J Geom Anal (2013) 23:2091–2157
DOI 10.1007/s12220-012-9322-4

Harmonic Analysis on Chord Arc Domains

Emmanouil Milakis · Jill Pipher · Tatiana Toro

Received: 31 December 2010 / Published online: 30 June 2012

Abstract In the present paper we study the solvability of the Dirichlet problem for second order divergence form elliptic operators with bounded measurable coefficients which are small perturbations of given operators in rough domains beyond the Lipschitz category. In our approach, the development of the theory of tent spaces on these domains is essential.

Keywords Chord arc domains · Elliptic measures · Perturbations of the Laplacian · Tent spaces

Mathematics Subject Classification 35B20 · 31B35 · 46E30 · 35J25

1 Introduction

In this paper, we establish fine properties of the elliptic measure associated with the solvability of the Dirichlet problem for certain small perturbations of elliptic operators in chord arc domains. The elliptic measure is that which arises naturally as

---

Communicated by Steven G. Krantz.

E. Milakis
Department of Mathematics & Statistics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus
e-mail: emilakis@ucy.ac.cy

J. Pipher (✉)
Mathematics Department, Brown University, P.O. Box 1917, Providence, RI 02912, USA
e-mail: jpipher@math.brown.edu

T. Toro
Department of Mathematics, University of Washington, P.O. Box 354350, Seattle, WA 98195-4350, USA
e-mail: toro@math.washington.edu

the representing measure associated with the solution of the Dirichlet problem for a second order elliptic operator with continuous boundary data. The "fine properties" of such measures are sharply described by the conditions defining the Muckenhoupt weight classes, in which these measures are compared to other natural measures, such as surface measure, which live on the boundary of the domain.

We will consider second order elliptic operators in divergence form, $L = \operatorname{div} A\nabla$, which are perturbations, in a sense to be made precise, of some given elliptic operators. The perturbation theory developed here for chord arc domains is the extension of that same theory for Lipschitz domains; see [3, 8, 9], and [10] for some prior literature.

A chord arc domain in $\mathbb{R}^n$ is a non-tangentially accessible (NTA) domain whose boundary is rectifiable and whose surface measure is Ahlfors regular (i.e., the surface measure on boundary balls of radius r grows like r^{n-1}). We refer the reader to the available literature, and specifically to [11] for the precise definition of NTA domains. NTA domains possess all of the following properties: (i) a quantified standard relationship between elliptic measure on the boundary of a domain and the Green's function for that domain, (ii) the doubling property of elliptic measure, and (iii) comparison principles for non-negative solutions to elliptic divergence form equations. These properties are consequences of the geometric definition of NTA domains and are stated precisely in the next section.

We briefly recall the Muckenhoupt weight classes (see [16] for a detailed discussion of these weight classes). If μ and ν are mutually absolutely continuous positive measures defined on the boundary of a domain, $\partial\Omega$, then there exists a weight function g such that $d\mu = g d\nu$. The measure $d\mu$ belongs to the weight class $B_q(d\nu)$ if there exists a constant $C > 0$ such that for all balls $B \subset \partial\Omega$, $(\nu(B)^{-1} \int_B g^q d\nu)^{1/q} \leq C\nu(B)^{-1} \int_B g d\nu$. The union of the B_q classes is the A_∞ class. By real variable methods, it is known that if elliptic measure and surface measure on a domain are related via a weight in the A_∞ class, then the Dirichlet problem with data in $L^p(d\sigma)$ is solvable *for some* $p < \infty$. There is a well-known relationship between the Muckenhoupt B_q weight classes, the existence of estimates for maximal functions and non-tangential maximal functions, and the solvability of Dirichlet problems for second order elliptic divergence form operators. We assume that the reader is familiar with these results in harmonic analysis/elliptic theory.

One specific and nontrivial result in this theory is Dahlberg's result of 1977: The harmonic measure ω on a Lipschitz domain is mutually absolutely continuous with respect to surface measure, σ, and the weight k relating the two measures ($d\omega = k d\sigma$) belongs to the $B_2(d\sigma)$ class. There is a further relationship between Muckenhoupt weights and the function space *BMO* of functions of bounded mean oscillation which then implies that $\log k \in BMO(d\sigma)$. On C^1 domains, Jerison and Kenig proved that $\log k \in VMO(d\sigma)$. *VMO* is the Sarason space of vanishing mean oscillation, a proper subspace of *BMO*, and arises as the predual of the Hardy space H^1. In [13], Kenig and Toro showed that $\log k$ belongs to $VMO(d\sigma)$ when the domain is merely chord arc (with a vanishing condition).

The theory of perturbation of elliptic operators on Lipschitz domains begins with a result of Dahlberg, [4], which measures the difference between coefficients of the matrices of two divergence form elliptic operators in a Carleson norm. Here is the setup for the general perturbation theory: If $L_0 = \operatorname{div} A_0 \nabla$ is elliptic in a domain Ω, then an elliptic operator $L_1 = \operatorname{div} A_1 \nabla$ is a perturbation of L_0 when the difference $\epsilon(X) = |A_1(X) - A_0(X)|$ is equal to zero when $X \in \partial\Omega$. How closely should these operators, L_0 and L_1, agree in the interior of the domain so that good properties of the elliptic measure associated with L_0 are preserved? The correct answer to this question is stated in terms of Carleson measures. The Carleson condition on $\epsilon(X)$ is a delicate measure of the rate at which $\epsilon(X)$ tends to zero as X approaches the boundary of Ω. In terms of such Carleson conditions, sharp results on perturbations were obtained in [10]. And in [8], Escauriaza showed that a (vanishing) Carleson condition on a perturbation of the Laplacian in C^1 domains preserved the Jerison–Kenig result, namely that $\log k \in VMO$. We will provide precise statements of some of these results in the next section.

Our aim is to extend the perturbation results of [10] to the setting of chord arc domains (CADs). Much of the technology of function spaces on the boundary which is available when the domain has Lipschitz boundary is not available in this setting. Therefore, a good portion of this paper is devoted to developing the theory of these function spaces for chord arc domains, especially the theory of tent spaces due to Coifman, Meyer, and Stein. These function spaces and their duals figure prominently in the theory of Hardy spaces and *BMO* spaces—the connection between them is established via Carleson measures. The development of the theory of tent spaces on chord arc domains is a purely geometric and independent aspect of this paper.

In [14], it was shown that a (vanishing) Carleson measure condition on perturbations of the Laplacian on CADs with vanishing constant preserves A_∞. In the last section of the paper we show that this result holds for perturbations from arbitrary elliptic divergence form operators on general CADs.

2 Preliminaries

In this section, we recall some definitions and give the necessary background on properties of solutions to elliptic equations in divergence form. We will also introduce some notation which will be used throughout the paper.

Definition 2.1 Let $\Omega \subset \mathbb{R}^n$. We say that Ω is a chord arc domain (CAD) if Ω is an NTA set of locally finite perimeter whose boundary is Ahlfors regular, i.e., the surface measure to the boundary satisfies the following condition: There exists $C \geq 1$ so that for $r \in (0, \operatorname{diam} \Omega)$ and $Q \in \partial\Omega$

$$C^{-1} r^{n-1} \leq \sigma\big(B(Q,r)\big) \leq C r^{n-1}. \tag{2.1}$$

Here $B(Q,r)$ denotes the n-dimensional ball of radius r and center Q, $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The best constant C above is referred to as the Ahlfors regularity constant.

Definition 2.2 Let $\Omega \subset \mathbb{R}^n$, $\delta > 0$, and $R > 0$. If D denotes Hausdorff measure and $\mathcal{L}(Q)$ denotes an $(n-1)$-plane containing a point $Q \in \Omega$, set

$$\theta(r) = \sup_{Q \in \partial\Omega} \inf_{\mathcal{L}(Q)} r^{-1} D\big[\partial\Omega \cap B(Q,r), \mathcal{L} \cap B(Q,r)\big] \tag{2.2}$$

We say that Ω is a (δ, R)-chord arc domain (CAD) if Ω is a set of locally finite perimeter such that

$$\sup_{0<r\le R} \theta(r) \le \delta \tag{2.3}$$

and

$$\sigma\big(B(Q,r)\big) \le (1+\delta)\omega_{n-1}r^{n-1} \quad \forall Q \in \partial\Omega \text{ and } \forall r \in (0, R]. \tag{2.4}$$

Here ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball in $\mathbb{R}^{n-1}$.

Definition 2.3 Let $\Omega \subset \mathbb{R}^n$. We say that Ω is a chord arc domain with vanishing constant if it is a (δ, R)-CAD for some $\delta > 0$ and $R > 0$,

$$\limsup_{r\to 0} \theta(r) = 0, \tag{2.5}$$

and

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} \frac{\sigma(B(Q,r))}{\omega_n r^{n-1}} = 1. \tag{2.6}$$

For the purpose of this paper, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. We consider elliptic operators L of the form

$$Lu = \operatorname{div}\big(A(X)\nabla u\big) \tag{2.7}$$

defined in the domain Ω with symmetric coefficient matrix $A(X) = (a_{ij}(X))$ and such that there are $\lambda, \Lambda > 0$ satisfying

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{2.8}$$

for all $X \in \Omega$ and $\xi \in \mathbb{R}^n$.

We say that a function u in Ω is a solution to $Lu = 0$ in Ω provided that $u \in W^{1,2}_{\text{loc}}(\Omega)$ and for all $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \big\langle A(x)\nabla u, \nabla\phi\big\rangle dx = 0.$$

A domain Ω is called regular for the operator L, if for every $g \in C(\partial\Omega)$, the generalized solution of the classical Dirichlet problem with boundary data g is a function $u \in C(\overline{\Omega})$. Let Ω be a regular domain for L as above and $g \in C(\partial\Omega)$. The Riesz Representation Theorem ensures that there exists a family of regular Borel probability measures $\{\omega_L^X\}_{X\in\Omega}$ such that

$$u(X) = \int_{\partial\Omega} g(Q)\,d\omega_L^X(Q).$$

For $X \in \Omega$, ω_L^X is called the L-elliptic measure of Ω with pole X. When no confusion arises, we will omit the reference to L and simply call it the elliptic measure.

To state our results, we introduce the notion of perturbation of an operator. Consider two elliptic operators $L_i = \operatorname{div}(A_i \nabla)$ for $i = 0, 1$ defined on a chord arc domain $\Omega \subset \mathbb{R}^n$. We say that L_1 is a perturbation of L_0 if the deviation function

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\} \tag{2.9}$$

where $\delta(X)$ is the distance of X to $\partial\Omega$, satisfies the following Carleson measure property: There exists a constant $C > 0$ such that

$$\sup_{0<r<\operatorname{diam}\Omega} \sup_{Q\in\partial\Omega} \left\{ \frac{1}{\sigma(B(Q,r))} \int_{B(Q,r)\cap\Omega} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \le C. \tag{2.10}$$

Note that in this case $L_1 = L_0$ on $\partial\Omega$. For $i = 0, 1$ we denote by $G_i(X, Y)$ the Green's function of L_i in Ω with pole at X and by ω_i^X the corresponding elliptic measure.

We now recall some of the results concerning the regularity of the elliptic measure of perturbation operators in Lipschitz domains. The results in the literature are more general than those quoted below.

Theorem 2.4 [4] *Let $\Omega = B(0, 1)$. If $L_0 = \Delta$, $a(X)$ is as in (2.9),*

$$h(Q, r) = \left\{ \frac{1}{\sigma(B(Q,r))} \int_{B(Q,r)\cap\Omega} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2}, \tag{2.11}$$

and

$$\lim_{r\to 0} \sup_{|Q|=1} h(Q, r) = 0,$$

then the elliptic kernel of L_1, $k = d\omega_{L_1}/d\sigma \in B_q(d\sigma)$ for all $q > 1$.

In [9], R. Fefferman investigated an alternative to the smallness condition on $h(Q, r)$ above, and considered a pointwise requirement on the quantity $A(a)(Q)$.

Theorem 2.5 [9] *Let $\Omega = B(0, 1)$ and $L_0 = \Delta$. Let $\Gamma(Q)$ denote a non-tangential cone with vertex $Q \in \partial\Omega$ and*

$$A(a)(Q) = \left(\int_{\Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right)^{1/2},$$

where $a(X)$ is as in (2.9). If $\|A(a)\|_{L^\infty} \le C$ then $\omega \in A_\infty(d\sigma)$.

The main results in [4] and in [9] are proved using Dahlberg's idea of introducing a differential inequality for a family of elliptic measures. In [10], Fefferman, Kenig, and Pipher presented a new direct proof of these results, and we will show here that this proof extends beyond the class of Lipschitz domains. This requires a careful reworking of many of the technical steps in the [10] proof, and the development of the required new analytic tools for CADs.

Theorem 2.6 [10] *Let Ω be a Lipschitz domain. Let L_1 be such that* (2.10) *holds. Then $\omega_1 \in A_\infty(d\sigma)$ whenever $\omega_0 \in A_\infty(d\sigma)$.*

Theorem 2.7 [10] *Let Ω be a Lipschitz domain. Let ω_0, ω_1 denote the L_0-elliptic measure and the L_1-elliptic measure, respectively, in Ω with pole $0 \in \Omega$. There exists an $\varepsilon_0 > 0$, depending on the ellipticity constants and the dimension, such that if*

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \varepsilon_0,$$

then $\omega_1 \in B_2(\omega_0)$. Here $T(\Delta) = \overline{B}(Q,r) \cap \Omega$ is the Carleson region associated with the surface ball $\Delta(Q,r) = B(Q,r) \cap \partial\Omega$, and $G_0(X) = G_0(0, X)$ denotes the Green's function for L_0 in Ω with pole at $0 \in \Omega$.

In the recent paper [14], Theorem 2.6 was generalized to chord arc domains with small constant in the case $L_0 = \Delta$. More precisely,

Theorem 2.8 [14] *Let Ω be a chord arc domain. Let $L_0 = \Delta$ and L_1 be such that* (2.10) *holds. There exists $\delta(n) > 0$ such that if $\Omega \subset \mathbb{R}^n$ is a (δ, R)-CAD with $0 < \delta \leq \delta(n)$, then $\omega_1 \in A_\infty(d\sigma)$.*

The purpose of the present paper is to extend the result above to perturbation operators on "rough domains". In particular, we will show the following result.

Theorem 2.9 *Let Ω be a chord arc domain. There exists an $\varepsilon_0 > 0$, depending also on the ellipticity constants and the dimension, such that if*

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \varepsilon_0, \tag{2.12}$$

then $\omega_1 \in B_2(\omega_0)$.

The various constants that will appear in the sequel may vary from formula to formula, although for simplicity we use the same letter(s). If we do not give any explicit dependence for a constant, we mean that it depends only on the usual parameters such as ellipticity constants, NTA constants, the Ahlfors regularity constant, the dimension, and the NTA character of the domain. Moreover, throughout the paper we shall use the notation $a \lesssim b$ to mean that there is a constant $c > 0$ such that $a \leq cb$. Similarly, $a \simeq b$ means that $a \lesssim b$ and $b \lesssim a$.

Next we recall the main theorems we will use about the boundary behavior of L-elliptic functions in non-tangentially accessible (NTA) domains for uniformly elliptic divergence form operators L with bounded measurable coefficients. We refer the reader to [12] for the definitions and more details regarding elliptic operators of divergence form defined in NTA domains.

Lemma 2.10 *Let Ω be an NTA domain, $Q \in \partial\Omega$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, 2r)$. Then*

$$C^{-1} < \frac{\omega^X(B(Q,r))}{r^{n-2}G(A(Q,r),X)} < C,$$

where $G(A(Q,r),X)$ is the L-Green function of Ω with pole X, and ω^X is the corresponding elliptic measure.

Lemma 2.11 *Let Ω be an NTA domain with constants $M > 1$ and $R > 0$, $Q \in \partial\Omega$, $0 < 2r < R$, and $X \in \Omega \backslash B(Q, 2Mr)$. Then for $s \in [0, r]$,*

$$\omega^X\big(B(Q,2s)\big) \le C\omega^X\big(B(Q,s)\big),$$

where C only depends on the NTA constants of Ω.

Lemma 2.12 *Let Ω be an NTA domain, and $0 < Mr < R$. Suppose that u, v vanish continuously on $\partial\Omega \cap B(Q, Mr)$ for some $Q \in \partial\Omega$, $u, v \ge 0$, and $Lu = Lv = 0$ in Ω. Then there exists a constant $C > 1$ (depending only on the NTA constants) such that for all $X \in B(Q,r) \cap \Omega$,*

$$C^{-1}\frac{u(A(Q,r))}{v(A(Q,r))} \le \frac{u(X)}{v(X)} \le C\frac{u(A(Q,r))}{v(A(Q,r))}.$$

Lemma 2.13 *Let $\mu \in A_\infty(d\omega)$, $0 \in \Omega$, and set*

$$S_\alpha(u)(Q) = \left(\int_{\Gamma_\alpha(Q)} \big|\nabla u(X)\big|^2 \delta(X)^{2-n}\,dX\right)^{1/2} \quad \text{and}$$

$$N_\alpha(u)(Q) = \sup\big\{\big|u(X)\big| : X \in \Gamma_\alpha(Q)\big\}$$

where for $Q \in \partial\Omega$

$$\Gamma_\alpha(Q) = \big\{X \in \Omega : |X - Q| < (1+\alpha)\delta(X)\big\}. \tag{2.13}$$

Then if $Lu = 0$ and $0 < p < \infty$,

$$\left(\int_{\partial\Omega} \big(S_\alpha(u)\big)^p d\mu\right)^{1/p} \le C_{\alpha,p}\left(\int_{\partial\Omega} \big(N_\alpha(u)\big)^p d\mu\right)^{1/p}.$$

If in addition $u(0) = 0$, then

$$\left(\int_{\partial\Omega} \big(N_\alpha(u)\big)^p d\mu\right)^{1/p} \le C_{\alpha,p}\left(\int_{\partial\Omega} \big(S_\alpha(u)\big)^p d\mu\right)^{1/p}.$$

3 Non-Tangential Behavior in CADs

In this section, we study the space of functions defined on chord arc domains whose non-tangential maximal function is well behaved. Our goal is to extend the theory of tent spaces developed by Coifman, Meyer, and Stein [2] in the upper half-space to chord arc domains. It will play a crucial role in the proof of Theorem 2.9.

We first study the notion of global γ-density with respect to a set $\mathcal{F} \subset \partial\Omega$ for a doubling measure μ supported on $\partial\Omega$. In this paper, μ will either be the elliptic measure of a divergence form operator defined on Ω or the surface measure to $\partial\Omega$. Please note that in contrast with the classical case we do not restrict the definition to the case where $\mathcal{F}$ is closed.

Definition 3.1 Let $\mathcal{F} \subset \partial\Omega$, and let $\gamma \in (0,1)$. A point $Q \in \partial\Omega$ has global γ-density with respect to $\mathcal{F}$ for a doubling measure μ if for $\rho \in (0, \operatorname{diam}\Omega)$

$$\frac{\mu(B(Q,\rho)\cap\mathcal{F})}{\mu(B(Q,\rho)} \geq \gamma. \tag{3.1}$$

Let $\mathcal{F}_\gamma^*$ be the set of points of global γ-density of $\mathcal{F}$.

Lemma 3.2 *Let Ω be a CAD with surface measure σ and let $\Delta(P,s) = B(P,s)\cap \partial\Omega$ be the surface ball centered at P. Given $\alpha > 0$, there exist $\gamma \in (0,1)$ close to 1 and $\lambda_0 > 0$ such that if $\mathcal{F} \subset \partial\Omega$ and $P \in \mathcal{F}_\gamma^*$, then for $P \in \Delta(P,(2+\alpha)r)$*

$$\sigma\big(\Delta(Q,\beta r)\cap\mathcal{F}\big) \geq \lambda_0(\beta r)^{n-1}, \tag{3.2}$$

where $\beta = \min\{1,\alpha\}$.

Proof Assume $\sigma(\Delta(Q,\beta r)\cap\mathcal{F}) < \lambda_0(\beta r)^{n-1}$. Since $\Delta(Q,\beta r) \subset \Delta(P,(2+\alpha+\beta)r) \subset \Delta(P,(3+2\alpha)r)$ and $P \in \mathcal{F}_\gamma^*$, then

$$\begin{aligned}
&\gamma\sigma\big(\Delta\big(P,(3+2\alpha)r\big)\big)\\
&\quad\leq \sigma\big(\Delta\big(P,(3+2\alpha)r\big)\cap\mathcal{F}\big)\\
&\quad\leq \sigma\big(\Delta\big(P,(3+2\alpha)r\big)\backslash\Delta(Q,\beta r)\big) + \sigma\big(\Delta(Q,\beta r)\big)\\
&\quad\leq \sigma(\Delta\big(P,(3+2\alpha)r\big)\bigg[1-\frac{1}{C^2}\cdot\bigg(\frac{\beta}{3+2\alpha}\bigg)^{n-1} + C^2\lambda_0\bigg(\frac{\beta}{3+2\alpha}\bigg)^{n-1}\bigg],
\end{aligned} \tag{3.3}$$

where C denotes the Ahlfors regularity constant. For $\lambda_0 = 1/2C^4$, (3.3) implies that $\gamma \leq 1 - \frac{1}{2C^2}\cdot(\frac{\beta}{3+2\alpha})^{n-1} < 1$, which is a contradiction if γ is close enough to 1. □

Please note that so far we have not assumed that the set $\mathcal{F}$ is closed. The following proposition requires the set $\mathcal{F}$ to be closed. It holds for a general doubling measure supported on $\partial\Omega$, but will only be applied to either surface measure or elliptic measure.

Proposition 3.3 *Let Ω be a CAD, and let μ be a doubling measure on $\partial\Omega$. Let $\mathcal{F} \subset \partial\Omega$ be a closed set. Then $\mathcal{F}_\gamma^* \subset \mathcal{F}$ and*

$$\mu\big(\big(\mathcal{F}_\gamma^*\big)^c\big) \leq C\mu\big(\mathcal{F}^c\big). \tag{3.4}$$

Here the constant C depends on γ and on the doubling constant of μ.

Proof Since $\mathcal{F}$ is closed, it is clear that $\mathcal{F}_\gamma^* \subset \mathcal{F}$. Let $\mathcal{O} = \mathcal{F}^c$ and $\mathcal{O}^* = (\mathcal{F}_\gamma^*)^c$. If $Q \in \mathcal{O}^*$, by definition there exists a radius $\varrho_Q > 0$ such that

$$\frac{\mu(\Delta(Q, \varrho_Q) \cap \mathcal{F})}{\mu(\Delta(Q, \varrho_Q))} < \gamma.$$

By Besicovitch (see [7]),

$$\mathcal{O}^* \subset \bigcup_{i=1}^{N_n} \bigcup_j \Delta(Q_j^i, \varrho_j^i)$$

where $\Delta(Q_j^i, \varrho_j^i) \cap \Delta(Q_l^i, \varrho_l^i) = \emptyset$ for $j \neq l$. Therefore,

$$\mu(\mathcal{O}^*) \leq \sum_{i=1}^{N_n} \sum_j \mu(\Delta(Q_j^i, \varrho_j^i))$$

and

$$\begin{aligned}\sum_{i=1}^{N_n} \sum_j \mu(\Delta(Q_j^i, \varrho_j^i)) &= \sum_{i=1}^{N_n} \sum_j \mu(\Delta(Q_j^i, \varrho_j^i) \cap \mathcal{F}) + \mu(\Delta(Q_j^i, \varrho_j^i) \cap \mathcal{O}) \\ &\leq \sum_{i=1}^{N_n} \sum_j \gamma\mu(\Delta(Q_j^i, \varrho_j^i)) + \mu(\Delta(Q_j^i, \varrho_j^i) \cap \mathcal{O}).\end{aligned}$$

Hence,

$$\mu(\mathcal{O}^*) \leq C \sum_{i=1}^{N_n} \sum_j \mu(\Delta(Q_j^i, \varrho_j^i) \cap \mathcal{O}) \leq C\mu(\mathcal{O}). \tag{3.5}$$

□

Definition 3.4 Let Ω be a CAD. We denote by $\mathcal{N}$ a linear space of Borel measurable functions F such that

$$\mathcal{N} = \left\{F : \Omega \to \mathbb{R} \text{ such that } N(F) \in L^1(d\sigma)\right\}$$

where $N(F)(Q) = \sup\{|F(X)| : X \in \Gamma(Q)\}$ and $\Gamma(Q) = \Gamma_1(Q)$ as defined in (2.13).

Remark 3.5 The set $\mathcal{N}$ with the norm given by $\|F\|_{\mathcal{N}} = \|N(F)\|_{L^1(\partial\Omega)}$ is a Banach space.

The following proposition shows that the definition of the space $\mathcal{N}$ above does not depend on the aperture of the cone used.

Proposition 3.6 *Let Ω be a CAD. Let μ be a doubling measure supported on $\partial\Omega$. For $Q \in \partial\Omega$ let*

$$N_\alpha F(Q) = \sup_{X \in \Gamma_\alpha(Q)} \left|F(X)\right|,$$

where $\Gamma_\alpha(Q)$ is as defined in (2.13). Then given $\alpha,\ \beta > 0$ there exists a constant C depending on α, β and the doubling constant of μ such that for all $\lambda > 0$

$$\mu\big(\big\{X \in \partial\Omega : N_\alpha F(X) > \lambda\big\}\big) \leq C\mu\big(\big\{X \in \partial\Omega : N_\beta F(X) > \lambda\big\}\big). \tag{3.6}$$

Hence for $1 \leq p < \infty$,

$$\int |N_\alpha F|^p d\mu \leq C \int |N_\beta F|^p d\mu. \tag{3.7}$$

Proof If $\alpha \leq \beta$, the inequality (3.6) is automatic. Thus, we may assume that $\alpha > \beta$. To prove (3.6), we would like to apply Proposition 3.3. We claim that for $\gamma \in (0, 1)$ close enough to 1, the set $\{X \in \partial\Omega : N_\alpha F(X) > \lambda\}$ is contained in the complement of the set of points of global γ-density with respect to $\{X \in \partial\Omega : N_\beta F(X) > \lambda\}^c$. It is straightforward to show that the set $\{X \in \partial\Omega : N_\beta F(X) > \lambda\}$ is open, which ensures that Proposition 3.3 applied to $\mathcal{F} = \{X \in \partial\Omega : N_\beta F(X) > \lambda\}^c$ combined with the previous claim yields (3.6). To prove the claim, assume that $N_\alpha F(Q) > \lambda$ for $Q \in \partial\Omega$ there exists $Y \in \Gamma_\alpha(Q)$ such that $F(Y) \geq \lambda$ and $|Q - Y| < (1 + \alpha)\delta(Y)$. Now let $Q_Y \in \partial\Omega$ such that $|Y - Q_Y| = \delta(Y)$; then $\Delta(Q_Y, \beta\delta(Y)) \subset \{P \in \partial\Omega : N_\beta F(P) > \lambda\} \cap \Delta(Q, (\alpha + \beta + 2)\delta(Y))$. In fact, if $P \in \Delta(Q_Y, \beta\delta(Y))$, then $|P - Y| \leq |P - Q_Y| + |Q_Y - Y| < (1 + \beta)\delta(Y)$ and $F(Y) > \lambda$. Therefore, since μ is doubling,

$$\begin{aligned}
&\frac{\mu(\{P : N_\beta F(P) > \lambda\} \cap \Delta(Q, (\alpha + \beta + 2)\delta(Y)))}{\mu(\Delta(Q, (\alpha + \beta + 2)\delta(Y)))} \\
&\geq \frac{\mu(\Delta(Q_Y, \beta\delta(Y)))}{\mu(\Delta(Q, (\alpha + \beta + 2)\delta(Y)))} \\
&\geq \frac{\mu(\Delta(Q_Y, (2 + \alpha + \beta)\delta(Y)))}{\mu(\Delta(Q, (\alpha + \beta + 2)\delta(Y)))} \\
&\geq \frac{\mu(\Delta(Q, \beta\delta(Y)))}{\mu(\Delta(Q, (\alpha + \beta + 2)\delta(Y)))} \\
&\geq C_0,
\end{aligned} \tag{3.8}$$

where C_0 depends on α, β and the doubling constant of μ. Note that (3.8) shows that for $Q \in \partial\Omega$ such that $N_\alpha F(Q) > \lambda$, Q is not a global γ-density point with respect to $\{P \in \partial\Omega : N_\beta F(P) > \lambda\}$ whenever $\gamma \in (1 - C_0/2, 1)$, which proves our claim. □

One of the goals of this section is to study the dual of the space $\mathcal{N}$. To achieve this, we still need to understand better the geometry of Ω and the structure of its boundary. To this effect, we first prove a Whitney decomposition type lemma for an open subset of $\partial\Omega$. Then we define the "tent" over an open subset of $\partial\Omega$. Finally, we define Carleson measures on Ω.

Lemma 3.7 *Let $F \subset \partial\Omega$ be a closed nonempty set on $\partial\Omega$. There exist a family of balls $\{B_k\}$ with $B_k = B(X_k, r_k)$, $X_k \in \partial\Omega$ and constants $1 < c^* < c^{**}$ such that if $B_k^* = c^* B_k = B(X_k, c^* r_k)$, $B_k^{**} = c^{**} B_k$, then*

- $B_k \cap B_j = \emptyset$, *for* $k \neq j$
- $\bigcup_k B_k^* = F^c \cap \partial\Omega = O$
- $B_k^{**} \cap F \neq \emptyset$.

