alpha}(f)(x) := \left\{\int_{\Gamma_\alpha^{\epsilon,R}(x)} \big|t\overline{\nabla}e^{-t\sqrt{L}}(f)(y)\big|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2},
$$

where $\Gamma_\alpha^{\epsilon,R}(x)$ is as in (7.2) and $\overline{\nabla} := (\nabla, \partial_t)$. When $\alpha = 1$, we denote $\widetilde{S}_P^{\epsilon,R,1}(f)$ simply by $\widetilde{S}_P^{\epsilon,R}(f)$. Obviously, for any $\alpha \in (0,\infty)$, $\epsilon, R \in (0,\infty)$ with $\epsilon < R$ and $f \in L^2(\mathbb{R}^n)$, $S_P^{\epsilon,R,\alpha}(f) \le \widetilde{S}_P^{\epsilon,R,\alpha}(f)$ pointwise. Now we give the following Lemma 7.8, which establishes a "good-λ inequality" concerning the truncated Lusin-area function $\widetilde{S}_P^{\epsilon,R,\alpha}$ and the non-tangential maximal function $\mathcal{N}_P$.

Lemma 7.8 *There exist positive constants C and $\epsilon_0 \in (0,1]$ such that, for all $\gamma \in (0,1]$, $\lambda \in (0,\infty)$, $\epsilon, R \in (0,\infty)$ with $\epsilon < R$, $f \in H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $t \in (0,\infty)$,*

$$
\begin{aligned}
&\int_{\{x\in\mathbb{R}^n: \widetilde{S}_P^{\epsilon,R,1/20}(f)(x)>2\lambda, \mathcal{N}_P(f)(x)\le\gamma\lambda\}} \varphi(x,t)\,dx \\
&\qquad \le C\gamma^{\epsilon_0}\int_{\{x\in\mathbb{R}^n: \widetilde{S}_P^{\epsilon,R,1/2}(f)(x)>\lambda\}} \varphi(x,t)\,dx. \qquad (7.6)
\end{aligned}
$$

Proof We prove this lemma by borrowing some ideas from [7, 8, 95]. Fix $0 < \epsilon < R < \infty$, $\gamma \in (0,1]$ and $\lambda \in (0,\infty)$. Let $f \in H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and

$$
O := \big\{x \in \mathbb{R}^n : \widetilde{S}_P^{\epsilon,R,1/2}(f)(x) > \lambda\big\}.
$$

It is easy to see that O is an open subset of $\mathbb{R}^n$. Let $O = \bigcup_k Q_k$ be the Whitney decomposition of O, where $\{Q_k\}_k$ are closed dyadic cubes of $\mathbb{R}^n$ with disjoint interiors and $2Q_k \subset O$, but $(4Q_k) \cap O^{\complement} \neq \emptyset$. To show (7.6), by $O = \bigcup_k Q_k$ and the disjoint property of $\{Q_k\}_k$, it suffices to show that there exists $\epsilon_0 \in (0, 1]$ such that, for all k,

$$\int_{\{x\in Q_k:\widetilde{S}_P^{\epsilon,R,1/20}(f)(x)>2\lambda,\mathcal{N}_P(f)(x)\le\gamma\lambda\}} \varphi(x,t)\,dx \lesssim \gamma^{\epsilon_0}\int_{Q_k}\varphi(x,t)\,dx. \tag{7.7}$$

From now on, we fix k and denote by l_k the *sidelength* of Q_k.

If $x \in Q_k$, then

$$\widetilde{S}_P^{\max\{10l_k,\epsilon\},R,1/20}(f)(x) \le \lambda. \tag{7.8}$$

Indeed, pick $x_k \in 4Q_k \cap O^{\complement}$. For any $(y,t) \in \mathbb{R}^n \times (0,\infty)$, if $|x-y| < \frac{t}{20}$ and $t \ge \max\{10l_k, \epsilon\}$, then $|x_k - y| \le |x_k - x| + |x - y| < 4l_k + \frac{t}{20} < \frac{t}{2}$, which implies that $\Gamma_{1/20}^{\max\{10l_k,\epsilon\},R}(x) \subset \Gamma_{1/2}^{\max\{10l_k,\epsilon\},R}(x_k)$. From this, it follows that

$$\widetilde{S}_P^{\max\{10l_k,\epsilon\},R,1/20}(f)(x) \le \widetilde{S}_P^{\max\{10l_k,\epsilon\},R,1/2}(f)(x_k) \le \lambda.$$

Thus, (7.8) holds true.

When $\epsilon \ge 10l_k$, by (7.8), we see that

$$\left\{x \in Q_k : \widetilde{S}_P^{\epsilon,R,1/20}(f)(x) > 2\lambda, \mathcal{N}_P(f)(x) \le \gamma\lambda\right\} = \emptyset$$

and hence (7.7) holds true. When $\epsilon < 10l_k$, to show (7.7), by the fact that $\widetilde{S}_P^{\epsilon,R,1/20}(f) \le \widetilde{S}_P^{\epsilon,10l_k,1/20}(f) + \widetilde{S}_P^{10l_k,R,1/20}(f)$ and (7.8), it remains to show that, for all $t \in (0,\infty)$,

$$\int_{\{x\in Q_k\cap F:g(x)>\lambda\}} \varphi(x,t)\,dx \lesssim \gamma^{\epsilon_0}\int_{Q_k}\varphi(x,t)\,dx, \tag{7.9}$$

where $g := \widetilde{S}_P^{\epsilon,10l_k,1/20}(f)$ and $F := \{x \in \mathbb{R}^n : \mathcal{N}_P(f)(x) \le \gamma\lambda\}$.

To prove (7.9), we claim that

$$\left|\left\{x \in Q_k \cap F : g(x) > \lambda\right\}\right| \lesssim \gamma^2|Q_k|. \tag{7.10}$$

If (7.10) holds true, it follows, from the fact that $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ and Lemma 2.8(v), that there exists $r \in (1,\infty)$ such that $\varphi \in \mathbb{RH}_r(\mathbb{R}^n)$, which, together with (7.10) and Lemma 2.8(viii), implies that, for all $t \in (0,\infty)$,

$$\begin{aligned}\frac{1}{\varphi(Q_k,t)}\int_{\{x\in Q_k\cap F:g(x)>\lambda\}}\varphi(x,t)\,dx &\lesssim \left\{\frac{|\{x\in Q_k\cap F:g(x)>\lambda\}|}{|Q_k|}\right\}^{(r-1)/r}\\ &\lesssim \gamma^{2(r-1)/r}.\end{aligned}$$

Let $\epsilon_0 := 2(r-1)/r$. Then $\int_{\{x\in Q_k\cap F:g(x)>\lambda\}}\varphi(x,t)\,dx \lesssim \gamma^{\epsilon_0}\varphi(Q_k,t)$, which implies that (7.9) holds true.

