Real Projective Iterated Function Systems

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Abstract This paper contains four main results associated with an attractor of a projective iterated function system (IFS). The first theorem characterizes when a projective IFS has an attractor which avoids a hyperplane. The second theorem establishes that a projective IFS has at most one attractor. In the third theorem the classical duality between points and hyperplanes in projective space leads to connections between attractors that avoid hyperplanes and repellers that avoid points, as well as hyperplane attractors that avoid points and repellers that avoid hyperplanes. Finally, an index is defined for attractors which avoid a hyperplane. This index is shown to be a nontrivial projective invariant.

1 Introduction

This paper provides the foundations of a surprisingly rich mathematical theory associated with the attractor of a real projective iterated function system (IFS). (A real projective IFS consists of a finite set of projective transformations $\{f_m : \mathcal{P} \to \mathcal{P}\}_{m=1}^M$

Communicated by Jeffrey Diller.

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where $\mathcal{P}$ is a real projective space. An attractor is a nonempty compact set $A \subset \mathcal{P}$ such that $\lim_{k\to\infty} \mathcal{F}^k(B) = \mathcal{F}(A) = A$ for all nonempty sets B in an open neighborhood of A, where $\mathcal{F}(B) = \bigcup_{m=1}^M f_m(B)$.) In addition to proving conditions which guarantee the existence and uniqueness of an attractor for a projective IFS, we also present several related concepts. The first connects an attractor which avoids a hyperplane with a hyperplane repeller. The second uses information about the hyperplane repeller to define a new index for an attractor. This index is both invariant under projective transformations and nontrivial, which implies that it joins the cross ratio and Hausdorff dimension as nontrivial invariants under the projective group. Thus, these attractors belong in a natural way to the collection of geometrical objects of classical projective geometry.

The definitions that support expressions such as "iterated function system", "attractor", "basin of attraction" and "avoids a hyperplane", used in this Introduction, are given in Sect. 3.

Iterated function systems are a standard framework for describing and analyzing self-referential sets such as deterministic fractals [2, 7, 22] and some types of random fractals [10]. Attractors of affine IFSs have many applications, including image compression [3, 8, 21] and geometric modeling [14]. They relate to the theory of the joint spectral radius [12] and to wavelets [11]. Projective IFSs have more degrees of freedom than comparable affine IFSs [5] while the constituent functions share geometrical properties such as preservation of straight lines and cross ratios. Projective IFSs have been used in digital imaging and computer graphics (see, for example, [6]), and they may have applications to image compression, as proposed in [4, p. 10]. Projective IFSs can be designed so that their attractors are smooth objects such as arcs of circles and parabolas, and rough objects such as fractal interpolation functions.

The behavior of attractors of projective IFSs appears to be complicated. In computer experiments conducted by the authors, attractors seem to come and go in a mysterious manner as parameters of the IFS are changed continuously. See Example 4 in Sect. 4 for an example that illustrates such phenomena. The intuition developed for affine IFSs regarding the control of attractors seems to be wrong in the projective setting. Our theorems provide insight into such behavior.

One key issue is the relationship between the existence of an attractor and the contractive properties of the functions of the IFS. In a previous paper [1] we investigated the relationship between the existence of attractors and the existence of contractive metrics for IFSs consisting of affine maps on $\mathbb{R}^n$. We established that an affine IFS $\mathcal{F}$ has an attractor if and only if $\mathcal{F}$ is contractive on all of $\mathbb{R}^n$. In the present paper we focus on the setting where $\mathbb{X} = \mathbb{P}^n$ is real n-dimensional projective space and each function in $\mathcal{F}$ is a projective transformation. In this case $\mathcal{F}$ is called a *projective IFS*.

Our first main result, Theorem 1, provides a set of equivalent characterizations of a projective IFS that possesses an attractor that avoids a hyperplane. The adjoint $\mathcal{F}^t$ of a projective IFS $\mathcal{F}$ is defined in Sect. 11, and convex body is defined in Definition 5. An IFS $\mathcal{F}$ is contractive on $S \subset X$ when $\mathcal{F}(S) \subset S$ and there is a metric on S with respect to which all the functions of the IFS are contractive; see Definition 3. For a set X in a topological space, $\overline{X}$ denotes its closure, and $\text{int}(X)$ denotes its interior.

Theorem 1 *If $\mathcal{F}$ is a projective IFS on $\mathbb{P}^n$, then the following statements are equivalent.*

(1) $\mathcal{F}$ *has an attractor A that avoids a hyperplane.*
(2) *There is a nonempty open set U that avoids a hyperplane such that* $\mathcal{F}(\overline{U}) \subset U$.
(3) *There is a nonempty finite collection of disjoint convex bodies* $\{C_i\}$ *such that* $\mathcal{F}(\bigcup_i C_i) \subset \mathrm{int}(\bigcup_i C_i)$.
(4) *There is a nonempty open set* $U \subset \mathbb{P}^n$ *such that* $\mathcal{F}$ *is contractive on* $\overline{U}$.
(5) *The adjoint projective IFS* $\mathcal{F}^t$ *has an attractor* A^t *that avoids a hyperplane.*

When these statements are true we say that $\mathcal{F}$ *is contractive.*

Statement (4) is of particular importance because if an IFS is contractive, then it possesses an attractor that depends continuously on the functions of the IFS; see, for example, [2, Sect. 3.11]. Moreover, if an IFS is contractive, then various canonical measures, supported on its attractor, can be computed by means of the "chaos game" algorithm [7], and diverse applications, such as those mentioned above, become feasible. Note that statement (4) of Theorem 1 immediately implies uniqueness of an attractor in the set U, but not uniqueness in $\mathbb{P}^n$. See also Remark 2 in Sect. 13.

Our second main result establishes uniqueness of attractors, independently of whether or not Theorem 1 applies.

**Theorem 2** *A projective IFS has at most one attractor.*

The classical projective duality between points and hyperplanes manifests itself in interesting ways in the theory of projective IFSs. Theorem 3 below, which depends on statement (5) in Theorem 1, is an example. It is a geometrical description of the dynamics of $\mathcal{F}$ as a set operator on $\mathbb{P}^n$. The terminology used is provided in Sect. 11.

Theorem 3 (1) *A projective IFS has an attractor that avoids a hyperplane if and only if it has a hyperplane repeller that avoids a point. The basin of attraction of the attractor is the complement of the union of the hyperplanes in the repeller.*

(2) *A projective IFS has a hyperplane attractor that avoids a point if and only if it has a repeller that avoids a hyperplane. The basin of attraction of the hyperplane attractor is the set of hyperplanes that do not intersect the repeller.*

Figure 1 illustrates Theorem 3. Here and in the other figures we use the disk model of the projective plane. Diametrically opposite points on the boundary of the disk are identified in $\mathbb{P}^2$. In the left-hand panel of Figure 1 the "leaf" is the attractor A of a certain projective IFS $\mathcal{F}$ consisting of four projective transformations on $\mathbb{P}^2$. The surrounding grainy region approximates the set R of points in the corresponding hyperplane repeller. The complement of R is the basin of attraction of A. The central green, red, and yellow objects in the right panel comprise the attractor of the adjoint IFS $\mathcal{F}^t$, while the grainy orange scimitar-shaped region illustrates the corresponding hyperplane repeller.

Theorem 3 enables us to associate a geometrical index with an attractor that avoids a hyperplane. More specifically, if an attractor A avoids a hyperplane then A lies in the complement of (the union of the hyperplanes in) the repeller. Since the connected components of this complement form an open cover of A and since A is compact, A is
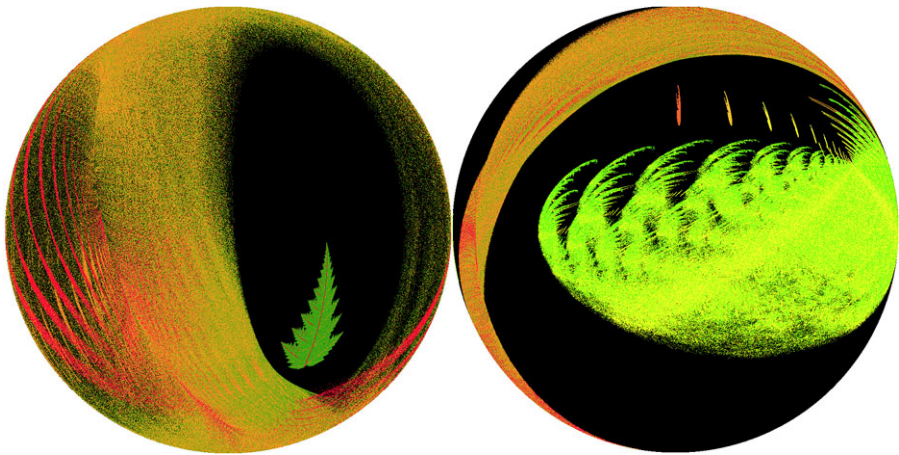
Fig. 1 (Color online) The *image* on the *left* shows the attractor and hyperplane repeller of a projective IFS. The basin of attraction of the leaf-like attractor is the black convex region together with the leaf. The *image* on the *right* shows the attractor and repeller of the adjoint system

actually contained in a finite set of components of the complement. These observations lead to the definition of a geometric index of A, index(A), as is made precise in Definition 13. This index is an integer associated with an attractor A, not any particular IFS that generates A. As shown in Sect. 12, as a consequence of Theorem 4, this index is nontrivial, in the sense that it can take positive integer values other than one. Moreover, it is invariant under PGL($n + 1, \mathbb{R}$), the group of real, dimension n, projective transformations. That is, index(A) = index($g(A)$) for all $g \in$ PGL($n + 1, \mathbb{R}$).

See Remark 3 of Sect. 13 concerning attractors and repellers in the case of affine IFSs. See Remark 4 in Sect. 13 concerning the fact that the Hausdorff dimension of the attractor is also an invariant under the projective group.

2 Organization

Since the proofs of our results are quite complicated, this section describes the structure of this paper, including an overview of the proof of Theorem 1.

Section 3 contains definitions and notation related to iterated function systems, and background information on projective space, convex sets in projective space, and the Hilbert metric.

Section 4 provides examples that illustrate the intricacy of projective IFSs and the value of our results. These examples also illustrate the role of the avoided hyperplane in statements (1), (2), and (5) of Theorem 1.

The proof of Theorem 1 is achieved by showing that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1) \Leftrightarrow (5).$$

Section 5 contains the proof that $(1) \Rightarrow (2)$, by means of a topological argument. Statement (2) states that the IFS $\mathcal{F}$ is a "topological contraction" in the sense that it sends a nonempty compact set into its interior.

Section 6 contains the proof of Proposition 4, which describes the action of a projective transformation on the convex hull of a connected set in terms of its action on the connected set. This is a key result that is used subsequently.

Section 7 contains the proof that (2) $\Rightarrow$ (3) by means of a geometrical argument, in Lemmas 2 and 3. Statement (3) states that the compact set, in statement (2), that is sent into its interior can be chosen to be the disjoint union of finitely many convex bodies. What makes the proof somewhat subtle is that, in general, there is no single convex body that is mapped into its interior.

Sections 8 and 9 contain the proof that (3) $\Rightarrow$ (4). Statement (4) states that, with respect to an appropriate metric, each function in $\mathcal{F}$ is a contraction. The requisite metric is constructed in two stages. On each of the convex bodies in statement (3), the metric is basically the Hilbert metric as discussed in Sect. 3. How to combine these metrics into a single metric on the union of the convex bodies is what requires the two sections.

Section 10 contains both the proof that (4) $\Rightarrow$ (1) and the proof of Theorem 2.

Section 11 contains the proof that (1) $\Leftrightarrow$ (5), namely that $\mathcal{F}$ has an attractor if and only if $\mathcal{F}^t$ has an attractor. The adjoint IFS $\mathcal{F}^t$ consists of those projective transformations which, when expressed as matrices, are the transposes of the matrices that represent the functions of $\mathcal{F}$. The proof relies on properties of an operation, called the complementary dual, that takes subsets of $\mathbb{P}^n$ to subsets of $\mathbb{P}^n$.

Section 11 also contains the proof of Theorem 3, which concerns the relationship between attractors and repellers. The proof relies on classical duality between $\mathbb{P}^n$ and its dual $\widehat{\mathbb{P}^n}$, as well as the equivalence of statement (4) in Theorem 1. Note that, if $\mathcal{F}$ has an attractor A then the orbit under $\mathcal{F}$ of any compact set in the basin of attraction of A will converge to A in the Hausdorff metric. Theorem 3 tells us that if A avoids a hyperplane, then there is also a set $\mathcal{R}$ of hyperplanes that repel, under the action of $\mathcal{F}$, hyperplanes "close" to $\mathcal{R}$. The hyperplane repeller $\mathcal{R}$ is such that the IFS $\mathcal{F}^{-1}$, consisting of all inverses of functions in $\mathcal{F}$, when applied to the dual space of $\mathbb{P}^n$, has $\mathcal{R}$ as an attractor. The relationship between the hyperplane repeller of an IFS $\mathcal{F}$ and the attractor of the adjoint IFS $\mathcal{F}^t$ is described in Proposition 10.

Section 12 considers properties of attractors that are invariant under the projective group $\mathrm{PGL}(n+1, \mathbb{R})$. In particular, we define index(A) of an attractor A that avoids a hyperplane, and establish Theorem 4 which shows that this index is a nontrivial group invariant.

Section 13 contains various remarks that add germane information that could interrupt the flow on a first reading. In particular, the topic of non-contractive projective IFSs that, nevertheless, have attractors is mentioned. Other areas open to future research are also mentioned.

3 Iterated Function Systems, Projective Space, Convex Sets, and the Hilbert Metric

3.1 Iterated Function Systems and Their Attractors

Definition 1 Let $\mathbb{X}$ be a complete metric space. If $f_m : \mathbb{X} \to \mathbb{X}$, $m = 1, 2, \ldots, M$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_M)$ is called an *iterated function system* (IFS).

To define the attractor of an IFS, first define

$$\mathcal{F}(B) = \bigcup_{f \in \mathcal{F}} f(B)$$

for any $B \subset \mathbb{X}$. By slight abuse of terminology we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and for the above mapping. For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k-fold composition of $\mathcal{F}$, the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k. Define $\mathcal{F}^0(B) = B$.

Definition 2 A nonempty compact set $A \subset \mathbb{X}$ is said to be an *attractor* of the IFS $\mathcal{F}$ if

(i) $\mathcal{F}(A) = A$ and
(ii) there is an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k \to \infty} \mathcal{F}^k(B) = A$, for all compact sets $B \subset U$, where the limit is with respect to the Hausdorff metric.

The largest open set U such that (ii) is true is called the *basin of attraction* [for the attractor A of the IFS $\mathcal{F}$].

See Remark 6 in Sect. 13 concerning a different definition of attractor.

Definition 3 A function $f : \mathbb{X} \to \mathbb{X}$ is called a *contraction* with respect to a metric d if there is $0 \le \alpha < 1$ such that $d(f(x), f(y)) \le \alpha\, d(x, y)$ for all $x, y \in \mathbb{R}^n$.

An IFS $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_M)$ is said to be *contractive on a set* $U \subset \mathbb{X}$ if $\mathcal{F}(U) \subset U$ and there is a metric $d : U \times U \to [0, \infty)$, giving the same topology as on U, such that, for each $f \in \mathcal{F}$ the restriction $f|_U$ of f to U is a contraction on U with respect to d.

3.2 Projective Space

Let $\mathbb{R}^{n+1}$ denote $(n + 1)$-dimensional Euclidean space and let $\mathbb{P}^n$ denote real *projective space*. Specifically, $\mathbb{P}^n$ is the quotient of $\mathbb{R}^{n+1} \backslash \{0\}$ by the equivalence relation which identifies $(x_0, \ldots, x_n)$ with $(\lambda x_0, \ldots, \lambda x_n)$ for all nonzero $\lambda \in \mathbb{R}$. Let

$$\phi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{P}^n$$

denote the canonical quotient map. The set $(x_0, \ldots, x_n)$ of coordinates of some $x \in \mathbb{R}^{n+1}$ such that $\phi(x) = p$ is referred to as *homogeneous coordinates* of p. If $p, q \in \mathbb{P}^n$ have homogeneous coordinates $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$, respectively, and $\sum_{i=0}^n p_i q_i = 0$, then we say that p and q are *orthogonal*, and write $p \perp q$. A *hyperplane* in $\mathbb{P}^n$ is a set of the form

$$H = H_p = \{q \in \mathbb{P}^n : p \perp q = 0\} \subset \mathbb{P}^n,$$

for some $p \in \mathbb{P}^n$.