In addition, if we define

$$Q_k = B_k^* \cap \left(\bigcup_{j<k} Q_j \right)^c \cap \left(\bigcup_{j>k} B_j \right)^c,$$

then $B_k \subset Q_k \subset B_k^*$, the Q_k's are disjoint, and $\bigcup_{k=1}^{\infty} Q_k = O$.

Proof Consider $0 < \varepsilon < 1/6$ and let $d(X) = \sup\{d : B(X, d) \cap \partial\Omega \subset O\}$ for $X \in \partial\Omega$. Let us choose a maximal disjoint subcollection of $\{B(X, \varepsilon d(X))\}_{X \in O}$. For this countable subcollection $\{B_k\}_{k=1}^{\infty}$, where $B_k := B(X_k, \varepsilon d(X_k))$ and $X_k \in \partial\Omega$, we consider $B_k^* = B(X_k, \frac{d(X_k)}{2})$ and $B_k^{**} = B(X_k, 2d(X_k))$. Clearly (i) and (iii) hold; moreover, $B_k^* \subset O$. To show that

$$O \subset \bigcup_{k \geq 1} B_k^*,$$

we take $Y \in O$. By the selection of $\{B_k\}$, there exists k such that

$$B\big(Y, \varepsilon d(Y)\big) \cap B\big(X_k, \varepsilon d(X_k)\big) \neq \emptyset. \tag{3.9}$$

Therefore, $|Y - X_k| < \varepsilon d(Y) + \varepsilon d(X_k)$. Moreover, $d(Y) \leq |X_k - Y| + d(X_k)$, which implies $d(Y) \leq \frac{1+\varepsilon}{1-\varepsilon} d(X_k)$, and as a consequence $|Y - X_k| < 3\varepsilon d(X_k) < \frac{d(X_k)}{2}$, since $\varepsilon < 1/6$.

By construction, $B_1 \subset Q_1 \subset B_1^*$. Assume that for $k \geq 2$ and $j \leq k-1$, $B_j \subset Q_j \subset B_j^*$ and note that $B_k \subset (\bigcup_{j>k} B_j)^c \cup B_k^*$. By definition and using the hypothesis of induction, we have for $j < k$

$$Q_j^c = \big(B_j^*\big)^c \cup \left(\bigcup_{i<j} Q_j \right) \cup \left(\bigcup_{i>j} B_i \right) \supset \left(\bigcup_{i \neq j} B_i \right) \supset B_k$$

and $\bigcap_{j<k} Q_j^c \supset \bigcup_{i \geq k} B_i \supset B_k$, which ensures that $B_k \subset Q_k \subset B_k^*$.

It is clear that $\bigcup_{k=1}^{\infty} Q_k \subset= \bigcup_{k=1}^{\infty} B_k^* = O$. For $X \in O$ consider two cases. Either there exists j such that $X \in B(X_j, \varepsilon d(X_j)) =: B_j \subset B_j^*$, $X \notin Q_i$ for $i < j$, $X \notin B_i$ for $i \neq j$ and therefore $X \in Q_j$. Or for all j, $X \notin B_j$. In this case, there exists a k_0 such that $X \in B_{k_0}^*$ but $X \notin B_k^*$, for $k < k_0$. Hence $X \notin Q_k$ with $k < k_0$, which implies $X \in Q_{k_0}^*$ and $O \subset \bigcup_k Q_k$. $\square$

To define the notion of "tent" over an open subset of $\partial\Omega$, we first look at "fans" of cones over subsets on $\partial\Omega$. Let $\mathcal{F} \subset \partial\Omega$. For $\alpha > 0$, define

$$R_\alpha(\mathcal{F}) = \bigcup_{Q\in\mathcal{F}} \Gamma_\alpha(Q), \tag{3.10}$$

where $\Gamma_\alpha(Q)$ is as defined in (2.13). We denote $R_1(\mathcal{F})$ by $R(\mathcal{F})$. Given an open set $O \subset \partial\Omega$, the tent over O is defined as

$$T(O) = \Omega \setminus R(\mathcal{F}). \tag{3.11}$$

Lemma 3.8 *Let O be the open set defined by*

$$O = \{Q \in \partial\Omega : N(F)(Q) > \alpha\}.$$

Then

$$T(O) \subseteq \bigcup_{P\in O} T(\Delta(P, \operatorname{dist}(P, O^c))).$$

Proof Recall that $T(\Delta(Q,r)) = \overline{B}(Q,r) \cap \Omega$. Let $Y \in T(O)$. Then $Y \notin R(\mathcal{F})$, hence $NF(Q_Y) > \alpha$, where by $Q_Y \in \partial\Omega$ we denote the boundary point satisfying $|Y - Q_Y| = \delta(Y)$. Now if $P \in \Delta(Q_Y, \delta(Y))$, then $|P - Y| < 2\delta(Y)$, $Y \in \Gamma(P)$, and since $Y \notin R(\mathcal{F})$, then $P \notin \mathcal{F}$, thus $P \in O$, i.e., $\Delta(Q_Y, \delta(Y)) \subset O$, which implies $\delta(Y) \le \operatorname{dist}(Q_Y, O^c)$. □

Definition 3.9 Let Ω be a CAD. For a Borel measure μ on $\overline{\Omega}$, we define for $Q \in \partial\Omega$,

$$C(\mu)(Q) = \sup_{Q\in\Delta} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d\mu$$

and denote by

$$\mathcal{C} = \left\{\mu \text{ is Borel in } \overline{\Omega} : \|\mu\|_{\mathcal{C}} = \sup_{Q\in\partial\Omega} C(\mu)(Q) < \infty\right\}.$$

$\mathcal{C}$ is the collection of Carleson measures on Ω.

Lemma 3.10 *Assume that μ is a positive measure on $\overline{\Omega}$ such that $\|\mu\|_{\mathcal{C}} \le 1$, i.e., $\mu(T(B)) \le \sigma(B)$ for all balls B. Then, for every open set $O \subset \partial\Omega$*

$$\mu(T(O)) \le C\sigma(O).$$

Proof As in the classical setting, we appeal to the Whitney decomposition, Lemma 3.7. If the B_k's are as in Lemma 3.7, then we claim that

$$T(O) \subset \bigcup_k T(B_k^{**}). \tag{3.12}$$

Thus

$$\mu\big(T(O)\big) \le \sum_k \mu\big(T\big(B_k^{**}\big)\big) \le C \sum_k \sigma\big(B_k^{**}\big) \le C \sum_k \sigma(B_k)$$
$$\lesssim \sum_k \sigma(Q_k) \lesssim \sigma(O)$$

where we have used the fact that σ is Ahlfors regular (in fact, the doubling properties of σ are enough). To prove (3.12), consider a point $Z \in T(O) \subseteq \bigcup_{P\in O} T(\Delta(P, \operatorname{dist}(P, O^c)))$, that is, $Z \in T(\Delta(P, \operatorname{dist}(P, O^c)))$ for some $P \in O$, and there exists a k such that $P \in B_k^*$ and $|P - X_k| < \frac{d_k}{2}$. Now if $Y \in O^c$ is such that $d_k = |Y - X_k|$ $|Y - P| \le |Y - X_k| + |X_k - P| < \frac{3d_k}{2}$, $\operatorname{dist}(P, O^c) < \frac{3d_k}{2}$ and $|Z - X_k| \le |Z - P| + |P - X_k| < \operatorname{dist}(P, O^c) + \frac{d_k}{2} < 2d_k$, that is, $Z \in T(\Delta(X_k, 2d_k)) = T(B_k^{**})$. □

We are now ready to study the relationship between the spaces $\mathcal{N}$ and $\mathcal{C}$. First we prove the analogue of Proposition 3 in [2].

Proposition 3.11 *Let Ω be a CAD. If $F \in \mathcal{N}$ and $\mu \in \mathcal{C}$, then*

$$\left|\int_\Omega F(X)d\mu(X)\right| \lesssim \int_{\partial\Omega} NF(Q)C(\mu)(Q)d\sigma. \tag{3.13}$$

Proof Assume that $F \ge 0$ and consider the open set $O = \{P \in \partial\Omega : NF(P) > \alpha\}$. Using the notation in Lemma 3.7 and the fact that σ is Ahlfors regular, we have for $X \in Q_k$

$$\mu\big(T\big(B_k^{**}\big)\big) \le C(\mu)(X)\sigma\big(B_k^{**}\big) \le C(\mu)(X)\sigma(B_k) \le \int_{Q_k} C(\mu)(X)d\sigma.$$

By Lemma 3.10 and the fact that the Q_k's are disjoint, we have that

$$\mu\big(\big\{\big|F(X)\big| > \alpha\big\}\big) \le \sum_k \mu\big(T\big(B_k^{**}\big)\big) \le C \sum_k \int_{Q_k} C(\mu)(X)d\sigma$$
$$\le C \int_{\{NF(X)>\alpha\}} C(\mu)(X)d\sigma.$$

Integrating over α and using Fubini yields (3.13). □

Corollary 3.12 *Let $\mu \in \mathcal{C}$. Let F be a function defined on Ω such that $NF \in L^p(d\sigma)$, for some $p \in (0, \infty)$ fixed. Then,*

$$\int_\Omega \big|F(X)\big|^p d\mu \le C \int_{\partial\Omega} \big|NF(Q)\big|^p C(\mu)(Q)d\sigma. \tag{3.14}$$

Proof Inequality (3.14) follows from Proposition 3.11 if we replace $|F(X)|$ by $|F(X)|^p$. □

We now present a couple of integration lemmas. They provide control of boundary integrals in terms of solid integrals on CAD, via Fubini. In what follows, the function $A(X)$ is a non-negative measurable function in Ω. In Sect. 4, we will take $A(X)$ to be the square function.

Lemma 3.13 *Let Ω be a CAD. Given $\alpha > 0$, if $\mathcal{F} \subset \partial\Omega$ and A is a non-negative measurable function in Ω, then*

$$\int_{\mathcal{F}}\left(\int_{\Gamma_\alpha(Q)} A(X)dX\right)d\sigma(Q) \leq C_\alpha \int_{R_\alpha(\mathcal{F})} A(X)\delta(X)^{n-1}dX, \tag{3.15}$$

where $R_\alpha(\mathcal{F})$ is given by (3.10).

Proof By Fubini's theorem,

$$\begin{aligned}\int_{\mathcal{F}}\left(\int_{\Gamma_\alpha(Q)} A(X)dX\right)d\sigma(Q) &= \int_{\mathcal{F}}\int_{\Omega} A(X)\chi_{\Gamma_\alpha(Q)}(X)dXd\sigma(Q)\\ &= \int_{\Omega}\int_{\mathcal{F}} A(X)\chi_{\Gamma_\alpha(Q)}(X)d\sigma(Q)dX. \end{aligned}\tag{3.16}$$

If $\chi_{\Gamma_\alpha(Q)}(X) = 1$, then $|X - Q| < (1 + \alpha)\delta(X)$, and if $Q_X \in \partial\Omega$ is such that $|X - Q_X| = \delta(X)$, then $|Q_X - Q| \leq |X - Q_X| + |X - Q| < (2+\alpha)\delta(X)$, and

$$\int_{\mathcal{F}} \chi_{\Gamma_\alpha(Q)}(X)d\sigma(Q) \leq \sigma\big(\Delta\big(Q_X, (2+\alpha)\delta(X)\big)\big) \leq C_\alpha\delta(X)^{n-1}. \tag{3.17}$$

Combining (3.17) and (3.16), we obtain

$$\begin{aligned}&\int_{\Omega}\int_{\mathcal{F}} A(X)\chi_{\Gamma_\alpha(Q)}(X)d\sigma(Q)dX\\ &\quad\leq \int_{\Omega}\int_{\mathcal{F}} A(X)\chi_{\Gamma_\alpha(Q)}(X)\chi_{\Delta(Q_X,(2+\alpha)\delta(X))}(Q)d\sigma(Q)dX\\ &\quad\leq \int_{R_\alpha(\mathcal{F})} A(X)\left(\int_{\mathcal{F}}\chi_{\Delta(Q_X,(2+\alpha)\delta(X))}(Q)d\sigma(Q)\right)dX\\ &\quad\leq C_\alpha\int_{R_\alpha(\mathcal{F})} A(X)\delta(X)^{n-1}dX. \end{aligned}$$

□

Lemma 3.14 *Let Ω be a CAD. Given $\alpha > 0$, there exists $\gamma \in (0, 1)$ close to 1 such that if $\mathcal{F} \subset \partial\Omega$ and A is a non-negative measurable function in Ω, then*

$$\int_{R_\alpha(\mathcal{F}^*_\gamma)} A(X)\delta(X)^{n-1}dX \leq C_\alpha \int_{\mathcal{F}}\left(\int_{\Gamma_\beta(Q)} A(X)dX\right)d\sigma(Q), \tag{3.18}$$

where $\beta = \min\{1, \alpha\}$.

Proof If $\chi_{\Gamma_\beta(Q)}(X) = 0$, then $|X - Q| \geq (1 + \beta)\delta(X)$, and $|Q - Q_X| \geq \beta\delta(X)$. Hence $\chi_{\Gamma_\beta(Q)}(X) \geq \chi_{\Delta(Q_X,\beta\delta(X))}(Q)$. Fubini's theorem yields

$$\begin{aligned}
&\int_{\mathcal{F}}\int_{\Gamma_\beta(Q)} A(X)dXd\sigma(Q) \\
&\quad = \int_{\partial\Omega}\int_{\mathcal{F}} A(X)\chi_{\Gamma_\beta(Q)}(X)d\sigma(Q)dX \\
&\quad \geq \int_{\Omega} A(X)\int_{\mathcal{F}} \chi_{\Delta(Q_X,\beta\delta(X))}(Q)d\sigma(Q)dX \\
&\quad \geq \int_{R_\alpha(\mathcal{F}^*_\gamma)} A(X)\int_{\mathcal{F}} \chi_{\Delta(Q_X,\beta\delta(X))}(Q)d\sigma(Q)dX. \qquad (3.19)
\end{aligned}$$

Note that if $X \in R_\alpha(\mathcal{F}^*_\gamma)$, there is $P \in \mathcal{F}^*_\gamma$ such that $X \in \Gamma_\alpha(P)$ and $Q_X \in \Delta(P, (2+\alpha)\delta(X))$, then applying (3.2) in (3.3) we obtain

$$\int_{\mathcal{F}}\int_{\Gamma_\beta(Q)} A(X)dXd\sigma(Q) \geq C_\alpha \int_{R_\alpha(\mathcal{F}^*_\gamma)} A(X)\delta(X)^{n-1}dX. \qquad (3.20)$$

□

Corollary 3.15 *Let Ω be a CAD. Given $\alpha > 0$ there exists $\gamma \in (0, 1)$ close to 1 such that if $\mathcal{F} \subset \partial\Omega$ and f is a measurable function in Ω, then*

$$\begin{aligned}
\int_{\mathcal{F}^*_\gamma}\int_{\Gamma_\alpha(Q)} \frac{f^2(x)}{\delta(X)^n}dXd\sigma(Q) &\leq C_\alpha \int_{R_\alpha(\mathcal{F}^*_\gamma)} \frac{f^2(X)}{\delta(X)}dX \\
&\leq C_\alpha \int_{\mathcal{F}}\int_{\Gamma(Q)} \frac{f^2(x)}{\delta(X)^n}dXd\sigma(Q). \qquad (3.21)
\end{aligned}$$

Proof Combining Lemma 3.14 applied to $\mathcal{F}$ and Lemma 3.13 applied to $\mathcal{F}^*_\gamma$ with $A(X) = \frac{f^2(X)}{\delta^n(X)}$, we obtain (3.21). □

4 Square Functions in CADs

Next we focus our attention on the tent spaces T^p defined for chord arc domains, following the theory developed by Coifman, Meyer, and Stein in [2]. Suppose that f is a measurable function defined on Ω. For $\alpha > 0$ and $Q \in \partial\Omega$, we define

$$A^{(\alpha)}(f)(Q) = \left(\int_{\Gamma_\alpha(Q)} f(X)^2\frac{dX}{\delta(X)^n}\right)^{1/2}. \qquad (4.1)$$

The square function of f is defined as $A(f) = A^{(1)}(f)$. By analogy with the space $\mathcal{N}$ defined in Sect. 3, we denote by T^p for $1 \leq p < \infty$ the space of all Borel measurable functions given by

$$T^p = \{ f \in L^2(\Omega) : A(f) \in L^p(\sigma) \}. \tag{4.2}$$

Remark 4.1 The space T^p as defined above with the norm $\|f\|_{T^p} = \|A(f)\|_{L^p(\sigma)}$ is a Banach space.

We define operator $C(f) : \partial\Omega \to \mathbb{R}$ by

$$C(f)(Q) := \sup_{Q\in\Delta} \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} f(X)^2 \frac{dX}{\delta(X)} \right)^{1/2} \tag{4.3}$$

where Δ is a surface ball and $T(\Delta)$ is the tent over it. We also introduce the space

$$T^\infty = \{ f \in L^2(\Omega) : C(f) \in L^\infty(\sigma) \}, \tag{4.4}$$

with the norm $\|f\|_{T^\infty} = \|C(f)\|_{L^\infty}$.

Theorem 4.2

(a) *Whenever* $g \in T^1$ *and* $C(f) \in L^\infty(\sigma)$, *then*

$$\int_\Omega |f(X)g(X)| \frac{dX}{\delta(X)} \le C \|C(f)\|_{L^\infty} \|g\|_{T^1}.$$

(b) *More precisely,*

$$\int_\Omega |f(X)g(X)| \frac{dX}{\delta(X)} \le C \int_{\partial\Omega} C(f)(Q)A(g)(Q)d\sigma(Q).$$

Proof Without loss of generality we may assume that both f and g are non-negative. For any $\tau > 0$, we define the truncated cone

$$\Gamma^\tau(Q) = \{ X \in \Omega : |X - Q| < 2\delta(X), \delta(X) \le \tau \} \tag{4.5}$$

and let

$$A_\tau(f)(Q) := \left(\int_{\Gamma^\tau(Q)} f(X)^2 \frac{dX}{\delta(X)^n} \right)^{1/2}.$$

Note that $A_\tau(f)$ increases with τ, is constant for $\tau > \operatorname{diam}\Omega$, and $A_\infty(f) = A(f)$. Given f, define the "stopping time" $\tau(Q)$, which is given for $Q \in \partial\Omega$ by

$$\tau(Q) = \sup\{ \tau > 0 : A_\tau(f)(Q) \le \Lambda C(f)(Q) \}$$

where Λ is a large constant to be determined later. Λ is only allowed to depend on n, the NTA constants, and the Ahlfors regularity constant.

Claim: There exists a constant $c_0 > 0$ *such that for every* $Q_0 \in \partial\Omega$ *and* $0 < r \le \operatorname{diam}\Omega$

$$\sigma(\{ Q \in \Delta(Q_0, r) : \tau(Q) \ge r \}) \ge c_0 \sigma(\Delta(Q_0, r)).$$

We assume that the previous claim holds and we prove that part (b) is satisfied by showing that for $H \geq 0$

$$\int_{\Omega} H(X)\delta(X)^{n-1} dX \leq C_1 \int_{\partial\Omega} \left\{ \int_{\Gamma^{\tau(Q)}(Q)} H(X) dX \right\} d\sigma(Q) \tag{4.6}$$

where C_1 depends on c_0 and the Ahlfors constant. Applying Fubini's theorem we obtain

$$\int_{\partial\Omega} \int_{\Gamma^{\tau(Q)}(Q)} H(X) dX d\sigma(Q) = \int_{\Omega} \int_{\partial\Omega} H(X) \chi_{\Gamma^{\tau(Q)}(Q)}(X) d\sigma(Q) dX.$$

Note that if $Q \in \Delta(Q_X, \delta(X))$ and $\delta(X) \leq \tau(Q)$ then $|Q - X| \leq |Q - Q_X| + |Q_X - X| < 2\delta(X)$, which implies that $X \in \Gamma^{\tau(Q)}(Q)$ and

$$\chi_{\Gamma^{\tau(Q)}(Q)}(X) \geq \chi_{\Delta(Q_X, \delta(X)) \cap \{\tau(Q) \geq \delta(X)\}}(Q).$$

Therefore, using the claim and the fact that σ is Ahlfors regular, we obtain

$$\begin{aligned}
&\int_{\Omega} \int_{\partial\Omega} H(X) \chi_{\Gamma^{\tau(Q)}(Q)}(X) d\sigma(Q) dX \\
&\quad \geq \int_{\Omega} \int_{\partial\Omega} H(X) \chi_{\Delta(Q_X, \delta(X)) \cap \{\tau(Q) \geq \delta(X)\}}(Q) d\sigma(Q) dX \\
&\quad \geq \int_{\Omega} H(X) \sigma\big(\{Q \in \Delta\big(Q_X, \delta(X)\big) : \tau(Q) \geq \delta(X)\}\big) dX \\
&\quad \geq \int_{\Omega} H(X) c_0 \sigma\big(B\big(Q_X, \delta(X)\big)\big) dX \\
&\quad \geq C_1^{-1} \int_{\Omega} H(X) \delta(X)^{n-1} dX.
\end{aligned}$$

To prove part (b), we take $H(X) = f(X)g(X)\delta(X)^{-n}$ in the inequality (4.6),

$$\int_{\Omega} f(X)g(X) \frac{dX}{\delta(X)} \leq C_1 \int_{\partial\Omega} \left(\int_{\Gamma^{\tau(Q)}(Q)} f(X)g(X)\delta(X)^{-n} dX \right) d\sigma(Q)$$

and then we use the Cauchy–Schwarz inequality in order to obtain

$$\begin{aligned}
&\int_{\Gamma^{\tau(Q)}(Q)} f(X)g(X)\delta(X)^{-n} dX \\
&\quad \leq \left(\int_{\Gamma^{\tau(Q)}(Q)} \frac{f^2(X)}{\delta(X)^n} dX \right)^{1/2} \left(\int_{\Gamma^{\tau(Q)}(Q)} \frac{g^2(X)}{\delta(X)^n} dX \right)^{1/2}.
\end{aligned}$$

Therefore,

$$\int_{\Omega} f(X)g(X) \frac{dX}{\delta(X)} \leq C_1 \int_{\partial\Omega} A_{\tau(Q)}(f)(Q) A_{\tau(Q)}(g)(Q) d\sigma(Q). \tag{4.7}$$

By the definition of $\tau(Q)$,

$$A_{\tau(Q)}(f)(Q) \leq \Lambda C(f)(Q) \quad \text{and} \quad A_{\tau(Q)}(g)(Q) \leq A(g)(Q).$$

Hence,

$$\int_{\Omega} f(X)g(X)\frac{dX}{\delta(X)} \leq C \int_{\partial\Omega} C(f)(Q)A(g)(Q)d\sigma(Q)$$

as required in part (b). In order to complete the proof, we need to prove the claim stated above.

Proof of Claim For $Q_0 \in \partial\Omega$, consider $\Delta = \Delta(Q_0, r)$ and $\tilde{\Delta} = \Delta(Q_0, 3r)$. Note that $\bigcup_{Q\in\Delta} \Gamma^r(Q) \subset T(\tilde{\Delta})$. Indeed, if $X \in \Gamma^r(Q)$ for $Q \in \Delta$, then $|X - Q| < 2\delta(X)$ and $\delta(X) \leq r$, that is, $|X - Q_0| < |Q_0 - Q| + |Q - X| < r + 2\delta(X) \leq 3r$, which implies $X \in B(Q_0, 3r) \cap \Omega = T(\tilde{\Delta})$. Thus, for $Q \in \Delta$,

$$\begin{aligned}
\int_{\Delta} A_r^2(f)(Q)d\sigma(Q) &= \int_{\Delta}\int_{\Gamma^r(Q)} \frac{f^2(X)}{\delta(X)^n} dX d\sigma(Q) \\
&= \int_{\Omega}\int_{\Delta} \frac{f^2(X)}{\delta(X)^n} \chi_{\Gamma^r(Q)}(X) d\sigma(Q) dX \\
&\leq \int_{\Omega} \frac{f^2(X)}{\delta(X)^n} \chi_{B(Q_0,3r)}(X) \sigma\big(\Delta\big(Q_X, 3\delta(X)\big)\big) dX \\
&\leq C \int_{T(\tilde{\Delta})} \frac{f^2(X)}{\delta(X)} dX.
\end{aligned}$$

Since $\sigma(\tilde{\Delta}) \leq c\sigma(\Delta)$ for any $Q \in \Delta$

$$\frac{1}{\sigma(\Delta)}\int_{\Delta} A_r^2(f)(Q)d\sigma(Q) \lesssim \frac{1}{\sigma(\tilde{\Delta})}\int_{T(\tilde{\Delta})} \frac{f^2(X)}{\delta(X)} dX \lesssim C^2(f)(Q) \leq C' \inf_{Q\in\Delta} C(f)(Q).$$

If $\sigma(\{Q \in \Delta : \tau(Q) \geq r\}) < c_0\sigma(B)$, then $\sigma(\{Q \in \Delta : \tau(Q) < r\}) > (1 - c_0)\sigma(\Delta)$ and

$$\begin{aligned}
\int_{\Delta} A_r^2(f)(Q)d\sigma(Q) &\geq \int_{\Delta\cap\{\tau(Q)<r\}} {A^2}_r(f)(Q)d\sigma(Q) \\
&> \Lambda^2 \int_{\Delta\cap\{\tau(Q)<r\}} C^2(f)(Q)d\sigma(Q) \\
&\geq \Lambda^2 \inf_{\Delta} C^2(f)(Q)\sigma\big(\Delta \cap \big\{\tau(Q) < r\big\}\big) \\
&\geq \Lambda^2(1 - c_0) \inf_{\Delta} C^2(f)(Q)\sigma(\Delta)
\end{aligned}$$

which would imply

$$\Lambda^2(1 - c_0) \inf_{\Delta} C^2(f)(Q) < C' \inf_{\Delta} C^2(f)(Q) \quad \text{or} \quad \Lambda^2(1 - c_0) \leq C',$$

which is a contradiction if we take Λ large and $c_0 = 3/4$ fixed. This concludes the proof of the claim, thus that of Theorem 4.2. □

Remark 4.3 As in [2], Theorem 4.2 can be used to identify the dual of T^1 with those F for which $C(F) \in L^{\infty}(\sigma)$.

Remark 4.4 Note that if $1 < p, q < \infty$, are such that $\frac{1}{p} + \frac{1}{q} = 1$, $f \in T^p$, $g \in T^q$, then using (4.7) and Hölder's inequality we have that

$$\int_{\Omega} \frac{f(X)g(X)}{\delta(X)} dX \lesssim \|f\|_{T^p} \|g\|_{T^q}.$$

Similarly, (b) in Theorem 4.2 ensures that

$$\int_{\Omega} \frac{f(X)g(X)}{\delta(X)} dX \lesssim \left\|C(f)\right\|_{L^p} \|g\|_{T^q}. \tag{4.8}$$

It will be proved in Theorem 6.1 that for $2 < p < \infty$, $A(f) \in L^p(\sigma)$ if and only if $C(f) \in L^p(\sigma)$.

As in [2], we prove that the definition of tent spaces is independent of the aperture of the cone used. The following proposition is also crucial for the forthcoming results in Sect. 7 (see, in particular, Remarks 7.3 and 7.2).

Proposition 4.5 *Using the notation in* (4.1), *we have that for* $0 < p < \infty$

$$\left\|A^{(\alpha)}(f)\right\|_{L^p(\sigma)} \approx \left\|A(f)\right\|_{L^p(\sigma)}. \tag{4.9}$$

To prove Proposition 4.5, we assume that $\alpha > 1$. We note that in this case $A^{(\alpha)}(f) \geq A(f)$. We show that there exists a constant $C(\alpha, p)$ such that $\|A^{(\alpha)}(f)\|_{L^p} \leq C(\alpha, p)\|A(f)\|_{L^p}$. This proves (4.9) for $\alpha > 1$. The case $\alpha \leq 1$ is proved the same way by reversing the roles of α and 1. The following lemma, which is straightforward on Lipschitz domains, requires proof on a CAD.