Now we show (7.10). By Chebychev's inequality, we know that (7.10) can be deduced from

$$\int_{Q_k\cap F}\big[g(x)\big]^2\,dx \lesssim (\gamma\lambda)^2|Q_k|. \tag{7.11}$$

From the Caccioppoli inequality associated with L (see, for example, [51, Lemma 8.3]), the differential structure of L and the divergence theorem, similar to the proof of [95, (3.9)], it follows that (7.11) holds true. We omit the details and hence complete the proof of Lemma 7.8. □

Now we prove Proposition 7.6 by using Lemmas 7.7 and 7.8.

Proof of Proposition 7.6 Assume that $f\in H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Take $p_2\in(0,i(\varphi))$ such that φ is uniformly lower type p_2. By Lemma 2.6(ii), we know that $\varphi(x,t)\sim\int_0^t\frac{\varphi(x,s)}{s}\,ds$ for all $x\in\mathbb{R}^n$ and $t\in(0,\infty)$, which, together with Fubini's theorem and Lemma 7.8, implies that, for all $\epsilon, R\in(0,\infty)$ with $\epsilon<R$ and $\gamma\in(0,1]$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\varphi\big(x,\widetilde{S}_P^{\epsilon,R,1/20}(f)(x)\big)\,dx\\
&\quad\sim\int_{\mathbb{R}^n}\int_0^{\widetilde{S}_P^{\epsilon,R,1/20}(f)(x)}\frac{\varphi(x,t)}{t}\,dt\,dx\\
&\quad\sim\int_0^\infty\frac{1}{t}\int_{\{x\in\mathbb{R}^n:\widetilde{S}_P^{\epsilon,R,1/20}(f)(x)>t\}}\varphi(x,t)\,dx\,dt\\
&\quad\lesssim\int_0^\infty\frac{1}{t}\int_{\{x\in\mathbb{R}^n:\mathcal{N}_P(f)(x)>\gamma t\}}\varphi(x,t)\,dx\,dt\\
&\qquad+\gamma^{\epsilon_0}\int_0^\infty\frac{1}{t}\int_{\{x\in\mathbb{R}^n:\widetilde{S}_P^{\epsilon,R,1/2}(f)(x)>t/2\}}\cdots\\
&\quad\lesssim\frac{1}{\gamma}\int_0^\infty\int_{\{x\in\mathbb{R}^n:\mathcal{N}_P(f)(x)>t\}}\frac{\varphi(x,t)}{t}\,dx\,dt\\
&\qquad+\gamma^{\epsilon_0}\int_0^\infty\int_{\{x\in\mathbb{R}^n:\widetilde{S}_P^{\epsilon,R,1/2}(f)(x)>t\}}\cdots\\
&\quad\sim\frac{1}{\gamma}\int_{\mathbb{R}^n}\varphi\big(x,\mathcal{N}_P(f)(x)\big)\,dx+\gamma^{\epsilon_0}\int_{\mathbb{R}^n}\varphi\big(x,\widetilde{S}_P^{\epsilon,R,1/2}(f)(x)\big)\,dx.
\end{aligned}\tag{7.12}$$

Furthermore, by (7.12), Lemma 7.7 and $\widetilde{S}_P^{\epsilon,R,1/2}(f)\le\widetilde{S}_P^{\epsilon,R}(f)$ pointwise, we conclude that, for all $\gamma\in(0,1]$, and $\epsilon, R\in(0,\infty)$ with $\epsilon<R$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\varphi\big(x,\widetilde{S}_P^{\epsilon,R}(f)(x)\big)\,dx\\
&\quad\sim\int_{\mathbb{R}^n}\varphi\big(x,\widetilde{S}_P^{\epsilon,R,1/20}(f)(x)\big)\,dx\\
&\quad\lesssim\frac{1}{\gamma}\int_{\mathbb{R}^n}\varphi\big(x,\mathcal{N}_P(f)(x)\big)\,dx+\gamma^{\epsilon_0}\int_{\mathbb{R}^n}\varphi\big(x,\widetilde{S}_P^{\epsilon,R}(f)(x)\big)\,dx,
\end{aligned}$$

which, together with the facts that, for all $\lambda \in (0,\infty)$, $\widetilde{S}_P^{\epsilon,R}(f/\lambda) = \widetilde{S}_P^{\epsilon,R}(f)/\lambda$ and $\mathcal{N}_P(f/\lambda) = \mathcal{N}_P(f)/\lambda$, implies that there exists a positive constant $\widetilde{C}$ such that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{S}_P^{\epsilon,R}(f)(x)}{\lambda}\right) dx$$
$$\le \widetilde{C}\left[\frac{1}{\gamma}\int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{N}_P(f)(x)}{\lambda}\right) dx + \gamma^{\epsilon_0}\int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{S}_P^{\epsilon,R}(f)(x)}{\lambda}\right) dx\right]. \quad (7.13)$$

Take $\gamma \in (0,1]$ such that $\widetilde{C}\gamma^{\epsilon_0} = 1/2$. Then from (7.13) and the fact that $S_P^{\epsilon,R}(f) \le \widetilde{S}_P^{\epsilon,R}(f)$ pointwise, we deduce that, for all $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{S_P^{\epsilon,R}(f)(x)}{\lambda}\right) dx \le \int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{S}_P^{\epsilon,R}(f)(x)}{\lambda}\right) dx$$
$$\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{N}_P(f)(x)}{\lambda}\right) dx.$$

By the Fatou lemma and letting $\epsilon \to 0$ and $R \to \infty$, we know that, for all $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{S_P(f)(x)}{\lambda}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{N}_P(f)(x)}{\lambda}\right) dx,$$

which implies that $\|S_P(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|\mathcal{N}_P(f)\|_{L^\varphi(\mathbb{R}^n)}$ and hence completes the proof of Proposition 7.6. □

To prove Theorem 7.4, we need the following Moser type local boundedness estimate from [51, Lemma 8.4].

Lemma 7.9 *Let u be a weak solution of $\widetilde{L}u := Lu - \partial_t^2 u = 0$ in the ball $B(Y_0, 2r) \subset \mathbb{R}^{n+1}_+$. Then for all $p \in (0,\infty)$, there exists a positive constant $C(n,p)$, depending on n and p, such that*

$$\sup_{Y \in B(Y_0,r)} |u(Y)| \le C(n,p)\left\{\frac{1}{r^{n+1}}\int_{B(Y_0,2r)} |u(Y)|^p\, dY\right\}^{1/p}.$$

Now we prove Theorem 7.4 by using Theorem 7.2, Lemma 7.9 and Proposition 7.6.