Definition 4 A set $X \subset \mathbb{P}^n$ is said to *avoid a hyperplane* if there exists a hyperplane $H \subset \mathbb{P}^n$ such that $H \cap X = \emptyset$.

We define the "round" metric $d_{\mathbb{P}}$ on $\mathbb{P}^n$ as follows. Each point p of $\mathbb{P}^n$ is represented by a line in $\mathbb{R}^{n+1}$ through the origin, or by the two points a_p and b_p where this line intersects the unit sphere centered at the origin. Then, in the obvious notation, $d_{\mathbb{P}}(p, q) = \min\{\|a_p - a_q\|, \|a_p - b_q\|\}$ where $\|x - y\|$ denotes the Euclidean distance between x and y in $\mathbb{R}^{n+1}$. In terms of homogeneous coordinates, the metric is given by

$$d_{\mathbb{P}}(p, q) = \sqrt{2 - 2\frac{|\langle p, q \rangle|}{\|p\|\|q\|}},$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product. The metric space $(\mathbb{P}^n, d_{\mathbb{P}})$ is compact.

A *projective transformation* f is an element of $\mathrm{PGL}(n + 1, \mathbb{R})$, the quotient of $\mathrm{GL}(n + 1, \mathbb{R})$ by the multiples of the identity matrix. A mapping $f : \mathbb{P}^n \to \mathbb{P}^n$ is well defined by $f(\phi x) = \phi(L_f x)$, where $L_f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is any matrix representing projective transformation f. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R}^{n+1} & \xrightarrow{\ L_f\ } & \mathbb{R}^{n+1} \\
\phi \downarrow & & \downarrow \phi \\
\mathbb{P}^n & \xrightarrow[\ f\ ]{} & \mathbb{P}^n.
\end{array}
$$

When no confusion arises we may designate an n-dimensional projective transformation f by a matrix $L_f \in \mathrm{GL}(n + 1, \mathbb{R})$ that represents it. An IFS $\mathcal{F} = (\mathbb{P}^n; f_1, f_2, \ldots, f_M)$ is called a *projective IFS* if each $f \in \mathcal{F}$ is a projective transformation on $\mathbb{P}^n$.

3.3 Convex Subsets of $\mathbb{P}^n$

We now define the notions of convex set, convex body, and convex hull of a set with respect to a hyperplane. In Proposition 4 we state an invariance property that plays a key role in the proof of Theorem 1.

If $H \subset \mathbb{P}^n$ is a hyperplane, then there is a unique hyperplane $\overline{H} \in \mathbb{R}^{n+1}$ such that $\phi(\overline{H}) = H$. If $p \in \mathbb{P}^n \backslash H$, there is a unique 1-dimensional subspace $\overline{p} \in \mathbb{R}^{n+1}$ such that $\phi(\overline{p}) = p$. Let u be a unit vector orthogonal to $\overline{H}$ and $W = \{x : \langle x, u \rangle = 1\}$ be the corresponding affine subspace of $\mathbb{R}^{n+1}$. Define a mapping $\theta : \mathbb{P}^n \backslash H \to W$ by letting $\theta(p)$ be the intersection of $\overline{p}$ with W. Now θ is a surjective mapping from $\mathbb{P}^n \backslash H$ onto the n-dimensional affine space W such that projective subspaces of $\mathbb{P}^n \backslash H$ go to affine subspaces of W. In light of the above, it makes sense to consider $\mathbb{P}^n \backslash H$ as an affine space.

Definition 5 A set $S \subset \mathbb{P}^n \backslash H$ is said to be *convex with respect to a hyperplane H* if S is a convex subset of $\mathbb{P}^n \backslash H$, considered as an affine space as described above. Equivalently, with notation as in the above paragraph, S is convex with respect to H if

$\theta(S)$ is a convex subset of W. A closed set that is convex with respect to a hyperplane and has nonempty interior is called a *convex body*.

It is important to distinguish this definition of "convex" from projective convex, which is the term often used to describe a set $S \subset \mathbb{P}^n$ with the property that if l is a line in $\mathbb{P}^n$ then $S \cap l$ is connected. (See [18, 19] for a discussion of related matters.)

Definition 6 Given a hyperplane $H \subset \mathbb{P}^n$ and two points $x, y \in \mathbb{P}^n \backslash H$, the unique line $\overline{xy}$ through x and y is divided into two closed line segments by x and y. The one that does *not* intersect H will be called the *line segment with respect to H* and denoted $\overline{xy}_H$.

Note that C is convex with respect to a hyperplane H if and only if $\overline{xy}_H \subset C$ for all $x, y \in C$.

Definition 7 Let $S \subset \mathbb{P}^n$ and let H be a hyperplane such that $S \cap H = \emptyset$. The *convex hull of S with respect to H* is

$$\text{conv}_H(S) = \text{conv}(S),$$

where $\text{conv}(S)$ is the usual convex hull of S, treated as a subset of the affine space $\mathbb{P}^n \backslash H$. Equivalently, with notation as above, if $S' = \text{conv}(\theta(S))$, where conv denotes the ordinary convex hull in W, then $\text{conv}_H(S) = \phi(S')$.

We can also describe $\text{conv}_H(S)$ as the smallest convex subset of $\mathbb{P}^n \backslash H$ that contains S, i.e., the intersection of all convex sets of $\mathbb{P}^n \backslash H$ containing S. The key result concerning convexity and projective transformations is Proposition 4 in Sect. 6.

3.4 The Hilbert Metric

In this section we define the Hilbert metric associated with a convex body.

Let $p, q \in \mathbb{P}^n$, with $p \neq q$ and with homogeneous coordinates $p = (p_0, \ldots, p_n)$ and $q = (q_0, \ldots, q_n)$. Any point r on the line $\overline{pq}$ has homogeneous coordinates $r_i = \alpha_1 p_i + \alpha_2 q_i$, $i = 0, 1, \ldots, n$. The pair (α_1, α_2) is referred to as the *homogeneous parameters* of r with respect to p and q. Since the homogeneous coordinates of p and q are determined only up to a scalar multiple, the same is true of the homogeneous parameters (α_1, α_2).

Let $a = (\alpha_1, \alpha_2)$, $b = (\beta_1, \beta_2)$, $c = (\gamma_1, \gamma_2)$, $d = (\delta_1, \delta_2)$ be any four points on such a line in terms of homogeneous parameters. Their *cross ratio* $R(a, b, c, d)$, in terms of homogeneous parameters on the projective line, is defined to be

$$R(a, b, c, d) = \frac{\begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}} - \frac{\begin{vmatrix} \delta_1 & \alpha_1 \\ \delta_2 & \alpha_2 \end{vmatrix}}{\begin{vmatrix} \delta_1 & \beta_1 \\ \delta_2 & \beta_2 \end{vmatrix}}. \tag{3.1}$$

The key property of the cross ratio is that it is invariant under any projective transformation and under any change of basis $\{p, q\}$ for the line. If none of the four points

is the first base point p, then the homogeneous parameters of the points are $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$, $(\delta, 1)$ and the cross ratio can be expressed as the ratio of (signed) distances:

$$R(a, b, c, d) = \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)}.$$

Definition 8 Let $K \subset \mathbb{P}^n$ be a convex body. Let $H \subset \mathbb{P}^n$ be a hyperplane such that $H \cap K = \varnothing$. Let x and y be distinct points in int(K). Let a and b be two distinct points in the boundary of K such that $\overline{xy}_H \subset \overline{ab}_H$, where the order of the points along the line segment $\overline{ab}_H$ is a, x, y, b. The *Hilbert metric* d_K on int(K) is defined by

$$d_K(x, y) = \log R(a, b, x, y) = \log\left(\frac{|ay|\,|bx|}{|ax|\,|by|}\right).$$

Here $|ay| = \|a' - y'\|$, $|bx| = \|b' - x'\|$, $|ax| = \|a' - x'\|$, $|by| = \|b' - y'\|$ denote Euclidean distances associated with any set of collinear points $a', x', y', b' \in \mathbb{R}^{n+1}$ such that $\phi(a') = a$, $\phi(x') = x$, $\phi(y') = y$, and $\phi(b') = b$.

A basic property of the Hilbert metric is that it is a projective invariant. See [16, p. 105] for a more complete discussion of the properties of this metric. See Remark 4 in Sect. 13 concerning the relationship between the metrics $d_\mathbb{P}$ and d_K and its relevance to the evaluation and projective invariance of the Hausdorff dimension.
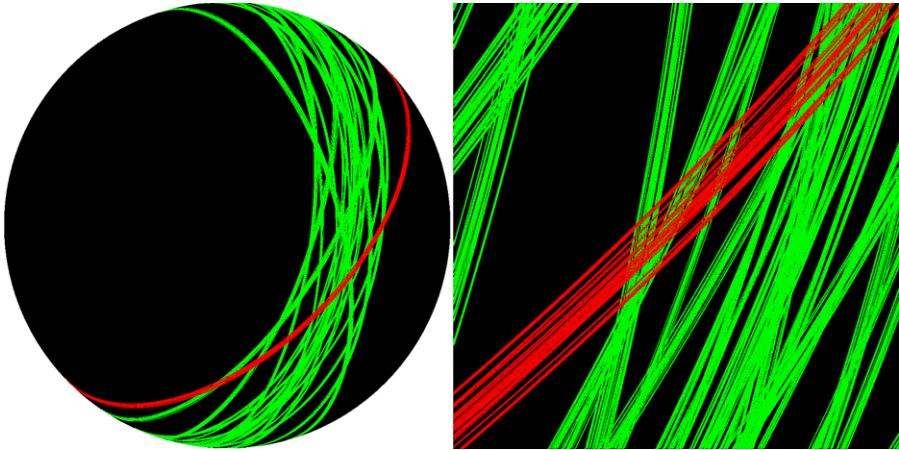
4 Examples

Example 1 (IFSs with one transformation) Let $\mathcal{F} = (\mathbb{P}^n; f)$ be a projective IFS with a single transformation. By Theorem 1 such an IFS has an attractor if and only if any matrix L_f representing f has a dominant eigenvalue. (The map L_f has a real eigenvalue λ_0 with corresponding eigenspace of dimension 1, such that $\lambda_0 > |\lambda|$ for every other eigenvalue λ.) For such an IFS the attractor is a single point whose homogeneous coordinates are the coordinates of the eigenvector corresponding to λ_0. The hyperplane repeller of $\mathcal{F}$ is the single hyperplane $\phi(E)$, where E is the span of the eigenspaces corresponding to all eigenvalues of L_f except λ_0. The attractor of the adjoint IFS is also a single point, $\phi(E^\perp)$, where $E^\perp$ is the unique line through the origin in $\mathbb{R}^{n+1}$ perpendicular to the hyperplane E.

Example 2 (Convex hull caveat) In Theorem 1 the implication $(2) \Rightarrow (3)$ contains a subtle issue. It may seem, at first sight, to be trivial because surely one could choose C simply to be the convex hull of U? The following example shows that this is not true. Let $\mathcal{F} = (\mathbb{P}^1; f_1, f_2)$ where

$$f_1 = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} -4 & 0 \\ 1 & 1 \end{pmatrix}.$$

In $\mathbb{P}^1$ a hyperplane is just a point. Let $H_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $H_\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be two hyperplanes and consider the four points $p = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$, $q = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $r = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $s = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$ in $\mathbb{P}^1$. Let

Fig. 2 (Color online) Projective attractor which includes a hyperplane, and a zoom. See Example 3

C_1 be the line segment $\overline{pq}_{H_0}$ and let $C_2 = \overline{rs}_{H_0}$. There are two possible convex hulls of $C_1 \cup C_2$, one with respect to the hyperplane H_0, for example and the other with respect to H_∞ for example. It is routine to check that $\mathcal{F}(\overline{C_1 \cup C_2}) \subset C_1 \cup C_2$ but $\mathcal{F}(\overline{\text{conv}_H(C_1 \cup C_2)}) \not\subset \text{conv}_H(C_1 \cup C_2)$, where H is either H_0 or H_∞. Thus the situation is fundamentally different from the affine case; see [1].

Example 3 (A non-contractive IFS with an attractor) Theorem 1 leaves open the possible existence of a non-contractive IFS that, nevertheless, has an attractor. According to Theorem 1 such an attractor must have nonempty intersection with every hyperplane. The following example shows that such an IFS does exist. Let $\mathcal{F} = (P^2; f_1, f_2)$ where

$$f_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\cos\theta & -2\sin\theta \\ 0 & 2\sin\theta & 2\cos\theta \end{pmatrix},$$

and θ/π is irrational. In terms of homogeneous coordinates (x, y, z), the attractor of $\mathcal{F}$ is the line $x = 0$.

Another example is illustrated in Figure 2, where

$$f_1 = \begin{pmatrix} 41 & -19 & 19 \\ -19 & 41 & 19 \\ 19 & 19 & 41 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} -10 & -1 & 19 \\ -10 & 21 & 1 \\ 10 & 10 & 10 \end{pmatrix}.$$

Neither function f_1 nor f_2 has an attractor, but the IFS consisting of both of them does. The union A of the points in the red and green lines is the attractor. Since any two lines in $\mathbb{P}^2$ have nonempty intersection, the attractor A has nonempty intersection with every hyperplane. Consequently, by Theorem 1 there exists no metric with respect to which both functions are contractive. In the right panel a zoom is shown which displays the fractal structure of the set of lines that comprise the attractor. The

color red is used to indicate the image of the attractor under f_1, while green indicates its image under f_2.

Example 4 (Attractor discontinuity) This example consists of a family $F = \{\mathcal{F}(t) : t \in \mathbb{R}\}$ of projective IFSs that depend continuously on a real parameter t. The example demonstrates how behavior of a projective family F may be more complicated than in the affine case. Let $\mathcal{F}(t) = (\mathbb{P}^2; f_1, f_2, f_3)$ where

$$f_1 = \begin{pmatrix} 198t + 199 & 198t + 198 & -198t^2 - 297t - 99 \\ 0 & 1 & 0 \\ 198 & 198 & -198t - 98 \end{pmatrix},$$

$$f_2 = \begin{pmatrix} 397 & 396 & -594 \\ 0 & 1 & 0 \\ 198 & 198 & -296 \end{pmatrix}, \quad \text{and} \quad f_3 = \begin{pmatrix} 595 & 594 & -1485 \\ 0 & 1 & 0 \\ 198 & 198 & -494 \end{pmatrix}.$$

This family interpolates quadratically between three IFSs, $\mathcal{F}(0)$, $\mathcal{F}(1)$, and $\mathcal{F}(2)$, each of which has an attractor that avoids a hyperplane. But the IFSs $\mathcal{F}(0.5)$ and $\mathcal{F}(1.5)$ do not have an attractor. This contrasts with the affine case, where similar interpolations yield IFSs that have an attractor at all intermediate values of the parameter. For example, if hyperbolic affine IFSs $\mathcal{F}$ and $\mathcal{G}$ each have an attractor, then so does the average IFS, $(t\mathcal{F} + (1 - t)\mathcal{G})$ for all $t \in [0, 1]$.