Lemma 4.6 *For* $f \in T^1$ *and* $\lambda > 0$, *the set* $\mathcal{F} = \{Q \in \partial\Omega : A^{(\alpha)}(f)(Q) \leq \lambda\}$ *is closed.*

Proof To prove that $\mathcal{F}^c = \{Q \in \partial\Omega : A^{(\alpha)}(f)(Q) > \lambda\}$ is open, we show that given $Q \in \partial\Omega$ such that $A^{(\alpha)}(f)(Q) > \lambda$, there exist $\eta > 0$ and $\epsilon > 0$ such that if $P \in \partial\Omega$ with $|P - Q| < \epsilon\eta$, then

$$\int_{\Gamma_\alpha(P)\setminus B(P,\eta)} \frac{f^2(X)}{\delta(X)^n} dX > \lambda^2.$$

Since $A^{(\alpha)}(f)(Q) > \lambda$, there exists $\eta > 0$ so that

$$\int_{\Gamma_\alpha(Q)\setminus B(Q,\eta)} \frac{f^2(X)}{\delta(X)^n} dX > \left(\frac{A^{(\alpha}(f)(Q) + \lambda}{2}\right)^2.$$

Observe that

$$\left|\int_{\Gamma_\alpha(P)\setminus B(P,\eta)} \frac{f^2(X)}{\delta(X)^n} dX - \int_{\Gamma_\alpha(Q)\setminus B(Q,\eta)} \frac{f^2(X)}{\delta(X)^n} dX\right| \leq \int_D \frac{f^2(X)}{\delta(X)^n} dX$$

where $D = (\Gamma_\alpha(Q) \setminus B(Q,\eta)) \triangle (\Gamma_\alpha(P) \setminus B(P,\eta))$. If $|X-Q| < (1+\alpha)\delta(X)$ and $|X-Q| \geq \eta$, then $\delta(X) \geq \frac{\eta}{1+\alpha}$. If $P \in B(Q,\epsilon\eta)$ and $X \notin \Gamma_\alpha(P) \setminus B(P,\eta)$, then $|X-Q| \geq (1+\alpha)(1-\epsilon)\delta(X)$. Thus we need to study sets of the form

$$V_P = \big\{X \in \Omega : |X-P| \geq \eta; \\ \delta(X)(1+\alpha)(1-\epsilon) \leq |X-P| < (1+\alpha)\delta(X)\big\} \tag{4.10}$$

for $P \in B(Q,\epsilon\eta)$ and prove that they have small $\mathcal{H}^n$-measure. Note that for $\epsilon < 1/2$

$$V_P \subset V'_\epsilon = \big\{X \in \Omega : |X-Q| \geq \eta/2; \\ \delta(X)(1+\alpha)(1-\epsilon)^2 \leq |X-Q| < (1+\alpha)^2\delta(X)\big\}. \tag{4.11}$$

Note that $D \subset V_P \cup V_Q \subset V'$. We show that given $\alpha > 0$ and $\delta > 0$, there exists $\beta > 0$ such that

$$\mathcal{H}^n\big(\big\{X \in \Omega : |X-Q| \geq \eta/2; \\ \delta(X)(1+\alpha-\beta) \leq |X-Q| \leq (1+\alpha+\beta)\delta(X)\big\}\big) < \delta, \tag{4.12}$$

which ensures that given $\alpha > 0$ and $\delta > 0$, there exists $\epsilon > 0$ such that $\mathcal{H}^n(V'_\epsilon) < \delta$. Fix $\alpha > 0$ and take $\beta > 0$ small, such that $\alpha - \beta > \alpha/2$. Consider the set

$$V = \Omega \cap \left\{|X-P| \geq \eta/2,\ \frac{1}{1+\alpha+\beta} \leq \frac{\delta(X)}{|X-P|} \leq \frac{1}{1+\alpha-\beta}\right\} \subset \Omega \setminus B(Q,\eta/2).$$

$F(X) = \frac{\delta(X)}{|X-Q|} - 1$ is a non-positive Lipschitz function on $\Omega \setminus B(Q,\eta/2)$. By the co-area formula, we have

$$\mathcal{H}^n(V) = \int_{-1}^{0}\left(\int_{F^{-1}(t)} \frac{1}{JF}\chi_V\, d\mathcal{H}^{n-1}\right)dt \\ = \int_{\frac{1}{1+\alpha+\beta}-1}^{\frac{1}{1+\alpha-\beta}-1}\left(\int_{F^{-1}(t)} \frac{1}{JF}\chi_V\, d\mathcal{H}^{n-1}\right)dt. \tag{4.13}$$

Since

$$\int_{-1}^{0}\left(\int_{F^{-1}(t)} \frac{1}{JF}\chi_{\Omega\setminus B(Q,\eta/2)}\, d\mathcal{H}^{n-1}\right)dt \leq \mathcal{H}^n(\Omega) \leq C(\Omega) < \infty,$$

given $\alpha > 0$, there exists $\beta > 0$ small, such that

$$\int_{\frac{1}{1+\alpha+\beta}-1}^{\frac{1}{1+\alpha-\beta}-1}\left(\int_{F^{-1}(t)} \frac{1}{JF}\chi_{\Omega\setminus B(Q,\eta/2)}\, d\mathcal{H}^{n-1}\right)dt < \delta. \tag{4.14}$$

Note that (4.11), (4.13), and (4.14) yield (4.12).

Since $f \in L^2(\Omega)$, given $\epsilon' > 0$ there exists $\delta > 0$ so that if (4.12) holds then

$$\int_D \frac{f^2(X)}{\delta(X)^n}dX \leq 2^n\eta^{-n}\int_D f^2(X)dX < \eta^{-n}\epsilon'.$$

Thus

$$\int_{\Gamma_\alpha(P)\setminus B(P,\eta)} \frac{f^2(X)}{\delta(X)^n} dX > \int_{\Gamma_\alpha(Q)\setminus B(Q,\eta)} \frac{f^2(X)}{\delta(X)^n} dX - \eta^{-n}\epsilon'\left(\frac{A^{(\alpha}(f)(Q)+\lambda}{2}\right)^2 - \eta^{-n}\epsilon'.$$

Since $A^{(\alpha)}(f)(Q) > \lambda$, we can choose $\epsilon' > 0$ so that

$$\left(A^{(\alpha)}(f)(P)\right)^2 \geq \int_{\Gamma_\alpha(P)\setminus B(P,\eta)} \frac{f^2(X)}{\delta(X)^n} dX > \lambda^2. \qquad \square$$

Proof of Proposition 4.5 We fix $\lambda > 0$ and let

$$\mathcal{F} = \left\{Q \in \partial\Omega : A(f)(Q) \leq \lambda\right\}, \quad O = \mathcal{F}^C = \left\{Q \in \partial\Omega : A(f)(Q) > \lambda\right\}.$$

Since $\mathcal{F}$ is a closed set, $\mathcal{F}^*_\gamma \subset \mathcal{F}$ (see Proposition 3.3). Let $O^* = (\mathcal{F}^*_\gamma)^C$. Since σ is Ahlfors regular, it is doubling, and (3.4) ensures that

$$\begin{aligned}
&\sigma\left(\left\{Q \in \partial\Omega : A^{(\alpha)}(f)(Q) > \lambda\right\}\right)\\
&\quad \leq \sigma\left(\left(\mathcal{F}^*_\gamma\right)^C\}\right) + \sigma\left(\left\{Q \in \mathcal{F}^*_\gamma : A^{(\alpha)}(f)(Q) > \lambda\right\}\right)\\
&\quad \leq \sigma\left(O^*\right) + \frac{C_{\alpha,\gamma}}{\lambda^2}\int_{\mathcal{F}} \left(A(f)(Q)\right)^2 d\sigma(Q)\\
&\quad \leq C\sigma\left(\left\{Q \in \partial\Omega : A(f)(Q) > \lambda\right\}\right) + \frac{C}{\lambda^2}\int_{\{A(f)\leq\lambda\}} \left(A(f)(Q)\right)^2 d\sigma(Q).
\end{aligned}$$

Multiplying both sides by $p\lambda^{p-1}$ and integrating, we obtain

$$\begin{aligned}
&p\int_0^\infty \sigma\left(\left\{Q \in \partial\Omega : A^{(\alpha)}(f) > \lambda\right\}\right)\lambda^{p-1} d\lambda\\
&\quad \leq Cp\int_0^\infty \sigma\left(\left\{Q \in \partial\Omega : A(f) > \lambda\right\}\right)\lambda^{p-1} d\lambda\\
&\qquad + Cp\int_0^\infty \lambda^{p-3}\int_{\{A(f)\leq\lambda\}} \left(A(f)(Q)\right)^2 d\sigma d\lambda. \qquad (4.15)
\end{aligned}$$

If $p < 1$, Fubini's theorem applied to the second term yields

$$\begin{aligned}
&\int_0^\infty \lambda^{p-3}\int_{\{A(f)\leq\lambda\}} \left(A(f)(Q)\right)^2 d\sigma d\lambda\\
&\quad = \int_0^\infty \int_{\partial\Omega} \chi_{\{Af\leq\lambda\}}\left(A(f)(Q)\right)^2 \lambda^{p-3} d\sigma d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega} \big(Af(Q)\big)^2 \int_{Af(Q)}^{\infty} \lambda^{p-3} d\lambda d\sigma \\
&= C_p \int_{\partial\Omega} \big(Af(Q)\big)^2 \big(Af(Q)\big)^{p-2} d\sigma(Q). \qquad (4.16)
\end{aligned}
$$

Thus for $p < 2$, $\|A^{(\alpha)} f\|_{L^p}^p \leq C\|Af\|_{L^p}^p$.

For the case $p \geq 2$, if $\frac{1}{r} + \frac{2}{p} = 1$, observe that

$$
\|A^{(\alpha)} f\|_p^2 = \sup_{\psi} \bigg\{ \int_{\partial\Omega} \big(A^{(\alpha)} f\big)^2 \psi d\sigma : \ \psi \in L^r(\sigma), \ \|\psi\|_{L^r} \leq 1 \bigg\}. \qquad (4.17)
$$

Note that

$$
\int_{\partial\Omega} \big(A^{(\alpha)} f\big)^2 \psi d\sigma \leq \bigg(\int_{\partial\Omega} \big(A^{(\alpha)} f\big)^p \bigg)^{2/p} \bigg(\int_{\partial\Omega} \psi^r \bigg)^{1/r}.
$$

Also, if $X \in \Gamma_\alpha(Q)$, then $|X - Q| < (1+\alpha)\delta(X)$ and $|Q - Q_X| < (2+\alpha)\delta(X)$. Therefore,

$$
\begin{aligned}
&\int_{\partial\Omega} \big(A^{(\alpha)}(f)(Q)\big)^2 \psi(Q) d\sigma(Q) \\
&\quad = \int_{\partial\Omega} \int_{\Gamma_\alpha(Q)} f^2(X) \frac{dX}{\delta(X)^n} \psi(Q) d\sigma(Q) \\
&\quad = \int_{\partial\Omega} \int_{\Omega} \frac{f^2(X)}{\delta(X)^n} \chi_{\Gamma_\alpha(Q)}(X) \chi_{\Delta(Q_X,(2+\alpha)\delta(X))}(Q) \psi(Q) dX d\sigma \\
&\quad = \int_{\Omega} \bigg(\int_{\partial\Omega} \chi_{\Delta(Q_X,(2+\alpha)\delta(X))}(Q) \psi(Q) \chi_{\Gamma_\alpha(Q)}(X) d\sigma(Q) \bigg) \frac{f^2(X)}{\delta(X)^n} dX \\
&\quad \leq \int_{\Omega} \frac{f^2(X)}{\delta(X)^n} \int_{\Delta(Q_X,(2+\alpha)\delta(X))} \psi(Q) d\sigma(Q) dX \\
&\quad \leq C_\alpha \int_{\Omega} M_{(2+\alpha)\delta(X)} \psi(Q_X) \frac{f^2(X)}{\delta(X)} dX \qquad (4.18)
\end{aligned}
$$

where

$$
M_s \psi(P) = \frac{1}{s^{n-1}} \int_{\Delta(P,s)} \psi(Q) d\sigma(Q).
$$

Let

$$
\psi^*(P) = \sup_{s>0} \frac{1}{s^{n-1}} \int_{\Delta(P,s)} \psi(Q) d\sigma(Q).
$$

Then

$$
M_s(M_{\beta s}\psi)(P) = \frac{1}{s^{n-1}} \int_{\Delta(P,s)} \frac{1}{(\beta s)^{n-1}} \int_{\Delta(Q,\beta s)} \psi(X) d\sigma(X) d\sigma(Q).
$$

If $\beta > 1$ and $Q \in \Delta(P,s)$, observe that $\Delta(P,(\beta-1)s) \subset \Delta(Q,\beta s)$ and

$$\begin{aligned} M_s(M_{\beta s}\psi)(P) &\geq \frac{1}{s^{n-1}}\int_{\Delta(P,s)}\frac{1}{(\beta s)^{n-1}}\int_{\Delta(P,(\beta-1)s)}\psi(X)d\sigma(X)d\sigma(Q) \\ &\geq C\frac{1}{(\beta s)^{n-1}}\int_{\Delta(P,(\beta-1)s)}\psi(X)d\sigma(X) \\ &\geq C_\beta M_{(\beta-1)s}\psi. \end{aligned}$$

That is, for $\beta > 1$,

$$M_{(\beta-1)s}\psi \leq C_\beta M_s(M_{\beta s}\psi) \leq C_\beta M_s\psi^* \tag{4.19}$$

since $M_{\beta s}\psi \leq C\psi^*$. Plugging (4.19) into (4.18) with $s = \delta(X)$, $\beta - 1 = 2 + \alpha$, we obtain

$$\begin{aligned} &\int_{\partial\Omega}\left(A^{(\alpha)}(f)(Q)\right)^2\psi(Q)d\sigma(Q) \\ &\quad\lesssim C_\alpha\int_\Omega M_{(2+\alpha)\delta(X)}\psi(Q_X)\frac{f^2(X)}{\delta(X)}dX \\ &\quad\leq C_\alpha\int_\Omega M_{\delta(X)}\psi^*(Q_X)\frac{f^2(X)}{\delta(X)}dX \\ &\quad\leq C_\alpha\int_\Omega\frac{f^2(X)}{\delta(X)^n}\int_{\Delta(Q_X,\delta(X))}\psi^*(Q)d\sigma(Q)dX \\ &\quad\leq C_\alpha\int_\Omega\int_{\partial\Omega}\frac{f^2(X)}{\delta(X)^n}\chi_{\Delta(Q_X,\delta(X))}(Q)\chi_{\Gamma(Q)}(X)\psi^*(Q)d\sigma(Q)dX \\ &\quad\leq C_\alpha\int_{\partial\Omega}\left(\int_{\Gamma(Q)}\frac{f^2(X)}{\delta(X)^n}dX\right)\psi^*(Q)d\sigma(Q) \\ &\quad\leq C_\alpha\int_{\partial\Omega}(Af)^2(Q)\psi^*(Q)d\sigma(Q) \\ &\quad\leq C_\alpha\|Af\|_{L^p}^2\left\|\psi^*\right\|_{L^r} \leq C_\alpha\|Af\|_{L^p}^2\|\psi\|_{L^r} \leq C_\alpha\|Af\|_{L^p}^2 \end{aligned}$$

where we have used the fact that if $Q \in \Delta(Q_X,\delta(X))$ then $|X - Q| \leq |Q - Q_X| + |X - Q_X| \leq 2\delta(X)$ and the fact that the maximal function of ψ is bounded in $L^r(\sigma)$. Taking the supremum over all ψ yields that $\|A^{(\alpha)}f\|_{L^p} \leq C_\alpha\|Af\|_{L^p}$ for $2 \leq p < \infty$. □

Definition 4.7 A T^1 atom is a function $a(X)$ which is supported in $T(\Delta)$, $\Delta = B(Q_0,r)\cap\partial\Omega$ for $Q_0 \in \partial\Omega$ and

$$\int_{T(\Delta)}a^2(X)\frac{dX}{\delta(X)} \leq \frac{1}{\sigma(\Delta)}. \tag{4.20}$$

Observe that if a is supported in $B(Q_0, r) \cap \Omega$, then $A(a)$ given by

$$A(a) = \left(\int_{\Gamma(Q)} \frac{a^2(X)}{\delta(X)^n} dX \right)^{1/2}$$

is supported in $\Delta(Q_0, 3r)$. Indeed, if $X \in \Gamma(Q)$ and $|Q - Q_0| \geq 3r$, then $|X - Q_0| \geq r$, which gives $a(X) = 0$ and thus $A(a)(Q) = 0$. Using (4.20) we estimate,

$$\begin{aligned}
\int_{\partial\Omega} A^2(a)(Q) d\sigma(Q) &= \int_{\partial\Omega} \int_{\Gamma(Q)} \frac{a^2(X)}{\delta(X)^n} dX d\sigma(Q) \\
&= \int_{\partial\Omega} \int_{\Omega} \chi_{\Gamma(Q)}(X) \frac{a^2(X)}{\delta(X)^n} dX d\sigma(Q) \\
&\leq \int_{\Omega} \int_{\partial\Omega} \chi_{\Delta(Q_X, 3\delta(X))}(Q) \chi_{\Gamma(Q)}(X) \frac{a^2(X)}{\delta(X)^n} d\sigma(Q) dX \\
&\leq C \int_{\Omega} \frac{a^2(X)}{\delta(X)} dX = C \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \lesssim \frac{1}{\sigma(\Delta)}
\end{aligned}$$

which yields

$$\int_{\partial\Omega} A(a)(Q) d\sigma(Q) \leq \left(\int_{\partial\Omega} A^2(a)(Q) d\sigma(Q) \right)^{1/2} \big(\sigma(\Delta)\big)^{1/2} \leq C_n$$

where C_n depends on the Ahlfors regularity constant. Thus, if a is a T^1 atom, then $a \in T^1$ and $\|a\|_{T^1} = \|A(a)\|_{L^1} \leq C_n$.

We are now ready to prove the duality relation between T^1 and T^∞ (see (4.2) and (4.4)).

Theorem 4.8 *If $G \in (T^1)^*$, then there exists a $g \in T^\infty$ such that for every $f \in T^1$*

$$\big|G(f)\big| \simeq \left| \int_{\Omega} f(x) g(x) \frac{dX}{\delta(X)} \right|.$$

Proof We first notice that Theorem 4.2 shows that every $g \in T^\infty$ induces an element in $(T^1)^*$. Let $G \in (T^1)^*$ and note that if K is a compact set in Ω and f is supported in K with $f \in L^2(K)$ then $f \in T^1$. First we consider $K = \overline{B(X_0, r)}$ with $\operatorname{dist}(K, \partial\Omega) \geq \epsilon_0$. If $X \in \Gamma(Q) \cap \overline{B(X_0, r)}$ then

$$\begin{aligned}
|Q - Q_{X_0}| &\leq |Q - X| + |X - X_0| + |X_0 - Q_{X_0}| < 2\delta(X) + r + \delta(X_0) \\
&\leq 2|X - Q_{X_0}| + r + \delta(X_0) \leq 2|X - X_0| + 2|X_0 - Q_{X_0}| + r + \delta(X_0) \\
&\leq 3r + 3\delta(X_0) < 4r + 4\delta(X_0).
\end{aligned}$$

Thus

$$\int_{\partial\Omega} A(f)(Q)d\sigma(Q) = \int_{\partial\Omega}\left(\int_{\Gamma(Q)\cap\overline{B(X_0,r)}} \frac{f^2(X)}{\delta(X)^n}dX\right)^{1/2} d\sigma(Q)$$
$$\leq \int_{\Delta(Q_{X_0,4r+4\delta(X_0)})}\left(\int_{\Gamma(Q)\cap B(X_0,r)} \frac{f^2(X)}{\delta(X)^n}dX\right)^{1/2} d\sigma(Q)$$
$$\lesssim \epsilon_0^{-n/2}\big(r+\delta(X_0)\big)^{n-1}\left(\int_{\overline{B(X_0,r)}} f^2(X)\right)^{1/2}.$$

If $K \subseteq \bigcup_{i=1}^m \overline{B(X_i,r_i)}$ with $\overline{B(X_i,r_i)} \subset\subset \Omega$, $\operatorname{dist}(B(X_i,r_i),\partial\Omega) \geq r_i > \frac{1}{2}\operatorname{dist}(K,\partial\Omega) = \varepsilon_K$ and $r_i \leq \operatorname{diam} K$, then $\delta(X_i) \leq \operatorname{diam} K + \operatorname{dist}(K,\partial\Omega)$ and

$$\int_{\partial\Omega} A(f)(Q)d\sigma(Q)$$
$$\leq \int_{\partial\Omega}\sum_{i=1}^m\left(\int_{\Gamma(Q)\cap\overline{B(X_i,r_i)}} \frac{f^2(X)}{\delta(X)^n}dX\right)^{1/2} d\sigma(Q)$$
$$\leq \sum_{i=1}^m C_K\varepsilon_K^{-n/2}\left(\int_{\overline{B(X_i,r_i)}} f^2(X)dX\right)^{1/2}\big(r_i+\delta(X_i)\big)^{n-1} \leq C_K\|f\|_{L^2(K)}.$$

Therefore, for f compactly supported,

$$\big|G(f)\big| \leq C\|f\|_{T^1} \leq C_K\|f\|_{L^2(K)}.$$

Thus G induces a bounded linear functional on $L^2(K)$ which can be represented by a $g_K \in L^2(K)$. Taking an increasing family of such K which exhaust Ω gives us a function $g \in L^2_{\mathrm{loc}}(\Omega)$ and

$$G(f) = \int_\Omega f(X)g(X)\frac{dX}{\delta(X)} \tag{4.21}$$

whenever $f \in T^1$ with compact support in Ω.

Let $a \in T^1$ be an atom supported on $T(\Delta)$. Then

$$\big|G(a)\big| \leq \|G\|\|a\|_{T^1} \leq C_n\|G\|.$$

For the atom

$$a_m = g\chi_{T(\Delta)\cap\{\delta(X)>r/m\}}\left(\sigma(\Delta)\int_{T(\Delta)\cap\{\delta(X)>r/m\}} \frac{g^2(X)}{\delta(X)}dX\right)^{-1/2}$$

where $\Delta = B(Q,r)\cap\partial\Omega$, we have

$$\big|G(a_m)\big| = \left(\frac{1}{\sigma(\Delta)}\int_{T(\Delta)\cap\{\delta(X)>r/m\}} \frac{g^2(X)}{\delta(X)}dX\right)^{1/2} \leq C_n\|G\|,$$

and if $m \to \infty$

$$\left(\frac{1}{\sigma(\Delta)}\int_{T(\Delta)}\frac{g^2(X)}{\delta(X)}dX\right)^{1/2} \le C_n\|G\|,$$

which shows that $C(g) \in L^\infty$. This representation of G (as in (4.21)) can be extended to all of T^1 since the subspace of the functions with compact support is dense in T^1. □

5 Duality of T^p Spaces

The main purpose of the present section is to study the dual spaces to T^p spaces for $1 < p < \infty$. The main result is contained in the following theorem.

Theorem 5.1 *Let $1 < p < \infty$. The dual of T^p is the space T^q with $\frac{1}{p}+\frac{1}{q}=1$. More precisely, if $G \in (T^p)^*$, then there exists $g \in T^q$ such that for every $f \in T^p$*

$$G(f) = \int_\Omega f(X)g(X)\frac{dX}{\delta(X)} \quad \textit{and} \quad \|G\| \simeq \|g\|_{T^q}.$$

Proof As in [2], we first study the case $p = 2$. Note that from Lemmas 3.13 and 3.14 there exists a constant C_n such that

$$C_n^{-1}\int_\Omega f^2(X)\frac{dX}{\delta(X)} \le \|f\|^2_{T^2} \le C_n\int_\Omega f^2(X)\frac{dX}{\delta(X)}. \tag{5.1}$$

Given $g \in T^2$, the proof of Theorem 4.2 (see (4.7)) yields that the operator G defined by

$$G(f) = \int_\Omega f(X)g(X)\frac{dX}{\delta(X)}$$

satisfies

$$\left|G(f)\right| \le C\int_{\partial\Omega} A(f)(Q)A(g)(Q)d\sigma(Q) \le C\|f\|_{T^2}\|g\|_{T^2}. \tag{5.2}$$

Thus $G \in (T^2)^*$.

Consider T^2 with the norm induced by the inner product

$$\langle f, g\rangle = \int_\Omega \frac{f(X)g(X)}{\delta(X)}dX;$$

then $(T^2, \langle,\rangle)$ is a Hilbert space. Given $G \in (T^2)^*$, by the Riesz Representation Theorem there exists $g \in (T^2, \langle\ ,\ \rangle)$ such that

$$G(f) = \int_\Omega f(X)g(X)\frac{dX}{\delta(X)}.$$

By (5.1) and (5.2), we have that

$$C_n^{-1}\|g\|_{T^2} \le \|G\| \le C_n\|g\|_{T^2}.$$

Consider the case $1 < p < 2$. Let $G \in (T^p)^*$. For $f \in L^2(\Omega)$ with compact support $K \subset\subset \Omega$, let $S = \{Q \in \partial\Omega : \Gamma(Q) \cap K \neq \emptyset\}$ and $\varepsilon_K = \operatorname{dist}(K, \partial\Omega)$. Then $|G(f)| \le \|G\|\|f\|_{T^p}$ and

$$\begin{aligned}\|f\|_{T^p}^p &\le \int_S \left(\int_{\Gamma(Q)\cap K} \frac{f^2(X)}{\delta(X)^n} dX\right)^{p/2} d\sigma \\ &\le \left(\int_K \frac{f^2(X)}{\delta(X)^n} dX\right)^{p/2} \sigma(S) \le C_K \left(\int_K \frac{f^2(X)}{\delta^2(X)} dX\right)^{1/2}. \qquad (5.3)\end{aligned}$$

Thus $|G(f)| \le C_K\|f/\delta\|_{T^2}$. By the Riesz Representation Theorem there exists g which is locally in $L^2(\Omega)$ such that

$$G(f) = \int_\Omega \frac{f(X)g(X)}{\delta(X)} dX$$

whenever $f \in L^2(\Omega)$ and has compact support in Ω. Note that for every $K \subset\subset \Omega$, $f \in L^2(\Omega)$, by Theorem 4.2(b)

$$\begin{aligned}\int_K f^2(X)dX &\le C_K \int_\Omega \frac{(f\chi_K)(X)(f\chi_K)(X)}{\delta(X)} dX \\ &\le C_K \int_{\partial\Omega} A(f\chi_K)(Q)C(f\chi_K)(Q)d\sigma(Q) \qquad (5.4) \\ &\le C_K \left(\int_{\partial\Omega} A^p(f\chi_K)(Q)d\sigma(Q)\right)^{1/p} \\ &\quad \times \left(\int_{\partial\Omega} C^q(f\chi_K)(Q)d\sigma(Q)\right)^{1/q}, \qquad (5.5)\end{aligned}$$

where $1/p+1/q = 1$. By the definition of $C(f\chi_K)$ and if $\delta_K(Q)$ denotes the distance of Q to the set K,

$$C(f\chi_K)(Q) = \sup_{Q\in\Delta}\left(\frac{1}{\sigma(\Delta)}\int_{T(\Delta)} \frac{(f\chi_K)^2(X)}{\delta(X)} dX\right)^{1/2} \lesssim \left(\int_{T(\Delta)} f^2(X)dX\right)^{1/2} \varepsilon_K^{-\frac{?}{2}}$$

and

$$\int_{\partial\Omega} \left(C(f\chi_K)(Q)\right)^q d\sigma(Q) \le C_K \left(\int_K f^2(X)dX\right)^{q/2}. \qquad (5.6)$$

Combining (5.4) and (5.6), we have for $p > 1$

$$\left(\int_K f^2(X)dX\right)^{1/2} \le C(K,p)\|f\chi_K\|_{T^p}.$$

Observe that the set $\{f \in T^p : f \text{ is compactly supported in } \Omega\}$ is dense in T^p. Indeed, let us choose an increasing family of compact sets $\{K_n\}$ which exhaust Ω. For $f \in T^p$, consider $f_m = f\chi_{K_m} \in T^p$. Then

$$\|f_m - f\|_{T^p} = \big\|A(f_m - f)\big\|_{L^p(\sigma)}$$

and

$$\begin{aligned}\big\|A(f_m - f)\big\|_{L^p(\sigma)} &= \left(\int_{\partial\Omega}\left[\int_{\Gamma(Q)} \frac{(f_m - f)^2(X)}{\delta(X)^n} dX\right]^{p/2} d\sigma\right)^{1/2} \\ &= \left(\int_{\partial\Omega}\big(E_m(f)(Q)\big)^{p/2} d\sigma(Q)\right)^{1/2},\end{aligned}$$

where

$$E_m(f)(Q) = \int_{\Gamma(Q)\cap K_m^c} \frac{f^2(X)}{\delta(X)^n} dX.$$

Note that $0 \le E_m(f)(Q) \le E_{m-1}(f)(Q) \le \cdots \le E(f)(Q) = \int_{\Gamma(Q)} \frac{f^2(X)}{\delta(X)^n} dX$. Since $f \in T^p$, $E_m(f) \to 0$ a.e. $Q \in \partial\Omega$, and by the Dominated Convergence Theorem

$$\lim_{m\to\infty}\int_{\partial\Omega} E_m(f)(Q)d\sigma(Q) = 0 \quad \text{and} \quad \lim_{m\to\infty}\big\|A(f_m - f)\big\|_{L^p} = 0.$$

We claim that for g as above there exists $C' > 0$ such that

$$\big\|A(g_K)\big\|_{L^q} \le C'\|G\|, \tag{5.7}$$

where $g_K = g\chi_K$, and K is any compact subset of Ω. The key point is that C' is a constant independent of the choice of the set K. Note that this ensures that

$$\|g\|_{T^q} \le C'\|G\|. \tag{5.8}$$

Let r denote the exponent dual to $q/2$, $\frac{1}{r} + \frac{2}{q} = 1$. Then, as in the proof of Proposition 4.5 (see (4.17)),

$$\begin{aligned}&\big\|A(g_K)\big\|_{L^q}^2 \\ &\quad = \sup_{\psi}\left\{\int_{\partial\Omega}\big(A(g_K)(Q)\big)^2\psi(Q)d\sigma(Q) : \psi \ge 0, \psi \in L^r(\partial\Omega), \|\psi\|_{L^r} \le 1\right\}.\end{aligned} \tag{5.9}$$

As in the proof of Proposition 4.5 to obtain (see (4.18))

$$\begin{aligned}&\sup_{\psi}\int_{\partial\Omega}\big(A(g_K)(Q)\big)^2\psi(Q)d\sigma(Q) \\ &\quad \le C\sup_{\psi}\int_{\Omega}\frac{g_K^2(X)}{\delta(X)} M_{3\delta(X)}\psi(Q_X)dX = C\sup_{\psi} G(h_\psi),\end{aligned} \tag{5.10}$$

where $h_\psi(X) = g_K(X) M_{3\delta(X)}\psi(Q_X)$. Note that

$$M_{3\delta(X)}\psi(Q_X) = \frac{1}{(3\delta(X))^{n-1}} \int_{\Delta(Q_X, 3\delta(X))} \psi(Y) d\sigma(Y)$$
$$\leq C M_{6\delta(X)}\psi(Q) \leq C M\psi(Q) \tag{5.11}$$
$$A(h_\psi)(Q) = \left(\int_{\Gamma(Q)} \frac{g_K^2(X) M_{3\delta(X)}^2 \psi(Q_X)}{\delta(X)^n} dX \right)^{1/2} \lesssim M\psi(Q) A(g_K)(Q)$$

where $M\psi$ denotes the maximal function of ψ.