Proof of Theorem 7.4 The proof of Theorem 7.4 is divided into the following six steps.

Step 1. $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let M be as in Theorem 7.2. By Theorem 7.2, we know that $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with equivalent quasi-norms. Thus, we only need to prove $H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset$

$H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. To this end, similar to the proof of (4.5), it suffices to show that, for any $\lambda\in\mathbb{C}$ and (φ,M)-atom a with $\operatorname{supp} a\subset B:=B(x_B,r_B)$,

$$\int_{\mathbb{R}^n}\varphi\big(x,\mathcal{N}_h(\lambda a)(x)\big)\,dx\lesssim\varphi\big(B,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big).$$

From the $L^2(\mathbb{R}^n)$-boundedness of $\mathcal{N}_h$ and (2.5), similar to the proof of (4.5), it follows that the above estimate holds true. We omit the details here.

Step 2. $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, which is deduced from the fact that, for all $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, $\mathcal{R}_h(f)(x)\le\mathcal{N}_h(f)(x)$.

Step 3. $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. By the subordination formula associated with L,

$$e^{-t\sqrt{L}}=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}L}e^{-u}u^{-1/2}\,du$$

with $t\in(0,\infty)$ (see, for example, [8]), we know that, for all $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}\mathcal{R}_P(f)(x)&\le\sup_{t\in(0,\infty)}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\big|e^{-\frac{t^2}{4u}L}(f)(x)\big|\,du\\&\lesssim\mathcal{R}_h(f)(x)\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\,du\\&\lesssim\mathcal{R}_h(f)(x),\end{aligned}$$

which implies that, for all $f\in H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, $\|f\|_{H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)}\lesssim\|f\|_{H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)}$. From this and the arbitrariness of f, we deduce that $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Step 4. $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\subset H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. For all $f\in L^2(\mathbb{R}^n)$, $x\in\mathbb{R}^n$ and $t\in(0,\infty)$, let $u(x,t):=e^{-tL^{1/2}}(f)(x)$. Then $\widetilde{L}u=Lu-\partial_t^2u=0$ in $\mathbb{R}^{n+1}_+$. Let $x\in\mathbb{R}^n$ and $t\in(0,\infty)$. Then by Lemma 7.9, we know that, for any $\gamma\in(0,1)$ and $y\in Q(x,t/4)$,

$$\begin{aligned}\big|e^{-t\sqrt{L}}(f)(y)\big|^\gamma&\lesssim\frac{1}{t^{n+1}}\int_{t/2}^{3t/2}\int_{Q(x,t/2)}\big|e^{-s\sqrt{L}}(f)(z)\big|^\gamma\,dz\,ds\\&\lesssim\frac{1}{t^n}\int_{Q(x,t)}|\mathcal{R}_P(f)(z)|^\gamma\,dz\\&\lesssim\mathcal{M}\big(\big[\mathcal{R}_P(f)\big]^\gamma\big)(x),\end{aligned}$$

which implies that, for all $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\mathcal{N}_P^{1/4}(f)(x)\lesssim\big\{\mathcal{M}\big(\big[\mathcal{R}_P(f)\big]^\gamma\big)(x)\big\}^{1/\gamma}.\tag{7.14}$$

Let $q_0 \in (q(\varphi), \infty)$, $p_2 \in (0, i(\varphi))$ and $\gamma_0 \in (0,1)$ such that $\gamma_0 q_0 < p_2$. Then we know that φ is of uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$. For any $\alpha \in (0,\infty)$ and $g \in L^{q_0}_{\mathrm{loc}}(\mathbb{R}^n)$, let $g = g\chi_{\{x\in\mathbb{R}^n:|g(x)|\le\alpha\}} + g\chi_{\{x\in\mathbb{R}^n:|g(x)|>\alpha\}} =: g_1 + g_2$. Then from Lemma 2.8(vi), we infer that, for all $t \in (0,\infty)$,

$$\begin{aligned}\int_{\{x\in\mathbb{R}^n:\mathcal{M}(g)(x)>2\alpha\}} \varphi(x,t)\,dx &\le \int_{\{x\in\mathbb{R}^n:\mathcal{M}(g_2)(x)>\alpha\}} \varphi(x,t)\,dx \\ &\le \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} \big[\mathcal{M}(g_2)(x)\big]^{q_0} \varphi(x,t)\,dx \\ &\lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} |g_2(x)|^{q_0} \varphi(x,t)\,dx \\ &\sim \frac{1}{\alpha^{q_0}} \int_{\{x\in\mathbb{R}^n:|g(x)|>\alpha\}} |g(x)|^{q_0} \varphi(x,t)\,dx,\end{aligned}$$

which implies that, for all $\alpha \in (0,\infty)$,

$$\begin{aligned}&\int_{\{x\in\mathbb{R}^n:[\mathcal{M}([\mathcal{R}_P(f)]^{\gamma_0})(x)]^{1/\gamma_0}>\alpha\}} \varphi(x,t)\,dx \\ &\quad\lesssim \frac{1}{\alpha^{\gamma_0 q_0}} \int_{\{x\in\mathbb{R}^n:[\mathcal{R}_P(f)(x)]^{\gamma_0}>\frac{\alpha^{\gamma_0}}{2}\}} \big[\mathcal{R}_P(f)(x)\big]^{\gamma_0 q_0} \varphi(x,t)\,dx \\ &\quad\lesssim \sigma_{\mathcal{R}_P(f),t}\left(\frac{\alpha}{2^{1/\gamma_0}}\right) + \frac{1}{\alpha^{\gamma_0 q_0}} \int_{\frac{\alpha}{2^{1/\gamma_0}}}^{\infty} \gamma_0 q_0 s^{\gamma_0 q_0 - 1} \sigma_{\mathcal{R}_P(f),t}(s)\,ds. \qquad (7.15)\end{aligned}$$