5 Proof that (1) $\Rightarrow$ (2) in Theorem 1

Lemma 1 (i) *If the projective IFS $\mathcal{F}$ has an attractor A then there is a nonempty open set U such that $A \subset U$, $\mathcal{F}(\overline{U}) \subset U$, and $\overline{U}$ is contained in the basin of attraction of A.*

(ii) (Theorem 1 (1) $\Rightarrow$ (2)) *If the projective IFS $\mathcal{F}$ has an attractor A and there is a hyperplane H such that $H \cap A = \emptyset$, then there is a nonempty open set U such that $A \subset U$, $\overline{U} \cap H = \emptyset$, $\mathcal{F}(\overline{U}) \subset U$, and $\overline{U}$ is contained in the basin of attraction of A.*

Proof We prove (ii) first. The proof will make use of the function $\mathcal{F}^{-1}(X) = \{x \in \mathbb{P}^n : f(x) \in X \text{ for all } f \in \mathcal{F}\}$. Note that $\mathcal{F}^{-1}$ takes open sets to open sets, $X \subset (\mathcal{F}^{-1} \circ \mathcal{F})(X)$ and $(\mathcal{F} \circ \mathcal{F}^{-1})(X) \subset X$ for all X.

Since A is an attractor contained in $\mathbb{P}^n \backslash H$, there is an open set V containing A such that $\overline{V}$ is compact, $\overline{V} \subset \mathbb{P}^n \backslash H$, and $A = \lim_{k \to \infty} \mathcal{F}^k(\overline{V})$. Hence there is an integer m such that $\mathcal{F}^k(\overline{V}) \subset V$ for $k \geq m$.

Define V_k, $k = 0, 1, \ldots, m$, recursively, going backwards from V_m to V_0, as follows. Let $V_m = V$ and for $k = m - 1, \ldots, 2, 1, 0$, let $V_k = V \cap \mathcal{F}^{-1}(V_{k+1})$. If $O = V_0$, then O has the following properties:

(1) O is open,
(2) $A \subset O$,
(3) $\mathcal{F}^k(O) \subset V$ for all $k \geq 0$.

To check property (2) notice that $\mathcal{F}(A) = A$ implies $A \subset (\mathcal{F}^{-1} \circ \mathcal{F})(A) = \mathcal{F}^{-1}(A)$. Then $A \subset V = V_m$ implies that $A \subset V_m$ for all m, in particular $A \subset$

$V_0 = O$. To check property (3) notice that $V_k \subset \mathcal{F}^{-1}(V_{k+1})$ implies $\mathcal{F}(V_k) \subset (\mathcal{F} \circ \mathcal{F}^{-1})(V_{k+1}) \subset V_{k+1}$. It then follows that $\mathcal{F}^k(O) \subset V_k \subset V$ for $0 \le k \le m$. Also $\mathcal{F}^k(O) \subset \mathcal{F}^k(V) \subset V$ for all $k > m$.

Since $A = \lim_{n \to \infty} \mathcal{F}^n(\overline{O})$, there is an integer K such that $\mathcal{F}^K(\overline{O}) \subset O$. Let O_k, $k = 0, 1, \dots, K$, be defined recursively, going backwards from O_K to O_0, as follows. Let $O_K = O$, and for $k = K - 1, \dots, 2, 1, 0$, let O_k be an open set such that

(4) $\mathcal{F}^k(\overline{O}) \subset O_k$,
(5) $\overline{O_k} \subset \mathbb{P}^n \backslash H$, and
(6) $\mathcal{F}(\overline{O_k}) \subset O_{k+1}$.

To verify that a set O_k with these properties exists, first note that property (4) holds for $k = K$. To verify the properties for all $k = K - 1, \dots, 2, 1, 0$ inductively, assume that O_k, $k \ge 1$, satisfies property (4). Using property (4) we have $\mathcal{F}^{k-1}(\overline{O}) \subset \mathcal{F}^{-1}(\mathcal{F}^k(\overline{O})) \subset \mathcal{F}^{-1}(O_k)$ and using property (3) we have $\mathcal{F}^{k-1}(\overline{O}) \subset \overline{V} \subset \mathbb{P}^n \backslash H$. Now choose O_{k-1} to be an open set such that $\mathcal{F}^{k-1}(\overline{O}) \subset O_{k-1}$ and $\overline{O_{k-1}} \subset \mathcal{F}^{-1}(O_k) \cap (\mathbb{P}^n \backslash H)$. The last inclusion implies $\mathcal{F}(\overline{O_{k-1}}) \subset O_k$.

We claim that

$$U = \bigcup_{k=0}^{K-1} O_k$$

satisfies the properties in the statement of part (ii) of the lemma. (*) By property (5) we have $\overline{U} \cap H = \emptyset$. By properties (2) and (4) we have $A = \mathcal{F}^k(A) \subset \mathcal{F}^k(O) \subset O_k$ for each k, which implies $A \subset U$. Last,

$$\mathcal{F}(\overline{U}) = \bigcup_{k=0}^{K-1} \mathcal{F}(\overline{O_k}) \subset \bigcup_{k=1}^{K} O_k = \bigcup_{k=1}^{K-1} O_k \cup O_K \subset U \cup O \subset U \cup O_0 \subset U,$$

the first inclusion coming from property (6) and the second-to-last inclusion coming from property (4) applied to $k = 0$. This completes the proof that there is a nonempty open set U such that $A \subset U$. $\overline{U} \cap H = \emptyset$, and $\mathcal{F}(\overline{U}) \subset U$. Now note that, by construction, $\overline{U}$ is such that $\mathcal{F}^K(\overline{U}) \subset O_K = O$ and that $\overline{O}$ lies in $\overline{V}$ which lies in the basin of attraction of A, which implies that $\overline{U}$ is contained in the basin of attraction of A. This completes the proof of (ii).

The proof of (i) is the same as the above proof of (ii), except that $\mathbb{P}^n \backslash H$ is replaced by $\mathbb{P}^n$ throughout, and the sentence (*) is omitted. $\qquad\square$

6 Projective Transformations of Convex Sets

This section describes the action of a projective transformation on a convex set. We develop the key result, Proposition 4, that is used subsequently.

Proposition 1 states that the property of being a convex subset (with respect to a hyperplane) of a projective space is preserved under a projective transformation.

Proposition 1 *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation. For any two hyperplanes H, H' with $S \cap H = \emptyset$ and $f(S) \cap H' = \emptyset$, the set $S \subset \mathbb{P}^n$ is a convex set with respect to H if and only if $f(S)$ is convex with respect to H'.*

Proof Assume that S is convex with respect to H. To show that $f(S)$ is convex with respect to H' it is sufficient to show, given any two points x', $y' \in f(S)$, that $\overline{x'y'}_{H'} \subseteq f(S)$. If $x = f^{-1}(x')$ and $y = f^{-1}(y')$, then by the convexity of S and the fact that $S \cap H = \emptyset$, we know that $\overline{xy}_H \subseteq S$. Hence $f(\overline{xy}_H) \subseteq f(S)$. Since $f(S) \cap H' = \emptyset$, and f takes lines to lines, $\overline{x'y'}_{H'} = f(\overline{xy}_H) \subseteq f(S)$.

The converse follows since f^{-1} is a projective transformation. $\qquad\square$

Proposition 2 states that $\mathrm{conv}_H(S)$ behaves well under projective transformation.

Proposition 2 *Let $S \subset \mathbb{P}^n$ and let H be a hyperplane such that $S \cap H = \emptyset$. If $f : \mathbb{P}^n \to \mathbb{P}^n$ is a projective transformation, then*

$$\mathrm{conv}_{f(H)} f(S) = f(\mathrm{conv}_H(S)).$$

Proof Since $S \subseteq \mathrm{conv}_H(S)$, we know that $f(S) \subseteq f(\mathrm{conv}_H(S))$. Moreover, by Proposition 1, we know that $f(\mathrm{conv}_H(S))$ is convex with respect to $f(H)$. To show that $\mathrm{conv}_{f(H)} f(S) = f(\mathrm{conv}_H(S))$ it is sufficient to show that $f(\mathrm{conv}_H(S))$ is the smallest convex subset containing $f(S)$, i.e., there is no set C such that C is convex with respect to $f(H)$ and $f(S) \subseteq C \subsetneq f(\mathrm{conv}_H(S))$. However, if such a set exists, then by applying the inverse f^{-1} to the above inclusion, we have $S \subseteq f^{-1}(C) \subsetneq \mathrm{conv}_H(S)$. Since $f^{-1}(C)$ is convex by Proposition 1, we arrive at a contradiction to the fact that $\mathrm{conv}_H(S)$ is the smallest convex set containing S. $\quad\square$

In general, $\mathrm{conv}_H(S)$ depends on the avoided hyperplane H. But, as Proposition 3 shows, it is independent of the avoided hyperplane when S is connected.

Proposition 3 *If $S \subset \mathbb{P}^n$ is a connected set such that $S \cap H = S \cap H' = \emptyset$ for hyperplanes H, H' of $\mathbb{P}^n$, then*

$$\mathrm{conv}_H(S) = \mathrm{conv}_{H'}(S).$$

Proof The fact that S is connected and $S \cap H' = \emptyset$ implies that $\mathrm{conv}_H(S) \cap H' = \emptyset$. Therefore $\mathrm{conv}_H(S)$ is the ordinary convex hull of S in $(\mathbb{P}^n \backslash H) \backslash H'$, which is an affine n-dimensional space with a hyperplane deleted. Likewise $\mathrm{conv}_{H'}(S)$ is the ordinary convex hull of S in $(\mathbb{P}^n \backslash H') \backslash H = (\mathbb{P}^n \backslash H) \backslash H'$. Therefore, $\mathrm{conv}_H(S) = \mathrm{conv}_{H'}(S)$. $\qquad\square$

The key result that will be needed, for example, in Sect. 7, is the following.

Proposition 4 *Let $S \subset \mathbb{P}^n$ be a connected set and let H be a hyperplane. If $S \cap H = \emptyset$ and $f : \mathbb{P}^n \to \mathbb{P}^n$ is a projective transformation such that $f(S) \cap H = \emptyset$, then*

$$\mathrm{conv}_H f(S) = f(\mathrm{conv}_H(S)).$$

Proof This follows at once from Propositions 2 and 3. $\qquad\square$

7 Proof that (2)⟹(3) in Theorem 1

The implication (2)⟹(3) in Theorem 1 is proved in two steps. We show that (2)⟹(2.5)⟹(3) where (2.5) is the following statement.

(2.5) *There is a hyperplane H and a nonempty finite collection of nonempty disjoint connected open sets $\{O_i\}$ such that $\mathcal{F}(\bigcup_i \overline{O}_i) \subset \cup_i O_i$ and $\bigcup_i \overline{O}_i \cap H = \emptyset$.*

Lemma 2 ((2)⟹(2.5)) *If there is a nonempty open set U and a hyperplane H with $\overline{U} \cap H = \emptyset$ such that $\mathcal{F}(\overline{U}) \subset U$, then there is a nonempty finite collection of nonempty disjoint connected open sets $\{O_i\}$ such that $\mathcal{F}(\bigcup_i \overline{O}_i) \subset \cup_i O_i$ and $\bigcup_i \overline{O}_i \cap H = \emptyset$.*

Proof Let $U = \bigcup_\alpha U_\alpha$, where the U_α are the connected components of U. Let $\tilde{A} = \bigcap_k \mathcal{F}^k(\overline{U})$ and let $\{O_i\}$ be the set of U_α that have nonempty intersection with $\tilde{A}$. This set is finite because the sets in $\{O_i\}$ are pairwise disjoint and $\tilde{A}$ is compact. Since $\mathcal{F}(\tilde{A}) \subset \tilde{A}$ and $\mathcal{F}(\overline{U}) \subset U$, we find that $\mathcal{F}(\bigcup \overline{O}_i) \subset \bigcup_i O_i$. Since $\bigcup_i \overline{O}_i \subset \overline{U}$ and $\overline{U} \cap H = \emptyset$, we have $\bigcup_i \overline{O}_i \cap H = \emptyset$.   □

Lemma 3 ((2.5)⟹(3)): *If there is a nonempty finite collection of nonempty disjoint connected open sets $\{O_i\}$ and a hyperplane H such that $\mathcal{F}(\bigcup_i \overline{O}_i) \subset \bigcup_i O_i$ and $\bigcup_i \overline{O}_i \cap H = \emptyset$, then there is a nonempty finite collection of disjoint convex bodies $\{C_i\}$ such that $\mathcal{F}(\bigcup_i C_i) \subset \text{int}(\bigcup_i C_i)$.*

Proof Assume that there is a nonempty finite collection of nonempty disjoint connected open sets $\{O_i\}$ such that $\mathcal{F}(\bigcup_i \overline{O}_i) \subset \bigcup_i O_i$ and $\bigcup_i \overline{O}_i$ avoids a hyperplane. Let $O = \bigcup_i O_i$. Since $\mathcal{F}(\overline{O}) \subset O$, it must be the case that, for each $f \in \mathcal{F}$ and each i, there is an index that we denote by $f(i)$, such that $f(\overline{O}_i) \subset O_{f(i)}$. Since $\overline{O}_i$ is connected and both $\overline{O}_i$ and $f(\overline{O}_i)$ avoid the hyperplane H it follows from Proposition 4 that

$$f(\text{conv}_H(\overline{O}_i)) = \text{conv}_H(f(\overline{O}_i)) \subset \text{conv}_H(O_{f(i)}) \subset \text{int}(\text{conv}_H(\overline{O}_{f(i)})).$$

For each i, let $C_i = \text{conv}_H(\overline{O}_i)$, so that each C_i is a convex body. Then we have

$$f(C_i) \subset \text{int}(C_{f(i)}).$$

However, it may occur, for some $i \neq j$, that $C_i \cap C_j \neq \emptyset$. In this case $C_i \cup C_j$ is a connected set that avoids the hyperplane H, and is such that $f(C_i \cup C_j)$ also avoids H. It follows again by Proposition 4 that

$$\text{conv}_H(f(C_i \cup C_j)) = f(\text{conv}_H(C_i \cup C_j)) \subset \text{int}(\text{conv}(C_{f(i)} \cup C_{f(j)})).$$

Define C_i and C_j to be related if $C_i \cap C_j \neq \emptyset$, and let $\sim$ denote the transitive closure of this relation. (That is, if C_i is related to C_j and C_j is related to C_k, then C_i is related to C_k.) From the set $\{C_i\}$ define a new set U' whose elements are

$$U' = \left\{ \text{conv}\left(\bigcup_{C \in Z} C \right) : Z \text{ is an equivalence class with respect to } \sim \right\}.$$

By abuse of language, let $\{C_i\}$ be the set of convex sets in U'. It may again occur, for some $i \neq j$, that $C_i \cap C_j \neq \emptyset$. In this case we repeat the equivalence process. In a finite number of such steps we arrive at a finite set of disjoint convex bodies $\{C_i\}$ such that $\mathcal{F}(\cup C_i) \subset \text{int}(\cup C_i)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemmas 2 and 3 taken together imply that $(2) \Rightarrow (3)$ in Theorem 1.

8 Part 1 of the Proof that $(3) \Rightarrow (4)$ in Theorem 1

The standing assumption in this section is that statement (3) of Theorem 1 is true. We begin to develop a metric with respect to which $\mathcal{F}$ is contractive. The final metric is defined in the next section.

Let $\mathcal{U} := \{C_1, C_2, \ldots, C_q\}$ be the set of nonempty convex connected components in statement (3) of Theorem 1. Define a directed graph (digraph) G as follows. The nodes of G are the elements of $\mathcal{U}$. For each $f \in \mathcal{F}$, there is an edge *colored* f directed from node U to node V if $f(U) \subset \text{int}(V)$. Note that, for each node U in G, there is exactly one edge of each color emanating from U. Note also that G may have *multiple edges* from one node to another and may have loops. (A *loop* is an edge from a node to itself.)

A *directed path* in a digraph is a sequence of nodes $U_0, U_1, \ldots, U_k$ such that there is an edge directed from U_{i-1} to U_i for $i = 1, 2, \ldots, k$. Note that a directed path is allowed to have repeated nodes and edges. Let $p = U_0, U_1, \ldots, U_k$ be a directed path. If $f_1, f_2, \ldots, f_k$ are the colors of the successive edges, then we will say that p has *type* $f_1 f_2 \cdots f_k$.