Integrating (5.11), noting that $\frac{p}{r} + \frac{p}{q} = 1$, and applying Hölder's inequality, we conclude that

$$\|h_\psi\|_{T^p} = \left(\int_{\partial\Omega} \big(A(h_\psi)(Q)\big)^p d\sigma(Q) \right)^{1/p}$$
$$\lesssim \left(\int_{\partial\Omega} \big(M\psi(Q)\big)^r d\sigma(Q) \right)^{1/r} \left(\int_{\partial\Omega} \big(A(g_K)(Q)\big)^q d\sigma(Q) \right)^{1/q}$$
$$\lesssim \|M\psi\|_{L^r} \big\|A(g_K)\big\|_{L^q} \lesssim \|\psi\|_{L^r} \big\|A(g_K)\big\|_{L^q}. \tag{5.12}$$

Since h_ψ is compactly supported, $h_\psi \in T^p$

$$\big|G(h_\psi)\big| \leq \|G\| \|h_\psi\|_{T^p} \lesssim \|G\| \cdot \|\psi\|_{L^r} \|g_K\|_{T^q} \tag{5.13}$$

Combining (5.10), (5.13), and (5.12)

$$\big\|A(g_K)\big\|_{L^q}^2 = \|g_K\|_{T^q}^2 \leq C\|G\| \cdot \|g_K\|_{T^q},$$

where C is independent of K. This proves (5.7) and (5.8). The density of compactly supported functions f in T^p ensures that $G(f) = \int_\Omega \frac{f(X)g(X)}{\delta(x)} dX$. Applying Hölder's inequality to (5.2), we conclude that $\|G\| \leq \|g\|_{T^q}$. This combined with (5.8) guarantees that $\|G\| \simeq \|g\|_{T^q}$, which completes the proof in the case $1 < p < 2$.

To prove Theorem 5.1 for any $p \in (2, \infty)$, it is enough to show that for $1 < p < 2$, T^p is reflexive. By the Eberlein–Smulyan Theorem (see [17]) it is enough to show that whenever $f_n \in T^p$, $\|f_n\|_{T^p} \leq 1$, there exists a subsequence which converges weakly in T^p. If $\{f_n\} \in T^p$ with $\|f_n\|_{T^p} \leq 1$, we have

$$\left(\int_K f_n^2(X) dX \right)^{1/2} \leq C_K \|f_n\|_{T^p} \leq C_K.$$

Therefore, taking a compact exhaustion of Ω, we show that there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \rightharpoonup f$ in $L^2(K)$, for all $K \subset\subset \Omega$. Let $G \in (T^p)^*$. By the proof above, there exists $g \in T^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that $G(f) = \int_\Omega \frac{f(X)g(X)}{\delta(x)} dX$. Given $\varepsilon > 0$, there exists a compact set K such that

$$\big\|A(g - g_K)\big\|_{T^q} = \|g - g_K\|_{T^q} < \varepsilon,$$

where $g_K = g\chi_K$. Note that

$$G(f_{n_j}) - G(f_{n_i}) = \int_\Omega \frac{(f_{n_j} - f_{n_i})(X)}{\delta(X)} g(X)dX$$
$$= \int_\Omega \frac{(f_{n_j} - f_{n_i})(X)}{\delta(X)} g_K(X)dX + \int_\Omega \frac{(f_{n_j} - f_{n_i})(X)}{\delta(X)} (g - g_K)(X)dX.$$

Since $f_{n_i} \rightharpoonup f$ in $L^2(K)$ and $g_K(X)/\delta(X) \in L^2(K)$,

$$\int_\Omega \frac{(f_{n_j} - f_{n_i})(X)}{\delta(X)} g_K(X)dX \longrightarrow 0 \quad \text{as } i, j \to \infty.$$

Also, by (4.7)

$$\left| \int_\Omega \frac{(f_{n_j} - f_{n_i})(X)}{\delta(X)} (g - g_K)(X)dX \right| \lesssim \int_{\partial\Omega} A(f_{n_j} - f_{n_i})(Q) A(g - g_K)(Q) d\sigma(Q)$$
$$\leq \|f_{n_j} - f_{n_i}\|_{T^p} \|g - g_K\|_{T^q} < 2\varepsilon.$$

Thus $\{G(f_{n_i})\}$ converges, which ensures that $\{f_{n_i}\}$ converges weakly in T^p. □

6 Relation Between Integrals on Cones A and Carleson's Function C

In this section, we study, as in [2], the relation between the functionals A and C. We show that if $2 < p < \infty$, then $\|A(f)\|_{L^p} \simeq \|C(f)\|_{L^p}$.

Theorem 6.1

(a) *If* $0 < p < \infty$, *then*

$$\|A(f)\|_{L^p} \leq C_p \|C(f)\|_{L^p}.$$

(b) *If* $2 < p \leq \infty$, *then*

$$\|C(f)\|_{L^p} \leq C_p \|A(f)\|_{L^p}.$$

The proof of Theorem 6.1 uses the following "good-λ" inequality.

Lemma 6.2 *There exist a fixed aperture* $\alpha > 1$ *and a constant* $C > 0$ *so that for* $0 < \gamma \leq 1$ *and* $0 < \lambda < \infty$

$$\sigma\left(\left\{Q \in \partial\Omega : A(f)(Q) > 2\lambda;\ C(f)(Q) \leq \gamma\lambda\right\}\right)$$
$$\leq C\gamma^2 \sigma\left(\left\{Q \in \partial\Omega : A^{(\alpha)}(f)(Q) > \lambda\right\}\right). \tag{6.1}$$

Proof Let $\bigcup Q_k$ be a Whitney decomposition of $\{A^{(\alpha)}(f)(Q) > \lambda\}$ as in Lemma 3.7. For each k, there exists $Y_k \in \{A^{(\alpha)}(f)(Q) \leq \lambda\}$ such that $\operatorname{dist}(Y_k, Q_k) \leq c_0 \operatorname{diam} Q_k$. Since $\alpha > 1$, $A^{(\alpha)}(f) \geq A(f)$ and the set $\{A(f)(Q) > 2\lambda\}$ is contained in the set $\{A^{(\alpha)}(f)(Q) > \lambda\}$.

To prove (6.1), it is enough to show that

$$\sigma\Big(\big\{X \in Q_k : A(f)(X) > 2\lambda;\ C(f)(X) \leq \gamma\lambda\big\}\Big) \leq c\gamma^2\sigma(Q_k).$$

The construction in Lemma 3.7 for $\{A^{(\alpha)}(f)(Q) \leq \lambda\}$ yields a family of balls $\{B_k\}$ such that $B_k = B(X_k, \frac{1}{24}d(X_k))$, $B_k^* = B(X_k, \frac{1}{2}d(X_k))$, $B_k^{**} = B(X_k, 2d(X_k))$, and $B_k \subset Q_k \subset B_k^*$ for $X_k \in \{A^{(\alpha)}(f) > \lambda\}$ and $d(X_K) = \operatorname{dist}(X_K, \{A^{(\alpha)}(f) \leq \lambda\})$. Note that $\frac{1}{12}d(X_k) \leq \operatorname{diam} Q_k \leq d(X_k) =: 2r_k$ and that there exists $Y_k \in \{A^{(\alpha)}(f)(Q) \leq \lambda\}$ such that $\operatorname{dist}(Y_k, Q_k) \leq 4r_k$. If $P \in Q_k$, then $|P - Y_k| \leq 5d(X_k)$. Define $f = f_1 + f_2$, where

$$\begin{cases} f_1(X) = f(X)\chi_{\{\delta(X) \geq r_k\}} \\ f_2(X) = f(X)\chi_{\{\delta(X) < r_k\}}. \end{cases}$$

Note that $A(f) \leq A(f_1) + A(f_2)$. For $P \in Q_k$, $|P - Y_k| \leq 5r_k$, where $Y_k \in \{A^{(\alpha)}(f)(Q) \leq \lambda\}$ and

$$A(f_1)(P)^2 = \int_{\Gamma(P)\cap\{\delta(X)\geq r_k\}} \frac{f^2(X)}{\delta(X)^n}dX \leq \int_{\Gamma_6(Y_k)\cap\{\delta(X)\geq r_k\}} \frac{f^2(X)}{\delta(X)^n}dX.$$

Thus for $\alpha \geq 6$ and $P \in Q_k$, we obtain

$$A(f_1)(P)^2 \leq A^{(\alpha)}(f)(Y_k)^2 \leq \lambda^2.$$

Thus for $P \in Q_k$, if $A(f)(P) \geq 2\lambda$, $A(f_1)(P) \leq \lambda$, and $2\lambda \leq A(f)(P) \leq A(f_1)(P) + A(f_2)(P)$, which ensures that $A(f_2)(P) \geq \lambda$, i.e., $\{P \in Q_k : A(f)(P) > 2\lambda\} \subset \{P \in Q_k : A(f_2)(P) \geq \lambda\}$. By the definition,

$$A(f_2)(P)^2 = \int_{\Gamma(P)\cap\{\delta(X)<r_k\}} \frac{f^2(X)}{\delta(X)^n}dX.$$

Lemma 3.13 combined the Ahlfors regularity of σ yields

$$\begin{aligned} \frac{1}{\sigma(B_k^*)}\int_{B_k^*}\big(A(f_2)(P)\big)^2 d\sigma(P) &\leq \frac{1}{\sigma(B_k^*)}\int_{B_k^{**}}\frac{f^2(X)}{\delta(X)}dX \leq \frac{C}{\sigma(B_k^{**})}\int_{B_k^{**}}\frac{f^2(X)}{\delta(X)}dX \\ &\leq C \inf_{P\in B_k^*}\big(C(f)(P)\big)^2. \end{aligned} \tag{6.2}$$

On the other hand, if the set $\{X \in Q_k : A(f)(X) > 2\lambda;\ C(f)(X) \leq \gamma\lambda\}$ is nonempty, there exists $P_0 \in Q_k \subset B_k^*$ such that $A(f)(P_0) > 2\lambda$ and $C(f)(P_0) \leq \gamma\lambda$. Thus, (6.2) yields

$$\frac{1}{\sigma(B_k^*)}\int_{B_k^*}\big(A(f_2)(P)\big)^2 d\sigma(P) \leq C\gamma^2\lambda^2.$$

In this case, using the Ahlfors regularity of σ,

$$\begin{aligned}\sigma\big(\big\{P\in Q_k : A(f_2)(P)>\lambda\big\}\big) &\le \sigma\big(\big\{P\in B_k^* : A(f_2)(P)>\lambda\big\}\big)C\gamma^2\sigma(B_k)\\ &\le C\gamma^2\sigma(Q_k).\end{aligned}$$

Hence

$$\sigma\big(\big\{P\in Q_k : Af(P)>2\lambda;\ C(f)(P)\le\gamma\lambda\big\}\big)\le C\gamma^2\sigma(Q_k),$$

and since $\{Q_k\}$ is a disjoint cover of $\{A^{(\alpha)}(f)>\lambda\}$

$$\begin{aligned}&\sigma\big(\big\{Q\in\partial\Omega : A(f)>2\lambda;\ C(f)\le\gamma\lambda\big\}\big)\\ &\quad\le\sum_k\sigma\big(\big\{X\in Q_k : A(f)>2\lambda;\ C(f)\le\gamma\lambda\big\}\big)\\ &\quad\le\sum_k C\gamma^2\sigma(Q_k)\le C\gamma^2\sigma\big(\big\{Q\in\partial\Omega : A^{(\alpha)}(f)>\lambda\big\}\big).\end{aligned}$$

□

Proof of Theorem 6.1 Note that Theorem 5.1 combined with (4.8) yield part (a) for $1<p<\infty$. Note that Lemma 6.2 ensures that for α big enough

$$\begin{aligned}\sigma\big(\big\{A(f)>2\lambda\big\}\big) &\le \sigma\big(\big\{A(f)>2\lambda;\ C(f)\le\gamma\lambda\big\}\big)+\sigma\big(\big\{C(f)>\gamma\lambda\big\}\big)\\ &\le C\gamma^2\sigma\big(\big\{A^{(\alpha)}(f)>\lambda\big\}\big)+\sigma\big(\big\{C(f)>\gamma\lambda\big\}\big).\end{aligned}$$

Multiplying both sides by $p\lambda^{p-1}$, integrating with respect to λ, and using Proposition 4.5, we obtain

$$\begin{aligned}2^{-p}\big\|A(f)\big\|_{L^p}^p &\le C\gamma^2\big\|A^{(\alpha)}(f)\big\|_{L^p}^p+C\gamma^{-p}\big\|C(f)\big\|_{L^p}^p C(\alpha,p)\gamma^2\big\|A(f)\big\|_{L^p}^p\\ &\quad+C\gamma^{-p}\big\|C(f)\big\|_{L^p}^p.\end{aligned}$$

Choosing $\gamma>0$ small enough so that $C\gamma^2C(\alpha,p)2^p<\frac{1}{2}$, we obtain

$$\big\|A(f)\big\|_{L^p}\le C\big\|C(f)\big\|_{L^p}$$

provided that $\|A(f)\|_{L^p}<\infty$. If $\|A(f)\|_{L^p}=\infty$, the result is obtained by applying the previous argument to $f\chi_K$, where K is selected from an increasing family of compact subsets which exhausts Ω.

To prove part (b) of Theorem 4.2, let $\Delta=\Delta(Q_0,r)$ and $t\Delta=\Delta(Q_0,tr)$ for $t>3$. Note that $X\in T(\Delta)$; then $\Delta(Q_X,\delta(X))\subset t\Delta$, as in Lemma 3.14 $\chi_{\Gamma(Q)}(X)\ge\chi_{\Delta(Q_X,\delta(Q))}(Q)$ thus

$$\begin{aligned}&\int_{t\Delta}\bigg(\int_{\Gamma(Q)}\frac{f^2(X)}{\delta(X)^n}dX\bigg)d\sigma(Q)\\ &\quad=\int_{\partial\Omega}\int_{\Omega}\frac{f^2(X)}{\delta(X)^n}\chi_{\Gamma(Q)}(X)\chi_{t\Delta}(Q)dXd\sigma(Q)\end{aligned}$$

$$\geq \int_{\partial\Omega}\int_{\Omega}\frac{f^2(X)}{\delta(X)^n}\chi_{\Delta(Q_X,\delta(X))}(Q)\chi_{T(\Delta)}(X)dXd\sigma(Q)$$
$$\geq \int_{\Omega}\frac{f^2(X)}{\delta(X)^n}\sigma\big(\Delta\big(Q_X,\delta(X)\big)\big)\chi_{T(\Delta)}(X)dXd\sigma(Q)$$
$$\geq \int_{T(\Delta)}\frac{f^2(X)}{\delta(X)}dX. \tag{6.3}$$

Equation (6.3) and the Ahlfors regularity of σ ensure that

$$\frac{1}{\sigma(\Delta)}\int_{T(\Delta)}\frac{f^2(X)}{\delta(X)}dX \leq \frac{C}{\sigma(\Delta)}\int_{t\Delta}\big(A(f)(Q)\big)^2 d\sigma(Q)$$
$$\leq \frac{C}{\sigma(t\Delta)}\int_{t\Delta}\big(A(f)(Q)\big)^2 d\sigma(Q).$$

Therefore, $(C(f)(Q))^2 \leq CM(A(f)(Q))^2$, which for $p>1$ ensures that

$$\left(\int_{\partial\Omega}\big(C(f)(Q)\big)^{2p}d\sigma(Q)\right)^{1/p} \leq C\left(\int_{\partial\Omega}\big(M\big(A(f)^2(Q)\big)\big)^p(Q)d\sigma(Q)\right)^{1/p}$$
$$\leq C\left(\int_{\partial\Omega}\big(A(f)(Q)\big)^{2p}d\sigma(Q)\right)^{1/2p}. \qquad \square$$

7 Solvability of the Dirichlet Problem in L^p for Perturbation Operators on CADs

In this section, we study the following question: Given a second order divergence form elliptic symmetric operator L_1 which is a perturbation of an operator L_0 for which the Dirichlet problem can be solved in L^p, what can be said about the solvability of the Dirichlet problem in L^q for L_1? As it was pointed out in the Introduction, this problem is well understood on Lipschitz domains. The goal of this section is to develop a similar theory for CADs. Given that we lack some of the tools available in the Lipschitz case, rather than following Dahlberg's steps we turn our attention to [10]. Proposition 7.1 below justifies this approach.

Assume that L_0 and L_1 are second order divergence form elliptic symmetric operators as in Sect. 2. Assume also that $0 \in \Omega$, and denote by $G_0(Y)$ the Green's function of L_0 in Ω with pole 0, and by ω_0 the corresponding elliptic measure. Let a be the deviation function defined in (2.9).

Proposition 7.1 *Let Ω be a CAD and assume that $\omega_0 \in B_p(\sigma)$ for some $p>1$. Given $\epsilon>0$, there exists $\delta>0$ such that if*

$$\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\sigma(\Delta)}\int_{T(\Delta)}\frac{a^2(X)}{\delta(X)}dX\right\}^{1/2} \leq \delta, \tag{7.1}$$

then

$$\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\omega_0(\Delta)}\int_{T(\Delta)}a^2(X)\frac{G_0(X)}{\delta^2(X)}dX\right\}^{1/2}\leq\epsilon. \tag{7.2}$$

Proof Let $\Delta_0=\Delta(Q_0,r_0)$ and $t\Delta_0=\Delta(Q_0,tr_0)$. Using Lemmas 3.14, 2.10, Fubini, and the notation for truncated cones introduced in (4.5), we have

$$\begin{aligned}\int_{T(\Delta_0)}a^2(X)\frac{G_0(X)}{\delta(X)^2}dX&\leq\int_{\partial\Omega}\int_{\Gamma(Q)}\frac{a^2(X)G_0(X)}{\delta(X)^{n+1}}\chi_{T(\Delta_0)}(X)dXd\sigma(Q)\\&\lesssim\int_{3\Delta_0}\int_{\Gamma^{r_0}(Q)}\frac{a^2(X)}{\delta(X)^n}\frac{\omega_0(\Delta(Q_X,\delta(X)))}{\delta(X)^{n-1}}dXd\sigma(Q)\\&\lesssim\int_{7\Delta_0}\int_{\Gamma_5^{r_0}(P)}\frac{a^2(X)}{\delta(X)^n}dXd\omega_0(P)\\&\lesssim\int_{7\Delta_0}\big(A_{r_0}^{(5)}(P)\big)^2d\omega_0(P).\end{aligned}\tag{7.3}$$

Since $\omega_0\in B_p(d\sigma)$ for some $p>1$, if $\frac{1}{p}+\frac{1}{q}=1$ and $k=\frac{d\omega_0}{d\sigma}$, then

$$\begin{aligned}&\int_{t\Delta_0}\big(A_{r_0}^{(\alpha)}(a)(P)\big)^2k(P)d\sigma(P)\\&\quad\leq\left(\int_{t\Delta_0}\big(A_{r_0}^{(\alpha)}(a)(P)\big)^{2q}d\sigma(P)\right)^{1/q}\left(\int_{t\Delta_0}k^pd\sigma\right)^{1/p}\\&\quad\leq\left(\fint_{t\Delta_0}\big(A_{r_0}^{(\alpha)}(a)(P)\big)^{2q}d\sigma(P)\right)^{1/q}\omega_0(t\Delta_0)\end{aligned}\tag{7.4}$$

because

$$\left(\int_{t\Delta_0}k^pd\sigma\right)^{1/p}\leq C\sigma(t\Delta_0)^{1/p}\fint_{t\Delta}kd\sigma\leq C\omega_0(t\Delta)\sigma(t\Delta_0)^{-1/q}.$$

Combining (7.3), (7.4), and Lemma 2.11, we obtain

$$\frac{1}{\omega_0(\Delta_0)}\int_{T(\Delta_0)}a^2(X)\frac{G_0(X)}{\delta(X)^2}dX\leq C\left(\fint_{7\Delta_0}\big(A_{r_0}^{(5)}(a)(P)\big)^{2q}d\sigma(P)\right)^{1/q}. \tag{7.5}$$

We estimate $(\fint_{7\Delta_0}(A_{r_0}^{(5)}(a)(P))^{2q}d\sigma(P))^{1/q}$ by duality. Let $g\in L^p(\sigma)$ with $\frac{1}{p}+\frac{1}{q}=1$ and g supported on $7\Delta_0$. Without loss of generality we may assume that $g\geq 0$.

$$\begin{aligned}&\int_{\partial\Omega}\big(A_{r_0}^{(5)}(a)(P)\big)^2g(P)d\sigma(P)\\&\quad=\int_{\partial\Omega}\int_{\Gamma_5(Q)}\frac{a^2(X)}{\delta(X)^n}\chi_{\{\delta(X)<r_0\}}(X)\chi_{7\Delta_0}(Q)g(Q)dXd\sigma(Q)\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\partial\Omega}\int_{\Omega}\frac{a^2(X)}{\delta(X)^n}\chi_{\{\delta(X)<r_0\}}(X)\chi_{T(13\Delta_0)}(X)\chi_{\Delta(Q_X,7\delta(X))}(Q)g(Q)dXd\sigma(Q)\\
&\leq \int_{T(13\Delta_0)}\chi_{\{\delta(X)<r_0\}}(X)\frac{a^2(X)}{\delta(X)^n}\left(\int_{\Delta(Q_X,7\delta(X))}g(Q)d\sigma(Q)\right)dX\\
&\leq C\int_{T(13\Delta_0)}\chi_{\{\delta(X)<r_0\}}(X)\frac{a^2(X)}{\delta(X)}\left(\fint_{\Delta(Q_X,7\delta(X))}g(Q)d\sigma(Q)\right)dX. \qquad (7.6)
\end{aligned}
$$

Letting $F(X)=\chi_{\{\delta(X)<r_0\}}(X)\fint_{\Delta(Q_X,7\delta(X))}g(Q)d\sigma(Q)$ and applying Proposition 3.11 to the last term in (7.6), we obtain

$$
\begin{aligned}
&\int_{\partial\Omega}\left(A_{r_0}^{(5)}(a)(P)\right)^2 g(P)d\sigma(P)\\
&\quad\leq C\int_{\partial\Omega}NF(Q)C\left(\frac{a^2}{\delta}\chi_{T(13\Delta_0)}\right)(Q)d\sigma(Q), \qquad (7.7)
\end{aligned}
$$

where

$$
NF(Q)=\sup_{X\in\Gamma(Q)}\left|F(X)\right|=\sup_{X\in\Gamma^{r_0}(Q)}\left|F(X)\right|\leq CM_{9r_0}g(Q). \qquad (7.8)
$$

Here $M_{9r_0}g$ denotes the truncated maximal function of g, i.e., $M_{9r_0}g(Q)=\sup_{0<r\leq 9r_0}\fint_{\Delta(Q,r)}|g|d\sigma$. Note that if $|Q-Q_0|\geq 10r_0$ and $X\in\Gamma^{r_0}(Q)$, then $|Q-Q_X|>7r_0$ and $NF(Q)=0$. Moreover, (7.1) yields $C(\frac{a^2}{\delta}\chi_{T(13\Delta_0)})(Q)\leq C(\frac{a^2}{\delta})(Q)\leq\delta$. This combined with (7.7), (7.8), Hölder's inequality, the fact that σ is Ahlfors regular, and the maximal function theorem ensures that

$$
\begin{aligned}
\int_{\partial\Omega}\left(A_{r_0}^{(5)}(a)(P)\right)^2 g(P)d\sigma(P) &\leq C\delta\int_{10\Delta_0}NF(Q)d\sigma(Q)\\
&\leq C\delta\int_{10\Delta_0}M_{9r_0}g(Q)d\sigma(Q)\\
&\leq C\delta\left(\int_{10\Delta_0}\left(M_{9r_0}g(Q)\right)^p d\sigma(Q)\right)^{1/p}\sigma(10\Delta_0)^{1/q}\\
&\leq C\delta\left(\int_{\partial\Omega}g(Q)^p d\sigma(Q)\right)^{1/p}\sigma(7\Delta_0)^{1/q}, \qquad (7.9)
\end{aligned}
$$

which implies

$$
\left(\fint_{7\Delta_0}\left(A_{r_0}^{(5)}(a)(P)\right)^{2q}d\sigma(P)\right)^{1/q}\leq C\delta. \qquad (7.10)
$$

Note that (7.10) combined with (7.5) yields (7.2), provided $C\delta<\epsilon$. □

In this section, we need to consider variants of the non-tangential maximal function of u. Define for $\alpha \in (0,1)$ and $\eta > 0$

$$\tilde{N}_\alpha^\eta F(Q) = \sup_{X \in \Gamma_\eta(Q)} \left(\fint_{B(X,\alpha\delta(X)/8)} F^2(Z)dZ \right)^{1/2}. \tag{7.11}$$

For simplicity, $\tilde{N}_\alpha^1 F = \tilde{N}_\alpha F$, $\tilde{N}_1^\eta F = \tilde{N}^\eta F$, and $\tilde{N}_1^1 F = \tilde{N} F$. Recall that $N_\eta F(Q) = \sup_{X\in\Gamma_\eta(Q)} |F(X)|$.

Remark 7.2 Let μ be a doubling measure on $\partial\Omega$. Then for $p \geq 1$, $\alpha, \beta \in (0,1)$, and $\eta > 0$

$$\|\tilde{N}_\alpha F\|_{L^p(\mu)} \sim \|\tilde{N}_\beta F\|_{L^p(\mu)} \sim \|\tilde{N}_\alpha^\eta F\|_{L^p(\mu)}.$$

Proof Note that Proposition 3.6 ensures that for $1 \leq p < \infty$, $\|\tilde{N}_\alpha^\eta F\|_{L^p(\mu)} \sim \|\tilde{N}_\alpha F\|_{L^p(\mu)}$. Moreover, for $\alpha > \beta$, $\tilde{N}_\beta F(Q) \leq (\alpha/\beta)^n \tilde{N}_\alpha F(Q)$. Thus it is enough to show $\|\tilde{N}_\alpha F\|_{L^p(\mu)} \leq C\|\tilde{N}_\beta F\|_{L^p(\mu)}$. We claim that for $\gamma = (2+\frac{\alpha}{8})(1-\frac{\alpha}{8})^{-1} - 1$

$$\tilde{N}_\alpha F(Q) \leq C_{n,\alpha,\beta} \tilde{N}_\beta^\gamma F(Q), \tag{7.12}$$

which yields the desired inequality. Note that

$$\begin{aligned}\fint_{B(X,\frac{\alpha}{8}\delta(X))} F^2(Z)dZ &= \frac{C_\alpha}{\delta(X)^n} \int_{B(X,\frac{\alpha}{8}\delta(X))} F^2(Z)dZ \\ &= \frac{C_\alpha}{\delta(X)^n} \int_{B(X,\frac{\alpha}{8}\delta(X))\setminus B(X,\frac{\beta}{8}\delta(X))} F^2(Z)dZ \\ &\quad + \frac{C_\alpha}{\delta(X)^n} \int_{B(X,\frac{\beta}{8}\delta(X))} F^2(Z)dZ.\end{aligned}$$

Covering the region $B(X,\frac{\alpha}{8}\delta(X))\setminus B(X,\frac{\beta}{8}\delta(X))$ by balls $B_i = B(Y_i, r)$ with radius $r = (1-\frac{\alpha}{8})\delta(x)\frac{\beta}{8}$ and $Y_i \in B(X,\frac{\alpha}{8}\delta(X))\setminus B(X,\frac{\beta}{8}\delta(X))$, and noting that the number of such balls only depends on α, β, n, we have

$$\begin{aligned}&\fint_{B(X,\frac{\alpha}{8}\delta(X))} F^2(Z)dZ \\ &\quad \leq C'_{\alpha,\beta,n}\left(\frac{1}{\delta(X)^n} \sum_i \int_{B_i} F^2(Z)dZ + \fint_{B(X,\frac{\beta}{8}\delta(X))} F^2(Z)dZ \right).\end{aligned} \tag{7.13}$$

If $X \in \Gamma(Q)$ and $Y \in B(X,\frac{\alpha}{8}\delta(X))$, then $(1-\frac{\alpha}{8})\delta(X) \leq \delta(Y) \leq (1+\frac{\alpha}{8})\delta(X)$ and $Y \in \Gamma_\gamma(Q)$. Hence

$$\begin{aligned}\fint_{B_i} F^2(Z)dZ &\leq C \sup_{Y\in\Gamma_\gamma(Q)} \fint_{B(Y,r)} F^2(Z)dZ \\ &\leq C \sup_{Y\in\Gamma_\gamma(Q)} \fint_{B(Y,\frac{\beta}{8}\delta(Y))} F^2(Z)dZ = \tilde{N}_\beta^\gamma F(Q),\end{aligned} \tag{7.14}$$

which combined with (7.13) yields (7.12). □

Remark 7.3 Assume that $L_i u = 0$ for $i = 0$ or $i = 1$. Then $\|\tilde{N}u\|_{L^p(\sigma)} \sim \|Nu\|_{L^p(\sigma)}$ for $1 \leq p < \infty$.