Here and in what follows, $\sigma_{\mathcal{R}_P(f),t}(\alpha) := \int_{\{x\in\mathbb{R}^n:\mathcal{R}_P(f)(x)>\alpha\}} \varphi(x,t)\,dx$. From this, (7.14), the uniformly upper type p_1 and lower type p_2 properties of φ and $\gamma_0 q_0 < p_2$, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n} \varphi\big(x, \mathcal{N}_P^{1/4}(f)(x)\big)\,dx \\ &\quad\lesssim \int_{\mathbb{R}^n} \varphi\Big(x, \big[\mathcal{M}\big(\big[\mathcal{R}_P(f)\big]^{\gamma_0}\big)(x)\big]^{1/\gamma_0}\Big)\,dx \\ &\quad\lesssim \int_{\mathbb{R}^n} \int_0^{\{\mathcal{M}([\mathcal{R}_P(f)]^{\gamma_0})(x)\}^{1/\gamma_0}} \frac{\varphi(x,t)}{t}\,dt\,dx \\ &\quad\sim \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n:[\mathcal{M}([\mathcal{R}_P(f)]^{\gamma_0})(x)]^{1/\gamma_0}>t\}} \varphi(x,t)\,dx\,dt \\ &\quad\lesssim \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n:\mathcal{R}_P(f)(x)>\frac{t}{2^{1/\gamma_0}}\}} \varphi(x,t)\,dx\,dt \\ &\qquad+ \int_0^\infty \frac{1}{t^{\gamma_0 q_0+1}} \left\{\int_{\frac{t}{2^{1/\gamma_0}}}^\infty \gamma_0 q_0 s^{\gamma_0 q_0 - 1} \sigma_{\mathcal{R}_P(f),t}(s)\,ds\right\} dt\end{aligned}$$

$$
\begin{aligned}
&\sim \mathrm{J}_{\mathcal{R}_P(f)} + \int_0^\infty \gamma_0 q_0 s^{\gamma_0 q_0 - 1} \left\{ \int_0^{2^{1/\gamma_0} s} \frac{1}{t^{\gamma_0 q_0 + 1}} \sigma_{\mathcal{R}_P(f),t}(s)\, dt \right\} ds \\
&\lesssim \mathrm{J}_{\mathcal{R}_P(f)} \\
&\quad + \int_0^\infty \gamma_0 q_0 s^{\gamma_0 q_0 - 1} \sigma_{\mathcal{R}_P(f),t}(s) \varphi\big(x, 2^{1/\gamma_0} s\big) \left\{ \int_0^{2^{1/\gamma_0} s} \left[\frac{t}{2^{1/\gamma_0} s} \right]^{p_2} \frac{1}{t^{\gamma_0 q_0 + 1}}\, dt \right\} ds \\
&\lesssim \mathrm{J}_{\mathcal{R}_P(f)} + \int_0^\infty \gamma_0 q_0 s^{\gamma_0 q_0 - 1} \sigma_{\mathcal{R}_P(f),t}(s) \frac{\varphi(x,s)}{(2^{\frac{1}{\gamma_0}} s)^{p_2}} \left\{ \int_0^{2^{1/\gamma_0} s} t^{p_2 - \gamma_0 q_0 - 1}\, dt \right\} ds \\
&\lesssim \mathrm{J}_{\mathcal{R}_P(f)} + \int_0^\infty \int_{\{x \in \mathbb{R}^n : \mathcal{R}_P(f)(x) > s\}} \frac{\varphi(x,s)}{s}\, ds \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, \mathcal{R}_P(f)(x)\big)\, dx,
\end{aligned}
$$

where

$$
\mathrm{J}_{\mathcal{R}_P(f)} := \int_0^\infty \int_{\{x \in \mathbb{R}^n : \mathcal{R}_P(f)(x) > t\}} \frac{\varphi(x,t)}{t}\, dx\, dt,
$$

which, together the fact that, for all $\lambda \in (0,\infty)$, $\mathcal{N}_P^{1/4}(f/\lambda) = \mathcal{N}_P^{1/4}(f)/\lambda$ and $\mathcal{R}_P(f/\lambda) = \mathcal{R}_P(f)/\lambda$, implies that, for all $\lambda \in (0,\infty)$,

$$
\int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{N}_P^{1/4}(f)(x)}{\lambda}\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{R}_P(f)(x)}{\lambda}\right) dx.
$$

From this, we further deduce that

$$
\big\| \mathcal{N}_P^{1/4}(f) \big\|_{L^\varphi(\mathcal{X})} \lesssim \big\| \mathcal{R}_P(f) \big\|_{L^\varphi(\mathcal{X})}. \tag{7.16}
$$

To end the proof of this step, we claim that, for all $g \in L^2(\mathbb{R}^n)$,

$$
\big\| \mathcal{N}_P^{1/4}(g) \big\|_{L^\varphi(\mathcal{X})} \sim \| \mathcal{N}_P(g) \|_{L^\varphi(\mathcal{X})}. \tag{7.17}
$$

Then by (7.16) and (7.17), we conclude that $\|\mathcal{N}_P(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|\mathcal{R}_P(f)\|_{L^\varphi(\mathcal{X})}$. From this and the arbitrariness of f, we deduce that $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Now we show (7.17). We borrow some ideas from [41, p. 166, Lemma 1]. By the change of variables, it suffices to prove that

$$
\int_{\mathbb{R}^n} \varphi\big(x, \mathcal{N}_P^N(f)(x)\big)\, dx \lesssim \int_{\mathbb{R}^n} \varphi\big(x, \mathcal{N}_P(f)(x)\big)\, dx, \tag{7.18}
$$

where N is a positive constant with $N \in (1,\infty)$. For any $\alpha \in (0,\infty)$, let

$$
E_\alpha := \big\{ x \in \mathbb{R}^n : \mathcal{N}_P(f)(x) > \alpha \big\} \quad \text{and} \quad E_\alpha^* := \big\{ x \in \mathbb{R}^n : \mathcal{M}(\chi_{E_\alpha})(x) > \widetilde{C}/N^n \big\},
$$

where $\widetilde{C} \in (0,1)$ is a positive constant. By $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$, we know that there exists $p \in (q(\varphi), \infty)$ such that $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$. From this and Lemma 2.8(vi), it follows that, for all $t \in [0, \infty)$,

$$\int_{E_\alpha^*} \varphi(x,t)\,dx \lesssim \frac{N^{np}}{\widetilde{C}^p} \int_{E_\alpha} \varphi(x,t)\,dx. \tag{7.19}$$

Moreover, we claim that $\mathcal{N}_P^N(f)(x) \le \alpha$ for all $x \notin E_\alpha^*$. Indeed, fix any given $(y,t) \in \mathbb{R}^n \times (0,\infty)$ satisfying $|y-x| < Nt$. Then $B(y,t) \not\subset E_\alpha$. If this is not true, then

$$\mathcal{M}(\chi_{E_\alpha})(x) \ge \frac{|B(y,t)|}{|B(y,Nt)|} = \frac{1}{N^n} > \frac{\widetilde{C}}{N^n}.$$