Lemma 4 *The graph G cannot have two directed cycles of the same type starting at different nodes.*

Proof By way of contradiction assume that $U \neq U'$ are the starting nodes of two paths p and p' of the same type $f_1 f_2 \cdots f_k$. Recall that the colors are functions of the IFS $\mathcal{F}$. If $g = f_k \circ f_{k-1} \circ \cdots \circ f_1 \circ f_0$, then the composition g takes the convex set U into $\text{int}(U)$ and the convex set U' into $\text{int}(U')$. By the Krein–Rutman theorem [23] this is impossible. More specifically, the Krein–Rutman theorem tells us that if K is a closed convex cone in $\mathbb{R}^{n+1}$ and $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear transformation such that $L(K) \subset \text{int}(K)$, then the spectral radius $r(L) > 0$ is a simple eigenvalue of L with an eigenvector $v \in \text{int}(K)$. $\qquad\qquad\qquad\qquad\square$

Each function $f \in \mathcal{F}$ acts on the set of nodes of G in this way: $f(U) = V$ where (U, V) is the unique edge of color f starting at U.

Lemma 5 *There exists a metric d_G on the set of nodes of G such that*

(1) $d_G(U, V) \geq 2$ *for all $U \neq V$ and*
(2) *each $f \in \mathcal{F}$ is a contraction with respect to d_G.*

Proof Starting from the graph G, construct a directed graph G_2 whose set of nodes consists of all unordered pairs $\{U, V\}$ of distinct nodes of G. In G_2 there is an edge from $\{U, V\}$ to $\{f(U), f(V)\}$ for all nodes $\{U, V\}$ in G_2 and for each $f \in \mathcal{F}$. Since G has no two directed cycles of the same type starting at different nodes, we know by Lemma 4 that G_2 has no directed cycle. Because of this, a partial order $\prec$ can be defined on the node set of G_2 by declaring that $\{U', V'\} \prec \{U, V\}$ if there is an edge from $\{U, V\}$ to $\{U', V'\}$ and then taking the transitive closure. Every finite partially ordered set has a linear extension (see [15], for example), i.e., there is an ordering $<$ of the nodes of G_2:

$$\{U_1, V_1\} < \{U_2, V_2\} < \cdots < \{U_m, V_m\}$$

such that if $\{U, V\} \prec \{U', V'\}$ then $\{U, V\} < \{U', V'\}$. Using $N(G)$ to denote the set of nodes of G, define a map $d_G : N(G) \times N(G) \to [0, \infty)$ in any way satisfying

(1) $d_G(U, U) = 0$ for all $U \in N(G)$,
(2) $d_G(U, V) = d_G(V, U)$ for all $U, V \in N(G)$, and
(3) $2 \le d_G(U_1, V_1) < d_G(U_2, V_2) < \cdots < d_G(U_m, V_m) \le 4$.

Properties (1), (2), and (3) guarantee that d_G is a metric on $N(G)$. The fact that $2 \le d_G(U_i, V_i) \le 4$ for all i guarantees the triangle inequality. If

$$s = \min_{1 \le i < m} \frac{d_G(U_i, V_i)}{d_G(U_{i+1}, V_{i+1})},$$

then $0 < s < 1$ and, for any $f \in \mathcal{F}$, we have

$$d_G(f(U), f(V)) \le s\, d_G(U, V)$$

because $\{f(U), f(V)\} \prec \{U, V\}$ by the definition of the partial order and $\{f(U), f(V)\} < \{U, V\}$ by the definition of linear extension. Hence f is a contraction with respect to d_G for any $f \in \mathcal{F}$.   □

9 Part 2 of the Proof that (3)⇒(4) in Theorem 1

In this section we construct a metric d_i on each component C_i of the collection $\{C_i\} = \{C_i : i = 1, 2, \ldots, q\}$ in statement (3) of Theorem 1. We will then combine the metrics d_i with the graph metric d_G in Sect. 8 to build a metric on $\bigcup_i C_i$ such that statement (4) in Theorem 1 is true. Proofs that a projective transformation is contractive with respect to the Hilbert metric go back to G. Birkhoff [13]; also see P.J. Bushell [17]. The next lemma is used to compute the contraction factor for projective maps under the Hilbert metric.

Lemma 6 *If $r \ge \alpha \ge 0, t \ge \alpha$, and $h, h', s, s' \in (0, 1)$, where $s' = 1 - s$, $h' = 1 - h$, and $s \le h$, then $\log(\frac{(r+h)(t+s')}{(r+s)(t+h')}) \le \log(\frac{(\alpha+h)(\alpha+s')}{(\alpha+s)(\alpha+h')}) \le \frac{1}{\alpha+1} \log(\frac{hs'}{sh'})$.*

Proof Since we are assuming that $s \le h$, $s(1 - h) > 0$, and $\alpha \ge 0$, it is an easy exercise to show that $\frac{(\alpha+h)(\alpha+s')}{(\alpha+s)(\alpha+h')} \ge 1$. A bit of algebra can be used to show that $N :=$

$\frac{(\alpha+h)(\alpha+s')}{(\alpha+s)(\alpha+h')} = \frac{(1-\frac{h'}{\alpha+1})(1-\frac{s}{\alpha+1})}{(1-\frac{s'}{\alpha+1})(1-\frac{h}{\alpha+1})}$. If we let $\alpha = 0$ in the above expression, we observe that $D := \frac{hs'}{sh'} = \frac{(1-h')(1-s)}{(1-s')(1-h)}$.

Since $\ln(1-x) = \log_e(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}$, whenever $|x| < 1$, for a logarithm of any base we see that

$$
\begin{aligned}
\frac{\log(N)}{\log(D)} &= \frac{\log(1-\frac{h'}{\alpha+1}) + \log(1-\frac{s}{\alpha+1}) - \log(1-\frac{s'}{\alpha+1}) - \log(1-\frac{h}{\alpha+1})}{\log(1-h') + \log(1-s) - \log(1-s') - \log(1-h)} \\
&= \frac{-\sum_{j=1}^{\infty} \left[\frac{h'^j}{j(\alpha+1)^j} + \frac{s^j}{j(\alpha+1)^j} - \frac{s'^j}{j(\alpha+1)^j} - \frac{h^j}{j(\alpha+1)^j} \right]}{-\sum_{j=1}^{\infty} \left[\frac{h'^j}{j} + \frac{s^j}{j} - \frac{s'^j}{j} - \frac{h^j}{j} \right]} \\
&= \frac{1}{\alpha+1} \frac{\sum_{j=1}^{\infty} \frac{1}{(\alpha+1)^{j-1}} \left[\frac{s'^j}{j} + \frac{h^j}{j} - \frac{h'^j}{j} - \frac{s^j}{j} \right]}{\sum_{j=1}^{\infty} \left[\frac{s'^j}{j} + \frac{h^j}{j} - \frac{h'^j}{j} - \frac{s^j}{j} \right]} \le \frac{1}{\alpha+1}.
\end{aligned}
$$

Note that the above inequality holds because the assumption $s \le h$ implies $s' = 1 - s \ge 1 - h = h'$ and $(1-s)^j + h^j \ge (1-h)^j + s^j$, for all positive integers j. Thus, the series in the numerator and denominator can be compared term by term. Finally, it is a straightforward argument to show the numerator $N(\alpha)$ has the property that if $r \ge \alpha \ge 0$ and $t \ge \alpha \ge 0$, then $\frac{(r+h)(t+s')}{(r+s)(t+h')} \le \frac{(\alpha+h)(\alpha+s')}{(\alpha+s)(\alpha+h')}$. Thus, $\log(\frac{(r+h)(t+s')}{(r+s)(t+h')}) \le \log(\frac{(\alpha+h)(\alpha+s')}{(\alpha+s)(\alpha+h')})$. $\square$

Proposition 5 *Let $\mathcal{F}$ be a projective IFS and let there be a nonempty finite collection of disjoint convex bodies $\{C_i : i = 1, 2, \ldots, q\}$ such that $\mathcal{F}(\bigcup_i C_i) \subset \text{int}(\bigcup_i C_i)$ as in statement (3) of Theorem 1. For $i \in \{1, 2, \ldots, q\}$ and $f \in \mathcal{F}$, let $f(i) \in \{1, 2, \ldots, q\}$ be defined by $f(C_i) \subset C_{f(i)}$. Then there is a metric d_i on C_i, giving the same topology on C_i as $d_{\mathbb{P}}$, such that*

1. *(C_i, d_i) is a complete metric space, for all $i = 1, 2, \ldots, q$;*
2. *there is a real $0 \le \alpha < 1$ such that*

$$
d_{f(i)}(f(x), f(y)) \le \alpha d_i(x, y)
$$

   for all $x, y \in C_i$, for all $i = 1, 2, \ldots, q$, for all $f \in \mathcal{F}$; and
3. *$d_i(x, y) \le 1$ for all $x, y \in C_i$ and all $i = 1, 2, \ldots, q$.*

Proof For each C_i there exists a hyperplane H_i such that $H_i \cap C_i = \emptyset$. Let $\widehat{C}_i = \{x \in \mathbb{P}^n : d_{\mathbb{P}}(x, y) \le \varepsilon, y \in C_i\}$ where ε is chosen so small that (i) $H_i \cap \widehat{C}_i = \emptyset$; and (ii) $f(\widehat{C}_i) \subset \text{int}(\widehat{C}_{f(i)}) \, \forall f \in \mathcal{F}, \forall i \in \{1, 2, \ldots, q\}$.

Given arbitrary $x, y \in \text{int}(\widehat{C}_i)$, let a, b be the points where the line $\overline{xy}$ intersects $\partial \widehat{C}_i$ and let a_f, b_f be the points where the line $\overline{f(x)f(y)}$ intersects $\partial \widehat{C}_{f(i)}$. Let $\hat{d}_i$ denote the Hilbert metric on the interior of $\widehat{C}_i$ for each i, and define

$$
\beta_{f,i} = \min\{|xy| : x \in \partial \widehat{C}_{f(i)}, y \in f(\widehat{C}_i)\} > 0, \quad \text{for } f \in \mathcal{F}, i \in \{1, 2, \ldots, q\}.
$$

We claim that

$$\hat{d}_{f(i)}(f(x), f(y)) = \ln\left(\frac{|a_f\, f(y)|\,|f(x)\,b_f|}{|a_f\, f(x)|\,|f(y)\,b_f|}\right)$$

$$\leq \frac{1}{\beta_{f,i}+1}\ln\left(\frac{|f(a)\,f(y)|\,|f(x)\,f(b)|}{|f(a)\,f(x)|\,|f(y)\,f(b)|}\right)$$

$$= \frac{1}{\beta_{f,i}+1}\ln\left(\frac{|a\,y|\,|x\,b|}{|a\,x|\,|y\,b|}\right) = \frac{1}{\beta_{f,i}+1}\hat{d}_i(x,y), \quad (9.1)$$

for all $x, y \in \text{int}(\widehat{C}_i)$, for all $f \in \mathcal{F}$, and all $i = 1, 2, \ldots$. Here $|\cdot|$ denotes Euclidean distance as discussed in Sect. 3. The second-to-last equality is the invariance of the cross ratio under a projective transformation. Concerning the inequality, without loss of generality, let $|f(a)\,f(b)| = 1$, $h := |f(a)\,f(y)|$, and $s := |f(a)\,f(x)|$. Moreover, let $r := |a_f\, f(x)|$ and $t := |f(y)\,b_f|$. Finally, let $s' = 1 - s$ and $h' = 1 - h$. Note that $s \leq h < 1$. The inequality is now the inequality of Lemma 6.

Now let $\alpha = \max\{\frac{1}{1+\beta_{f,i}} : f \in \mathcal{F}, \forall i = 1, 2, \ldots, q\} < 1$. It follows that

$$\hat{d}_{f(i)}(f(x), f(y)) \leq \alpha \hat{d}_i(x, y)$$

for all $x, y \in \widehat{C}_i$, for all $i = 1, 2, \ldots, q$, for all $f \in \mathcal{F}$. Since $C_i \subset \text{int}(\widehat{C}_i)$ it follows that statement (2) in Proposition 5 is true.

Statement (1) follows at once from the fact that the topology generated by the Hilbert metric $\hat{d}_i$ on C_i as defined above is bi-Lipschitz equivalent to $d_{\mathbb{P}}$; see Remark 4.

Since $\hat{d}_i : C_i \times C_i \to \mathbb{R}$ is continuous and $C_i \times C_i$ is compact, it follows that there is a constant J_i such that $\hat{d}_i(x, y) \leq J_i$ for all $x, y \in C_i$. Let $J = \max_i J_i$, and define a new metric d_i by $d_i(x, y) = \hat{d}_i(x, y)/J$ for all $x, y \in C_i$. We have that d_i satisfies (1), (2), and (3) in the statement of Proposition 5. $\qquad\square$

Lemma 7 (Theorem 1 (3)$\Rightarrow$(4)) *If there is a nonempty finite collection of disjoint convex bodies $\{C_i\}$ such that $\mathcal{F}(\bigcup_i C_i) \subset \text{int}(\bigcup_i C_i)$, as in statement (3) of Theorem 1, then there is a nonempty open set $U \subset \mathbb{P}^n$ and a metric $d : \overline{U} \to [0, \infty)$, generating the same topology as $d_{\mathbb{P}}$ on $\overline{U}$, such that $\mathcal{F}$ is contractive on $\overline{U}$.*

Proof Let $U = \bigcup_i \text{int}(C_i)$. Define $d : \overline{U} \times \overline{U}$ by

$$d(x, y) = \begin{cases} d_i(x, y) & \text{if } (x, y) \in C_i \times C_i \text{ for some } i, \\ d_G(C_i, C_j) & \text{if } (x, y) \in C_i \times C_j \text{ for some } i \neq j, \end{cases}$$

where the metrics d_i and d_G are defined in Lemma 5 and Proposition 5.

First we show that d is a metric on $\overline{U}$. We only need to check the triangle inequality. If x, y and z lie in the same connected component of C_i, the triangle inequality follows from Proposition 5. If x, y, and z lie in three distinct components, the triangle inequality follows from Lemma 5. If $x, y \in C_i$ and $z \in C_j$ for some $i \neq j$, then

$$d(x, y) + d(y, z) = d_i(x, y) + d_G(C_i, C_j) \geq d_G(C_i, C_j) = d(x, z),$$

$$d(x, z) + d(z, y) = d_G(C_i, C_j) + d_G(C_j, C_i) \geq 2 \geq d_i(x, y) = d(x, y).$$

Second we show that $\mathcal{F}$ is contractive with respect to d. By Proposition 5 there is $0 \leq \alpha < 1$ such that, if x and y lie in the same connected component of U and $f \in \mathcal{F}$, then

$$d(f(x), f(y)) \leq \alpha \, d(x, y).$$

If x and y lie in different connected components of U, then there are two cases. If $f(x)$ and $f(y)$ lie in different connected components, then by Lemma 5,

$$d(f(x), f(y)) = d_G(f(x), f(y)) \leq \alpha_G \, d_G(x, y) = d(x, y),$$

where α_G is the constant guaranteed by Lemma 5. If $f(x)$ and $f(y)$ lie in the same connected component U_i, then

$$d(f(x), f(y)) = d_i(f(x), f(y)) \leq 1 \leq \frac{1}{2} d_G(x, y) = \frac{1}{2} d(x, y).$$

Third, and last, the metric d generates the same topology on $\overline{U}$ as the metric $d_{\mathbb{P}}$, because, for any convex body K, the Hilbert metric d_K and the metric $d_{\mathbb{P}}$ are bi-Lipschitz equivalent on any compact subset of the interior of K; see Remark 4 in Sect. 13. □

10 Proof that (4)⟹(1) in Theorem 1 and the Proof of the Uniqueness of Attractors

This section contains a proof that statement (4) implies statement (1) in Theorem 1 and a proof of Theorem 2 on the uniqueness of the attractor.

A point $p_f \in \mathbb{P}^n$ is said to be an *attractive fixed point* of the projective transformation f if $f(p_f) = p_f$, and f is a contraction with respect to the round metric on some open ball centered at p_f. If f has an attractive fixed point, then the real Jordan canonical form [26] can be used to show that any matrix $L_f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ representing f has a dominant eigenvalue. In the case that f has an attractive fixed point, let E_f denote the n-dimensional L_f-invariant subspace of $\mathbb{R}^{n+1}$ that is the span of the eigenspaces corresponding to all the other eigenvalues. Let $H_f := \phi(E_f)$ be the corresponding hyperplane in $\mathbb{P}^n$. Note that H_f is invariant under f and $p_f \notin H_f$. Moreover, the basin of attraction of p_f for f is $\mathbb{P}^n \backslash H_f$.