Proof Since $\tilde{N}u(Q) \leq CN_{10/7}u(Q)$, Proposition 3.6 ensures that $\|\tilde{N}u\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}$. Since u is a solution for L_i, u^2 is a subsolution for L_i, and Lemma 1.1.8 of [12] guarantees that

$$u^2(X) \leq \sup_{B(X, \frac{\delta(X)}{16})} u^2(Y) \leq C \fint_{B(X, \frac{\delta(X)}{8})} u^2(Z) dZ.$$

Hence $Nu(Q) \leq C\tilde{N}u(Q)$, and $\|Nu\|_{L^p(\sigma)} \leq C\|\tilde{N}u\|_{L^p(\sigma)}$ follows. □

We still need a few preliminaries before we can get to the proof of Theorem 2.9. Recall that by assumption $0 \in \Omega$. Let $R_0 = \frac{1}{2^{30}} \min\{\delta(0), 1\}$. The following calculation shows that we may assume that $a(X) = 0$ for all $X \in \Omega$ such that $\delta(X) > 4R_0$. Cover the boundary $\partial\Omega$ by balls $\{B(Q_i, R_0/2)\}_{i=1}^M$ such that $Q_i \in \partial\Omega$ and $|Q_i - Q_j| \geq \frac{R_0}{2}$ for $i \neq j$. Note that M depends only on n, R_0, and diam Ω. Let $\{\varphi_i\}_{i=1}^M$ be a partition of unity associated with this covering satisfying $0 \leq \varphi_i \leq 1$, $\operatorname{spt} \varphi_i \subset B(Q_i, 2R_0)$, $\varphi_i \equiv 1$ on $B(Q_i, R_0)$, and $|\nabla\varphi_i| \leq 4/R_0$. Define

$$\psi_i(X) = \begin{cases} (\sum_{j=1}^M \varphi_j(X))^{-1}\varphi_i(X) & \text{if } \sum_{j=1}^M \varphi_j(X) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $X \in (\partial\Omega, \frac{1}{2}R_0) := \{Y \in \mathbb{R}^n : \exists Q_Y \in \partial\Omega \text{ with } |Q_Y - Y| = \delta(Y) \leq R_0/2\}$ there exists $Q_X \in \partial\Omega$ with $|Q_X - X| \leq R_0/2$ and $i \in \{1, \ldots, M\}$ such that $|Q_X - Q_i| < R_0/2$. Thus $X \in B(Q_i, R_0)$ and $\varphi_i(X) = 1$; therefore $\sum_{j=1}^M \psi_j(X) = 1$. If $X \in \mathbb{R}^n \setminus (\partial\Omega, 2R_0)$ then $\varphi_i(X) = 0$ and $\sum_{j=1}^M \psi_j(X) = 0$. Consider the matrix

$$A'(X) = \left(\sum_{j=1}^M \psi_j(X)\right) A_1(X) + \left(1 - \sum_{j=1}^M \psi_j(X)\right) A_0(X) \tag{7.15}$$

and the corresponding operator $L' = \operatorname{div} A'\nabla$. Note that A' is symmetric and L' is an elliptic second order divergence form operator with bounded coefficients in Ω. Denote by a' the deviation function

$$a'(X) = \sup_{B(X, \delta(X)/2)} \left|A'(Y) - A_0(Y)\right|.$$

Lemma 7.4 *Let A' be as in (7.15). Then $a'(X) = 0$ for $X \in \Omega$, with $\delta(X) > 4R_0$.*

Proof For $X \in \Omega$ with $\delta(X) > 4R_0$, if $Y \in B(X, \delta(X)/2)$, then $\delta(Y) \geq \frac{\delta(X)}{2} > 2R_0$, $A'(Y) = A_0$, and $a'(X) = 0$. □

Lemma 7.5 *If ω' denotes the elliptic measure associated with L' with pole at 0, then $\omega_1 \in B_p(\omega_0)$ if and only if $\omega' \in B_p(\omega_0)$.*

Proof Let G' be the Green's function for L' in Ω. Note that for $X \in (\partial\Omega, \frac{R_0}{2})$, $A'(X) = A_1(X)$. For $r < R_0/4$ and $Q \in \partial\Omega$, the comparison principle for NTA domains yields that for $i = 0, 1$

$$\frac{G_i(0, A(Q,r))}{r} \sim \frac{\omega_i(\Delta(Q,r))}{r^{n-1}}, \quad \text{and} \quad \frac{G'(0, A(Q,r))}{r} \sim \frac{\omega'(\Delta(Q,r))}{r^{n-1}}. \tag{7.16}$$

Moreover,

$$\frac{G_1(0, A(Q,r))}{G'(0, A(Q,r))} \sim 1. \tag{7.17}$$

Combining (7.16) and (7.17), we have

$$\begin{aligned} &\frac{G_1(0, A(Q,r))}{G_0(0, A(Q,r))} \sim \frac{\omega_1(\Delta(Q,r))}{\omega_0(\Delta(Q,r))} \quad \text{and} \\ &\frac{G_1(0, A(Q,r))}{G'(0, A(Q,r))} \sim \frac{\omega_1(\Delta(Q,r))}{\omega'(\Delta(Q,r))} \sim 1, \end{aligned} \tag{7.18}$$

which yields for every $Q \in \partial\Omega$ and for every $r < R_0/2$

$$\frac{\omega'(\Delta(Q,r))}{\omega_0(\Delta(Q,r))} \sim \frac{\omega_1(\Delta(Q,r))}{\omega_0(\Delta(Q,r))} \tag{7.19}$$

with constants that only depend on the NTA constants of Ω. Letting r tend to 0, we obtain that for every $Q \in \partial\Omega$

$$\frac{d\omega'}{d\omega_0}(Q) \sim \frac{d\omega_1}{d\omega_0}(Q). \tag{7.20}$$

□

Lemma 7.6 *Assume that*

$$\sup_{\Delta\subset\partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(Y) \frac{G_0(Y)}{\delta(Y)^2} dY \right\}^{1/2} < \varepsilon_0 \tag{7.21}$$

with $a(Y) = 0$ for $Y \in \Omega$ and $\delta(Y) > 4R_0$, where $R_0 = \frac{1}{2^{30}} \min\{\delta(0), 1\}$. Then there exists $C > 0$ such that for $X \in \Omega$ with $\delta(X) > 5R_0$

$$\sup_{\Delta\subset\partial\Omega} \left\{ \frac{1}{\omega_0^X(\Delta)} \int_{T(\Delta)} a^2(Y) \frac{G_0(X,Y)}{\delta(Y)^2} dY \right\}^{1/2} \le C\varepsilon_0. \tag{7.22}$$

Here C depends on NTA constants of Ω, the NTA character of Ω its diameter and R_0.

Proof If $\Delta = \Delta(Q,r)$ with $r \le 9/2R_0$ and $\delta(X) > 5R_0$, then for $Y \in T(\Delta)$ by the comparison principle and (7.16) we have

$$\frac{G_0(X,Y)}{G_0(Y)} \sim \frac{G_0(X, A(Q,r))}{G_0(A(Q,r))} \sim \frac{\omega_0^X(\Delta(Q,r))}{\omega_0(\Delta(Q,r))} \tag{7.23}$$

hence

$$\frac{1}{\omega_0^X(\Delta)}\int_{T(\Delta)} a^2(Y)\frac{G_0(X,Y)}{\delta(Y)^2}dY \sim \frac{1}{\omega_0(\Delta)}\int_{T(\Delta)} a^2(Y)\frac{G_0(Y)}{\delta(Y)^2}dY. \tag{7.24}$$

If $r \geq 9/2R_0$, then

$$\frac{1}{\omega_0^X(\Delta)}\int_{T(\Delta)} a^2(Y)\frac{G_0(X,Y)}{\delta(Y)^2}dY = \frac{1}{\omega_0^X(\Delta)}\int_{T(\Delta)\cap(\partial\Omega,4R_0)} a^2(Y)\frac{G_0(X,Y)}{\delta(Y)^2}dY.$$

Covering $\partial\Omega$ by balls $\{B(Q,R_0/2)\}_{i=1}^M$, if $\Delta_i = B(Q_i,9/2R_0)\cap\partial\Omega$ we have, using (7.24), that

$$\begin{aligned}
&\frac{1}{\omega_0^X(\Delta)}\int_{T(\Delta)\cap(\partial\Omega,4R_0)} a^2(Y)\frac{G_0(X,Y)}{\delta(Y)^2}dY \\
&\leq \frac{1}{\omega_0^X(\Delta)}\sum_{i=1}^M\int_{T(\Delta_i)} a^2(Y)\frac{G_0(X,Y)}{\delta(Y)^2}dY \\
&\lesssim \sum_{i=1}^M\left(\int_{T(\Delta_i)} a^2(Y)\frac{G_0(Y)}{\delta(Y)^2}dY\right)\frac{\omega_0^X(\Delta_i)}{\omega_0(\Delta_i)}\frac{1}{\omega_0^X(\Delta)} \lesssim \varepsilon_0 \qquad (7.25)
\end{aligned}$$

because ω_0, ω_0^X are doubling, $\omega^X(\Delta) \sim C\omega^X(\Delta_i)$, and by (7.23). □

The last preliminary concerns the existence of a family of dyadic cubes in $\partial\Omega$ whose "projections" in Ω provide a good covering of $\Omega \cap (\partial\Omega, 4R_0)$, with R_0 as above. Since Ω is a CAD in $\mathbb{R}^n$, both $\sigma = \mathcal{H}^{n-1}\llcorner\partial\Omega$ and ω_0 are doubling measures, and therefore $(\partial\Omega, |\,|, \sigma)$ and $(\partial\Omega, |\,|, \omega_0)$ are spaces of homogeneous type. Here $|\,|$ denotes the Euclidean distance in $\mathbb{R}^n$. M. Christ's construction (see [1]) ensures that there exists a family of dyadic cubes $\{Q_\alpha^k \subset \partial\Omega : k \in \mathbb{Z}, \alpha \in I_k\}$, $I_k \subset \mathbb{N}$ such that for every $k \in \mathbb{Z}$

$$\sigma\left(\partial\Omega \setminus \bigcup_\alpha Q_\alpha^k\right) = 0, \qquad \omega_0\left(\partial\Omega \setminus \bigcup_\alpha Q_\alpha^k\right) = 0. \tag{7.26}$$

Furthermore, the following properties are satisfied:

1. If $l \geq k$, then either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$.
2. For each (k,α) and each $l < k$, there is a unique β so that $Q_\alpha^k \subset Q_\beta^l$.
3. There exists a constant $C_0 > 0$ such that $\operatorname{diam} Q_\alpha^k \leq C_0 8^{-k}$.
4. Each Q_α^k contains a ball $B(Z_\alpha^k, 8^{-k-1})$.

The fact that $B(Z_\alpha^k, 8^{-k-1}) \subset Q_\alpha^k$ implies that $\operatorname{diam} Q_\alpha^k \geq 8^{-k-1}$. The Ahlfors regularity property of σ, combined with properties 3 and 4, ensure that there exists $C_1 > 1$ such that

$$C_1^{-1}8^{-k(n-1)} \leq \sigma\left(Q_\alpha^k\right) \leq C_1 8^{-k(n-1)}. \tag{7.27}$$

In addition, the doubling property of ω_0 yields

$$\omega_0\big(B\big(Z_\alpha^k, 8^{-k-1}\big)\big) \sim \omega_0\big(Q_\alpha^k\big). \tag{7.28}$$

For $k \in \mathbb{Z}$ and $\alpha \in I_k$, we define

$$\begin{aligned} I_\alpha^k = \big\{ & Y \in \Omega : \lambda 8^{-k-1} < \delta(Y) < \lambda 8^{-k+1}, \\ & \exists P \in Q_\alpha^k \text{ so that } \lambda 8^{-k-1} < |P - Y| < \lambda 8^{-k+1} \big\}, \end{aligned} \tag{7.29}$$

where $\lambda > 0$ is chosen so that for each k, the $\{I_\alpha^k\}_{\alpha \in I_k}$'s have finite overlaps and

$$\Omega \cap (\partial\Omega, 4R_0) \subset \bigcup_{\alpha, k \leq k_0} I_\alpha^k. \tag{7.30}$$

Here k_0 is chosen so that $4R_0 < \lambda 8^{-k-1}$; i.e., $k_0 = [\frac{\log\lambda - \log 32R_0}{\log 8}] + 1$. To see that such a $\lambda > 0$ can be found, note that if $I_\alpha^k \cap I_\beta^k \neq \emptyset$ there exist $Y \in I_\alpha^k \cap I_\beta^k$, $P_\alpha \in Q_\alpha^k$, and $P_\beta \in Q_\alpha^k$ so that

$$\lambda 8^{-k-1} < \delta(Y), \quad |P_\alpha - Y|, \quad |P_\beta - Y| < \lambda 8^{-k+1}.$$

Thus $|P_\alpha - P_\beta| \leq 2\lambda 8^{-k+1}$ and for $P \in Q_\beta^k$,

$$\begin{aligned} |P_\alpha - P| &\leq |P_\alpha - P_\beta| + |P_\beta - P| \leq 2\delta(Y) + \operatorname{diam} Q_\beta^k \\ &\leq 2\lambda 8^{-k+1} + C_0 8^{-k} \leq 8^{-k}(16\lambda + C_0). \end{aligned} \tag{7.31}$$

Thus (7.31) yields that given I_α^k, if Q_β^k is such that $I_\alpha^k \cap I_\beta^k \neq \emptyset$ then $Q_\beta^k \subset B(P_\alpha, 8^{-k}(16\lambda + C_0))$ for some $P_\alpha \in Q_\alpha^k$. Since $\{Q_\beta^k\}_{\beta \in I_k}$ is a disjoint collection, (7.27) yields that the number N of cubes Q_β^k so that $I_\alpha^k \cap I_\beta^k \neq \emptyset$ satisfies $NC^{-1}8^{-k(n-1)} \leq C8^{-k(n-1)}(16\lambda + C_0)^{n-1}$, i.e., $N \leq C^2(16\lambda + C_0)^{n-1}$. To show that the I_α^k's cover $(\partial\Omega, 4R_0)$, let $Y \in (\partial\Omega, 4R_0)$, $\delta(Y) \leq 4R_0 < \frac{1}{2^{28}} \min\{\delta(0), 1\}$ by choosing $\lambda \geq \frac{1}{8}\max\{\delta(0), 1\} + 1 + 64C_0$ we have that $\delta(Y) < \frac{\lambda}{8}$. Thus there exists $k \geq 2$ so that $\lambda 8^{-k-1} < \delta(Y) < \lambda 8^{-k+1}$ and $Q_Y \in \partial\Omega$ so that $|Q_Y - Y| = \delta(Y)$. Let $\rho_0 = \frac{1}{2}\min\{\delta(Y) - \lambda 8^{-k-1}, \lambda 8^{-k+1} - \delta(Y)\} > 0$. Since $\sigma(\partial\Omega \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k) = 0$ and $\sigma(\Delta(Q_Y, \rho_0)) \geq C^{-1}\rho_0^{n-1} > 0$ there exists $\alpha \in I_k$ so that $\Delta(Q_Y, \rho_0) \cap Q_\alpha^k \neq \emptyset$. Let $P_\alpha \in \Delta(Q_Y, \rho_0) \cap Q_\alpha^k$; then

$$\begin{aligned} \delta(Y) - \rho_0 &\leq |Q_Y - Y| - |P_\alpha - Q_Y| \leq |P_\alpha - Y| \leq |P_\alpha - Q_Y| + |Q_Y - Y| \\ &\leq \rho_0 + \delta(Y). \end{aligned} \tag{7.32}$$

Hence by the selection of ρ_0,

$$\frac{\delta(Y) + \lambda 8^{-k-1}}{2} \leq |P_\alpha - Y| \leq \frac{\delta(Y) + \lambda 8^{-k+1}}{2}.$$

Thus $Y \in I_\alpha^k$, provided that λ is chosen as above.

Next we proceed with the proof of Theorem 2.9, following the approach presented in [10]. Note that (2.12) implies that $\varepsilon(X) = A_1(X) - A_0(X) \equiv 0$ on $\partial\Omega$, i.e., $L_0 = L_1$ on $\partial\Omega$. Thus L_1 is regarded as a perturbation of L_0. Hence as in [10] the strategy consists of regarding the solution to L_1 with given boundary data as a perturbation of the solution to L_0 with the same boundary data. We consider the Dirichlet problem

$$\begin{cases} L_1 u_1 = 0 & \text{in } \Omega \\ u_1|_{\partial\Omega} = f \in L^2(\omega_0). \end{cases} \tag{7.33}$$

We need to show the following a priori estimate

$$\left\| N(u_1) \right\|_{L^2(\omega_0)} \le \|f\|_{L^2(\omega_0)}, \tag{7.34}$$

which is equivalent to the statement that $\omega_1 \in B_2(\omega_0)$. Assume that $f \in C(\partial\Omega)$ and u_1 is a solution of (7.33). Let u_0 satisfy

$$\begin{cases} L_0 u_0 = 0 & \text{in } \Omega \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{7.35}$$

Then

$$\left\| N(u_0) \right\|_{L^2(\omega_0)} \le \|f\|_{L^2(\omega_0)}$$

since $Nu_0(Q) \le CM_{\omega_0}(f)(Q)$ and u_1 is related to u_0 by the formula

$$u_1(X) = u_0(X) + \int_\Omega G_0(X,Y) L_0 u_1(Y)\,dY = u_0(X) + F(X).$$

Integration by parts shows that

$$F(X) = \int_\Omega G_0(X,Y)(L_0 - L_1)u_1(Y)\,dY = \int_\Omega \nabla_Y G_0(X,Y)\varepsilon(Y)\nabla u_1(Y)\,dY$$

where $\varepsilon(Y) = A_1(Y) - A_0(Y)$.

As in [10], the proof of Theorem 2.9 follows from the two lemmas below (Lemmas 7.7 and 7.8). We start with the analogue of Lemma 2.9 of [10].

Lemma 7.7 *Let Ω be a CAD and assume that (2.12) holds. Then there exist $C > 1$ and $M > 1$ such that for $Q_0 \in \partial\Omega$*

$$\tilde{N}F(Q_0) \le C\varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q_0) \tag{7.36}$$

and

$$\tilde{N}_{1/2}\big(\delta|\nabla F|\big)(Q_0) \le C\varepsilon_0\big[M_{\omega_0}\big(S_M(u_1)\big)(Q_0) + \tilde{N}\big(\delta|\nabla F|\big)(Q_0)\big]. \tag{7.37}$$

Therefore,

$$\int_{\partial\Omega}\big[\tilde{N}F(Q)^2 + \tilde{N}\big(\delta|\nabla F|\big)(Q)^2\big]d\omega_0(Q) \le C\varepsilon_0^2 \int_{\partial\Omega} S^2(u_1)(Q)\,d\omega_0(Q). \tag{7.38}$$

Here M_{ω_0} denotes the Hardy–Littlewood maximal function with respect to ω_0, and $S_\alpha(u)$ denotes the square function of u given by

$$S_\alpha^2(u)(Q) = \int_{\Gamma_\alpha(Q)} \left|\nabla u(X)\right|^2 \delta(X)^{2-n} dX. \tag{7.39}$$

Proof The proof follows the same guidelines of Lemma 2.9 in [10]. We estimate each term separately. First we show that there exists $M > 1$ so that for $Q_0 \in \partial\Omega$

$$\tilde{N} F(Q_0) \leq C\varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q_0). \tag{7.40}$$

Let $X \in \Gamma(Q_0)$ and set $B(X) = B(X, \delta(X)/4)$. We split the potential F into two pieces

$$F(Z) = F_1(Z) + F_2(Z) \tag{7.41}$$

where

$$F_1(Z) = \int_{B(X)} \nabla_Y G_0(Z,Y)\varepsilon(Y)\nabla u_1(Y)dY \tag{7.42}$$

and

$$F_2(Z) = \int_{\Omega\setminus B(X)} \nabla_Y G_0(Z,Y)\varepsilon(Y)\nabla u_1(Y)dY. \tag{7.43}$$

To estimate $\tilde{N}F(Q_0)$, let $X \in \Gamma(Q_0)$ and note that

$$\int_{B(X,\frac{\delta(X)}{8})} F^2(Z)\frac{dZ}{\delta(X)^n} \lesssim \fint_{B(X,\frac{\delta(X)}{8})} F_1^2(Z)dZ + \fint_{B(X,\frac{\delta(X)}{8})} F_2^2(Z)dZ.$$

We look at each term on the right-hand side separately. For $Y \in B(X)$, $\frac{3\delta(X)}{4} \leq \delta(Y) \leq \frac{5\delta(X)}{4}$, and either $\delta(X) < 8R_0$ or $\delta(X) \geq 8R_0$. If $\delta(X) \geq 8R_0$ then $\delta(Y) \geq 6R_0$ thus $\varepsilon(Y) = 0$. If $\delta(X) < 8R_0$ then $\delta(Y) < 10R_0$ and $|Y| \geq 8R_0$. In this case the Harnack principle ensures that $G_0(X) \sim G_0(Y)$. Furthermore, since ω_0 is doubling, for $Y \in \Gamma_{5/4}(Q_0)$ the relationship between the Green's function and the elliptic measure on NTA domains yields

$$\frac{G_0(X)}{\delta(X)} \sim \frac{\omega_0(\Delta(Q_0,\delta(X))}{\delta(X)^{n-1}} \sim \frac{\omega_0(\Delta(Q_0,\delta(Y))}{\delta(Y)^{n-1}} \sim \frac{G_0(Y)}{\delta(Y)}. \tag{7.44}$$

Therefore for $Y_0 \in B(X)$ either $\varepsilon(Y_0) = 0$ or for $\delta(X) < 8R_0$. In this case (7.44) and the doubling properties of ω_0 imply

$$\begin{aligned} \left|\varepsilon(Y_0)\right| &\lesssim \left(\fint_{B(X,\frac{\delta(X)}{8})} a^2(Y)dY\right)^{1/2} \\ &\lesssim \left(\fint_{B(X)} a^2(Y)dY\right)^{1/2} \end{aligned}$$

$$\lesssim \left(\int_{B(X)} \frac{a^2(Y)}{\delta(Y)^2} G_0(Y) \frac{\delta(Y)}{G_0(Y)} \frac{dY}{\delta^{n-1}(X)} \right)^{1/2}$$

$$\lesssim \left(\frac{1}{\delta(X)^{n-1}} \int_{B(X)} \frac{a^2(Y)}{\delta(Y)^2} G_0(Y) \frac{\delta(X)^{n-1}}{\omega_0(\Delta(Q_0, \delta(X)))} dY \right)^{1/2}$$

$$\lesssim \left(\frac{1}{\omega_0(\Delta(Q_0, \delta(X)))} \int_{B(X,\delta(X)/4)} \frac{a^2(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2}$$

$$\lesssim \left(\frac{1}{\omega_0(\Delta(Q_0, \delta(X)))} \int_{T(\Delta(Q_0,3\delta(X)))} \frac{a^2(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2}$$

$$\lesssim \left(\frac{1}{\omega_0(\Delta(Q_0, 3\delta(X)))} \int_{T(\Delta(Q_0,3\delta(X)))} \frac{a^2(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2} \lesssim \varepsilon_0. \quad (7.45)$$

Let $\tilde{G}_0(Z, Y)$ be the Green's function for L_0 in $2B(X) = B(X, \delta(X)/2)$. Let

$$\begin{aligned} K(Z, Y) &= G_0(Z, Y) - \tilde{G}_0(Z, Y), \\ \tilde{F}_1(Z) &= \int_{B(X)} \nabla_Y \tilde{G}_0(Z, Y) \varepsilon(Y) \nabla u_1(Y) dY \end{aligned} \quad (7.46)$$

and

$$\hat{F}_1(Z) = F_1(Z) - \tilde{F}_1(Z), \quad (7.47)$$

$$\begin{cases} L_0 \tilde{F} = \operatorname{div}[\varepsilon \nabla u_1 \chi_{B(X)}] & \text{in } 2B(X) \\ \tilde{F} = 0 & \text{on } \partial(2B(X)). \end{cases} \quad (7.48)$$

Using (7.45), as in [10], we have that

$$\begin{aligned} \int_{2B(X)} |\nabla \tilde{F}_1|^2 dZ &\le C \int_{2B(X)} A_0 \nabla \tilde{F}_1 \nabla \tilde{F}_1 dZ = C \int \nabla \tilde{F}_1 \varepsilon \nabla u_1 \chi_B dZ \\ &\le \frac{1}{2} \int_{B(X)} |\nabla \tilde{F}_1|^2 dZ + C\varepsilon_0^2 \int_{B(X)} |\nabla u_1|^2 dZ. \end{aligned} \quad (7.49)$$

Combining Sobolev inequality and (7.49) we obtain

$$\int_{2B(X)} |\tilde{F}_1|^2 dZ \le C\delta(X)^2 \int_{2B(X)} |\nabla \tilde{F}_1|^2 dZ \le C\varepsilon_p^2 \delta(X)^2 \int_{B(X)} |\nabla u_1|^2 dZ. \quad (7.50)$$

Thus since for $Z \in B(X)$, $\delta(Z) \sim \delta(X)$ (7.50) yields

$$\begin{aligned} \left(\fint_{B(X, \frac{\delta(X)}{8})} |\tilde{F}_1|^2 dZ \right)^{1/2} &\le C \left(\fint_{B(X, \frac{\delta(X)}{2})} |\tilde{F}_1|^2 dZ \right)^{1/2} \\ &\le C\varepsilon_0 \left(\int_{B(X)} |\nabla u_1|^2 \delta(Z)^{2-n} dZ \right)^{1/2}. \end{aligned} \quad (7.51)$$

If $X \in \Gamma(Q_0)$ and $Z \in B(X)$ then $Z \in \Gamma_2(Q_0)$ and from (7.51) we conclude

$$\left(\fint_{B(X,\frac{\delta(X)}{8})} |\tilde{F}_1|^2 dZ\right)^{1/2} \leq C\left(\fint_{B(X,\frac{\delta(X)}{2})} |\tilde{F}_1|^2 dZ\right)^{1/2} \leq C\varepsilon_0 S_2(u_1)(Q_0). \quad (7.52)$$

We now estimate $\hat{F}_1$ by writing

$$\hat{F}_1 = F_1 - \tilde{F}_1 = \int_{B(X)} \nabla_Y K(Z,Y)\varepsilon \nabla u_1(Y) dY. \quad (7.53)$$

That is,

$$\left|\hat{F}_1(Z)\right| \leq \varepsilon_0 \int_{B(X)} \left|\nabla_Y K(Z,Y)\right| \left|\nabla u_1(Y)\right| dY.$$

For fixed $Z \in B(X)$ we have that $L_0 K(Z,Y) = 0$ in $2B(X)$. Applying Cauchy–Schwarz and Cacciopoli's inequality (to K), we obtain

$$\left|\hat{F}_1(Z)\right| \leq \frac{C\varepsilon_0}{\delta(X)} \left(\int_{\frac{3}{2}B(X)} \left|K(Z,Y)\right|^2 dY\right)^{1/2} \left(\int_{B(X)} \left|\nabla u_1(Y)\right|^2 dY\right)^{1/2}. \quad (7.54)$$

Since $K(Z,-) \geq 0$ Harnack's inequality yields,

$$\begin{aligned} \left(\fint_{\frac{3}{2}B(X)} K(Z,Y)^2 dY\right)^{1/2} &\leq C\left(\fint_{\frac{3}{2}B(X)} K(Z,Y) dY\right) \\ &\leq C \fint_{\frac{3}{2}B(X)} |Z-Y|^{2-n} dY \end{aligned} \quad (7.55)$$

since $G_0(Z,Y) \lesssim \frac{1}{|Z-Y|^{n-2}}$. Thus since for $Y \in B(X)$, $\delta(X) \sim \delta(Y)$ combining (7.54) and (7.55) we have

$$\begin{aligned} \left(\fint_{2B(X)} \left|\hat{F}_1(Z)\right|^2 dZ\right)^{1/2} &\leq \frac{C\varepsilon_0}{\delta(X)} \left(\int_{2B(X)} \left(\fint_{\frac{3}{2}B(X)} \frac{dY}{|Z-Y|^{n-2}}\right)^2 dZ\right)^{1/2} \\ &\quad \times \left(\int_{B(X)} \left|\nabla u_1(Y)\right|^2 dY\right)^{1/2} \\ &\leq C\varepsilon_0 \delta(X)^{1-n/2} \left(\int_{B(X)} \left|\nabla u_1(Y)\right|^2 dY\right)^{1/2} \\ &\leq C\varepsilon_0 \left(\int_{B(X)} \left|\nabla u_1(Y)\right|^2 \delta(Y)^{2-n} dY\right)^{1/2} \\ &\leq C\varepsilon_0 S_2(u_1)(Q_0). \end{aligned} \quad (7.56)$$

Combining (7.47), (7.52) and (7.56) we obtain

$$\left(\fint_{B(X,\frac{\delta(X)}{2})} \left|F_1(Z)\right|^2 dZ\right)^{1/2} \leq C\varepsilon_0 S_2(u_1)(Q_0). \quad (7.57)$$

Next we give a pointwise estimate for $F_2(Z)$ when $Z \in B(X, \delta(X)/8)$. Note that in this case Z is away from the pole of the Green's function that appears as an integrand in the definition of F_2. To estimate $F_2(Z)$ for $Z \in B(X, \delta(X)/8)$ we consider two cases: $\delta(X) \le 4R_0$ and $\delta(X) > 4R_0$. In the second case we use Lemma 7.6.