This gives a contradiction with $x \notin E_\alpha^*$, and hence the claim holds true. From the claim, we deduce that there exists $z \in B(y,t)$ such that $\mathcal{N}_P(f)(z) \le \alpha$, which implies that $|e^{-t\sqrt{L}}(f)(y)| \le \mathcal{N}_P(f)(z) \le \alpha$. By this and the choice of (y,t), we conclude that, for all $x \notin E_\alpha^*$, $\mathcal{N}_P^N(f)(x) \le \alpha$, which, together with Lemma 2.6(ii), Fubini's theorem and (7.19), implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\big(x, \mathcal{N}_P^N(f)(x)\big)\,dx &\sim \int_{\mathbb{R}^n} \int_0^{\mathcal{N}_P^N(f)(x)} \frac{\varphi(x,t)}{t}\,dt\,dx \\
&\sim \int_0^\infty \int_{\{x\in\mathbb{R}^n:\mathcal{N}_P^N(f)(x)>t\}} \frac{\varphi(x,t)}{t}\,dx\,dt \\
&\lesssim \int_0^\infty \int_{E_t^*} \frac{\varphi(x,t)}{t}\,dx\,dt \\
&\lesssim \int_0^\infty \int_{E_t} \frac{\varphi(x,t)}{t}\,dx\,dt \sim \int_{\mathbb{R}^n} \varphi\big(x, \mathcal{N}_P(f)(x)\big)\,dx.
\end{aligned}$$

Thus, the claim (7.18) holds true.

Step 5. $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. This is just the conclusion of Proposition 7.6.

Step 6. $H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. This is directly deduced from Theorem 7.2.

From Steps 1 though 6, we deduce that

$$\begin{aligned}
H_{\varphi,L}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big) &= H_{\varphi,\mathcal{N}_h}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big) = H_{\varphi,\mathcal{R}_h}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big) \\
&= H_{\varphi,\mathcal{R}_P}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big) = H_{\varphi,\mathcal{N}_P}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big) \\
&= H_{\varphi,S_P}\big(\mathbb{R}^n\big) \cap L^2\big(\mathbb{R}^n\big)
\end{aligned}$$

with equivalent quasi-norms, which, together with the fact that $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $H_{\varphi,S_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are, respectively, dense in $H_{\varphi,L}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$,

$H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$ and $H_{\varphi,S_P}(\mathbb{R}^n)$, and a density argument, then implies that the spaces $H_{\varphi,L}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_h}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_h}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{R}_P}(\mathbb{R}^n)$, $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$ and $H_{\varphi,S_P}(\mathbb{R}^n)$ coincide with equivalent quasi-norms, which completes the proof of Theorem 7.4. □

Now we consider the boundedness of the Riesz transform $\nabla L^{-1/2}$ associated with L. By the functional calculus of L, we know that, for all $f \in L^2(\mathbb{R}^n)$,

$$\nabla L^{-1/2} f = \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-tL} f \frac{dt}{\sqrt{t}}. \tag{7.20}$$

It is well known that $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$ (see, for example, [51, (8.20)]). To establish the main results in this subsection about the boundedness of the Riesz transform $\nabla L^{-1/2}$ on $H_{\varphi,L}(\mathbb{R}^n)$, we need the following conclusion, which is just [51, Lemma 8.5] (see also [57, Lemma 6.2]).

Lemma 7.10 *There exist two positive constants C and c such that, for all closed sets E and F in $\mathbb{R}^n$ and $f \in L^2(E)$,*

$$\left\| t\nabla e^{-t^2 L} f \right\|_{L^2(F)} \le C \exp\left\{ -\frac{[\operatorname{dist}(E,F)]^2}{ct^2} \right\} \|f\|_{L^2(E)}.$$

Theorem 7.11 *Let φ and L be as in Theorem 7.2. Then the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.*

Proof First let $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}] + \frac{1}{2}$, where n, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.12) and (2.11). Then there exist $p_2 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2}) + \frac{1}{2}$, φ is uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$. Moreover, by Proposition 4.7, we know that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that $f = \sum_j \lambda_j \alpha_j$ in $L^2(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|f\|_{H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n)}$, which, together with the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, implies that

$$\nabla L^{-1/2}(f) = \sum_j \lambda_j \nabla L^{-1/2}(\alpha_j) \tag{7.21}$$

in $L^2(\mathbb{R}^n)$.

To finish the proof of Theorem 7.11, it suffices to show that, for any $\lambda \in \mathbb{C}$ and (φ, M)-atom α supported in $B := B(x_B, r_B)$,

$$\int_{\mathbb{R}^n} \varphi\big(x, \nabla L^{-1/2}(\alpha)(x)\big)\, dx \lesssim \varphi\big(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})}\big). \tag{7.22}$$

If (7.22) holds true, then it follows, from this and (7.21), that

$$\int_{\mathbb{R}^n} \varphi\big(x, \nabla L^{-1/2}(f)(x)\big)\, dx \lesssim \sum_j \varphi\big(B_j, |\lambda_j| \|\chi_{B_j}\|^{-1}_{L^\varphi(\mathcal{X})}\big),$$

where, for each j, $\operatorname{supp}\alpha_j \subset B_j$. By this and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|f\|_{H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n)}$, we conclude that $\|\nabla L^{-1/2}(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}$, which, together with the fact that $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_{\varphi,L}(\mathbb{R}^n)$ and a density argument, implies that $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.

Now we prove (7.22). By the definition of α, we know that there exists $b \in \mathcal{D}(L^M)$ such that $\alpha = L^M b$ and (ii) and (iii) of Definition 4.3 hold true. First we see that

$$\int_{\mathbb{R}^n} \varphi\big(x, \lambda\nabla L^{-1/2}(\alpha)(x)\big)\,dx = \sum_{j=0}^{\infty} \int_{U_j(B)} \varphi\big(x, \lambda\nabla L^{-1/2}(\alpha)(x)\big)\,dx$$
$$=: \sum_{j=0}^{\infty} \mathrm{I}_j. \tag{7.23}$$

From the assumption $\varphi \in \mathbb{RH}_{2/[2-I(\varphi)]}(\mathbb{R}^n)$, Lemma 2.8(iv) and the definition of $I(\varphi)$, we infer that, there exists $p_1 \in [I(\varphi), 1]$ such that φ is of uniformly upper type p_1 and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathbb{R}^n)$. When $j \in \{0, \dots, 4\}$, by the uniformly upper type p_1 property of φ, Hölder's inequality, the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathbb{R}^n)$ and Lemma 2.8(vii), we conclude that

$$\begin{aligned}
\mathrm{I}_j &\lesssim \int_{U_j(B)} \varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big)\big(1 + \big[|\nabla L^{-1/2}(\alpha)(x)|\|\chi_B\|_{L^\varphi(\mathcal{X})}\big]^{p_1}\big)\,dx \\
&\lesssim \varphi\big(2^j B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big) + \|\chi_B\|^{p_1}_{L^\varphi(\mathbb{R}^n)} \big\|\nabla L^{-1/2}(\alpha)\big\|^{p_1}_{L^2(\mathbb{R}^n)} \\
&\quad \times \left\{\int_{2^j B} \big[\varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big)\big]^{\frac{2}{2-p_1}}\,dx\right\}^{\frac{2-p_1}{2}} \\
&\lesssim \varphi\big(2^j B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big) \lesssim \varphi\big(B, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big).
\end{aligned} \tag{7.24}$$