Lemma 8 (Theorem 1 (4)⟹(1)) *If there is a nonempty open set $U \subset \mathbb{P}^n$ such that $\mathcal{F}$ is contractive on $\overline{U}$, then $\mathcal{F}$ has an attractor A that avoids a hyperplane.*

Proof We are assuming statement (4) in Theorem 1 that the IFS $\mathcal{F}$ is contractive on $\overline{U}$ with respect to some metric d. Since $\overline{U}$ is compact and $(\mathbb{P}^n, d_{\mathbb{P}})$ is a complete metric space, $(\overline{U}, d)$ is a complete metric space. It is well known in this case [22] that $\mathcal{F}$ has an attractor $A \subset U$. It only remains to show that there is a hyperplane H such that $A \subset \mathbb{P}^n \backslash H$.

Let f be any function in $\mathcal{F}$. Since f is a contraction on $\overline{U}$, we know by the Banach contraction mapping theorem that f has an attractive fixed point x_f. We claim that $x_f \in A$. If $x \in \mathbb{P}^n \setminus H_f$ lies in the basin of attraction of A, then $x_f = \lim_{k \to \infty} f^k(x) \in A$. It now suffices to show that $A \cap H_f = \emptyset$. By way of contradiction, assume that $x \in A \cap H_f$. Since $\mathcal{F}$ is contractive on $\overline{U}$, it is contractive on A. Since $x_f \in A$, we have $d(f^k(x), x_f) = d(f^k(x), f^k(x_f)) \to 0$ as $k \to \infty$, which is impossible since $f^k(x) \in H_f$ and $x_f \notin H_f$.   □

So now we have that Statements (1), (2), (3), and (4) in Theorem 1 are equivalent. The proof of Lemma 8 also shows the following.

Corollary 1 *If $\mathcal{F}$ is a contractive IFS, then each $f \in \mathcal{F}$ has an attractive fixed point x_f and an invariant hyperplane H_f.*

Proposition 6 *Let $\mathcal{F}$ be a projective IFS containing at least one map that has an attractive fixed point. If $\mathcal{F}$ has an attractor A, then A is the unique attractor in $\mathbb{P}^n$.*

Proof Assume that there are two distinct attractors A, A', and let U, U' be their respective basins of attraction. If $U \cap U' \neq \emptyset$, then $A = A'$, because if there is $x \in U \cap U'$ then $A' = \lim_{k \to \infty} \mathcal{F}^k(x) = A$, where the limit is with respect to the Hausdorff metric. Therefore $U \cap U' = \emptyset$ and $A \cap A' = \emptyset$.

If $f \in \mathcal{F}$ has an attractive fixed point p_f and $p \in U \setminus H_f$, and $p' \in U' \setminus H_f$, then both

$$p_f = \lim_{k \to \infty} f^k(p) \subseteq \lim \mathcal{F}^k(p) = A, \quad \text{and}$$

$$p_f = \lim_{k \to \infty} f^k(p') \subseteq \lim \mathcal{F}^k(p') = A'.$$

But this is impossible since $A \cap A' = \emptyset$. So Proposition 6 is proved.   □

We can now prove Theorem 2—that a projective IFS has at most one attractor.

Proof of Theorem 2 Assume, by way of contradiction, that A and A' are distinct attractors of $\mathcal{F}$ in $\mathbb{P}^n$. As in the proof of Proposition 6, it must be the case that $A \cap A' = \emptyset$ and hence that their respective basins of attraction are disjoint.

By Lemma 1 there exist open sets U and U' such that $A \subset U$, $A' \subset U'$, and $\mathcal{F}(\overline{U}) \subset U$ and $\mathcal{F}(\overline{U'}) \subset U'$. Since $\overline{U}$ belongs to the basin of attraction of A and $\overline{U'}$ belongs to the basin of attraction of A', we have $U \cap U' = \emptyset$. If $f \in \mathcal{F}$ and $x \in U$, then in the Hausdorff topology

$$A(x) := \lim_{k \to \infty} \overline{\bigcup_{m \geq k} f^m(x)} \subset A$$

and $A(x)$ is nonempty. Similarly, if $x' \in U'$, then

$$A(x') := \lim_{k \to \infty} \overline{\bigcup_{m \geq k} f^m(x')} \subset A'$$

and $A(x')$ is nonempty.

Let L_f be a matrix for $f \in \mathcal{F}$ in real Jordan canonical form and such that the largest modulus of an eigenvalue is 1. Let W denote the L_f-invariant subspace of $\mathbb{R}^{n+1}$ corresponding to the eigenvalues of modulus 1, and let L denote the restriction of L_f to W. If E is the subspace of $\mathbb{P}^n$ corresponding to the subspace W of $\mathbb{R}^{n+1}$, then, by use of the Jordan canonical form, $A(x) \subset E$ and $A(x') \subset E$. Together with the inclusions above, this implies that $A \cap E \neq \emptyset$ and $A' \cap E \neq \emptyset$. Hence $U_E := U \cap E \neq \emptyset$ and $U'_E := U' \cap E \neq \emptyset$ and if $f|_E$ denotes the restriction of f to E, then

$$f|_E(\overline{U_E}) = f(\overline{U} \cap E) = f(\overline{U}) \cap E \subset U \cap E = U_E, \qquad (10.1)$$

and similarly $f|_E(\overline{U'_E}) \subset U'_E$.

Each Jordan block of L can have one of the following forms

(a) $\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$, (b) $\begin{pmatrix} R & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & R & \cdots & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \cdots & R \end{pmatrix}$,

(c) $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$, (d) $\begin{pmatrix} R & I & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & R & I & \cdots & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & R \end{pmatrix}$,

where R is a rotation matrix of the form $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix} \right)$, $\mathbf{0}$ denotes the 2×2 zero matrix, and I denotes the 2×2 identity matrix. Let $V_W = \phi^{-1}(U_E)$ and $V' = \phi^{-1}(U')$.

Case 1. $L : W \to W$ is an isometry. This is equivalent to saying that each Jordan block of L is of type (a) or (b). The fact that $|\det L| = 1$, and regarding L as acting on the unit sphere in W, implies that $L(\overline{V_W}) \subset V_W$ is not possible unless $V_W = E$, which in turn implies that $f|_E(\overline{U_E}) \subset U_E$ is not possible unless $U_E = E$. Therefore, by equation (10.1) we have $U_E = E$ and similarly $U'_E = E$, which implies that $U \cap U' \neq \emptyset$, contradicting what was stated above.

Case 2. There is at least one Jordan block in L_E of the form (c) or (d). Define the *size* of an $m \times m$ Jordan block B as m if B is of type (c) and $m/2$ if B is of type (d). Let s be the maximum of the sizes of all (c) and (d) type Jordan blocks. Let $\widehat{W}$ be the subspace of $\mathbb{R}^{n+1}$ consisting of all points $(x_0 x_1, \ldots, x_n)$ in homogeneous coordinates with $x_i = 0$ for all i not corresponding to the first row of a Jordan block of type (c) and size s or to the first two rows of a Jordan block of type (d) and size s. Let $\widehat{E}$ be the projective subspace corresponding to $\widehat{W}$. If $x \in U$, then it is routine to check, by iterating the Jordan canonical form and scaling so that the maximum modulus of an eigenvalue is 1, that $A(x) \subset \widehat{E}$. Similarly, if $x' \in U'$, then $A(x') \subset \widehat{E}$. Therefore $U_{\widehat{E}} := U \cap \widehat{E} \neq \emptyset$ and $U'_{\widehat{E}} := U' \cap \widehat{E} \neq \emptyset$. As done above for E, if $f|_{\widehat{E}}$ denotes the restriction of f to $\widehat{E}$, then $f|_{\widehat{E}}(\overline{U_{\widehat{E}}}) \subset U_{\widehat{E}}$, and $f|_{\widehat{E}}(\overline{U'_{\widehat{E}}}) \subset U'_{\widehat{E}}$. But $\widehat{W}$ is invariant under L and, if $\widehat{L}$ is the restriction of L to $\widehat{W}$, then $\widehat{L}$ is an isometry. We now arrive at a contradiction exactly as was done in Case 1. $\qquad \square$

11 Duals and Adjoints

Recall that $d_{\mathbb{P}}(\cdot, \cdot)$ is the metric on $\mathbb{P}^n$ defined in Sect. 3.2. The *hyperplane orthogonal to* $p \in \mathbb{P}$ is defined and denoted by

$$p^{\perp} = \{q \in \mathbb{P}^n : q \perp p\}.$$

If $(\mathbb{X}, d_{\mathbb{X}})$ denotes a compact metric space $\mathbb{X}$ with metric $d_{\mathbb{X}}$, then $(\mathbb{H}(\mathbb{X}), h_{\mathbb{X}})$ denotes the corresponding compact metric space that consists of the nonempty compact subsets of $\mathbb{X}$ with the Hausdorff metric $h_{\mathbb{X}}$ derived from $d_{\mathbb{X}}$, defined by

$$h_{\mathbb{X}}(B, C) = \max \left\{ \sup_{b \in B} \inf_{c \in C} d_{\mathbb{X}}(b, c), \sup_{c \in C} \inf_{b \in B} d_{\mathbb{X}}(b, c) \right\}$$

for all $B, C \in \mathbb{H}$. It is a standard result that if $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_M)$ is a contractive IFS, then $\mathcal{F} : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ is a contraction with respect to the Hausdorff metric.

Definition 9 The *dual space* $\widehat{\mathbb{P}^n}$ of $\mathbb{P}^n$ is the set of all hyperplanes of $\mathbb{P}^n$, equivalently $\widehat{\mathbb{P}^n} = \{p^{\perp} : p \in \mathbb{P}^n\}$. The dual space is endowed with a metric $d_{\widehat{\mathbb{P}}}$ defined by

$$d_{\widehat{\mathbb{P}}}(p^{\perp}, q^{\perp}) = d_{\mathbb{P}}(p, q)$$

for all $p^{\perp}, q^{\perp} \in \widehat{\mathbb{P}}$. The map $\mathcal{D} : \mathbb{P}^n \to \widehat{\mathbb{P}^n}$ defined by

$$\mathcal{D}(p) = p^{\perp}$$

is called the *duality map*. The duality map can be extended to a map $\mathcal{D} : \mathbb{H}(\mathbb{P}^n) \to \mathbb{H}(\widehat{\mathbb{P}^n})$ between compact subsets of $\mathbb{P}^n$ and $\widehat{\mathbb{P}^n}$ in the usual way.

Given a projective transformation f and any matrix L_f representing it, the matrix $L_{f^{-1}} := L_f^{-1}$ represents the projective transformation f^{-1} that is the inverse of f. In a similar fashion, define the *adjoint* f^t and the adjoint inverse transformation f^{-t} as the projective transformations represented by the matrices

$$L_{f^t} := L_f^t \quad \text{and} \quad L_{f^{-t}} := (L_f^{-1})^t = (L_f^t)^{-1},$$

respectively, where t denotes the transpose matrix. It is easy to check that the adjoint and adjoint inverse are well defined. For a projective IFS $\mathcal{F}$, the following related iterated function systems will be used in this section.

(1) The *adjoint of the projective IFS* $\mathcal{F}$ is denoted by $\mathcal{F}^t$ and defined to be

$$\mathcal{F}^t = (\mathbb{P}^n; f_1^t, f_2^t, \ldots, f_M^t).$$

(2) The *inverse of the projective IFS* $\mathcal{F}$ is the projective IFS

$$\mathcal{F}^{-1} = (\mathbb{P}^n; f_1^{-1}, f_2^{-1}, \ldots, f_M^{-1}).$$

(3) If $\mathcal{F} = (\mathbb{P}^n; f_1, f_2, \ldots, f_M)$ is a projective IFS then the *corresponding hyperplane IFS* is

$$\widehat{\mathcal{F}} = (\widehat{\mathbb{P}^n}; f_1, f_2, \ldots, f_M),$$

where $f_m : \widehat{\mathbb{P}^n} \to \widehat{\mathbb{P}^n}$ is defined by $f_m(H) = \{f_m(q) \,|\, q \in H\}$. Notice that, whereas $\mathcal{F}$ is associated with the compact metric space $(\mathbb{P}^n, d_\mathbb{P})$, the hyperplane IFS $\widehat{\mathcal{F}}$ is associated with the compact metric space $(\widehat{\mathbb{P}^n}, d_{\widehat{\mathbb{P}}})$.

(4) The *corresponding inverse hyperplane IFS* is

$$\widehat{\mathcal{F}^{-1}} = (\widehat{\mathbb{P}^n}; f_1^{-1}, f_2^{-1}, \ldots, f_M^{-1}),$$

where $f_m^{-1} : \widehat{\mathbb{P}^n} \to \widehat{\mathbb{P}^n}$ is defined by $f_m^{-1}(H) = \{f_m^{-1}(q) \,|\, q \in H\}$.

Proposition 7 *The duality map $\mathcal{D}$ is a continuous, bijective, inclusion preserving isometry between compact metric spaces $(\mathbb{P}^n, d_\mathbb{P})$ and $(\widehat{\mathbb{P}^n}, d_{\widehat{\mathbb{P}}})$ and also a continuous, bijective, inclusion preserving isometry between $(\mathbb{H}(\mathbb{P}^n), h_\mathbb{P})$ and $(\mathbb{H}(\widehat{\mathbb{P}^n}), h_{\widehat{\mathbb{P}}})$. Moreover, the following diagrams commute for any projective transformation f and any projective IFS $\mathcal{F}$:*

$$
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\;\mathcal{D}\;} & \widehat{\mathbb{P}^n} \\
{\scriptstyle f^t}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\
\mathbb{P}^n & \xrightarrow[\;\mathcal{D}\;]{} & \widehat{\mathbb{P}^n}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{H}(\mathbb{P}^n) & \xrightarrow{\;\mathcal{D}\;} & \mathbb{H}(\widehat{\mathbb{P}^n}) \\
{\scriptstyle \mathcal{F}^t}\downarrow & & \downarrow{\scriptstyle \widehat{\mathcal{F}^{-1}}} \\
\mathbb{H}(\mathbb{P}^n) & \xrightarrow[\;\mathcal{D}\;]{} & \mathbb{H}(\widehat{\mathbb{P}^n}).
\end{array}
$$

Proof Clearly $\mathcal{D}$ maps $\mathbb{P}^n$ bijectively onto $\widehat{\mathbb{P}^n}$ and $\mathbb{H}(\mathbb{P}^n)$ bijectively onto $\mathbb{H}(\widehat{\mathbb{P}^n})$. The continuity of $\mathcal{D}$ and the inclusion preserving property are also clear. The definition of $d_{\widehat{\mathbb{P}}}$ in terms of $d_\mathbb{P}$ implies that $\mathcal{D}$ is an isometry from $\mathbb{P}^n$ onto $\widehat{\mathbb{P}^n}$. The definition of $h_{\widehat{\mathbb{P}}}$ in terms of $d_{\widehat{\mathbb{P}}}$ and the definition of $h_{\mathbb{P}^n}$ in terms of $d_\mathbb{P}$ implies that $\mathcal{D}$ is an isometry from $\mathbb{H}(\mathbb{P}^n)$ onto $\mathbb{H}(\widehat{\mathbb{P}^n})$. The compactness of $(\mathbb{P}^n, d_\mathbb{P})$ implies that $(\widehat{\mathbb{P}^n}, d_{\widehat{\mathbb{P}}})$ is a compact metric space.