Assume that $\delta(X) \le 4R_0$ and let $Q_X \in \partial\Omega$ be such that $|X - Q_X| = \delta(X)$. Let $\Omega_0 = \Omega \cup B(Q_X, \frac{\delta(X)}{2})$ and $\Delta_0 = \partial\Omega \cap B(Q_X, \delta(X)/2)$. For $j \ge 1$ define $\Omega_j = \Omega \cap B(Q_X, 2^{j-1}\delta(X))$ with $j = 1, \dots, N$ and $2^{14}R_0 \le 2^{N-1}\delta(X) < 2^{15}R_0$. Let $\tilde{R}_j = \Omega_{2j} \setminus \Omega_{2j-2}$, $\Delta_j = \partial\Omega \cap B(Q_X, 2^{j-1}\delta(X))$ and $A_j = A(Q_X, 2^{j-1}\delta(X)) \in \Omega_j$. We now follow the argument that appears in [10] using the dyadic surface cubes constructed by M. Christ and described above (see (7.26)) and their interior projections (see (7.29)). Note that $\Omega_0 \subset \bigcup_{Q^k_\alpha \subset 3\Delta_0} I^k_\alpha$. In fact if $Y \in \Omega_0$ then $\delta(Y) \le |Y - Q_X| < \frac{\delta(X)}{2} < 2R_0$. As in the proof of (7.30) there exists $k \ge 2$ so that $\frac{\lambda}{8} < 8^k\delta(Y) < 8\lambda$ and $Q_Y \in \partial\Omega$ with $|Y - Q_Y| = \delta(Y)$. For $\rho_0 = \min\{\frac{\delta(Y)-\lambda 8^{-k-1}}{2}, \frac{\lambda 8^{-k+1}-\delta(Y)}{2}\}$ there exists Q^k_α so that $P_\alpha \in Q^k_\alpha \cap \Delta(Q_Y, \rho_0)$ and $Y \in I^k_\alpha$. For any $P \in Q^k_\alpha$,

$$
\begin{aligned}
|P - Q_X| &\le |P - P_\alpha| + |P_\alpha - Q_Y| + |Q_Y - Y| + |Y - Q_X| \\
&\le \operatorname{diam} Q^k_\alpha + \rho_0 + \delta(Y) + \frac{\delta(X)}{2} \\
&< C_0 8^{-k} + \rho_0 + \delta(Y) + \frac{\delta(X)}{2} \\
&< \frac{8C_0}{\lambda}\delta(Y) + \delta(Y) + \frac{\delta(X)}{2} + \rho_0 \\
&< \frac{9}{8}\delta(Y) + \frac{\delta(X)}{2} + \frac{\delta(Y)}{2} \\
&< 2\delta(Y) + \frac{\delta(X)}{2} < \delta(X) + \frac{\delta(X)}{2} < \frac{3\delta(X)}{2}
\end{aligned} \tag{7.58}
$$

which implies that $Q^k_\alpha \subset 3\Delta_0$. We now estimate $F_2(Z)$ for $Z \in B(X, \delta(X)/8)$ as follows.

$$
\begin{aligned}
|F_2(Z)| \le &\left|\int_{\Omega_0} \nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y)dY\right| \\
&+ \sum_{j=1}^{N}\left|\int_{\tilde{R}_j \cap (\Omega \setminus B(X))} \nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y)dY\right| \\
&+ \int_{(\Omega\setminus B(X))\cap(\partial\Omega, 4R_0)\setminus B(Q_X, 2^{15}R_0)} \left|\nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y)\right|dY.
\end{aligned} \tag{7.59}
$$

We estimate each term separately. To estimate the first term we note that

$$
\begin{aligned}
|F^0_2(Z)| &= \left|\int_{\Omega_0} \nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y)dY\right| \\
&\le \int_{\Omega_0} |\nabla_Y G_0(Z, Y)||\varepsilon(Y)||\nabla u_1(Y)|dY \\
&\le \lim_{\varepsilon\to 0^+} \int_{\Omega_0\setminus(\partial\Omega, \varepsilon)} |\nabla_Y G_0(Z, Y)||\varepsilon(Y)||\nabla u_1(Y)|dY.
\end{aligned} \tag{7.60}
$$

The goal is to estimate

$$F_2^\varepsilon(Z)=\int_{\Omega_0\setminus(\partial\Omega,\varepsilon)}\big|\nabla_Y G_0(Z,Y)\big|\big|\varepsilon(Y)\big|\big|\nabla u_1(Y)\big|dY$$

independently of $\varepsilon>0$. In particular

$$F_0^\varepsilon(Z)\le\sum_{\substack{Q_\alpha^k\subset 3\Delta_0\\ \varepsilon<\lambda 8^{-k-1}}}\sup_{I_\alpha^k}\big|\varepsilon(Y)\big|\bigg(\int_{I_\alpha^k}\big|\nabla_Y G_0(Z,Y)\big|^2dY\bigg)^{1/2}$$
$$\times\bigg(\int_{I_\alpha^k}\big|\nabla u_1(Y)\big|^2dY\bigg)^{1/2}. \tag{7.61}$$

By Cacciopoli's inequality

$$\bigg(\int_{I_\alpha^k}\big|\nabla_Y G_0(Z,Y)\big|^2dY\bigg)^{1/2}\le\frac{C}{\operatorname{diam}Q_\alpha^k}\bigg(\int_{\hat I_\alpha^k}\big|G_0(Z,Y)\big|^2dY\bigg)^{1/2} \tag{7.62}$$

where $\hat I_\alpha^k=\{Y\in\Omega:\exists Z\in I_\alpha^k,\ |Z-Y|<\frac{\delta(Z)}{8^4}\}$. By the comparison principle for NTA domains, the Harnack principle, and the doubling properties of ω^Z and ω_0 we have for $Y\in\hat I_\alpha^k$

$$\frac{G_0(Z,Y)}{G_0(Y)}\sim\frac{G_0(Z,A_0)}{G_0(A_0)}\sim\frac{\omega^Z(\Delta_0)}{\omega_0(\Delta_0)}. \tag{7.63}$$

Thus for $Y\in\hat I_\alpha^k$ with $Q_\alpha^k\subset 3\Delta_0$

$$\frac{G_0(Z,Y)}{G_0(Y)}\le\frac{C}{\omega_0(\Delta_0)}. \tag{7.64}$$

Combining (7.61), (7.62), (7.63), and (7.64) we have that

$$\big|F_2^\varepsilon(Z)\big|\lesssim\sum_{\substack{Q_\alpha^k\subset 3\Delta_0\\ k\le k_\varepsilon}}\frac{1}{\omega_0(\Delta_0)}\bigg(\int_{\hat I_\alpha^k}\frac{G_0^2(Y)a^2(Y)}{\delta(Y)^2}dY\bigg)^{1/2}$$
$$\times\bigg(\int_{I_\alpha^k}\big|\nabla u_1(Y)\big|^2dY\bigg)^{1/2}. \tag{7.65}$$

Note that for $Y\in\hat I_\alpha^k$, there exist $Z\in I_\alpha^k$ and $P_\alpha\in Q_\alpha^k$ so that $\frac{\lambda}{8}<|Z-P_\alpha|<\lambda 8$, $\lambda 8^{-k-1}-\lambda 8^{-k-3}<|Y-P_\alpha|<\lambda 8^{-k+1}+\lambda 8^{-k-3}$ and $\delta(Z)(1-8^{-4})<\delta(Y)<\delta(Z)(1+8^{-4})$. That is, $|P_\alpha-Q_Y|\le|P_\alpha-Z|+|Z-Y|\le\lambda 8^{-k+1}+\delta(Z)8^{-4}\le\delta(Z)(64+8^{-4})\le 65\delta(Y)$. Now using the doubling property of ω_0 we have

$$G_0(Y)\sim\frac{\omega_0(\Delta(Q_Y,\delta(Y)))}{\delta(Y)^{n-2}}\sim\frac{\omega_0(\Delta(P_\alpha,\delta(Y)))}{\delta(Y)^{n-2}}. \tag{7.66}$$

Recall that there exists $Z_\alpha^k \in \partial\Omega$ such that $\Delta(Z_\alpha^k, 8^{-k-1}) \subset Q_\alpha^k \subset \Delta(Z_\alpha^k, 2C_0 8^{-k})$ (see the construction of the Q_α^k) and $|P_\alpha - Z_\alpha^k| \le \operatorname{diam} Q_\alpha^k \le C_0 8^{-k} \sim \delta(Y)$. Again by the doubling property of ω_0 we have that (7.66) yields for $Y \in I_\alpha^k$

$$G_0(Y) \sim \frac{\omega_0(\Delta(Z_\alpha^k, 8^{-k-1}))}{(8^{-k})^{n-2}} \sim \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \tag{7.67}$$

and combining (7.65) with (7.67) we have

$$\begin{aligned}\left|F_2^\varepsilon(Z)\right| \le \sum_{\substack{Q_\alpha^k \subset 3\Delta_0 \\ k+1 \le \frac{\log\lambda - \log\varepsilon}{8}}} \frac{1}{\omega_0(\Delta_0)} \left(\int_{\hat{I}_\alpha^k} \frac{G_0(Y)a^2(Y)}{\delta(Y)^2} dY\right)^{1/2} \\ \times \left(\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} \left|\nabla u_1(Y)\right|^2 dY\right)^{1/2}.\end{aligned} \tag{7.68}$$

To finish the estimate of $F_2^\varepsilon(Z)$ for $Z \in B(X, \delta(X)/8)$ we use a "stopping time" argument. For $j \in \mathbb{Z}$ let $M \ge 64(1 + C_0)$ and

$$O_j = \left\{Q \in 3\Delta_0 : T_\varepsilon u_1(Q) = \left(\int_{(\Gamma_M(Q)\setminus B_{2\varepsilon}(Q))\cap(\partial\Omega, 4R_0)} \left|\nabla u_1(Y)\right|^2 \delta(Y)^{2-n} dY\right)^{1/2} > 2^j\right\}.$$

We say that a surface cube Q_α^k in the dyadic grid belongs to J_j if

$$\omega_0\left(Q_\alpha^k \cap O_j\right) \ge \frac{1}{2}\omega_0\left(Q_\alpha^k\right) \quad \text{and} \quad \omega_0\left(Q_\alpha^k \cap O_{j+1}\right) < \frac{1}{2}\omega_0\left(Q_\alpha^k\right) \tag{7.69}$$

and it belongs to J_∞ if

$$\omega_0\left(Q_\alpha^k \cap \left\{T_\varepsilon u_1(Q) = 0\right\}\right) \ge \frac{1}{2}\omega_0\left(Q_\alpha^k\right). \tag{7.70}$$

Note that there exists $0 < c_0 < 1$ depending on the doubling constant of ω_0 so that for $\tilde{O}_j = \{M_{\omega_0}(\chi_{O_j}) > c_0\}$; if $Q_\alpha^k \in J_j$ then $Q_\alpha^k \subset \tilde{O}_j$ and

$$\omega_0\left(Q_\alpha^k \cap \tilde{O}_j \setminus O_{j+1}\right) \ge \frac{1}{2}\omega_0\left(Q_\alpha^k\right). \tag{7.71}$$

In fact, for Q_α^k there exists $Z_\alpha^k \in Q_\alpha^k$ so that

$$\Delta\left(Z_\alpha^k, 8^{-k-1}\right) \subset Q_\alpha^k \subset \Delta\left(Z_\alpha^k, 2C_0 8^{-k}\right). \tag{7.72}$$

Moreover, if $Q_\alpha^k \in J_j$ for $P \in Q_\alpha^k$, $|Z_\alpha^k - P| \le \operatorname{diam} Q_\alpha^k \le C_0 8^{-k}$ thus

$$\Delta\left(Z_\alpha^k, 2C_0 8^{-k}\right) \subset \Delta\left(P, 3C_0 8^{-k}\right) \subset \Delta\left(Z_\alpha^k, 4C_0 8^{-k}\right) \tag{7.73}$$

and by (7.72) and the doubling property of ω_0 we have

$$\begin{aligned} M_{\omega_0}(\chi_{O_j})(P) &\ge \frac{\omega_0(\Delta(P, 3C_0 8^{-k}) \cap O_j)}{\omega_0(\Delta(P, 3C_0 8^{-k})} \\ &\ge \frac{\omega_0(\Delta(Z_\alpha^k, 2C_0 8^{-k}) \cap O_j)}{\omega_0(\Delta(Z_\alpha^k, 4C_0 8^{-k}))} \end{aligned}$$

$$\gtrsim \frac{\omega_0(Q_\alpha^k \cap O_j)}{\omega_0(\Delta(Z_\alpha^k, 8^{-k-1}))}$$

$$\gtrsim \frac{\omega_0(Q_\alpha^k \cap O_j)}{\omega_0(Q_\alpha^k)} \geq c_0. \tag{7.74}$$

We conclude that if $Q_\alpha^k \in J_j$ then $Q_\alpha^k \subset \tilde{O}_j$. Since $O_{j+1} \subset O_j \subset \tilde{O}_j$

$$\omega_0\big(Q_\alpha^k \cap \tilde{O}_j \setminus O_{j+1}\big) = \omega_0\big(Q_\alpha^k \cap O_{j+1}^c\big)$$
$$= \omega_0\big(Q_\alpha^k\big) - \omega_0\big(Q_\alpha^k \cap O_{j+1}\big) > \frac{1}{2}\omega_0\big(Q_\alpha^k\big), \tag{7.75}$$

which ensures that $Q_\alpha^k \subset \{Q \in \partial\Omega : M_{\omega_0}(\chi_{\tilde{O}_j \setminus O_{j+1}})(Q) > c_0\} = U_j$. A weak type inequality for M_{ω_0} applied to $\chi_{\tilde{O}_j \setminus O_{j+1}}$ and χ_{O_j} yields

$$\omega_0(U_j) \leq C\omega_0(\tilde{O}_j \setminus O_{j+1}) \leq C\omega_0(O_j). \tag{7.76}$$

Note that for each $\varepsilon > 0$ $T_\varepsilon u_1(Q)$ is bounded. Thus for $Q_\alpha^k \subset 3\Delta_0$ either $T_\varepsilon u_1 \equiv 0$ or there exists j_0 so that

$$2^{j_0-1} \leq ⨍_{Q_\alpha^k} T_\varepsilon u_1(Q) d\omega_0(Q) < 2^{j_0}.$$

In the first case $Q_\alpha^k \in J_\infty$, in the second $\omega_0(Q_\alpha^k \cap O_j) < \frac{1}{2}\omega_0(Q_\alpha^k)$ for $j \geq j_0$. Furthermore either there exists $j < j_0$ so that (7.69) is satisfied or for all $l \in \mathbb{Z}$, $\omega_0(Q_\alpha^k \cap O_l) < \frac{1}{2}\omega_0(Q_\alpha^k)$ which implies that $\omega_0(Q_\alpha^k \cap \{T_\varepsilon u_1(Q) = 0\}) \geq \frac{1}{2}\omega(Q_\alpha^k)$. In this case $Q_\alpha^k \in J_\infty$ and

$$Q_\alpha^k \subset \big\{Q \in \partial\Omega : M_{\omega_0}(\chi_{\{T_\varepsilon u_1 = 0\}})(Q) > c_0\big\} = U_\infty.$$

As above a weak type inequality on the maximal function yields that $\omega_0(U_\infty) \leq C\omega_0(O_\infty)$ where $O_\infty = \{Q \in \partial\Omega : T_\varepsilon u_1(Q) = 0\}$. Note that if $Q_\alpha^k \in J_\infty$, $\omega_0(Q_\alpha^k \cap O_\infty) \geq \frac{1}{2}\omega_0(Q_\alpha^k)$.

We now go back to our estimate of F_2^ε for $Z \in B(X, \delta(X/8))$. Combining (7.68), the Cauchy–Schwarz inequality, and letting $k_\varepsilon = \frac{\log\lambda - \log\varepsilon}{8} - 1$ we have

$$\big|F_2^\varepsilon(Z)\big| \leq \frac{1}{\omega_0(\Delta_0)} \sum_j \sum_{k \leq k_\varepsilon} \sum_{Q_\alpha^k \in J_j} \left(\int_{I_\alpha^k} \frac{G_0(Y)a^2(Y)}{\delta(Y)^2} dY\right)^{1/2}$$
$$\times \left(\frac{\omega_0(Q_\alpha^k)}{(\text{diam } Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} \big|\nabla u_1(Y)\big|^2 dY\right)^{1/2}$$
$$+ \frac{1}{\omega_0(\Delta_0)} \sum_{k \leq k_\varepsilon} \sum_{Q_\alpha^k \in J_\infty} \left(\int_{I_\alpha^k} \frac{G_0(Y)a^2(Y)}{\delta(Y)^2} dY\right)^{1/2}$$

$$\times \left(\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} \left|\nabla u_1(Y)\right|^2 dY \right)^{1/2}$$

$$\lesssim \frac{1}{\omega_0(\Delta_0)} \sum_j \left(\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_j} \int_{I_\alpha^k} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY \right)^{1/2}$$

$$\times \left(\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_j} \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} \left|\nabla u_1(Y)\right|^2 dY \right)^{1/2}$$

$$+ \frac{1}{\omega_0(\Delta_0)} \left(\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_\infty} \int_{I_\alpha^k} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY \right)^{1/2}$$

$$\times \left(\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_\infty} \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} \left|\nabla u_1(Y)\right|^2 dY \right)^{1/2}. \qquad (7.77)$$

Note that if $Q_\alpha^k, Q_\beta^l \in J_j$, and $Q_\alpha^k \cap Q_\beta^l \neq \emptyset$ then either $Q_\alpha^k \subset Q_\beta^l$ or $Q_\beta^l \subset Q_\alpha^k$. Since $Q_\alpha^k \subset \Delta(Z_\alpha^k, C_0 8^{-k})$ by construction, then for $Y \in I_\beta^l$ there exists $P \in Q_\beta^l$ so that $\lambda/8 < 8^k|P-Y| < 8\lambda$ and $|Y - Z_\alpha^k| \le C_0 8^{-k} + \lambda 8^{-k+1}$ thus $Y \in T(\Delta(Z_\alpha^k, (C_0 + 8\lambda)8^{-k}))$. Furthermore since $\Delta(Z_\alpha^k, 8^{-k-1}) \subset Q_\alpha^k$, ω_0 is doubling, the I_α^k's have finite overlap, and (7.75) we conclude that

$$\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_j} \int_{I_\alpha^k} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY \lesssim \sum_{\substack{Q_\alpha^k \in J_j \\ Q_\alpha^k \text{ disjoint}}} \int_{T(\Delta(Z_\alpha^k, (C_0+8\lambda)8^{-k}))} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY$$

$$\lesssim \varepsilon_p^2 \sum_{\substack{Q_\alpha^k \in J_j \\ Q_\alpha^k \text{ disjoint}}} \omega_0\big(Q_\alpha^k\big) \lesssim \varepsilon_p^2 \omega_0(O_j). \qquad (7.78)$$

Similarly we obtain

$$\sum_{k \le k_\varepsilon} \sum_{Q_\alpha^k \in J_\infty} \int_{I_\alpha^k} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY \lesssim \varepsilon_p^2 \sum_{\substack{Q_\alpha^k \in J_\infty \\ Q_\alpha^k \text{ disjoint}}} \omega_0\big(Q_\alpha^k\big) \lesssim \varepsilon_p^2 \omega_0(O_\infty). \qquad (7.79)$$

To estimate the other term note that $I_\alpha^k \subset \Gamma_M(P)$ for all $P \in Q_\alpha^k$ and $M > 64 + 8C_0$. In fact if $Y \in I_\alpha^k$, $\delta(Y) > \lambda 8^{-k-1}$ and there is $P' \in Q_\alpha^k$ so that $|Y - P'| < \lambda 8^{-k+1}$, thus for $P \in Q_\alpha^k$ $|Y-P| < \lambda 8^{-k+1} + \operatorname{diam} Q_\alpha^k \le \lambda 8^{-k+1} + C_0 8^{-k} < \delta(Y)(1+M)$ and $Y \in \Gamma_M(P)$. So if $Q_\beta^l \subset Q_\alpha^k$, $I_\alpha^k \subset \Gamma_M(P)$ for every $P \in Q_\beta^l$ and since the I_α^k's have finite overlap then denoting by $S_M^{\varepsilon,l} = (\Gamma_M(Q) \setminus B_\varepsilon(Q)) \cap (\partial\Omega, 4R_0) \cap$

$\{\frac{\lambda}{8} < 8^l \delta(Y) < 8\lambda\}$ we have using (7.75)

$$\begin{aligned}
&\sum_{k\le k_\varepsilon}\sum_{Q_\alpha^k\in J_j}\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}}\int_{I_\alpha^k}\left|\nabla u_1(Y)\right|^2 dY\\
&\lesssim \sum_{k\le k_\varepsilon}\sum_{Q_\alpha^k\in J_j}\omega_0\big(Q_\alpha^k\big)\int_{I_\alpha^k}\left|\nabla u_1(Y)\right|^2\delta(Y)^{2-n}dY\\
&\lesssim \sum_{\substack{Q_\alpha^k\in J_j\\ Q_\alpha^k \text{ disjoint}\\ k\le k_\varepsilon}}\ \sum_{\substack{Q_\beta^l\subset Q_\alpha^k\\ Q_\beta^l\in J_j}}\omega_0\big(Q_\beta^l\big)\int_{I_\beta^l}\left|\nabla u_1(Y)\right|^2\delta(Y)^{2-n}dY\\
&\lesssim \sum_{\substack{Q_\alpha^k\in J_j\\ Q_\alpha^k \text{ disjoint}\\ k\le k_\varepsilon}}\ \sum_{\substack{Q_\beta^l\subset Q_\alpha^k\\ Q_\beta^l\in J_j}}\omega_0\big(\tilde{O}_j\setminus O_{j+1}\cap Q_\beta^l\big)\int_{I_\beta^l}\left|\nabla u_1(Y)\right|^2\delta(Y)^{2-n}dY\\
&\lesssim \sum_{\substack{Q_\alpha^k\in J_j\\ Q_\alpha^k \text{ disjoint}\\ k\le k_\varepsilon}}\ \sum_{\substack{Q_\beta^l\subset Q_\alpha^k\\ Q_\beta^l\in J_j}}\int_{\tilde{O}_j\setminus O_{j+1}\cap Q_\beta^l}\int_{S_M^{\varepsilon,l}}\left|\nabla u_1(Y)\right|^2\delta(Y)^{2-n}dY\,d\omega_0(Q)\\
&\lesssim \sum_{\substack{Q_\alpha^k\in J_j\\ Q_\alpha^k \text{ disjoint}\\ k\le k_\varepsilon}}\ \sum_{l}\int_{\tilde{O}_j\setminus O_{j+1}\cap Q_\alpha^k}\int_{S_M^{\varepsilon,l}}\left|\nabla u_1(Y)\right|^2\delta(Y)^{2-n}dY\,d\omega_0(Q)\\
&\lesssim \sum_{\substack{Q_\alpha^k\in J_j\\ Q_\alpha^k \text{ disjoint}\\ k\le k_\varepsilon}}\int_{\tilde{O}_j\setminus O_{j+1}\cap Q_\alpha^k}T_\varepsilon u_1(Q)^2 d\omega_0(Q)\\
&\lesssim \int_{\tilde{O}_j\setminus O_{j+1}}T_\varepsilon u_1(Q)^2 d\omega_0(Q).
\end{aligned}\tag{7.80}$$

Note that if $Q_\alpha^k \in J_\infty$ for $k \le k_\varepsilon$ since $I_\alpha^k \subset \Gamma_M(P)\setminus B_\varepsilon(P)$ then

$$\int_{I_\alpha^k}|\nabla u_1|^2\delta(Y)^{2-n}dY \le T_\varepsilon u_1(Q)^2 = 0.$$

Combining this remark with (7.76), (7.77), (7.78), and (7.80) we conclude that for $Z \in B(X, \delta(X)/8)$

$$\begin{aligned} \left|F_2^\varepsilon(Z)\right| &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j \omega_0(O_j)^{1/2} \left(\int_{\tilde{O}_j \setminus O_{j+1}} T_\varepsilon u_1(Q)^2 d\omega_0(Q) \right)^{1/2} \\ &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j 2^j \omega_0(O_j)^{1/2} \omega_0(\tilde{O}_j \setminus O_{j+1})^{1/2} \\ &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j 2^j \omega_0(O_j)^{1/2} \omega_0(U_j)^{1/2} \\ &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j 2^j \omega_0(O_j) \\ &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \int_{\Delta_0} T_\varepsilon u_1(Q) d\omega_0(Q). \end{aligned} \tag{7.81}$$

The last inequality comes from the fact that $\sum 2^j \omega_0(O_j) = \sum 2^j \omega_0(O_j \setminus O_{j+1}) + \sum_j 2^j \omega_0(O_j)$ which ensures that $\sum 2^j \omega_0(O_j) = \frac{1}{2} \sum 2^j \omega_0(O_j \setminus O_{j+1})$. It is important to note that at each step the constants involved are independent of $\varepsilon > 0$. Combining (7.60), (7.81), as well as the doubling property of ω_0 we have that

$$\begin{aligned} \left|F_2^0(Z)\right| &\le C \lim_{\varepsilon \to 0} \frac{\varepsilon_0}{\omega_0(\Delta_0)} \int_{\Delta_0} T_\varepsilon u_1(Q) d\omega_0(Q) \\ &\le C \frac{\varepsilon_0}{\omega_0(\Delta_0)} \int_{\Delta_0} S_M(u_1)(Q) d\omega_0(Q) \\ &\le C \varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q). \end{aligned} \tag{7.82}$$

To estimate the second part of (7.59) recall that for $j \ge 1$, $\tilde{R}_j = \Omega_{2j} \setminus \Omega_{2j-2}$, $\Delta_j = \Delta(Q_X, 2^{j-1}\delta(X))$, $\Omega_j = B(Q_X, 2^{j-1}\delta(X)) \cap \Omega$ and

$$A_j = A\big(Q_X, 2^{j-1}\delta(X)\big) \in \Omega_j.$$

Denote $R_j = \tilde{R}_j \setminus B(X)$. To estimate

$$F_2^j(Z) = \int_{R_j} \nabla_Y G_0(Z, Y) \varepsilon(Y) \nabla u_1(Y) dY \tag{7.83}$$

for $Z \in B(X, \delta(X)/8)$ divide R_j as follows

$$R_j = R_j \cap \big(\partial\Omega, 2^{2j-6}\delta(X)\big) \cup R_j \setminus \big(\partial\Omega, 2^{2j-6}\delta(X)\big). \tag{7.84}$$

Let $V_j = R_j \cap (\partial\Omega, 2^{2j-6}\delta(X))$ and $W_j = R_j \setminus (\partial\Omega, 2^{2j-6}\delta(X))$. Note that $V_j \subset \bigcup_{Q_\alpha^k \subset 3\Delta_{2j} \setminus \frac{1}{3}\Delta_{2j-2}} I_\alpha^k$. In fact if $Y \in V_j$ then $2^{2j-3}\delta(X) \le |Y - Q_X| < 2^{2j-1}\delta(X)$, $\delta(Y) < 4R_0$ and there exists k such that $8^{-k-1}\lambda < \delta(Y) < 8^{-k+1}\lambda$ and $Y \in I_\alpha^k$ for some α. For $\rho_0 = \frac{1}{2}\min\{\delta(Y) - \lambda 8^{-k-1}, \lambda 8^{-k+1} - \delta(Y)\}$ there exists $P \in Q_\alpha^k \cap \Delta(Q_Y, \rho_0)$ such that

$$\begin{aligned}|P - Q_X| &\geq |Q_X - Q_Y| - |P - Q_Y| \geq |Y - Q_X| - |Y - Q_Y| - |P - Q_Y| \\ &\geq 2^{2j-3}\delta(X) - \rho_0 - \delta(Y) \geq 2^{2j-3}\delta(X) - \frac{3}{2}\delta(Y) \geq \left(2^{2j-3} - 2^{2j-5}\right)\delta(X) \\ &> \frac{2^{2j-3}}{3}\delta(X).\end{aligned}$$