When $j \in \mathbb{N}$ with $j \geq 5$, from the uniformly upper type p_1 and the lower type p_2 properties of φ, it follows that

$$\begin{aligned}
\mathrm{I}_j &\lesssim \|\chi_B\|^{p_1}_{L^\varphi(\mathbb{R}^n)} \int_{U_j(B)} \varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big)\big|\nabla L^{-1/2}(\alpha)(x)\big|^{p_1}\,dx \\
&\quad + \|\chi_B\|^{p_2}_{L^\varphi(\mathbb{R}^n)} \int_{U_j(B)} \varphi\big(x, |\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big)\big|\nabla L^{-1/2}(\alpha)(x)\big|^{p_2}\,dx \\
&=: \mathrm{E}_j + \mathrm{F}_j.
\end{aligned} \tag{7.25}$$

To deal with E_j and F_j, we first estimate $\int_{U_j(B)} |\nabla L^{-1/2}(\alpha)(x)|^2\,dx$. By (7.20), the change of variables and Minkowski's inequality, we see that, for each $j \in \mathbb{N}$ with $j \geq 5$,

$$\int_{U_j(B)} \big|\nabla L^{-1/2}(\alpha)(x)\big|^2\,dx \lesssim \int_0^\infty \left\{\int_{U_j(B)} \big|\nabla e^{-t^2 L}\alpha(x)\big|^2\,dx\right\}^{1/2} dt$$

$$\sim \int_0^{r_B} \left\{ \int_{U_j(B)} \left| t\nabla e^{-t^2 L}\alpha(x) \right|^2 dx \right\}^{1/2} \frac{dt}{t}$$
$$+ \int_{r_B}^{\infty} \left\{ \int_{U_j(B)} \left| t\nabla \left(t^2 L\right)^M e^{-t^2 L} b(x) \right|^2 dx \right\}^{1/2} \frac{dt}{t^{2M+1}}$$
$$=: \mathrm{H}_{j,1} + \mathrm{H}_{j,2}. \tag{7.26}$$

We first estimate $\mathrm{H}_{j,1}$. From Lemma 7.10, we infer that

$$\mathrm{H}_{j,1} \lesssim \int_0^{r_B} \exp\left\{ -\frac{(2^j r_B)^2}{ct^2} \right\} \|\alpha\|_{L^2(B)} \frac{dt}{t}$$
$$\lesssim \left\{ \int_0^{r_B} \frac{t^{2M-1}}{(2^j r_B)^{2M-1}} \frac{dt}{t} \right\} \|\alpha\|_{L^2(B)} \sim 2^{-(2M-1)j} \|\alpha\|_{L^2(B)}$$
$$\lesssim 2^{-(2M-1)j} |B|^{1/2} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{7.27}$$

For $\mathrm{H}_{j,2}$, by Lemma 7.10, we see that

$$\mathrm{H}_{j,2} \lesssim \int_{r_B}^{\infty} \exp\left\{ -\frac{(2^j r_B)^2}{ct^2} \right\} \|b\|_{L^2(B)} \frac{dt}{t^{2M+1}}$$
$$\lesssim \int_{r_B}^{\infty} \frac{t^{(2M-1)}}{(2^j r_B)^{(2M-1)}} \frac{dt}{t^{2M+1}} \|b\|_{L^2(B)} \lesssim 2^{-(2M-1)j} |B|^{1/2} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)},$$

which, together with (7.26) and (7.27), implies that, for all $j \in \mathbb{N}$ with $j \geq 5$,

$$\left\{ \int_{U_j(B)} \left| \nabla L^{-1/2}(\alpha)(x) \right|^2 dx \right\}^{1/2} \lesssim 2^{-(2M-1)j} |B|^{1/2} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{7.28}$$

Thus, from Hölder's inequality, (7.28) and $\varphi \in \mathbb{RH}_{2/(2-p_1)}(\mathbb{R}^n) \subset \mathbb{RH}_{2/(2-p_2)}(\mathbb{R}^n)$, similar to the proof of (6.8), we infer that

$$\mathrm{E}_j \lesssim 2^{-jp_1[(2M-1+\frac{n}{2})-\frac{nq_0}{p_1}]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right). \tag{7.29}$$

Similarly, by using Hölder's inequality, (7.28) and $\varphi \in \mathbb{RH}_{2/(2-p_2)}(\mathbb{R}^n)$, we see that

$$\mathrm{F}_j \lesssim 2^{-jp_2[(2M-1+\frac{n}{2})-\frac{nq_0}{p_2}]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right),$$

which, together with (7.25), (7.29) and $p_1 \geq p_2$, implies that, for each $j \in \mathbb{N}$ with $j \geq 5$,

$$\mathrm{I}_j \lesssim 2^{-jp_2[(2M-1+\frac{n}{2})-\frac{nq_0}{p_2}]} \varphi\left(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right).$$

From this, $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2}) + \frac{1}{2}$, (7.23) and (7.24), we infer that (7.22) holds true, which completes the proof of Theorem 7.11. □

Now we recall the definition of the Musielak–Orlicz–Hardy space $H_\varphi(\mathbb{R}^n)$ introduced by Ky [63].

Definition 7.12 Let φ be as in Definition 2.4. The *Musielak–Orlicz–Hardy space* $H_\varphi(\mathbb{R}^n)$ is the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{G}(f) \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm* $\|f\|_{H_\varphi(\mathbb{R}^n)} := \|\mathcal{G}(f)\|_{L^\varphi(\mathbb{R}^n)}$, where $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{G}(f)$ denote, respectively, the *dual space* of the Schwartz functions space (namely, the *space of tempered distributions*) and the *grand maximal function of* f.

To state the atomic characterization of $H_\varphi(\mathbb{R}^n)$ established by Ky, we recall the notion of atoms introduced by Ky [63].

Definition 7.13 Let φ be as in Definition 2.4.

(I) For each ball $B \subset \mathbb{R}^n$, the *space* $L^q_\varphi(B)$ with $q \in [1, \infty]$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ supported in B such that

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup_{t\in(0,\infty)}[\frac{1}{\varphi(B,t)} \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx]^{1/q} < \infty, & q \in [1,\infty), \\ \|f\|_{L^\infty(B)} < \infty, & q = \infty. \end{cases}$$

(II) A triplet (φ, q, s) is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{Z}_+$ satisfies $s \geq \lfloor n[\frac{q(\varphi)}{i(\varphi)} - 1]\rfloor$. A measurable function a on $\mathbb{R}^n$ is called a (φ, q, s)-*atom* if there exists a ball $B \subset \mathbb{R}^n$ such that
  (i) $\operatorname{supp} a \subset B$;
  (ii) $\|a\|_{L^q_\varphi(B)} \leq \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;
  (iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s$.