To verify that the diagrams commute it is sufficient to show that, for all $x \in \mathbb{P}^n$ and any projective transformation f, we have $L_f^{-1}(x^\perp) = [L_f^t(x)]^\perp$. But, using the ordinary Euclidean inner product,

$$L_f^{-1}(x^\perp) = \{L_f^{-1} y : \langle x, y \rangle = 0\} = \{z : \langle x, L_f z \rangle = 0\}$$

$$= \{z : \langle L_f^t x, z \rangle = 0\} = [L_f^t(x)]^\perp. \qquad \square$$

Let $\mathbb{S}(\mathbb{P}^n)$ denote the set of all subsets of $\mathbb{P}^n$ (including the empty set).

Definition 10 The *complementary dual* of a set $X \subset \mathbb{P}^n$ is

$$X^* = \{q \in \mathbb{P}^n : q \perp x \text{ for no } x \in X\}.$$

For an IFS $\mathcal{F}$ define the operator $\overline{\mathcal{F}} : \mathbb{S}(\mathbb{P}^n) \to \mathbb{S}(\mathbb{P}^n)$ by

$$\overline{\mathcal{F}}(X) = \bigcap_{f \in \mathcal{F}} f^{-t}(X),$$

for any $X \in \mathbb{S}(\mathbb{P}^n)$.

Proposition 8 *The map* $* : \mathbb{S}(\mathbb{P}^n) \to \mathbb{S}(\mathbb{P}^n)$ *is an inclusion reversing function with these properties*:

1. *The following diagram commutes*

$$
\begin{array}{ccc}
\mathbb{S}(\mathbb{P}^n) & \xrightarrow{\ *\ } & \mathbb{S}(\mathbb{P}^n) \\
{\scriptstyle \mathcal{F}}\Big\downarrow & & \Big\downarrow{\scriptstyle \overline{\mathcal{F}}} \\
\mathbb{S}(\mathbb{P}^n) & \xrightarrow{\ *\ } & \mathbb{S}(\mathbb{P}^n).
\end{array}
$$

2. *If* $\mathcal{F}(X) \subset Y$, *then* $\mathcal{F}^t(Y^*) \subset X^*$.
3. *If* X *is open*, *then* X^* *is closed. If* X *is closed*, *then* X^* *is open*.
4. $\overline{X^*} \subset X^*$.

Proof The fact that the diagrams commute is easy to verify. Since the other assertions are also easy to check, we prove only statement (3). Since $*$ is inclusion reversing, $\mathcal{F}(X) \subset Y$ implies that $Y^* \subset [\mathcal{F}(X)]^* = \overline{\mathcal{F}}(X^*)$, the equality coming from the commuting diagram. The definition of $\overline{\mathcal{F}}$ then yields $\mathcal{F}^t(Y^*) \subset X^*$. □

Proposition 9 *If* $\mathcal{F}$ *is a projective IFS,* $U \subset \mathbb{P}^n$ *is open, and* $\mathcal{F}(\overline{U}) \subset U$, *then* $V = \overline{U}^*$ *is open and* $\mathcal{F}^t(\overline{V}) \subset V$.

Proof From statement (3) of Proposition 8 it follows that V is open. From $\mathcal{F}(\overline{U}) \subset U$ and from statement (2) of Proposition 8 it follows that $\mathcal{F}^t(U^*) \subset \overline{U}^*$. By statement (4) we have $\mathcal{F}^t(\overline{V}) = \mathcal{F}^t(\overline{\overline{U}^*}) \subset \mathcal{F}^t(U^*) \subset \overline{U}^* = V$. □

Lemma 9 (Theorem 1 (1)⇔(5)) *A projective IFS* $\mathcal{F}$ *has an attractor* A *that avoids a hyperplane if and only if* $\mathcal{F}^t$ *has an attractor* A^t *that avoids a hyperplane*.

Proof Suppose statement (1) of Theorem 1 is true. By statement (2) of Theorem 1 there is a nonempty open set U and a hyperplane H such that $\mathcal{F}(\overline{U}) \subset U$ and $H \cap \overline{U} = \emptyset$. By Proposition 9 we have $\mathcal{F}^t(\overline{V}) \subset V$ where $V = \overline{U}^*$ is open. Moreover, there is a hyperplane H^t such that $H^t \cap \overline{V} = \emptyset$: simply choose $H^t = a^\perp$ for any $a \in A \subset U$, where A is the attractor of $\mathcal{F}$. By the definition of the dual complement, $a^\perp \cap U^* = \emptyset$ which, by statement (4) of Proposition 8, implies that $a^\perp \cap \overline{V} = a^\perp \cap \overline{\overline{U}^*} = \emptyset$. So, as long as $V \neq \emptyset$, $\mathcal{F}^t$ also satisfies statement (2) of Theorem 1. In this case it follows that statement (1) of Theorem 1 is true for $\mathcal{F}^t$, and hence statement (5) is true.

We show that $V \neq \emptyset$ by way of contradiction. If $V = \emptyset$, then by the definition of the dual complement, every $y \in \mathbb{P}^n$ is orthogonal to some point in $\overline{U}$, i.e.,

$$\overline{U}^{\perp} := \{y : y \perp x \text{ for some } x \in \overline{U}\} = \mathbb{P}^n.$$

On the other hand, since $\overline{U}$ avoids some hyperplane $y^{\perp}$, we arrive at the contradiction $y \notin \overline{U}^{\perp}$.

The converse in Lemma 9 is immediate because $(\mathcal{F}^t)^t = \mathcal{F}$. $\qquad\square$

Definition 11 A set $\mathcal{A} \subset \widehat{\mathbb{P}^n}$ is called a *hyperplane attractor* of the projective IFS $\mathcal{F}$ if it is an attractor of the IFS $\widehat{\mathcal{F}}$. A set $R \subset \mathbb{P}^n$ is said to be a *repeller* of the projective IFS $\mathcal{F}$ if R is an attractor of the inverse IFS $\mathcal{F}^{-1}$. A set $\mathcal{R} \subset \widehat{\mathbb{P}^n}$ is said to be a *hyperplane repeller* of the projective IFS $\mathcal{F}$ if it is a hyperplane attractor of the inverse hyperplane IFS $\widehat{\mathcal{F}^{-1}}$.

Proposition 10 *The compact set $A \subset \mathbb{P}^n$ is an attractor of the projective IFS $\mathcal{F}^t$ that avoids a hyperplane if and only if $\mathcal{D}(A)$ is a hyperplane repeller of $\mathcal{F}$ that avoids a point.*

Proof Concerning the first of the two conditions in the definition of an attractor, we have from the commuting diagram in Proposition 7 that $\mathcal{F}^t(A) = A$ if and only if $\widehat{\mathcal{F}^{-1}}(\mathcal{D}(A)) = \mathcal{D}(\mathcal{F}^t(A)) = \mathcal{D}(A)$.

Concerning the second of the two conditions in the definition of an attractor, let B be an arbitrary subset contained in the basin of attraction U of $\mathcal{F}^t$. With respect to the Hausdorff metric, $\lim_{k \to \infty} (\mathcal{F}^t)^k(B) = A$ if and only if

$$\lim_{k \to \infty} \widehat{\mathcal{F}^{-1}}^k (\mathcal{D}(B)) = \lim_{k \to \infty} \mathcal{D}((\mathcal{F}^t)^k(B)) = \mathcal{D}(\lim_{k \to \infty} (\mathcal{F}^t)^k(B)) = \mathcal{D}(A).$$

Also, the attractor $\mathcal{D}(A)$ of $\widehat{\mathcal{F}^{-1}}$ avoids the point p if and only if the attractor A of $\mathcal{F}^t$ avoids the hyperplane $p^{\perp}$. $\qquad\square$

Lemma 10 *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation with attractive fixed point p_f and corresponding invariant hyperplane H_f. If $f^{-1} : \widehat{\mathbb{P}^n} \to \widehat{\mathbb{P}^n}$ has an attractive fixed point $\widehat{H}_f$, then $\widehat{H}_f = H_f$.*

Proof There is some basis with respect to which f has matrix $\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}$. If f is represented by matrix L_f with respect to the standard basis, then there is an invertible matrix M such that

$$L_f = M \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} M^{-1},$$

where L is a non-singular $n \times n$ matrix whose eigenvalues λ satisfy $|\lambda| < 1$. Then

$$L_f^{-1} = M \begin{pmatrix} L^{-1} & 0 \\ 0 & 1 \end{pmatrix} M^{-1} \quad \text{and} \quad L_f^t = M^{-t} \begin{pmatrix} L^t & 0 \\ 0 & 1 \end{pmatrix} M^t.$$

If $x = (0, 0, \ldots, 0, 1)$, then by Proposition 7

$$\widehat{H_f} = (M^{-t}x)^{\perp} = M(x^{\perp}) = H_f. \qquad \square$$

Proposition 11 *If $\mathcal{F}$ is a projective IFS and U is an open set such that $\overline{U}$ avoids a hyperplane and $\mathcal{F}(\overline{U}) \subset U$, then $\mathcal{F}$ has an attractor A and U is contained in the basin of attraction of A.*

Proof We begin by noting that $\mathcal{F}(\overline{U}) \subset U$ implies that $\{\mathcal{F}^k(\overline{U})\}_{k=1}^{\infty}$ is a nested sequence of nonempty compact sets. So

$$\widetilde{A} := \bigcap_{k=1}^{\infty} \mathcal{F}^k(\overline{U})$$

is also a nonempty compact set. Using the continuity of $\mathcal{F} : \mathbb{H}(\mathbb{P}^n) \to \mathbb{H}(\mathbb{P}^n)$, we have $\mathcal{F}(\widetilde{A}) = \widetilde{A}$.

If $B \in \mathbb{H}(P^n)$ is such that $B \subset U$, then, given any $\varepsilon > 0$, there is a positive integer $K := K(\varepsilon)$ such that $\mathcal{F}^{K(\varepsilon)}(B) \subset \widetilde{A}_{\varepsilon}$, the set $\widetilde{A}$ dilated by an open ball of radius ε.

In the next paragraph we are going to show that, for sufficiently small $\varepsilon > 0$, there is a metric on $\widetilde{A}_{\varepsilon}$ such that $\mathcal{F}$ is contractive on $\widetilde{A}_{\varepsilon}$. For now, assume that $\mathcal{F}$ is contractive on $\widetilde{A}_{\varepsilon}$. This implies, by Theorems 1 and 2, that $\mathcal{F}$ has a unique attractor A and it is contained in $\widetilde{A}_{\varepsilon}$. We now show that $A = \widetilde{A}$. That $\mathcal{F}$ is contractive on $\widetilde{A}_{\varepsilon}$ implies that $\mathcal{F}$, considered as a mapping on $\mathbb{H}(\widetilde{A}_{\varepsilon})$, is a contraction with respect to the Hausdorff metric. By the contraction mapping theorem, $\mathcal{F}$ has a unique fixed point, so $A = \widetilde{A}$. By choosing ε small enough that $\widetilde{A}_{\varepsilon} = A_{\varepsilon}$ lies in the basin of attraction of A, the fact that $\mathcal{F}^K(B) \subset \widetilde{A}_{\varepsilon}$ implies that $\lim_{k\to\infty} \mathcal{F}^k(B) = A$. Hence U lies in the basin of attraction of A, which concludes the proof of Proposition 11.

To prove that $\mathcal{F}$ is contractive on $\widetilde{A}_{\varepsilon}$ for sufficiently small $\varepsilon > 0$, we follow the steps in the construction of the metric in statement (4) of Theorem 1, starting from the proof of Lemma 2. As in the proof of Lemma 2, let $U = \bigcup_{\alpha} U_{\alpha}$, where the U_{α} are the connected components of U. Let $\{O_i\}$ be the set of U_{α} that have nonempty intersection with $\widetilde{A}$. Since $\widetilde{A}$ is compact and nonempty, we must have

$$\widetilde{A}_{\varepsilon} \subset \bigcup_i O_i$$

for all ε sufficiently small. We now follow the steps in the proof of Lemmas 2, 3, up to and including Lemma 7, to construct a metric on a finite set of convex bodies $\{C_i\}$ such that $\bigcup_i O_i \subset \bigcup_i C_i$ and such that $\mathcal{F}$ is contractive on $\bigcup_i C_i$. Note that the metric is constructed on a set containing $\bigcup_i O_i$, which in turn contains $\widetilde{A}_{\varepsilon}$. This completes the proof. $\qquad \square$

We can now prove Theorem 3.

Proof of Theorem 3 We prove the first statement of the theorem. The proof of the second statement is identical with $\mathcal{F}$ replaced by $\mathcal{F}^{-1}$.

Assume that projective IFS $\mathcal{F}$ has an attractor that avoids a hyperplane. By statement (4) of Theorem 1, the IFS $\mathcal{F}^t$ has an attractor that avoids a hyperplane. Then, according to Proposition 10, $\widehat{\mathcal{F}^{-1}}$ has an attractor that avoids a point. By the definition of hyperplane repeller, $\mathcal{F}$ has a hyperplane repeller that avoids a point.

Concerning the basin of attraction, let R denote the union of the hyperplanes in $\mathcal{R}$ and let $Q = \mathbb{P}^n \setminus R$. We must show that $Q = O$, where O is the basin of attraction of the attractor A of $\mathcal{F}$.

First we show that $O \subset Q$, i.e., $O \cap R = \emptyset$. Consider any $f : \mathbb{P}^n \to \mathbb{P}^n$ with $f \in \mathcal{F}$ and $f^{-1} : \widehat{\mathbb{P}^n} \to \widehat{\mathbb{P}^n}$. Since we have already shown that $\widehat{\mathcal{F}^{-1}}$ has an attractor, it satisfies all statements of Theorem 1. It then follows, exactly as in the proof of Lemma 8, that $f^{-1} : \widehat{\mathbb{P}^n} \to \widehat{\mathbb{P}^n}$ has an attractive fixed point, a hyperplane $\widehat{H_f} \in \mathcal{R} \subset \widehat{\mathbb{P}^n}$. Let

$$\mathcal{B} = \overline{\bigcup_{k=1}^{\infty} \bigcup_{f \in \mathcal{F}} (\widehat{\mathcal{F}^{-1}})^k (\widehat{H_f})} \subset \widehat{\mathbb{P}^n} \quad \text{and} \quad B = \bigcup_{H \in \mathcal{B}} H.$$

The fact that $\widehat{H_f} = H_f$ (Lemma 10) and $H_f \cap O = \emptyset$ for all $f \in \mathcal{F}$ implies that $O \cap B = \emptyset$. We claim that $\mathcal{B} = \mathcal{R}$ and hence $B = R$, which would complete the proof that $O \cap R = \emptyset$. Concerning the claim, because $\mathcal{R}$ is the attractor of $\widehat{\mathcal{F}^{-1}}$, we have that

$$\mathcal{R} = \lim_{k \to \infty} (\widehat{\mathcal{F}^{-1}})^k \left(\bigcup_{f \in \mathcal{F}} \widehat{H_f} \right) \subset \mathcal{B}.$$

Since $\widehat{H_f} \in \mathcal{R}$ for all $f \in \mathrm{F}$, also $\mathcal{B} \subset \mathcal{R}$, which completes the proof of the claim.