Following the same pattern of the proof above we have for $Z \in B(X, \delta(X)/8)$

$$\begin{aligned}&\int_{V_j} \left|\nabla_Y G_0(Z, Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1(Y)\right| dY \\ &\quad \leq \lim_{\varepsilon \to 0} \int_{V_j \setminus (\partial\Omega, \varepsilon)} \left|\nabla_Y G_0(Z, Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1(Y)\right| dY \end{aligned} \tag{7.85}$$

and

$$\begin{aligned}&\int_{V_j \setminus (\partial\Omega, \varepsilon)} \left|\nabla_Y G_0(Z, Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1\right| dY \\ &\lesssim \sum_{\substack{Q_\alpha^k \subset 3\Delta_{2j} \setminus \frac{1}{3}\Delta_{2j-2} \\ k \leq k_\varepsilon}} \left(\int_{I_\alpha^k} \frac{G_0(Z, Y)^2 a^2(Y)}{\delta(Y)^2} dY\right)^{1/2} \left(\int_{I_\alpha^k} |\nabla u_1|^2 dY\right)^{1/2}. \end{aligned} \tag{7.86}$$

For $Y \in I_\alpha^k \cap V_j$ and $Z \in B(X, \delta(X)/8)$, (7.64) and the vanishing properties of the Green's function at the boundary of an NTA domain yield

$$G_0(Z, Y) \lesssim \left(\frac{\delta(Z)}{\delta(X)2^{2j-1}}\right)^\beta G_0(A_{2j}, Y) \lesssim 2^{-2\beta j} G_0(Y) \frac{1}{\omega_0(\Delta_{2j})}. \tag{7.87}$$

Moreover for $Y \in I_\alpha^k$, $G_0(Y) \sim \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}}$ as in (7.67). Thus combining (7.86), (7.87), and the remark above we have

$$\begin{aligned}&\int_{V_j \setminus (\partial\Omega, \varepsilon)} \left|\nabla_Y G_0(Z, Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1\right| dY \\ &\lesssim \sum_{\substack{Q_\alpha^k \subset 3\Delta_{2j} \setminus \frac{1}{3}\Delta_{2j-2} \\ k \leq k_\varepsilon}} 2^{-2\beta j} \frac{1}{\omega_0(\Delta_{2j})} \left(\int_{I_\alpha^k} \frac{G_0(Y) a^2(Y)}{\delta(Y)^2} dY\right)^{1/2} \\ &\quad \times \left(\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \int_{I_\alpha^k} |\nabla u_1|^2 dY\right)^{1/2}. \end{aligned} \tag{7.88}$$

A "stopping time" argument yields, as in (7.82), that

$$\int_{V_j} \left|\nabla_Y G_0(Z,Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1(Y)\right| dY \le C\varepsilon_0 2^{-2\beta j} M_{\omega_0}\big(S_M(u_1)\big)(Q). \quad (7.89)$$

To estimate the corresponding integral over W_j, cover W_j with balls $B(X_{jl}, 2^{2j-8}\delta(X))$ such that $X_{jl} \in W_j$ and the $B(X_{jl}, 2^{2j-10}\delta(X))$'s are disjoint. Since $X_{jl} \in W_j$ $2^{2j-6}\delta(X) \le \delta(X_{jl}) \le 2^{2j-1}\delta(X)$, the $B_{jl} = B(X_{jl}, 2^{2j-8}\delta(X))$'s are non-tangential balls and

$$\begin{aligned}
&\int_{W_j} \left|\nabla_Y G_0(Z,Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1\right| dY \\
&\quad\le \sum_l \sup_{B_{jl}}\left|\varepsilon(Y)\right|\left(\int_{B_{jl}} \frac{G_0(Z,Y)^2}{\delta(Y)^2} dY\right)^{1/2}\left(\int_{B_{jl}} |\nabla u_1|^2 dY\right)^{1/2} \\
&\quad\le \sum_l \left(\int_{B_{jl}} \frac{a^2(Y)G_0(Z,Y)^2}{\delta(Y)^2} dY\right)^{1/2}\left(\int_{B_{jl}} |\nabla u_1|^2 dY\right)^{1/2} \qquad (7.90)
\end{aligned}$$

For $Y \in B_{jl}$, $2^{2j-7}\delta(X) \le \delta(Y) \le 2^{2j}\delta(X)$ and $Z \in B(X, \delta(X)/8)$

$$G_0(Z,Y) \le C2^{-2j\beta} G_0(A_{2j}, Y) \le C2^{-2j\beta}\frac{G_0(Y)}{\omega_0(\Delta_{2j})} \qquad (7.91)$$

and for $Y \in B_{jl}$, $G_0(Y) \le G_0(A_{2j})$. Thus combining this with (7.90) and (7.91) we obtain

$$\begin{aligned}
&\int_{W_j} \left|\nabla_Y G_0(Z,Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1\right| dY \\
&\quad\le C\frac{2^{-2\beta j}}{\omega_0(\Delta_j)} \sum_l \left(\int_{B_{jl}} \frac{a^2(Y)G_0(Y)}{\delta(Y)^2} dY\right)^{1/2}\left(\frac{G_0(A_{2j})}{(2^{2j}\delta(X))^{2-n}}\right)^{1/2} \\
&\qquad\times\left(\int_{B_{jl}} |\nabla u_1|^2\delta(Y)^{2-n} dY\right)^{1/2} \\
&\quad\le C\frac{2^{-2\beta j}}{\omega_0(\Delta_j)}\left(G_0(A_{2j})\big(2^{2j}\delta(X)\big)^{n-2}\right)^{1/2}\left(\sum_l \int_{B_{jl}} \frac{a^2(Y)G_0(Y)}{\delta(Y)^2} dY\right)^{1/2} \\
&\qquad\times\left(\sum_l \int_{B_{jl}} |\nabla u_1|^2\delta(Y)^{2-n} dY\right)^{1/2} \\
&\quad\le C2^{-2\beta j}\left(\frac{1}{\omega_0(\Delta_{2j+1})}\int_{\Omega_{2j+1}} \frac{a^2(Y)G_0(Y)}{\delta(Y)^2} dY\right)^{1/2} \\
&\qquad\times\left(\int_{(\Omega_{2j+1}\setminus\Omega_{2j-2})\setminus(\partial\Omega, 2^{2j-4}\delta(X))} |\nabla u_1|^2\delta(Y)^{2-n} dY\right)^{1/2} \qquad (7.92)
\end{aligned}$$

For $Y \in (\Omega_{2j+1} \setminus \Omega_{2j-2}) \setminus (\partial\Omega, 2^{2j-4}\delta(X))$ we have $|Y - Q_0| \le |Y - Q_X| + |Q_X - Q_0| \le 2^{2j}\delta(X) + |Q_X - X| + |X - Q_0| \le 2^{2j}\delta(X) + \delta(X) + 2\delta(X) \le 2^4\delta(Y) + 2^{-2j+4}\delta(Y) + 2^{-2j+5}\delta(Y)$ thus $|Y - Q_0| \le 64\delta(Y)$ and $Y \in \Gamma_{64}(Q_0)$, therefore for $M \ge 64$, (7.92) yields

$$\int_{W_j} \big|\nabla_Y G_0(Z, Y)\big|\big|\varepsilon(Y)\big|\big|\nabla u_1\big| dY \le C2^{-2\beta j}\varepsilon_0 S_M(u_1)(Q_0). \tag{7.93}$$

Combining (7.82), (7.89), and (7.93) we have for $\delta(X) \le 4R_0$ and $Z \in B(X, \delta(X/8))$

$$\begin{aligned}
&\left|\int_{(\Omega\setminus B(X))\cap B(Q_X,2^{15}R_0)} \nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y) dY\right| \\
&\le \int_{\Omega_0} \big|\nabla_Y G_0(Z, Y)\big|\big|\varepsilon(Y)\big|\big|\nabla u_1\big| dY + \sum_{j=1}^{N} \int_{R_j} \big|\nabla_Y G_0(Z, Y)\big|\big|\varepsilon(Y)\big|\big|\nabla u_1\big| dY \\
&\le C\varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q_0) + C\varepsilon_0 \sum_{j=1}^{N} 2^{-2\beta j} M_{\omega_0}\big(S_M(u_1)\big)(Q_0) \\
&\le C\varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q_0)
\end{aligned} \tag{7.94}$$

To complete the estimate for $F_2(Z)$ with $Z \in B(X, \delta(X)/8)$ it only remains to consider the integral

$$\int_{(\Omega\setminus B(X))\cap(\partial\Omega,4R_0)\setminus B(Q_X,2^{15}R_0)} \nabla_Y G_0(Z, Y)\varepsilon(Y)\nabla u_1(Y) dY. \tag{7.95}$$

Note that

$$\big(\Omega \setminus B(X)\big) \cap (\partial\Omega, 4R_0) \setminus B\big(Q_X, 2^{15}R_0\big) \subset \bigcup_{Q_\alpha^k \subset \partial\Omega\setminus\Delta(Q_X,2^{14}R_0)} I_\alpha^k.$$

If $Y \in (\Omega \setminus B(X)) \cap (\partial\Omega, 4R_0) \setminus B(Q_X, 2^{15}R_0)$ then $|Y - Q_X| \ge 2^{15}R_0$ and there are I_α^k and $P_\alpha^k \in Q_\alpha^k$ so that $Y \in I_\alpha^k$, $8/\lambda < |P_\alpha^k - Y|8^k < 8\lambda$. Given $Y' \in I_\alpha^k$ note that $|P_\alpha^k - Y'| \le \operatorname{diam} Q_\alpha^k + \lambda 8^{-k+1} \le C_0 8^{-k} + \lambda 8^{-k+1} \le 8C_0\delta(Y)/\lambda + 64\delta(Y) \le 65\delta(Y) \le 2^{10}R_0$. Thus $I_\alpha^k \subset B(P_\alpha^k, 2^{10}R_0) \cap \Omega$, and for $Y' \in B(P_\alpha^k, 2^{10}R_0) \cap \Omega$ we have $|Y' - Q_X| \ge |Y - Q_X| - |Y' - Y| \ge |Y - Q_X| - |Y - P_\alpha^k| - |Y' - P_\alpha^k| \ge 2^{15}R_0 - \lambda 8^{-k+1} - 2^{10}R_0 \ge 2^{15}R_0 - 2^{10}R_0 - 2^6\delta(Y) \ge 2^{14}R_0$. Moreover for $P \in Q_\alpha^k$, $|P - Q_X| \ge |P_\alpha^k - Q_X| - |P - P_\alpha^k| \ge |Y - Q_X| - |Y - P_\alpha^k| - |P - P_\alpha^k| \ge 2^{15}R_0 - \lambda 8^{-k+1} - \operatorname{diam} Q_\alpha^k \ge 2^{14}R_0$. If $Z \in B(X, \delta(X)/8)$ and $Y \in (\Omega \setminus B(X)) \cap (\partial\Omega, 4R_0) \setminus B(Q_X, 2^{15}R_0)$ then since $\delta(X) \le 4R_0$ we have $|Z - Y| \ge |Y - Q_X| - |Q_X - X| - |X - Z| \ge 2^{15}R_0 - \delta(X) - \delta(X)/8 \ge 2^{14}R_0$. Similarly if $Y \in B(P_\alpha^k, 2^{10}R_0) \cap \Omega$ we have $|Z - Y| \ge |Y - Q_X| - |Q_X - X| - |X - Z| \ge 2^{13}R_0$. We mimic the stopping time argument used when the integration over the region Ω_0 was considered. The key point is that for $Z \in B(X, \delta(X)/8)$ the pole of the Green's function is far away from the I_α^k's considered in the integration. Since $\partial\Omega \setminus \Delta(Q_X, 2^{14}R_0) \subset \partial\Omega \setminus \Delta(Q_0, 2^{13}R_0)$ and

$\omega_0(\partial\Omega) \le C_{R_0}\omega_0(\Delta(P, 2^9R_0))$ for any $P \in \partial\Omega$ by the doubling properties of ω_0, estimate (7.64) becomes $G_0(Z, Y) \le C_{R_0}G_0(Y)$ and (7.82) becomes

$$\begin{aligned}
&\int_{(\Omega\setminus B(X))\cap(\partial\Omega,4R_0)\setminus B(Q_X,2^{15}R_0)} \left|\nabla_Y G_0(Z, Y)\right|\left|\varepsilon(Y)\right|\left|\nabla u_1(Y)\right| \\
&\le C\varepsilon_0 \int_{\partial\Omega\setminus\Delta(Q_0,2^{13}R_0)} S_M(u_1)(Q)d\omega_0 \\
&\le C\varepsilon_0 M_{\omega_0} S_M(u_1)(Q_0). \qquad (7.96)
\end{aligned}$$

Combining (7.94) and (7.96) and noting that when $\delta(X) > 8R_0$ the integration over $\Omega \cap (\partial\Omega, 4R_0)$ is treated as that over Ω_0 or $\Omega\setminus B(X) \cap (\partial\Omega, 4R_0)\setminus B(Q_X, 2^{15}R_0)$ as the pole $Z \in B(X, \frac{\delta(X)}{8})$ is very far from the I_α^k's we obtain that

$$\left|F_2(Z)\right| \le C\varepsilon_0 M_{\omega_0} S_M(u_1)(Q_0), \quad \forall Z \in B\left(X, \frac{\delta(X)}{8}\right). \qquad (7.97)$$

Hence (7.41), (7.57), and (7.97)) yield for M large and fixed

$$\begin{aligned}
\tilde{N}F(Q_0) &= \sup_{X\in\Gamma(Q_0)} ⨍_{B(X,\frac{\delta(X)}{8})} F^2(Z)dZ \\
&\lesssim \sup_{X\in\Gamma(Q_0)} ⨍_{B(X,\frac{\delta(X)}{8})} F_1^2(Z)dZ + \sup_{X\in\Gamma(Q_0)} \int_{B(X,\frac{\delta(X)}{8})} F_2^2(Z)dZ \\
&\lesssim C\varepsilon_0 M_{\omega_0}\big(S_M(u_1)\big)(Q_0). \qquad (7.98)
\end{aligned}$$

We now estimate the second term in Lemma 7.7. Fix $Q_0 \in \partial\Omega$, $X \in \Gamma(Q_0)$, let $B(X) = B(X, \frac{\delta(X)}{16})$. Note that $B(X, \frac{\delta(X)}{8}) \subset \Gamma_2(Q_0)$, then

$$\begin{aligned}
⨍_{B(X)} \big(\delta|\nabla F|\big)^2(Z)dZ &\le C\delta^2(X)\frac{1}{\delta(X)^n}\int_{B(X)} |\nabla F|^2(Z)dZ \\
&\le C\frac{1}{\delta(X)^{n-1}}\int_{\frac{\delta(X)}{16}}^{\frac{\delta(X)}{8}} \int_{B(X,\rho)} |\nabla F|^2(Z)dZd\rho. \qquad (7.99)
\end{aligned}$$

The same argument as in [10] which only uses interior estimates yields

$$\begin{aligned}
\tilde{N}_{1/2}^2\big(\delta|\nabla F|\big)(Q_0) &\le C\tilde{N}F(Q_0)\tilde{N}\big(\delta|\nabla F|\big)(Q_0) + \varepsilon_0\tilde{N}\big(\delta|\nabla F|\big)S_2(u_1)(Q_0) \\
&\quad + \varepsilon_0\tilde{N}(F)(Q_0)S_2(u_1)(Q_0). \qquad (7.100)
\end{aligned}$$

Combining (7.100) with (7.36) and using the fact $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ we obtain (7.37). Integrating (7.37) and applying Remark 7.2 we obtain

$$\begin{aligned}
\int \tilde{N}\big(\delta|\nabla F|\big)^2 d\omega_0 &\le C\int \tilde{N}_{1/2}\big(\delta|\nabla F|\big)^2 d\omega_0 \\
&\le C\varepsilon_0 \int \big(M_{\omega_0} S_M(u_1)\big)^2 d\omega_0 + C\varepsilon_0 \int \tilde{N}\big(\delta|\nabla F|\big)^2 d\omega_0
\end{aligned}$$

$$\leq C\varepsilon_0 \int \big(S_M(u_1)\big)^2 d\omega_0 + C\varepsilon_0 \int \tilde{N}\big(\delta|\nabla F|\big)^2 d\omega_0, \tag{7.101}$$

which yields

$$\int \tilde{N}\big(\delta|\nabla F|\big)^2 d\omega_0 \leq C\varepsilon_0 \int S_M^2(u_1)(Q) d\omega_0. \tag{7.102}$$

Combining (7.102), the integration of (7.36) and the maximal function theorem we obtain (7.38) which concludes the proof of Lemma 7.7. □

Lemma 7.8 *Let Ω be a CAD and assume that* (2.12) *holds. Then there exists $C > 1$ so that*

$$\|SF\|^2_{L^2(\omega_0)} \leq C\big(\big\|\tilde{N}\big(\delta|\nabla F|\big)\big\|^2_{L^2(\omega_0)} + \|NF\|^2_{L^2(\omega_0)} + \|\tilde{N}F\|^2_{L^2(\omega_0)} + \|f\|^2_{L^2(\omega_0)}\big). \tag{7.103}$$

Proof For $s \in [1,2]$, let $\Omega_s = B(0, s\tilde{R}_0)$ where $\tilde{R}_0 = \frac{\delta(0)}{2^{30}}$, then

$$\begin{aligned}
\int_{\partial\Omega} S^2F(Q)d\omega_0 &= \int_{\partial\Omega}\int_{\Gamma(Q)\cap\Omega_s} \delta(Z)^{2-n}\big|\nabla F(Z)\big|^2 dZ d\omega_0 \\
&\quad + \int_{\partial\Omega}\int_{\Gamma(Q)\setminus\Omega_s} \delta(Z)^{2-n}\big|\nabla F(Z)\big|^2 dZ d\omega_0 \\
&= \int_{\partial\Omega}\int_{\Gamma(Q)\cap\Omega_s} \delta(Z)^{-n}\big(\delta(Z)\big|\nabla F(Z)\big|\big)^2 dZ d\omega_0 \\
&\quad + \int_{\Omega\setminus\Omega_s}\int_{\partial\Omega} \delta(Z)^{2-n}\big|\nabla F(Z)\big|^2 \chi_{\{Z\in\Gamma(Q)\}}(Q) d\omega_0 dZ \\
&\leq \int_{\partial\Omega}\int_{\Gamma(Q)\cap\Omega_s} \delta(Z)^{-n}\big(\delta(Z)\big|\nabla F(Z)\big|\big)^2 dZ d\omega_0(Q) \\
&\quad + \int_{\Omega\setminus\Omega_s} \big|\nabla F(Z)\big|^2 \delta(Z)^{2-n}\omega_0\big(\Delta\big(Q_Z, 3\delta(Z)\big)\big) dZ.
\end{aligned} \tag{7.104}$$

Note that if $0 \in \Gamma_2(Q)$ then $B(0, s\tilde{R}_0) \subset \Gamma_3(Q)$ and if $0 \notin \Gamma_2(Q)$ then $B(0, s\tilde{R}_0) \cap \Gamma(Q) = \emptyset$. Thus

$$\begin{aligned}
&\int_{\partial\Omega}\int_{\Gamma(Q)\cap\Omega_s} \delta(Z)^{-n}\big(\delta(Z)|\nabla F|\big)^2 dZ d\omega_0 \\
&\quad \leq \int_{\partial\Omega}\int_{\Gamma(Q)\cap\Omega_s} \delta(Z)^{-n}\big(\delta(Z)|\nabla F|\big)^2 \chi_{\{0\in\Gamma_2(Q)\}}(Z) dZ d\omega_0
\end{aligned}$$

$$\lesssim \int_{\partial\Omega} \fint_{B(0,s\tilde{R}_0)} \big(\delta(Z)\big|F(Z)\big|\big)^2 \chi_{\{0\in\Gamma_2(Q)\}}(Z)dZd\omega_0$$

$$\lesssim \int_{\partial\Omega} \tilde{N}\big(\delta\big|\nabla F(Z)\big|\big)^2(Q)d\omega_0(Q). \tag{7.105}$$

We now estimate the second term in (7.104). Since $\omega_0(\Delta(Q_Z, 3\delta(Z)))\delta(Z)^{2-n} \sim G_0(Z)$ using the ellipticity of L_0 we have

$$\int_{\Omega\setminus\Omega_s} \delta(Z)^{2-n}\big|\nabla F(Z)\big|^2 dZd\omega_0(Q)$$

$$\lesssim \int_{\Omega\setminus\Omega_s} \big|\nabla F(Z)\big|^2 G_0(Z)dZ$$

$$\lesssim \int_{\Omega\setminus\Omega_s} \big\langle A_0\nabla F(Z), \nabla F(Z)\big\rangle G_0(Z)dZ$$

$$\lesssim \int_{\Omega\setminus\Omega_s} \operatorname{div}(A_0\nabla F)F)G_0dZ - \int_{\Omega\setminus\Omega_s} \operatorname{div}((A_0\nabla F)FG_0dZ$$

$$\lesssim \frac{1}{2}\int_{\Omega\setminus\Omega_s} L_0\big(F^2\big)G_0dZ - \int_{\Omega\setminus\Omega_s} (L_0F)FG_0dZ. \tag{7.106}$$

Integration by parts on the second term in (7.106) yields

$$\int_{\Omega\setminus\Omega_s} (L_0F)(Z)F(Z)G_0(Z)dZ$$

$$= -\int_{\Omega\setminus\Omega_s} \operatorname{div}(\varepsilon\nabla u_1)(FG_0)(Z)dZ$$

$$= \int_{\Omega\setminus\Omega_s} \nabla(FG_0)\varepsilon\nabla u_1 dZ$$

$$= \int_{\Omega\setminus\Omega_s} G_0\nabla F\varepsilon\nabla u_1 dZ + \int_{\Omega\setminus\Omega_s} \nabla G_0 F\varepsilon\nabla u_1 dZ. \tag{7.107}$$

since $G_0 = 0$ on $\partial\Omega$ and $\varepsilon = 0$ on $\partial\Omega_s$ (recall $\varepsilon(Y) = 0$ when $\delta(Y) \geq 4R_0$). We use the dyadic decomposition of $\partial\Omega$ to estimate each term. Recall that for $Y \in I_\alpha^k$, $G_0(Y) \sim \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}}$ then given $\varepsilon > 0$

$$\int_{\Omega\setminus\Omega_s\setminus(\partial\Omega,\varepsilon)} |\nabla F|G_0\big|\varepsilon(Z)\big||\nabla u_1|dZ$$

$$\leq \sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ k\leq k_\varepsilon}} \sup_{I_\alpha^k}\big|\varepsilon(Z)\big| \int_{I_\alpha^k} G_0|\nabla F||\nabla u_1|dZ$$

$$\lesssim \sum_{\substack{Q_\alpha^k \subset \partial\Omega \\ k \le k_\varepsilon}} \sup_{I_\alpha^k} \big|\varepsilon(Z)\big| \frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}} \bigg(\int_{I_\alpha^k} |\nabla F||\nabla u_1| dZ \bigg). \tag{7.108}$$

To estimate $\int_{I_\alpha^k} |\nabla F||\nabla u_1| dZ$ cover I_α^k by balls $\{B(X_i, \lambda 8^{-k-3})\}_{1\le i\le N}$ with $X_i \in I_\alpha^k$ such that $|X_i - X_l| \ge \lambda 8^{-k-3}/2$. Here N is independent of k and the balls $B(X_i, \lambda 8^{-k-3})$ have finite overlap (also independent of k).

$$\begin{aligned}
&\int_{I_\alpha^k} |\nabla F||\nabla u_1| dZ \\
&\quad \le \sum_{i=1}^N \int_{B(X_i,\lambda 8^{-k-3})} |\nabla F||\nabla u_1| dZ \\
&\quad \le \sum_{i=1}^N \bigg(\int_{B(X_i,\lambda 8^{-k-3})} |\nabla F|^2 dZ \bigg)^{1/2} \bigg(\int_{B(X_i,\lambda 8^{-k-3})} |\nabla u_1|^2 dZ \bigg)^{1/2} \\
&\quad \le \sum_{i=1}^N \bigg(\int_{B(X_i,\lambda 8^{-k-3})} |\nabla u_1|^2 \bigg(\int_{B(Z,\lambda 8^{-k-2})} |\nabla F|^2 dY \bigg) dZ \bigg)^{1/2} \\
&\quad \lesssim \big(\operatorname{diam} Q_\alpha^k\big)^{\frac{n-2}{2}} \sum_{i=1}^N \bigg(\int_{B(X_i,\lambda 8^{-k-3})} |\nabla u_1|^2 \bigg(\fint_{B(Z,\frac{\delta(Z)}{8})} \big(\delta(Y)|\nabla F|\big)^2 dY \bigg) dZ \bigg)^{1/2} \\
&\quad \lesssim \bigg(\big(\operatorname{diam} Q_\alpha^k\big)^{n-2} \int_{I_\alpha^k} |\nabla u_1|^2 \bigg(\fint_{B(Z,\frac{\delta(Z)}{8})} \big(\delta(Y)|\nabla F|\big)^2 dY \bigg) dZ \bigg)^{1/2}.
\end{aligned} \tag{7.109}$$

Combining (7.108) and (7.109) we have

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_s\setminus(\partial\Omega,\varepsilon)} |\nabla F| G_0 \big|\varepsilon(Z)\big| |\nabla u_1| dZ \\
&\quad \lesssim \sum_{\substack{Q_\alpha^k \subset \partial\Omega \\ k \le k_\varepsilon}} \sup_{I_\alpha^k} \big|\varepsilon(Z)\big| \omega_0\big(Q_\alpha^k\big) \\
&\qquad \times \bigg(\int_{I_\alpha^k} |\nabla u_1|^2 \delta(Z)^{2-n} \bigg(\fint_{B(Z,\frac{\delta(Z)}{8})} \big(\delta(Y)|\nabla F|\big)^2 dY \bigg) dZ \bigg)^{1/2} \\
&\quad \lesssim \sum_{\substack{Q_\alpha^k \subset \partial\Omega \\ k \le k_\varepsilon}} \bigg(\int_{I_\alpha^k} \frac{a^2(Y) G_0(Y)}{\delta(Y)^2} dY \bigg)^{1/2} \omega_0\big(Q_\alpha^k\big)^{1/2} \\
&\qquad \times \bigg(\int_{I_\alpha^k} |\nabla u_1|^2 \delta(Z)^{2-n} \bigg(\fint_{B(Z,\frac{\delta(Z)}{8})} \big(\delta(Y)|\nabla F|\big)^2 dY \bigg) dZ \bigg)^{1/2}.
\end{aligned} \tag{7.110}$$

Applying a stopping time argument similar to the one used in the proof of Lemma 7.7 to estimate F_0^ε, to the function

$$\tilde{T}_\varepsilon(Q) = \left(\int_{\Gamma_M(Q)\setminus B_{2\varepsilon}(Q)} |\nabla u_1|^2 \delta(Z)^{2-n} \left(⨍_{B(Z,\delta(Z)/8)} \delta^2 |\nabla F|^2\right) dZ\right)^{1/2}$$

and letting ε tend to 0 we obtain

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_s} |\nabla F| G_0 |\varepsilon| |\nabla u_1| \\
&\quad\lesssim \varepsilon_0 \int_{\partial\Omega} \left(\int_{\Gamma_M(Q)} |\nabla u_1|^2 \delta(Z)^{2-n} ⨍_{B(Z,\delta(Z)/8)} \delta^2 |\nabla F|^2 dZ\right)^{1/2} d\omega_0(Q) \\
&\quad\lesssim \varepsilon_0 \int_{\partial\Omega} \tilde{N}^M\big(\delta|\nabla F|\big)(Q) S_M(u_1)(Q) d\omega_0(Q).
\end{aligned} \tag{7.111}$$

Now we turn our attention to the second term in (7.107). Applying Caccioppoli's inequality (see (7.62)) we have

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_s} |\nabla G_0| |F| |\varepsilon| |\nabla u_1| dZ \\
&\quad\le \sum_{Q_\alpha^k\subset\partial\Omega} \sup_{I_\alpha^k} |\varepsilon| \left(\int_{I_\alpha^k} |\nabla G_0|^2 dZ\right)^{1/2} \left(\int_{I_\alpha^k} \left|\nabla u_1(Z)\right|^2 F^2(Z) dZ\right)^{1/2} \\
&\quad\lesssim \sum_{Q_\alpha^k\subset\partial\Omega} \sup_{I_\alpha^k} |\varepsilon| \left(\int_{\hat{I}_\alpha^k} \frac{G_0(Y)^2}{\delta(Y)^2} dY\right)^{1/2} \left(\int_{I_\alpha^k} \left|\nabla u_1(Z)\right|^2 F^2(Z) dZ\right)^{1/2}.
\end{aligned} \tag{7.112}$$

Once again a similar argument to the one that appears in the proof of Lemma 7.7 with a stopping time argument applied to a truncation of

$$\left(\int_{\Gamma_M(Q)} |\nabla u_1|^2 \delta(Y)^{2-n} F(Y) dY\right)^{1/2}$$

yields the following estimate

$$\begin{aligned}
\int_{\Omega\setminus\Omega_s} |\nabla G_0| |F| |\varepsilon| |\nabla u_1| dZ &\lesssim \varepsilon_0 \int_{\partial\Omega} \left(\int_{\Gamma_M(Q)} |\nabla u_1|^2 \delta(Z)^{2-n} F(Y) dY\right)^{1/2} d\omega_0(Q) \\
&\lesssim \varepsilon_0 \int_{\partial\Omega} S_M(u_1)(Q) N_M F(Q) d\omega_0(Q).
\end{aligned} \tag{7.113}$$