(III) The *atomic Musielak–Orlicz–Hardy space*, $H^{\varphi,q,s}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{b_j\}_j$ is a sequence of multiples of (φ, q, s)-atoms with $\operatorname{supp} b_j \subset B_j$ and $\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty$. Moreover, letting

$$\Lambda_q\big(\{b_j\}_j\big) := \inf\left\{\lambda \in (0,\infty) : \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \leq 1\right\},$$

the *quasi-norm* of $f \in H^{\varphi,q,s}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{\varphi,q,s}(\mathbb{R}^n)} := \inf\big\{\Lambda_q\big(\{b_j\}_j\big)\big\},$$

where the infimum is taken over all the decompositions of f as above.

To establish the boundedness of $\nabla L^{-1/2}$ from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$, we need the atomic characterization of the space $H_\varphi(\mathbb{R}^n)$ obtained by Ky [63].

Lemma 7.14 *Let φ be as in Definition 2.4 and (φ, q, s) admissible. Then $H_\varphi(\mathbb{R}^n) = H^{\varphi,q,s}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Now we prove that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$ by using Proposition 4.7 and Lemma 7.14.

Theorem 7.15 *Let φ be as in Definition 2.4 with $i(\varphi) \in (\frac{n}{n+1}, 1]$, L as in (7.1), $q(\varphi)$ and $r(\varphi)$ as in (2.12) and (2.13), respectively. Assume that $q(\varphi) < \frac{(n+1)i(\varphi)}{n}$ and $r(\varphi) > \frac{2}{2-q(\varphi)}$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$.*

Proof Let $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $M \in \mathbb{N}$ with $M > \frac{n}{2}[\frac{q(\varphi)}{i(\varphi)} - \frac{1}{2}]$. Then there exist $p_2 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2})$, φ is uniformly lower type p_2 and $\varphi \in \mathbb{A}_{q_0}(\mathbb{R}^n)$. Moreover, by Proposition 4.7, we know that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ, M)-atoms such that $f = \sum_j \lambda_j \alpha_j$ in $L^2(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|f\|_{H^M_{\varphi,\mathrm{at}}(\mathbb{R}^n)}$. Moreover, we know that (7.21) also holds true in this case.

Let α be a (φ, M)-atom with $\operatorname{supp}\alpha \subset B := B(x_B, r_B)$. For $k \in \mathbb{Z}_+$, let $\chi_k := \chi_{U_k(B)}$, $\widetilde{\chi}_k := |U_k(B)|^{-1}\chi_k$, $m_k := \int_{U_k(B)} \nabla L^{-1/2}(\alpha)(x)\,dx$ and $M_k := \nabla L^{-1/2}(\alpha)\chi_k - m_k\widetilde{\chi}_k$. Then we have

$$\nabla L^{-1/2}(\alpha) = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} m_k \widetilde{\chi}_k. \tag{7.30}$$

For $j \in \mathbb{Z}_+$, let $N_j := \sum_{k=j}^{\infty} m_k$. By an argument similar to that used in the proof of [57, Theorem 6.3], we know that $\int_{\mathbb{R}^n} \alpha(x)\,dx = 0$, which, together with (7.30), yields that

$$\nabla L^{-1/2}(\alpha) = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k). \tag{7.31}$$

Obviously, for all $k \in \mathbb{Z}_+$,

$$\operatorname{supp} M_k \subset 2^{k+1}B \quad \text{and} \quad \int_{\mathbb{R}^n} M_k(x)\,dx = 0. \tag{7.32}$$

When $k \in \{0, \dots, 4\}$, by Hölder's inequality and the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, we conclude that

$$\begin{aligned}\|M_k\|_{L^2(\mathbb{R}^n)} &\le \left\{\int_{U_k(B)} |\nabla L^{-1/2}\alpha(x)|^2\,dx\right\}^{1/2} + \left\{\int_{U_k(B)} |m_k\widetilde{\chi}_k(x)|^2\,dx\right\}^{1/2} \\ &\lesssim \|\alpha\|_{L^2(\mathbb{R}^n)} + |m_k||U_k(B)|^{-1/2} \\ &\lesssim \|\alpha\|_{L^2(\mathbb{R}^n)} \lesssim |B|^{1/2}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}.\end{aligned} \tag{7.33}$$

From the Davies–Gaffney estimates (2.5) and the H_∞-functional calculi for L, similar to the proof of [52, Theorem 3.4], it follows that there exists $K \in \mathbb{N}$ with $K > n/4$ such that, for all $t \in (0, \infty)$, closed sets E, F in $\mathbb{R}^n$ with $\operatorname{dist}(E, F) > 0$ and $g \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} g \subset E$,

$$\left\|\nabla L^{-1/2}\left(I - e^{-tL}\right)^K g\right\|_{L^2(F)} \lesssim \left(\frac{t}{[\operatorname{dist}(E, F)]^2}\right)^K \|g\|_{L^2(E)}$$

and

$$\left\|\nabla L^{-1/2}\left(tLe^{-tL}\right)^K g\right\|_{L^2(F)} \lesssim \left(\frac{t}{[\operatorname{dist}(E,F)]^2}\right)^K \|g\|_{L^2(E)}.$$

By this, we conclude that, when $k \in \mathbb{N}$ with $k \geq 5$,

$$\begin{aligned}\left\|\nabla L^{-1/2}\alpha\right\|_{L^2(U_k(B))} &\lesssim \left\|\nabla L^{-1/2}\left(I - e^{-r_B^2 L}\right)^M \alpha\right\|_{L^2(U_k(B))} \\ &\quad + \sum_{k=1}^{M}\left\|\nabla L^{-1/2}\left(r_B^2 L e^{-\frac{k}{M} r_B^2 L}\right)^M r_B^{-2M} b\right\|_{L^2(U_k(B))} \\ &\lesssim 2^{-2kM}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}, \end{aligned} \tag{7.34}$$

which, together with Hölder's inequality, implies that, when $k \in \mathbb{N}$ with $k \geq 5$,

$$\|M_k\|_{L^2(\mathbb{R}^n)} \lesssim \left\|\nabla L^{-1/2}\alpha\right\|_{L^2(U_k(B))} \lesssim 2^{-2kM}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{7.35}$$