Finally we show that $Q \subset O$. By statements (2) and (5) of Theorem 1, $\mathcal{F}^t$ has an attractor A^t that avoids a hyperplane. Consequently there is an open neighborhood V of A^t and a metric such that $\mathcal{F}^t$ is contractive on $\overline{V}$, and $\overline{V}$ avoids a hyperplane. In particular $\mathcal{F}^t$ is a contraction on $\mathbb{H}(\overline{V})$ with respect to the Hausdorff metric. Let λ denote a contractivity factor for $\mathcal{F}^t|_{\overline{V}}$. Let $\varepsilon > 0$ be small enough that the closed set A_ε^t (the dilation of A^t by a closed ball of radius ε, namely the set of all points whose distance from A^t is less than or equal to ε) is contained in V. If $h_{\mathbb{P}}(A_\varepsilon^t, A^t) = \varepsilon$, then

$$h_{\mathbb{P}}(\mathcal{F}^t(A_\varepsilon^t), A^t) = h_{\mathbb{P}}(\mathcal{F}^t(A_\varepsilon^t), \mathcal{F}^t(A^t)) \le \lambda h_{\mathbb{P}}(A_\varepsilon^t, A^t)) = \lambda \varepsilon.$$

It follows that $\mathcal{F}^t(A_\varepsilon^t) \subset \mathrm{int}(A_\varepsilon^t)$ and from Proposition 8 (2, 3) that

$$\mathcal{F}(\overline{(A_\varepsilon^t)^*}) \subseteq \mathcal{F}(\mathrm{int}((A_\varepsilon^t)^*)) \subset (A_\varepsilon^t)^*.$$

Let $Q_\varepsilon := (A_\varepsilon^t)^*$. It follows from $\mathcal{F}(\overline{Q_\varepsilon}) \subset Q_\varepsilon$ and Proposition 11 that $Q_\varepsilon \subset O$. Let $\mathcal{R}_\varepsilon = \mathcal{D}(A_\varepsilon^t)$ and let $R_\varepsilon \subset \mathbb{P}^n$ be the union of the hyperplanes in $\mathcal{R}_\varepsilon$. By Proposition 10 and the definition of the dual complement, $Q_\varepsilon = \mathbb{P}^n \setminus R_\varepsilon$ and $Q = \mathbb{P}^n \setminus R$. Since $Q_\varepsilon \subset O$ it follows that $R_\varepsilon \subset \mathbb{P}^n \setminus O$. Since $\mathcal{D}$ is continuous (Proposition 7) and $A_\varepsilon^t \to A^t$ as $\varepsilon \to 0$, it follows that $\mathcal{R}_\varepsilon = \mathcal{D}(A_\varepsilon^t) \to \mathcal{D}(A^t) = \mathcal{R}$. Consequently $R \subset \mathbb{P}^n \setminus O$, and therefore $Q = \mathbb{P}^n \setminus R \subset O$. $\square$

12 Geometrical Properties of Attractors

The Hausdorff dimension of the attractor of a projective IFS is invariant under the
projective group $\mathrm{PGL}(n + 1, \mathbb{R})$. This is so because any projective transformation is
bi-Lipschitz with respect to $d_\mathbb{P}$, that is, if $f : \mathbb{P}^n \to \mathbb{P}^n$ is a projective transformation,
then there exist two constants $0 < \lambda_1 < \lambda_2 < \infty$ such that

$$\lambda_1 d_\mathbb{P}(x, y) \leq d_\mathbb{P}(f(x), f(y)) \leq \lambda_2 d_\mathbb{P}(x, y).$$

We omit the proof as it is a straightforward geometrical estimate.

The main focus of this section is another type of invariant that depends both on
the attractor and on a corresponding hyperplane repeller. It is a type of Conley in-
dex and is relevant to the study of parameter dependent families of projective IFSs
and the question of when there exists a continuous family of IFS's whose attractors
interpolate a given set of projective attractors, as discussed in Example 4 in Sect. 4.
Ongoing studies suggest that this index has stability properties with respect to small
perturbations and that there does not exist a family of projective IFSs whose attractors
continuously interpolate between attractors with different indices.

Definition 12 Let $\mathcal{F}$ be a projective IFS with attractor A that avoids a hyperplane
and let R denote the union of the hyperplanes in the hyperplane repeller of $\mathcal{F}$. The
index of $\mathcal{F}$ is

$$\mathrm{index}(\mathcal{F}) = \# \text{ connected components } O \text{ of } \mathbb{P}^n \backslash R \text{ such that } A \cap O \neq \emptyset.$$

Namely, the index of a contractive projective IFS is the number of components of
the open set $\mathbb{P}^n \backslash R$ which have nonempty intersection with its attractor. By statement
(1) of Theorem 3, we know that $\mathrm{index}(\mathcal{F})$ will always equal a positive integer.

Definition 13 Let A denote a nonempty compact subset of $\mathbb{P}^n$ that avoids a hyper-
plane. If $\mathbb{F}_A$ denotes the collection of all projective IFSs for which A is an attractor,
then the *index of A* is defined by the rule

$$\mathrm{index}(A) = \min_{\mathcal{F} \in \mathbb{F}_A} \{\mathrm{index}(\mathcal{F})\}.$$

If the collection $\mathbb{F}_A$ is empty, then define $\mathrm{index}(A) = 0$.

Note that an attractor A not only has a multitude of projective IFSs associated with
it, but it may also have a multitude of repellers associated with it. Clearly $\mathrm{index}(A)$
is invariant under $\mathrm{PGL}(n + 1, \mathbb{R})$, the group of real projective transformations. The
following lemma shows that, for any positive integer, there exists a projective IFS $\mathcal{F}$
that has that integer as index.

Proposition 12 *Let $\mathcal{F} = (\mathbb{P}^1; f_1, f_2, f_3, \ldots, f_M)$ be a projective IFS where, for
each m, the projective transformation f_m is represented by the matrix*

$$L_m := \begin{pmatrix} 2m\lambda - 2m + 1 & 2m(m - \frac{1}{2}) - m\lambda(2m - 1) \\ 2\lambda - 2 & 2m - \lambda(2m - 1) \end{pmatrix}.$$

Fig. 3 (Color online)
A projective IFS with index
equal to four. The attractor is
sketched in *white*, while the
union of the hyperplanes in the
hyperplane repeller is indicated
in *red*, *blue*, *green*, and *gray*
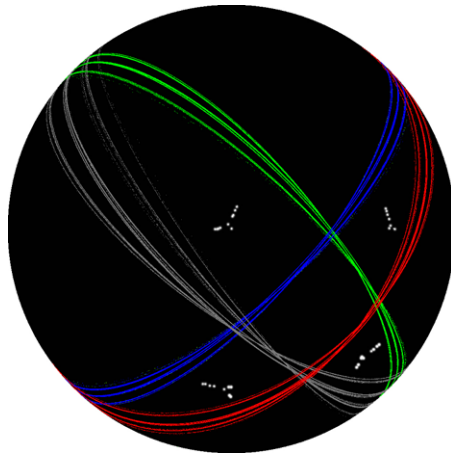
For any integer $M > 1$ and sufficiently large λ, the projective IFS has $\text{index}(\mathcal{F}) = M$.

Proof Topologically, the projective line $\mathbb{P}^1$ is a circle. It is readily verified that

$$L_m = \begin{pmatrix} \lambda m & m - \frac{1}{2} \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} m & m - \frac{1}{2} \\ 1 & 1 \end{pmatrix}^{-1},$$

from which it can be easily checked that, for λ is sufficiently large, f_m has attractive fixed point $x_m = \begin{pmatrix} m \\ 1 \end{pmatrix}$ and repulsive fixed point $y_m = \begin{pmatrix} m - \frac{1}{2} \\ 1 \end{pmatrix}$. In particular $L_m \begin{pmatrix} m \\ 1 \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} m \\ 1 \end{pmatrix}$ and $L_m \begin{pmatrix} m - \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} m - \frac{1}{2} \\ 1 \end{pmatrix}$. Note that the points x_i, $i = 1, 2, \ldots, M$, and y_i, $i = 1, 2, \ldots, M$, interlace on the circle (projective line). Also, as λ increases, the attractive fixed points x_m become increasingly attractive.

Let I_k denote a very small interval that contains the attractive fixed point x_k of f_k, for $k = 1, 2, \ldots, M$. When λ is sufficiently large, $f_m(\bigcup I_k) \subset I_m \subset \bigcup I_k$. It follows that the attractor of $\mathcal{F}$ is a Cantor set contained in $\bigcup I_k$. Similarly, the hyperplane repeller of $\mathcal{F}$ consists of another Cantor set that lies very close to the set of points $\{k - 0.5 : k = 1, 2, \ldots, M\}$. It follows that $\text{index}(\mathcal{F}) = M$.   □

Another example is illustrated in Figure 3. In this case the underlying space has dimension two and the IFS $\mathcal{F}$ has $\text{index}(\mathcal{F}) = 4$.

The previous result shows that the index of a contractive IFS can be any positive integer. It does not state that the same is true for the index of an attractor. The following Theorem 4 shows that the index of an attractor is a nontrivial invariant in that it is not always the case that $\text{index}(A) = 1$. To prove it we need the following definition and result.

Definition 14 A set $C \subset \mathbb{P}^n$ is called a *Cantor set* if it is the attractor of a contractive IFS $(\mathbb{P}^n; f_1, f_2, \ldots, f_M)$, $M \geq 2$, such that each point of C corresponds to an unique string $\sigma = \sigma_1 \sigma_2 \cdots \in \{1, 2, \ldots, M\}^\infty$ such that

$$x = \varphi_F(\sigma) = \lim_{k \to \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x_0), \tag{12.1}$$

where x_0 is any point in C.

Lemma 11 *Let* $\mathcal{F} = (\mathcal{P}^n; f_1, f_2, \ldots, f_M)$ *be a projective IFS whose attractor is a Cantor set* C. *Let the projective IFS*

$$\mathcal{G} = (\mathbb{P}^n; f_{\omega_1}, f_{\omega_2}, \ldots, f_{\omega_L})$$

have the same attractor C, *where each* f_{ω_l} *is a finite composition of functions in* $\mathcal{F}$, *i.e.,*

$$f_{\omega_l} = f_{\sigma_1^l} \circ f_{\sigma_2^l} \circ \cdots \circ f_{\sigma_{j_l}^l}$$

in the obvious notation. Then $\mathcal{F}$ *and* $\mathcal{G}$ *have the same hyperplane repeller and* $\mathrm{index}(\mathcal{F}) = \mathrm{index}(\mathcal{G})$.

Proof We must show that $\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{F}}$, where $\mathcal{R}_{\mathcal{F}}$ is the hyperplane repeller of $\mathcal{F}$ and $\mathcal{R}_{\mathcal{G}}$ is the hyperplane repeller of $\mathcal{G}$. Let $\sigma = \sigma_1 \sigma_2 \cdots$ and $\omega_{l_1} \omega_{l_2} \cdots$ be strings of symbols in $\{1, 2, \ldots, M\}^{\infty}$ and $\{\omega_1, \omega_2, \ldots, \omega_L\}^{\infty}$, respectively. Define

$$\psi : \{\omega_1, \omega_2, \ldots, \omega_L\}^{\infty} \to \{1, 2, \ldots, M\}^{\infty}$$

by

$$\psi(\omega_{l_1} \omega_{l_2} \cdots) = \zeta(\omega_{l_1}) \zeta(\omega_{l_2}) \cdots \quad \text{where } \zeta(\omega_l) = \sigma_1^l \sigma_2^l \cdots \sigma_{j_l}^l.$$

We claim that ψ is surjective. It is well known that the mapping $\varphi_F : \{1, 2, \ldots, M\}^{\infty} \to C$ in (12.1) is a continuous bijection; see, for example, [2, Chap. 4]. Let $\sigma = \sigma_1 \sigma_2 \cdots \in \{1, 2, \ldots, M\}^{\infty}$ and let $x = \lim_{k \to \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x_0)$. Since C is also the attractor of $\mathcal{G}$ it is likewise true that there is at least one string $\omega = \omega_{l_1} \omega_{l_2} \cdots \in \{\omega_1, \omega_2, \ldots, \omega_L\}^{\infty}$ such that

$$x = \lim_{k \to \infty} f_{\omega_{l_1}} \circ f_{\omega_{l_2}} \circ \cdots \circ f_{\omega_{l_k}}(x_0)$$
$$= \lim_{k \to \infty} (f_{\sigma_1^{l_1}} \circ \cdots \circ f_{\sigma_{j_{l_1}}^{l_1}}) \circ (f_{\sigma_1^{l_2}} \circ \cdots \circ f_{\sigma_{j_{l_2}}^{l_2}}) \circ \cdots \circ (f_{\sigma_1^{l_k}} \circ \cdots \circ f_{\sigma_{j_{l_k}}^{l_k}})(x_0).$$

By the uniqueness of σ in (12.1), we have $\psi(\omega) = \sigma$, showing that ψ is surjective.

We are now going to show that $\mathcal{R}_{\mathcal{F}} \subseteq \mathcal{R}_{\mathcal{G}}$. Let $r \in \mathcal{R}_{\mathcal{F}}$. Note that the hyperplanes of $\mathbb{P}$ are simply the points of $\mathbb{P}$. Moreover, the hyperplane repeller $\mathcal{R}_{\mathcal{F}}$ of $\mathcal{F}$ is simply the attractor of the IFS $\widehat{\mathcal{F}^{-1}} := (\widehat{\mathbb{P}^n}; f_1^{-1}, f_2^{-1}, \ldots, f_M^{-1})$ and the hyperplane repeller $\mathcal{R}_{\mathcal{G}}$ of $\mathcal{G}$ is the attractor of $\widehat{\mathcal{G}^{-1}} := (\widehat{\mathbb{P}^n}; f_{\omega_1}^{-1}, f_{\omega_2}^{-1}, \ldots, f_{\omega_L}^{-1})$. Let r_0 be the attractive fixed point of $f_{\omega_1}^{-1}$. Note that r_0 lies in both $\mathcal{R}_{\mathcal{G}}$ and in $\mathcal{R}_{\mathcal{F}}$. According to Theorems 1 and 3, both $\widehat{\mathcal{F}^{-1}}$ and $\widehat{\mathcal{G}^{-1}}$ are contractive. Therefore

$$r = \lim_{k \to \infty} f_{\sigma_1}^{-1} \circ f_{\sigma_2}^{-1} \circ \cdots \circ f_{\sigma_k}^{-1}(r_0)$$

for some $\sigma = \sigma_1 \sigma_2 \cdots \in \{1, 2, \ldots, M\}^{\infty}$. Since ψ is surjective, there is a string $\omega_{l_1} \omega_{l_2} \cdots \in \{\omega_1, \omega_2, \ldots, \omega_L\}^{\infty}$ such that

$$r = \lim_{k \to \infty} f_{\sigma_1}^{-1} \circ f_{\sigma_2}^{-1} \circ \cdots \circ f_{\sigma_k}^{-1}(r_0)$$

$$= \lim_{k \to \infty} (f_{\sigma_k}^{-1} \circ f_{\sigma_{k-1}} \circ \cdots \circ f_{\sigma_1})^{-1}(r_0)$$

$$= \lim_{m \to \infty} (f_{\omega_{l_m}} \circ f_{\omega_{l_{m-1}}} \circ \cdots \circ f_{\omega_{l_1}})^{-1}(r_0)$$

$$= \lim_{k \to \infty} f_{\omega_{l_1}}^{-1} \circ f_{\omega_{l_2}}^{-1} \circ \cdots \circ f_{\omega_{l_k}}^{-1}(r_0) \in \lim_{k \to \infty} (\widehat{\mathcal{G}^{-1}})^k (r_0) = \mathcal{R}_{\mathcal{G}}.$$

A similar, but easier, argument shows that $\mathcal{R}_{\mathcal{G}} \subseteq \mathcal{R}_{\mathcal{F}}$. Hence $\mathcal{F}$ and $\mathcal{G}$ have the same hyperplane repeller. Since the attractors and hyperplane repellers of both are the same we have index$(\mathcal{F}) =$ index$(\mathcal{G})$ by the definition of the index. $\qquad \square$

Theorem 4 *If $\mathcal{F} = (\mathbb{P}^1; f_1, f_2)$ is the projective IFS in Proposition* 12 *with $M = 2$, $\lambda = 10$, and A is the attractor of $\mathcal{F}$, then* index$(A) = 2$.

Proof Let $\widehat{\mathcal{F}} = (\mathbb{P}^1; \hat{f}_1, \hat{f}_2)$, where

$$\hat{f}_1 = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{f}_2 = \begin{pmatrix} 37 & -18 \\ 54 & -26 \end{pmatrix}.$$

It is easy to check that $\hat{f}_1 = f \circ f_1 \circ f^{-1}$ and $\hat{f}_2 = f \circ f_2 \circ f^{-1}$ where f_1 and f_2 are the functions in Proposition 12 when $\lambda = 10$, and f is the projective transformation represented by the matrix $L_f = \begin{pmatrix} 1 & -1 \\ 1 & -\frac{1}{2} \end{pmatrix}$. It is sufficient to show that if $\widehat{A}$ is the attractor of $\widehat{\mathcal{F}}$, then index$(\widehat{A}) = 2$. From here on the IFS $\mathcal{F}$ is not used, so we drop the "hat" from $\widehat{\mathcal{F}}, \hat{f}_1, \hat{f}_2, \widehat{A}$. Also to simplify notation, the set of points of the projective line are taken to be $\mathbb{P} = \mathbb{R} \cup \{\infty\}$, where $\begin{pmatrix} x \\ 1 \end{pmatrix}$ is denoted as the fraction x and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is denoted as ∞. In this notation $f_1(x) = \frac{1}{10}x$ and $f_2(x) = \frac{37x-18}{54x-26}$ when restricted to $\mathbb{R}$. The following are properties of $\mathcal{F}$.