Putting together (7.107), (7.111), and (7.113) we obtain

$$\begin{aligned}
\left|\int_{\Omega\setminus\Omega_s} L_0 F \cdot F G_0 dZ\right| &\lesssim \varepsilon_0 \int_{\partial\Omega} \tilde{N}_M\big(\delta|\nabla F|\big)(Q) S_M(u_1)(Q) d\omega_0 \\
&\quad + \varepsilon_0 \int_{\partial\Omega} N_M F(Q) S_M(u_1)(Q) d\omega_0.
\end{aligned} \tag{7.114}$$

To estimate the first term in (7.106) observe that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Omega_s} G_0 L_0\big(F^2\big)dZ \\
&\quad=\frac{1}{2}\int_{\Omega\setminus\Omega_s}\operatorname{div}\big(G_0A_0\nabla F^2\big)dZ-\frac{1}{2}\int_{\Omega\setminus\Omega_s}A_0\nabla G_0\nabla F^2(Z)dZ \\
&\quad=\int_{\partial\Omega_s}G_0A_0F\nabla F\cdot\vec{v}d\sigma-\frac{1}{2}\int_{\Omega\setminus\Omega_s}\operatorname{div}\big(F^2A_0\nabla G_0\big)dZ \\
&\quad=\int_{\partial\Omega_s}G_0A_0F\nabla F\cdot\vec{v}d\sigma-\frac{1}{2}\int_{\partial\Omega_s}F^2A_0\nabla G_0\cdot\vec{v}d\sigma. \qquad (7.115)
\end{aligned}$$

Integrating over $s\in[1,2]$ we obtain

$$\begin{aligned}
&\frac{1}{2}\int_1^2\left|\int_{\Omega\setminus B(0,s\tilde{R}_0)}G_0L_0F^2dZ\right|ds \\
&\quad=\int_1^2\int_{\partial B(0,s\tilde{R}_0)}|G_0||A_0||F||\nabla F|d\sigma ds \\
&\qquad+\frac{1}{2}\int_1^2\int_{\partial B(0,s\tilde{R}_0)}F^2|A_0||\nabla G_0|d\sigma ds \\
&\quad=\int_{B(0,2\tilde{R}_0)\setminus B(0,\tilde{R}_0)}G_0|A_0||F||\nabla F|dZ \\
&\qquad-\frac{1}{2}\int_{B(0,2\tilde{R}_0)\setminus B(0,\tilde{R}_0)}F^2|A_0||\nabla G_0|dZ. \qquad (7.116)
\end{aligned}$$

Looking at each term in (7.116) separately we have that

$$\begin{aligned}
&\int_{B(0,2\tilde{R}_0)\setminus B(0,\tilde{R}_0)}G_0|A_0||F||\nabla F|dZ \\
&\quad\lesssim\sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset}}\int_{I_\alpha^k\cap(\Omega_2\setminus\Omega_1)}G_0|F||\nabla F|dZ. \qquad (7.117)
\end{aligned}$$

Note that if $I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset$ there is $Y\in\Omega$ so that $\lambda/8<8^k\delta(Y)<8\lambda$ and $|\delta(Y)-\delta(0)|\lneqq 2\tilde{R}_0=\delta(0)2^{-29}$. Thus $(1+2^{-29})^{-1}\lambda/8<8^k\delta(0)<(1-2^{-29})^{-1}8\lambda$. Since $\operatorname{diam} Q_\alpha^k\sim 8^{-k}\sim\delta(0)$ then $\omega(Q_\alpha^k)\geq C_1$ an absolute constant only depending on the NTA constants of Ω. Thus for $Y\in I_\alpha^k\cap(\Omega_2\setminus\Omega_1)$ $G_0(Y)\lesssim\frac{1}{\delta(0)^{n-2}}\lesssim\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}}$. These combined with a computation like the one that appears in (7.108), (7.109), and

(7.110) yields

$$\begin{aligned}
&\int_{\Omega_2\setminus\Omega_1} G_0|A_0||F||\nabla F|dZ\\
&\quad\lesssim \sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset}} \left(\int_{I_\alpha^k}|F||\nabla F|dZ\right)\frac{\omega_0(Q_\alpha^k)}{(\operatorname{diam} Q_\alpha^k)^{n-2}}\\
&\quad\lesssim \sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset}} \omega_0\big(Q_\alpha^k\big)^{1/2}\left(\omega_0\big(Q_\alpha^k\big)\int_{I_\alpha^k}\delta^{2-n}|F|^2dZ\fint_{B(Z,\frac{\delta(Z)}{8})}\delta^2|\nabla F|^2dZ\right)^{1/2}\\
&\quad\lesssim \int_{\partial\Omega}\left(\int_{\Gamma_M(Q)\cap\Omega_2\setminus\Omega_1}\delta^{2-n}(Z)|F|^2\left(\fint_{B(Z,\frac{\delta(Z)}{8})}\delta^2|\nabla F|^2dY\right)dZ\right)^{1/2}d\omega_0\\
&\quad\lesssim \delta(0)^{(2-n)/2}\int_{\partial\Omega}N_MF(Q)\tilde N^M\big(\delta|\nabla F|\big)(Q)\delta(0)^{n/2}d\omega_0\\
&\quad\lesssim \delta(0)\int_{\partial\Omega}N_MF(Q)\tilde N^M\big(\delta|\nabla F|\big)(Q)d\omega_0(Q).
\end{aligned}\tag{7.118}$$

We control the second term in (7.116) by recalling that if $Y\in I_\alpha^k\cap\Omega_2\setminus\Omega_1$ then $\delta(0)\sim\operatorname{diam} Q_\alpha^k$ and $|\nabla G_0(Y)|\lesssim\frac{G_0(Y)}{\delta(Y)}\lesssim\frac{\omega_0(Q_\alpha^k)}{\delta(0)^{n-1}}$. Thus as in (7.118) and using the doubling properties of ω_0 we have

$$\begin{aligned}
&\int_{\Omega_2\setminus\Omega_1} F^2|A_0||\nabla G_0|dZ\\
&\quad\lesssim \sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset}} \int_{I_\alpha^k\cap\Omega_2\setminus\Omega_1}F^2|\nabla G_0|dZ\\
&\quad\lesssim \sum_{\substack{Q_\alpha^k\subset\partial\Omega\\ I_\alpha^k\cap\Omega_2\setminus\Omega_1\neq\emptyset}} \omega_0\big(Q_\alpha^k\big)^{1/2}\left(\omega_0\big(Q_\alpha^k\big)\int_{I_\alpha^k}\delta^{2-n}|F|^2\fint_{B(Z,\frac{\delta(Z)}{8})}|F|^2\right)^{1/2}\\
&\quad\lesssim \delta(0)\int_{\partial\Omega}N_MF(Q)\tilde N^M F(Q)d\omega_0(Q).
\end{aligned}\tag{7.119}$$

Combining (7.116), (7.118), and (7.119) we obtain

$$\begin{aligned}
&\int_1^2\left|\int_{\Omega\setminus B(0,s\tilde R_0)}G_0(Z)L_0F^2(Z)dZ\right|ds\\
&\quad\lesssim \delta(0)\int_{\partial\Omega}N_MF(Q)\tilde N^M\big(\delta|\nabla F|\big)(Q)d\omega_0(Q)\\
&\qquad+\delta(0)\int_{\partial\Omega}N_MF(Q)\tilde N^M F(Q)d\omega_0(Q).
\end{aligned}\tag{7.120}$$

Combining (7.104), (7.105), (7.106), (7.113), and (7.120) plus recalling the fact that $S_M(u_1) \leq S_M(F) + S_M(u_0)$ and $\|S_M F\|_{L^2(\omega_0)} \sim \|SF\|_{L^2(\omega_0)}$ and Remark 7.2 we obtain

$$
\begin{aligned}
&\int_{\partial\Omega} S^2 F(Q) d\omega_0(Q) \\
&\quad = \int_1^2 \int_{\partial\Omega} S^2 F(Q) d\omega_0(Q) ds \\
&\quad = \int_1^2 \int_{\partial\Omega} \int_{\Gamma(Q)\cap\Omega_s} \delta^{-n}(Z)\big(\delta(Z)|\nabla F|\big)^2 dZ d\omega_0 ds \\
&\qquad + \int_1^2 \int_{\Omega\setminus\Omega_s} |\nabla F|^2 \delta^{2-n}(Z)\omega_0\big(\Delta\big(Q_Z, 3\delta(Z)\big)\big) dZ ds \\
&\quad \lesssim \int_{\partial\Omega} \int_{\Gamma(Q)\cap\Omega_s} \delta^{-n}(Z)\big(\delta(Z)|\nabla F|\big)^2 dZ d\omega_0 + \int_1^2 \left| \int_{\Omega\setminus\Omega_s} L_0 F^2 G_0 \right| ds \\
&\qquad + \int_1^2 \left| \int_{\Omega\setminus\Omega_s} (L_0 F) F G_0 \right| ds \\
&\quad \lesssim \int_{\partial\Omega} \tilde{N}\big(\delta|\nabla F|\big)^2(Q) d\omega_0(Q) + \varepsilon_0 \int_{\partial\Omega} \tilde{N}^M\big(\delta|\nabla F|\big)(Q) S_M(u_1)(Q) d\omega_0(Q) \\
&\qquad + \varepsilon_0 \int_{\partial\Omega} S_M(u_1)(Q) N_M F(Q) d\omega_0 + \int_{\partial\Omega} N_M F(Q) \tilde{N}^M\big(\delta|\nabla F|\big) d\omega_0 \\
&\qquad + \int_{\partial\Omega} N_M F(Q) \tilde{N}^M(F)(Q) d\omega_0 \\
&\quad \lesssim \big\|\tilde{N}\big(\delta|\nabla F|\big)\big\|^2_{L^2(\omega_0)} + \|Su_0\|^2_{L^2(\omega_0)} + \|NF\|^2_{L^2(\omega_0)} \\
&\qquad + \big\|\tilde{N}(F)\big\|^2_{L^2(\omega_0)} + \varepsilon_0 \|SF\|^2_{L^2(\omega_0)}.
\end{aligned}
\tag{7.121}
$$

Since by Lemma 2.13 $\|Su_0\|^2_{L^2(\omega_0)} \lesssim \|Nu_0\|^2_{L^2(\omega_0)} \lesssim \|f\|^2_{L^2(\omega_0)}$ we obtain from (7.121)

$$
\begin{aligned}
\|SF\|^2_{L^2(\omega_0)} &\lesssim \big\|\tilde{N}\big(\delta|\nabla F|\big)\big\|^2_{L^2(\omega_0)} + \|NF\|^2_{L^2(\omega_0)} \\
&\quad + \big\|\tilde{N}(F)\big\|^2_{L^2(\omega_0)} + \|f\|^2_{L^2(\omega_0)}
\end{aligned}
\tag{7.122}
$$

which yields Lemma 7.8. □

Proof of Theorem 2.9 Since $S(u_1) \leq S(F) + S(u_0)$, (7.38), (7.103), and the argument above, (7.122) yields

$$
\begin{aligned}
&\int_{\partial\Omega} \tilde{N}F(Q)^2 d\omega_0 + \int_{\partial\Omega} \tilde{N}\big(\delta|\nabla F|\big)^2(Q) d\omega_0 \\
&\quad \leq C\varepsilon_0^2 \int_{\partial\Omega} S^2 u_1 d\omega_0
\end{aligned}
$$

$$\leq C\varepsilon_0^2 \int_{\partial\Omega} (SF)^2(Q)d\omega_0 + C\varepsilon_0^2 \int_{\partial\Omega} (Su_0)^2 d\omega_0$$
$$\leq C\varepsilon_0^2 \Big(\int_{\partial\Omega} \tilde{N}F(Q)^2 d\omega_0 + \int_{\partial\Omega} \tilde{N}\big(\delta|\nabla F|\big)^2(Q)d\omega_0 \Big)$$
$$+ C\varepsilon_0^2 \int_{\partial\Omega} NF(Q)^2 d\omega_0 + C\varepsilon_0^2 \int_{\partial\Omega} f^2 d\omega_0. \tag{7.123}$$

Thus

$$\int_{\partial\Omega} \tilde{N}F(Q)^2 d\omega_0 + \int_{\partial\Omega} \tilde{N}\big(\delta|\nabla F|\big)^2(Q)d\omega_0$$
$$\leq C\varepsilon_0^2 \int_{\partial\Omega} NF(Q)^2 d\omega_0 + C\varepsilon_0^2 \int_{\partial\Omega} f^2 d\omega_0. \tag{7.124}$$

Note that since $\|\tilde{N}u_i\|_{L^2(\omega_0)} \sim \|Nu_i\|^2_{L^2(\omega_0)}$, $i = 0, 1$ then by (7.124)

$$\int_{\partial\Omega} NF(Q)^2 d\omega_0 \leq \int_{\partial\Omega} Nu_1^2(Q)d\omega_0 + \int_{\partial\Omega} Nu_0^2(Q)d\omega_0$$
$$\leq C \int_{\partial\Omega} \tilde{N}u_1^2(Q)d\omega_0 + C \int_{\partial\Omega} f^2 d\omega_0$$
$$\leq C \int_{\partial\Omega} \tilde{N}F(Q)^2 d\omega_0 + C \int_{\partial\Omega} f^2 d\omega_0$$
$$\leq C\varepsilon_0^2 \int_{\partial\Omega} NF(Q)^2 d\omega_0 + C \int_{\partial\Omega} f^2 d\omega_0 \tag{7.125}$$

which ensures that

$$\int_{\partial\Omega} NF(Q)^2 d\omega_0 \leq C \int_{\partial\Omega} f^2 d\omega_0 \tag{7.126}$$

which yields

$$\int_{\partial\Omega} Nu_1^2(Q)d\omega_0 \leq C \int_{\partial\Omega} NF(Q)^2 d\omega_0 + C \int_{\partial\Omega} Nu_0^2(Q)d\omega_0$$
$$\leq C \int_{\partial\Omega} f^2 d\omega_0. \tag{7.127}$$

This concludes the proof of Theorem 2.9. □

8 Regularity for the Elliptic Kernel on CADs

Theorem 8.1 *Let Ω be a CAD—assume there exists a constant $C > 0$ such that*

$$\sup_{\Delta\subset\partial\Omega} \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right)^{\frac{1}{2}} < C. \tag{8.1}$$

Then $\omega_1 \in A_\infty(\sigma)$ if $\omega_0 \in A_\infty(\sigma)$.

The argument used to prove the result above is similar to the one used in [10] to prove Theorem 2.3. In our case it relies on a generalization of Fefferman's result to the CAD setting, as follows.

Theorem 8.2 *Let Ω be a CAD and let*

$$A(a)(Q) = \left(\int_{\Gamma(Q)} \frac{a^2(X)}{\delta(X)^n} dX \right)^{\frac{1}{2}}. \tag{8.2}$$

If $\|A(a)\|_{L^\infty(\sigma)} \leq C_0 < \infty$ *and* $\omega_0 \in A_\infty(\sigma)$ *then* $\omega_1 \in A_\infty(\sigma)$.

Proof of Theorem 8.2 This is a corollary of Theorem 2.9. In fact note that for $\Delta = B(Q_0, r) \cap \partial\Omega$ with $Q_0 \in \partial\Omega$ the fact that $\frac{\omega_0(\Delta(Q_X,\delta(X)))}{\delta(X)^{n-1}} \sim \frac{G_0(X)}{\delta(X)}$ combined with Fubini's theorem and the doubling properties of ω_0 yields

$$\begin{aligned}
\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} \frac{a^2(X) G_0(X)}{\delta(X)^2} dX &\lesssim \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} \frac{\omega_0(\Delta(Q_X \delta(X)))}{\delta(X)^{n-1}} dX \\
&= \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} \int_{\partial\Omega} \frac{a^2(X)}{\delta(X)^n} \chi_{\Delta(Q_X,\delta(X))}(Q) d\omega dX \\
&\leq \frac{1}{\omega_0(\Delta)} \int_{3\Delta} \left(\int_{T(\Delta)} \frac{a^2(X)}{\delta(X)^n} \chi_{\Gamma(Q)}(X) dX \right) d\omega_0 \\
&\leq \frac{1}{\omega_0(\Delta)} \int_{3\Delta} A^2(a)(Q) d\omega_0(Q) \\
&\leq \frac{1}{\omega_0(\Delta)} \int_{3\Delta} A^2(a)(Q) d\omega_0(Q).
\end{aligned} \tag{8.3}$$

Hence there exists $\delta > 0$ depending on n and the NTA constants of Ω such that if $\|A(a)\|_{L^\infty(\sigma)} \leq \delta$, and $\omega_0 \in A_\infty(\sigma)$ then $\omega_1 \in B_2(\omega_0)$. In fact since $\omega_0 \in A_\infty(\sigma)$ the fact that $\|A(a)\|_{L^\infty(\sigma)} \leq \delta$ implies that $\|A(a)\|_{L^\infty(\omega_0)} \leq \delta$. Estimate (8.3) guarantees that there exists $C > 0$ depending on n and the NTA constants of Ω such that

$$\sup_{\Delta \subset \partial\Omega} \left(\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right)^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}. \tag{8.4}$$

Choosing $\delta > 0$ small enough so that $C\delta^{\frac{1}{2}} < \epsilon_0$ in Theorem 2.9 we conclude that $\omega_1 \in B_2(\omega_0)$. To finish the proof of Theorem 8.2 consider the family of operators $L_t = (1-t)A_0 + tA_1$ for $0 \leq t \leq 0$. Consider a partition of $[0, 1]$ $\{t_i\}_{i=0}^m$ such that $0 < t_{i+1} - t_i < \frac{\delta}{C_0}$ where C_0 is as in the statement of Theorem 8.2. Let a_i be the deviation function corresponding to $L_{t_{i+1}} = L_{i+1}$ and $L_{t_i} = L_i$, here $\epsilon_i(X) = A_{t_{i+1}}(X) = (t_{i+1} - t_i)\epsilon(X)$, and $a_i(X) = (t_{i+1} - t_i)a(X)$. Hence $\|A(a_i)\|_{L^\infty(\sigma)} = (t_{i+1} - t_i)\|A(a)\|_{L^\infty(\sigma)} < \delta$. An iteration of the argument above ensures that for $i \in \{0, \ldots, m\}$ $\omega_i \in A_\infty(\sigma)$ and $\omega_{i+1} \in B_2(\omega_i)$. Hence $\omega_1 \in A_\infty(\sigma)$. □

Proof of Theorem 8.1 We are assuming the Carleson condition (8.1) on $\frac{a^2(X)}{\delta(X)}dX$ and that $\omega_0 \in A_\infty(\sigma)$. We will show that $\omega_1 \in A_\infty(\sigma)$ by showing that there exist $0 < \alpha < 1$ and $0 < \beta < 1$ such that if $\Delta = B(Q_0, r) \cap \partial\Omega$ and $E \subset \Delta$ then $\frac{\sigma(E)}{\sigma(\Delta)} > \alpha$ implies that $\frac{\omega_1(E)}{\omega_1(\Delta)} > \beta$.

For $r > 0$ and $\gamma > 0$ we denote by $\Gamma_{\gamma,r}(Q)$ the truncated cone of radius r and aperture γ, i.e., $\Gamma_{\gamma,r}(Q) = \{X \in \Omega : |X - Q| < (1+\gamma)\delta(X), 0 < \delta(X) < r\}$. We define the truncated square function with aperture determined by γ for the deviation function $a(X)$ by

$$A_{\gamma,r}(Q) = \left(\int_{\Gamma_{\gamma,r}(Q)} \frac{a^2(X)}{\delta(X)^n} dX \right)^{1/2}.$$

The appropriate constant γ will be chosen later.

Applying Lemma 3.13 to $A(X) = \frac{a^2(X)}{\delta(X)} \chi_{B(Q_0;(2+\gamma)r)}(X)$ we conclude that

$$\frac{1}{\sigma(\Delta)} \int_\Delta A^2_{\gamma,r}(Q) d\sigma(Q) \leq \frac{1}{\sigma(\Delta)} \int_{T((2+\gamma)\Delta)} \frac{a^2(X)}{\delta(X)} dX \leq C_\gamma$$

because σ is doubling and hypothesis (8.1). Thus there is a closed set $S \subset \Delta$ so that $\frac{\sigma(S)}{\sigma(\Delta)} \geq \frac{1}{2}$ and $A_{\gamma,r}(Q) \leq C'_\gamma$ for $Q \in S$.

Recall that there exist constants $0 < \beta < \gamma$ and $C_1 < C_2 < 0$ and a sawtooth domain Ω_S such that

(i) $\bigcup_{Q\in S} \Gamma_{\beta,C_1 r}(Q) \subset \Omega_S \subset \bigcup_{Q\in S} \Gamma_{\gamma,C_2 r}(Q)$
(ii) $\partial\Omega_S \cap \partial\Omega = S$
(iii) The NTA character of Ω_S is independent of S

(see [12]).

Without loss of generality we may assume that $\frac{3}{2}\beta + \frac{1}{2} < \gamma$. Let

$$\Omega' = \bigcup_{Q\in S} \Gamma_{\beta,C_1 r}(Q) \quad \text{and} \quad \tilde{\Omega} = \bigcup_{Q\in S} \Gamma_{\gamma,C_2 r}(Q).$$

For $X \in \Omega$ with $\delta(X) < C_1 r$ if $B(X, \frac{\delta(X)}{2}) \cap \Omega' \neq \emptyset$ then there exists $\tilde{Q} \in S$ so that $B(X, \frac{\delta(X)}{2}) \subset \Gamma_{\gamma,C_2 r}(\tilde{Q})$. In fact if $Y \in B(X, \frac{\delta(X)}{2}) \cap \Omega'$ there is $\tilde{Q} \in S$ so that $|\tilde{Q} - Y| < (1+\beta)\delta(Y)$ and $|\tilde{Q} - X| \leq |\tilde{Q} - Y| + |Y - X| < (1+\beta)\delta(Y) + \frac{\delta(X)}{2} \leq (1+\beta)\frac{3\delta(X)}{2} + \frac{\delta(X)}{2} = \delta(X)(1 + (\frac{3}{2}\beta + \frac{1}{2}))$. Thus $X \in \Gamma_{\gamma,r}(\tilde{Q})$. Define the operator $\tilde{L}_1 = \operatorname{div} \tilde{A}_1 \nabla$ by setting

$$\tilde{A}_1(X) = \begin{cases} A_1(X) & \text{if } X \in \Omega' \\ A_0(X) & \text{if } X \in \Omega \backslash \Omega' \end{cases}$$

Let $\tilde{a}(X) = \sup_{Y \in B(X, \frac{\delta(X)}{2})} |\tilde{A}_1(Y) - A_0(Y)|$ be the deviation function for $\tilde{L}_1$ and L_0. Observe that $\tilde{a}(X) \leq a(X)$. For γ and β as above consider

$$\tilde{A}_\beta(Q) = \left(\int_{\Gamma_\beta(Q)} \frac{\tilde{a}(X)^2}{\delta(X)^n} dX \right)^{\frac{1}{2}}.$$

Note that by the definition of $\tilde{A}_1$ and $\tilde{a}$,

$$\tilde{A}_\beta(Q) = \left(\int_{\Gamma_{\beta, C_1 r}(Q)} \frac{\tilde{a}(X)^2}{\delta(X)^n} dX \right)^{\frac{1}{2}}.$$

If $B(X, \frac{\delta(X)}{2}) \cap \Omega' = \emptyset$ then $\tilde{a}(X) = 0$. On the other hand, if $X \in \Gamma_{\beta, C_1 r}(Q)$ and $B(X, \frac{\delta(X)}{2}) \cap \Omega' \neq \emptyset$ then there exists $\tilde{Q} \in S$ so that $B(X, \frac{\delta(X)}{2}) \subset \Gamma_{\gamma, C_2 r}(\tilde{Q})$.

Thus $\tilde{A}_\beta(Q) \leq A_{\gamma, r}(\tilde{Q}) \leq C_{\gamma'}$. By Theorem 8.2 $\tilde{\omega}_1 = \omega_{\tilde{L}_1} \in A_\infty(\sigma)$. Choose $0 < \alpha < 1$ close to 1 so that $\frac{\sigma(E \cap S)}{\sigma(\Delta)} \geq \frac{1}{4}$ whenever $\frac{\sigma(E)}{\sigma(\Delta)} > \alpha$. Let $F = S \cap E$ since $\tilde{\omega}_1 \in A_\infty(\sigma)$ there exist constants $C > 0$ and $\eta > 0$ so that

$$\frac{\tilde{\omega}_1(F)}{\tilde{\omega}_1(\Delta)} > C \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^\eta \geq C'. \tag{8.5}$$

By [5] and [11] there exist constants $C > 0$ and $Q > 0$ depending on the ellipticity constants, the NTA constants of Ω, and n so that for $F \subset S$

$$\frac{1}{C} \left(\tilde{\omega}_1^{\Omega_S}(F) \right)^{\frac{1}{\theta}} \leq \frac{\tilde{\omega}_1(F)}{\tilde{\omega}_1(\Delta)} \leq C \left(\tilde{\omega}_1^{\Omega_S}(F) \right)^\theta \tag{8.6}$$

and

$$\frac{1}{C} \left(\omega_1^{\Omega_S}(F) \right)^{\frac{1}{\theta}} \leq \frac{\omega_1(F)}{\omega_1(\Delta)} \leq C \left(\omega_1^{\Omega_S}(F) \right)^\theta \tag{8.7}$$

(see Lemma 1.4.14 in [12]). Since $\Omega_S \subset \Omega'$, $\tilde{L}_1 = L_1$ on Ω_S and $\tilde{\omega}_1^{\Omega_S} = \omega_1^{\Omega_s}$. Combining (8.5), (8.6), and (8.7) we obtain since $F \subset E$

$$\begin{aligned} \frac{\omega_1(E)}{\omega_1(\Delta)} \geq \frac{\omega_1(F)}{\omega_1(\Delta)} \geq \frac{1}{C} \left(\omega_1^{\Omega_S}(F) \right)^{\frac{1}{\theta}} &= \frac{1}{C} \left(\tilde{\omega}_1^{\Omega_S}(F) \right)^{\frac{1}{\theta}} \\ &\geq C' \left(\frac{\tilde{\omega}_1(F)}{\tilde{\omega}_1(\Delta)} \right)^{\frac{1}{\theta^2}} \geq C'. \end{aligned} \tag{8.8}$$

□

Remark 8.3 Recall that by the work of David & Jerison [6] and Semmes [15], we have that if Ω is a CAD and ω denotes the harmonic measure then $\omega \in A_\infty(\sigma)$. Theorem 8.1 shows that the elliptic measure of operators which are perturbations of the Laplacian in the sense of (1) is also in $A_\infty(\sigma)$.

Acknowledgements T. Toro was partially supported by NSF DMS grants 0600915 and 0856687. E. Milakis was supported by Marie Curie International Reintegration Grant No. 256481 within the 7th European Community Framework Programme and NSF DMS grant 0856687. J. Pipher was partially supported by NSF DMS grant 0901139.

References

1. Christ, M.: A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **60/61**(2), 601–628 (1990)
2. Coifman, R., Meyer, Y., Stein, E.: Some new function spaces and their applications to harmonic analysis. J. Funct. Anal. **62**(2), 304–335 (1985)
3. Dahlberg, B.: Estimates of harmonic measure. Arch. Ration. Mech. Anal. **65**(3), 275–288 (1977)
4. Dahlberg, B.: On the absolute continuity of elliptic measure. Am. J. Math. **108**, 1119–1138 (1986)
5. Dahlberg, B., Jerison, D., Kenig, C.: Area integral estimates for elliptic differential operators with non-smooth coefficients. Ark. Math. **22**, 97–107 (1984)
6. David, G., Jerison, D.: Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. Indiana Univ. Math. J. **39**, 831–845 (1990)
7. Evans, L., Gariepy, R.: Measure theory and fine properties of functions. Studies in Advanced Mathematics (1992). viii+268 pp.
8. Escauriaza, L.: The L^p Dirichlet problem for small perturbations of the Laplacian. Isr. J. Math. **94**, 353–366 (1996)
9. Fefferman, R.: A criterion for the absolute continuity of the harmonic measure associated with an elliptic operator. J. Am. Math. Soc. **2**(1), 127–135 (1989)
10. Fefferman, R., Kenig, C., Pipher, J.: The theory of weights and the Dirichlet problem for elliptic equations. Ann. Math. **134**(1), 65–124 (1991)
11. Jerison, D., Kenig, C.: Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv. Math. **46**, 80–147 (1982)
12. Kenig, C.: Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems. CBMS Regional Conference Series in Mathematics, vol. 83. AMS, Providence (1994). xii+146 pp.
13. Kenig, C., Toro, T.: Harmonic measure on locally flat domains. Duke Math. J. **87**(3), 509–551 (1997)
14. Milakis, E., Toro, T.: Divergence form operators in Reifenberg flat domains. Math. Z. **264**(1), 15–41 (2010)
15. Semmes, S.: Analysis vs. Geometry on a class of rectifiable hypersurfaces. Indiana Univ. Math. J. **39**, 1005–1035 (1990)
16. Stein, E.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series, vol. 43. Princeton University Press, Princeton (1993). Monographs in Harmonic Analysis, III. xiv+695 pp.
17. Yosida, K.: Functional Analysis. Classics in Mathematics. Springer, Berlin (1980). Reprint of the sixth edn. (1980). xii+501 pp.