Furthermore, by $q(\varphi) < 2$ and $r(\varphi) > 2/[2 - q(\varphi)]$, we know that there exists $q \in (q(\varphi), 2)$ such that $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ and $\mathbb{RH}_{2/(2-q)}(\mathbb{R}^n)$. From this, Hölder's inequality, (7.33) and (7.35), it follows that, for all $k \in \mathbb{Z}_+$ and $t \in (0, \infty)$,

$$\begin{aligned}&\left[\varphi\left(2^{k+1}B, t\right)\right]^{-1}\int_{2^{k+1}B}|M_k(x)|^q\varphi(x,t)\,dx \\ &\quad\leq \left[\varphi\left(2^{k+1}B, t\right)\right]^{-1}\left\{\int_{2^{k+1}B}|M_k(x)|^2\,dx\right\}^{\frac{q}{2}}\left\{\int_{2^{k+1}B}\left[\varphi(x,t)\right]^{\frac{2}{2-q}}\,dx\right\}^{\frac{2-q}{2}} \\ &\quad\lesssim 2^{-2qkM}|B|^{\frac{q}{2}}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-q}|2^{k+1}B|^{-\frac{q}{2}}, \end{aligned} \tag{7.36}$$

which implies that

$$\|M_k\|_{L^q_\varphi(2^{k+1}B)} \lesssim 2^{-(2M+\frac{n}{2})k}\|\chi_B\|_{L^\varphi(B)}^{-1}. \tag{7.37}$$

Then by (7.37) and (7.32), we conclude that, for each $k \in \mathbb{Z}_+$, M_k is a multiple of a $(\varphi, q, 0)$-atom. Moreover, from (7.35), it follows that $\sum_{k=0}^{\infty} M_k$ converges in $L^2(\mathbb{R}^n)$.

Now we estimate $\|N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)\|_{L^2(\mathbb{R}^n)}$ with $k \in \mathbb{Z}_+$. By Hölder's inequality and (7.34), we see that

$$\begin{aligned}\left\|N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)\right\|_{L^2(\mathbb{R}^n)} &\lesssim |N_{k+1}||2^kB|^{-\frac{1}{2}} \lesssim \sum_{j=k+1}^{\infty}|m_{j+1}||2^kB|^{-\frac{1}{2}} \\ &\lesssim \sum_{j=k+1}^{\infty}|2^kB|^{-\frac{1}{2}}|2^jB|^{\frac{1}{2}}\|\nabla L^{-1/2}\alpha\|_{L^2(U_j(B))} \\ &\lesssim 2^{-2kM}|B|^{\frac{1}{2}}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \end{aligned} \tag{7.38}$$

From this and Hölder's inequality, similar to the proof of (7.37), we deduce that, for all $k \in \mathbb{Z}_+$,

$$\left\| N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k) \right\|_{L^q_\varphi(2^{k+1}B)} \lesssim 2^{-(2M+\frac{n}{2})k} \|\chi_B\|^{-1}_{L^\varphi(B)}, \tag{7.39}$$

which, together with $\int_{\mathbb{R}^n} (\widetilde{\chi}_{k+1}(x) - \widetilde{\chi}_k(x))\,dx = 0$ and $\operatorname{supp}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k) \subset 2^{k+1}B$, implies that, for each $k \in \mathbb{Z}_+$, $N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ is a multiple of a $(\varphi, q, 0)$-atom. Moreover, by (7.38), we see that $\sum_{k=0}^\infty N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ converges in $L^2(\mathbb{R}^n)$.

Thus, (7.31) is an atomic decomposition of $\nabla L^{-1/2}\alpha$ and, further by (7.37), (7.39), the uniformly lower type p_2 property of φ and $M > \frac{n}{2}(\frac{q_0}{p_2} - \frac{1}{2})$, we know that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_+} \varphi\big(2^{k+1}B, \|M_k\|_{L^q_\varphi(2^{k+1}B)}\big) + \sum_{k\in\mathbb{Z}_+} \varphi\big(2^{k+1}B, \|N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)\|_{L^q_\varphi(2^{k+1}B)}\big) \\
&\quad \lesssim \sum_{k\in\mathbb{Z}_+} \varphi\big(2^{k+1}B, 2^{-(2M+\frac{n}{2})k}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\big) \\
&\quad \lesssim \sum_{k\in\mathbb{Z}_+} 2^{-(2M+\frac{n}{2})p_2} 2^{knq_0} \lesssim 1.
\end{aligned} \tag{7.40}$$

Replacing α by α_j, consequently, we then denote M_k, N_k and $\widetilde{\chi}_k$ in (7.31), respectively, by $M_{j,k}$, $N_{j,k}$ and $\widetilde{\chi}_{j,k}$. Similar to (7.31), we know that

$$\nabla L^{-1/2} f = \sum_j \sum_{k=0}^\infty \lambda_j M_{j,k} + \sum_j \sum_{k=0}^\infty \lambda_j N_{j,k+1}(\widetilde{\chi}_{j,k+1} - \widetilde{\chi}_{j,k}),$$

where, for each j and k, $M_{j,k}$ and $N_{j,k+1}(\widetilde{\chi}_{j,k+1} - \widetilde{\chi}_{j,k})$ are multiples of $(\varphi, q, 0)$-atoms and both summations hold true in $L^2(\mathbb{R}^n)$, and hence in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, from (7.40) with B, M_k, $N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ replaced by B_j, $M_{j,k}$, $N_{j,k+1}(\widetilde{\chi}_{j,k+1} - \widetilde{\chi}_{j,k})$, respectively, we deduce that

$$\Lambda_q\big(\{M_{j,k}\}_{j,k}\big) + \Lambda_q\big(\{N_{j,k+1}(\widetilde{\chi}_{j,k+1} - \widetilde{\chi}_{j,k})\}_{j,k}\big) \lesssim \Lambda\big(\{\lambda_j\alpha_j\}_j\big) \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}.$$

From this and Lemma 7.14, we deduce that $\|\nabla L^{-1/2} f\|_{H_\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}$, which, together with the fact that $H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_{\varphi,L}(\mathbb{R}^n)$ and a density argument, implies that $\nabla L^{-1/2}$ is bounded from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$. This finishes the proof of Theorem 7.15. □

Remark 7.16 (i) Theorem 7.15 completely covers [51, Theorem 8.6] by taking $\varphi(x,t) := t$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$.

(ii) Theorem 7.11 completely covers [57, Theorem 6.2] by taking φ as in (1.2) with $\omega \equiv 1$ and Φ concave, and Theorem 7.15 completely covers [57, Theorem 6.3] by taking φ as in (1.2) with $\omega \equiv 1$, Φ concave and $p_\Phi \in (\frac{n}{n+1}, 1]$, where p_Φ is as in (2.8).

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