(1) The attractor C of $\mathcal{F}$ is a Cantor set.
(2) index$(\mathcal{F}) = 2$.
(3) The origin $a = 0$ is the attractive fixed point of f_1 while its repulsive hyperplane is ∞.
(4) The attractive fixed point of f_2 is at $c = 2/3$ and its repulsive hyperplane is at $1/2$.
(5) $C \subset [a, b] \cup [c, d]$, where $b = \frac{11}{40} - \frac{1}{120}\sqrt{609}$ $(= 0.069351)$ and $d = \frac{11}{4} - \frac{1}{12}\sqrt{609}$ $(= 0.69351)$ are the attractive fixed points of $f_1 \circ f_2$ and $f_2 \circ f_1$, respectively.
(6) If h is any projective transformation taking C into itself, then $h([a, b] \cup [c, d]) \subset [a, b] \cup [c, d]$.
(7) The symmetry group of C is trivial, i.e., the only projective transformation h such that $h(C) = C$ is the identity.

Property (1) is in the proof of Proposition 12, and property (2) is a consequence of Proposition 12. Properties (3) and (4) are easily verified by direct calculation. Property (5) can be verified by checking that $\mathcal{F}([a, b] \cup [c, d]) \subset [a, b] \cup [c, d]$.

To prove property (6), let I denote a closed interval (on the projective line, topologically a circle) that contains C. Its image $h^{-1}(I)$ is also a closed interval. Since $h(C) \subset C$, it follows that $C \subset h^{-1}(C)$. Since C contains $\{a, b, c, d\}$

and some points between a and b, $h^{-1}(I)$ must contain a, b and some points be-
tween a and b. It follows that $h^{-1}(I) \supset [a, b]$. Similarly $h^{-1}(I) \supset [c, d]$. There-
fore $h^{-1}(I) \supset [a, b] \cup [c, d]$, and hence $h([a, b] \cup [c, d]) \subset I$. Now choose I to
be $[a, d]$ to get (A) $h([a, b] \cup [c, d]) \subset [a, d]$. Choose I to be $[c, b]$ (by which we
mean the line segment that goes from c through d then $\infty = -\infty$ then through a to
end at b) to obtain (B) $h([a, b] \cup [c, d]) \subset [c, b]$. It follows from (A) and (B) that
$h([a, b] \cup [c, d]) \subset [a, d] \cap [c, b] = [a, b] \cup [c, d]$.

To prove property (7), assume that $h(C) = C$. We will show that h must be the
identity. By property (6) $h([a, b] \cup [c, d]) = [a, b] \cup [c, d]$. Taking the complement,
we have $h((b, c) \cup (d, a)) = (b, c) \cup (d, a)$, and so $h([b, c] \cup [d, a]) = [b, c] \cup [d, a]$.
Hence

$$h([a, b] \cup [c, d]) \cap h([b, c] \cup [d, a])$$

$$= ([a, b] \cup [c, d]) \cap ([b, c] \cup [d, a]).$$

It follows that $h(\{a, b, c, d\}) = \{a, b, c, d\}$. Any projective transformation that maps
$\{a, b, c, d\}$ to itself must preserve the cross ratio of the four points, so the only possi-
bilities are (i) $h(a) = a$, $h(b) = b$, $h(c) = c$, $h(d) = d$, in which case h is the identity
map; (ii) $h(a) = b$, $h(b) = a$, $h(c) = d$, $h(d) = c$; (iii) $h(a) = c$, $h(b) = d$, $h(c) = a$,
$h(d) = b$, and (iv) $h(a) = d$, $h(b) = c$, $h(c) = b$, $h(d) = a$. In each case one can write
down the specific projective transformation, for example, (iii) is achieved by

$$h(x) = \frac{(d - c)(b - c)(x - a)}{(b - a + d - c)(x - c) - (d - c)(b - c)} + c.$$

The other two specific transformations can be deduced by permuting the symbols a,
b, c, d. In each of the cases (ii), (iii), and (iv) it is straightforward to check numerically
that $h(x)$ does not map C into C. (One compares the union of closed intervals

$$[f_1(a), f_1(b)] \cup [f_1(c), f_1(d)] \cup [f_2(a), f_2(b)] \cup [f_2(c), f_2(d)],$$

whose endpoints belong to C and which contains C, with the union

$$[h(f_1(a)), h(f_1(b))] \cup [h(f_1(c)), h(f_1(d))]$$

$$\cup [h(f_2(a)), h(f_2(b))] \cup [h(f_2(c)), h(f_2(d))].)$$

It follows that h must be the identity map, as claimed.

Let $\mathcal{G} = (\mathbb{P}^1; g_1, g_2, \ldots, g_L)$ be any projective IFS with attractor equal to C. The
proof proceeds by showing the following: (‡) for any $g \in \mathcal{G}$ we have $g = f_{\sigma_1} \circ f_{\sigma_2} \circ$
$\cdots \circ f_{\sigma_k}$, for some k, where each σ_i is either 1 or 2. Then, by Lemma 11, $\mathcal{G}$ has the
same hyperplane repeller as $\mathcal{F}$, and hence index$(\mathcal{G}) = $ index$(\mathcal{F}) = 2$. So any IFS with
attractor C has index 2. This completes the proof of Theorem 4 because it shows that
any IFS with attractor C has index 2, i.e., index$(C) = 2$.

To prove claim (‡), consider the IFS $\mathcal{H} = ([a, b] \cup [c, d]; f_1, f_2, g)$ where g is any
function in IFS $\mathcal{G}$. By property (6) $g([a, b] \cup [c, d]) \subset [a, b] \cup [c, d]$. So $\mathcal{H}$ is indeed a
well-defined IFS. It follows immediately from the fact that both $\mathcal{F}$ and $\mathcal{G}$ have attrac-
tor equal to C that $\mathcal{H}$ also has attractor C. It cannot be the case that $g([a, b]) \subset [a, b]$

and $g([c, d]) \subset [c, d])$ since then g would have two attractive fixed points, which is impossible. Similarly, it cannot occur that $g([c, d]) \subset [a, b]$ and $g([a, b]) \subset [c, d]$ for then g^2 would have two attractive fixed points, which is also impossible. It cannot occur that $g(a) \subset [a, b]$ and $g(b) \subset [c, d]$ for then $g([a, b] \cup [c, d])$ would not be contained in $[a, b] \cup [c, d]$, contrary to property (6). Similarly, we rule out the possibilities that $g(a) \subset [c, d]$ and $g(b) \subset [a, b]$; that $g(c) \subset [a, b]$ and $g(d) \subset [c, d]$; and that $g(d) \subset [a, b]$ and $g(c) \subset [c, d]$. It follows that either $g([a, b] \cup [c, d]) \subset [a, b] \subset f_1([a, d])$, or $g([a, b] \cup [c, d]) \subset [c, d] \subset f_2([c, b])$ where $[c, b]$ denotes the interval from c to ∞ then from $-\infty$ to b. (Here, the containments $[a, b] \subset f_2([c, b])$ and $[c, d] \subset f_2([c, b])$ are readily verified by direct calculation.) It now follows that either $g(C) \subset C \cap f_1([a, d]) = f_1(C)$ or $g(C) \subset C \cap f_2([c, b]) = f_2(C)$. Hence

$$g(C) \subset C_{\sigma_1} := f_{\sigma_1}(C)$$

for $\sigma_1 \in \{1, 2\}$. If $g(C) = C_{\sigma_1}$ then $h(C) = C$ where h is the projective transformation $f_{\sigma_1}^{-1} \circ g$. In this case property (7) implies that h must be the identity map. Therefore

$$g = f_{\sigma_1}.$$

If, on the other hand, $g(C) \subsetneq f_{\sigma_1}(C)$, then we consider the IFS

$$\mathcal{H}_{\sigma_1} = (f_{\sigma_1}([a, b] \cup [c, d]); f_{\sigma_1} \circ f_1 \circ f_{\sigma_1}^{-1}, f_{\sigma_1} \circ f_2 \circ f_{\sigma_1}^{-1}, g \circ f_{\sigma_1}^{-1}).$$

It is readily checked that the functions that comprise this IFS indeed map $f_{\sigma_1}([a, b] \cup [c, d])$ into itself. The attractor of $\mathcal{H}_{\sigma_1}$ is $C_{\sigma_1} = f_{\sigma_1}(C)$ because

$$\mathcal{H}_{\sigma_1}(C_{\sigma_1}) = f_{\sigma_1} \circ f_1 \circ f_{\sigma_1}^{-1}(f_{\sigma_1}(C)) \cup f_{\sigma_1} \circ f_2 \circ f_{\sigma_1}^{-1}(f_{\sigma_1}(C)) \cup g \circ f_{\sigma_1}^{-1}(f_{\sigma_1}(C))$$

$$= f_{\sigma_1}(f_1(C) \cup f_2(C)) \cup g(C) = f_{\sigma_1}(C) \cup g(C)$$

$$= C_{\sigma_1} \quad (\text{because } g(C) \subset f_{\sigma_1}(C)).$$

Let

$$a_{\sigma_1} < b_{\sigma_1} < c_{\sigma_1} < d_{\sigma_1}$$

denote the endpoints of the two intervals $f_{\sigma_1}([a, b])$ and $f_{\sigma_1}([c, d])$, and write our new IFS as

$$\mathcal{H}_{\sigma_1} = ([a_{\sigma_1}, b_{\sigma_1}] \cup [c_{\sigma_1}, d_{\sigma_1}]; f_{(\sigma_1)1}, f_{(\sigma_1)2}, g_{\sigma_1}),$$

where

$$f_{(\sigma_1)\sigma_2} = f_{\sigma_1} \circ f_{\sigma_2} \circ f_{\sigma_1}^{-1}, \quad \text{and} \quad g_{\sigma_1} = g \circ f_{\sigma_1}^{-1}.$$

Repeat our earlier argument to obtain

$$g_{\sigma_1}([a_{\sigma_1}, b_{\sigma_1}] \cup [c_{\sigma_1}, d_{\sigma_1}]) \subset f_{\sigma_2}([a_{\sigma_1}, b_{\sigma_1}] \cup [c_{\sigma_1}, d_{\sigma_1}]),$$

and in particular that

$$g_{\sigma_1}(C_{\sigma_1}) \subset C_{\sigma_1\sigma_2} := f_{(\sigma_1)\sigma_2}(C_{\sigma_1}) = f_{\sigma_1} \circ f_{\sigma_2} \circ f_{\sigma_1}^{-1} \circ f_{\sigma_1}(C) = f_{\sigma_1} \circ f_{\sigma_2}(C)$$

for some $\sigma_2 \in \{1, 2\}$. If $g_{\sigma_1}(C_{\sigma_1}) = C_{\sigma_1\sigma_2}$ then $g_{\sigma_1}(f_{\sigma_1}(C)) = f_{\sigma_1} \circ f_{\sigma_2}(C)$ which implies $g \circ f_{\sigma_1}^{-1} \circ f_{\sigma_1}(C) = f_{\sigma_1} \circ f_{\sigma_2}(C)$ which implies, as above, that

$$g = f_{\sigma_1} \circ f_{\sigma_2}.$$

If $g_{\sigma_1}(C_{\sigma_1}) \subsetneq C_{\sigma_1\sigma_2}$ then we construct a new projective IFS $H_{\sigma_1\sigma_2}$ in the obvious way and continue the argument. If the process does not terminate with

$$g = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}$$

for some k, then $g(C)$ is a singleton, which is impossible because g is invertible. We conclude that

$$\mathcal{G} = (\mathbb{P}; f_{\omega_1}, f_{\omega_2}, \ldots, f_{\omega_L})$$

where

$$f_{\omega_l} = f_{\sigma_1^l} \circ f_{\sigma_2^l} \circ \cdots \circ f_{\sigma_{k_l}^l}$$

in the obvious notation. This concludes the proof of claim (‡).

Now Lemma 11 implies index$(\mathcal{G}) =$ index$(\mathcal{F})$. So the index of any projective IFS that has C as its attractor is 2. It follows that index$(A) = 2$. □

13 Remarks

Various remarks are placed in this section so as to avoid interrupting the flow of the main development.

Remark 1 Example 3 in Sect. 4 illustrates that there exist non-contractive projective IFSs that, nevertheless, have attractors. Such IFSs are not well understood and invite further research.

Remark 2 It is well known [27] that if each function of an IFS is a contraction on a complete metric space X, then $\mathcal{F}$ has a unique attractor in X. So statement (4) of the Theorem 1 immediately implies the existence of an attractor A, but not that there is a hyperplane H such that $A \cap H = \emptyset$.

Remark 3 Let $\mathcal{F}$ be a contractive IFS. By Corollary 1, each $f \in \mathcal{F}$ has an invariant hyperplane H_f. If all these invariant hyperplanes are identical, say $H_f = H$ for all $f \in \mathcal{F}$, then the projective IFS $\mathcal{F}$ is equivalent to an affine IFS acting on the embedded affine space $\mathbb{P}^n \backslash H$. More specifically, let $G = (\mathbb{R}^n; g_1, g_2, \ldots, g_M)$ be an affine IFS where $g_i(x) = L_i'(x) + t_i$ and where L_i' is the linear part and t_i the translational part. A corresponding projective IFS is $\mathcal{F} = (\mathbb{P}^n; f_1, f_2, \ldots, f_M)$ where, for each i the projective transformation f_i is represented by the matrix L_{f_i}:

$$L_{f_i} \begin{pmatrix} x_0 \\ x_1 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_i & L_i' \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ . \\ x_n \end{pmatrix}.$$

Here $\mathbb{R}^n$ corresponds to $\mathbb{P}\backslash H$ with H the hyperplane $x_0 = 0$. In this case the hyperplane repeller of $\mathcal{F}$ is H.

Remark 4 Straightforward geometrical comparisons between $d_K(x, y)$ and $d_{\mathbb{P}}(x, y)$ show that (i) the two metrics are bi-Lipschitz equivalent on any convex body contained in $\mathrm{int}(K)$ and (ii) if f is any projective transformation on $\mathbb{P}^n$ then the metric $d_{f(\mathbb{P})}$ defined by $d_{f(\mathbb{P})}(x, y) = d_{\mathbb{P}}(f(x), f(y))$ for all $x, y \in \mathbb{P}^n$ is bi-Lipschitz equivalent to $d_{\mathbb{P}}$. A consequence of assertions (i) and (ii) is that the value of the Hausdorff dimension of any compact subset of $\mathrm{int}(K)$ is the same if it is computed using the round metric $d_{\mathbb{P}}$ or the Hilbert metric d_K; see [20, Corollary 2.4, p. 30], and its value is invariant under the group of projective transformations on $\mathbb{P}^n$. In particular, the Hausdorff dimension of an attractor of a projective IFS is a projective invariant.

Remark 5 Theorem 1 provides conditions for the existence of a metric with respect to which a projective IFS is contractive. In so doing, it invites other directions of development, including IFS with place-dependent probabilities [9], graph-directed IFS theory [24], projective fractal interpolation, and so on. In subsequent papers we hope to describe a natural generalization of the joint spectral radius and applications to digital imaging.

Remark 6 Definition 2 of the attractor of an IFS is a natural generalization of the definition [2, p. 82] of the attractor of a contractive IFS. Another general definition, in the context of iterated closed relations on a compact Hausdorff space, has been given by McGehee [25]. He proves that his definition is equivalent to Definition 2, for the case of contractive iterated function systems. However, readily constructed examples show that McGehee's definition of attractor is weaker than Definition 2.

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