J Geom Anal (2012) 22:74–89
DOI 10.1007/s12220-010-9191-7

Semiclassical L^p Estimates of Quasimodes on Curved Hypersurfaces

Andrew Hassell · Melissa Tacy

Received: 25 March 2010 / Published online: 9 November 2010

Abstract Let M be a compact manifold of dimension n, $P = P(h)$ a semiclassical pseudodifferential operator on M, and $u = u(h)$ an L^2 normalized family of functions such that $P(h)u(h)$ is $O(h)$ in $L^2(M)$ as $h \downarrow 0$. Let $H \subset M$ be a compact submanifold of M. In a previous article, the second-named author proved estimates on the L^p norms, $p \geq 2$, of u restricted to H, under the assumption that the u are semiclassically localized and under some natural structural assumptions about the principal symbol of P. These estimates are of the form $Ch^{-\delta(n,k,p)}$ where $k = \dim H$ (except for a logarithmic divergence in the case $k = n - 2$, $p = 2$). When H is a hypersurface, i.e., $k = n - 1$, we have $\delta(n, n - 1, 2) = 1/4$, which is sharp when M is the round n-sphere and H is an equator.

In this article, we assume that H is a hypersurface, and make the additional geometric assumption that H is *curved* (in the sense of Definition 2.6 below) with respect to the bicharacteristic flow of P. Under this assumption we improve the estimate from $\delta = 1/4$ to $1/6$, generalizing work of Burq–Gérard–Tzvetkov and Hu for Laplace eigenfunctions. To do this we apply the Melrose–Taylor theorem, as adapted by Pan and Sogge, for Fourier integral operators with folding canonical relations.

Keywords Eigenfunction estimates · L^p estimates · Semiclassical analysis · Pseudodifferential operators · Restriction to hypersurfaces

Communicated by Michael Taylor.

This research was supported in part by Australian Research Council Discovery Grant DP0771826, and an Australian Postgraduate Award.

A. Hassell
Department of Mathematics, Mathematical Sciences Institute, Australian National University, Canberra, 0200 ACT, Australia
e-mail: Andrew.Hassell@anu.edu.au

M. Tacy (✉)
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
e-mail: mtacy@ias.edu

Mathematics Subject Classification (2000) 35Pxx · 58J40

1 Introduction

Let M be a compact manifold of dimension n and $P = P(h)$ a semiclassical pseudodifferential operator on M parameterized by the positive number $h \in (0, h_0]$. Suppose that $u = u(h)$ is an $O(h)$ quasimode, i.e., an L^2-normalized family of functions, defined for some subset of $(0, h_0]$ accumulating at 0, such that $P(h)u(h)$ is $O(h)$ in $L^2(M)$. We assume P has real principal symbol $p(x, \xi)$ and that its full symbol is smooth in h. We also put technical assumptions on $p(x, \xi)$ (see Definitions 2.5 and 2.6) and assume u is localized (see Definition 2.1). One important special case is when $P(h) = h^2\Delta - 1$ where Δ is the Laplacian with respect to a Riemannian metric on M. Then $u(h)$ is an approximate eigenfunction with eigenvalue h^{-2}:

$$(\Delta - h^{-2})u(h) = O(h^{-1}) \quad \text{in } L^2(M).$$

Other cases of interest are discussed in [13], where this framework was introduced.

The aim of this paper is to bound the extent to which $u(h)$ can concentrate as $h \to 0$ by estimating the L^p norm of u restricted to hypersurfaces, in a manner that is sharp (up to a constant independent of h) as $h \to 0$. In particular, we wish to relate the degree of concentration to the geometry of the hypersurface to the bicharacteristic flow of $P(h)$.

There are a number of ways to study concentration of eigenfunctions. One can, for example, study semiclassical measures as in Gérard–Leichtnam [9], Zelditch [19], Zelditch–Zworski [20], Anantharaman [1], Anantharaman–Koch–Nonnenmacher [3], Anantharaman–Nonnenmacher [2]. The aim of these studies is generally to prove nonconcentration theorems under geometric conditions on the geodesic flow (such as Anosov flow).

In 1988 Sogge [16] produced sharp L^p estimates for spectral clusters (and therefore eigenfunctions) of elliptic operators, comparing the size of the L^p norm over the full manifold to the L^2 norm in terms of powers of the eigenvalue λ. Tataru [18] in 1998 proved estimates for restrictions of solutions of second-order hyperbolic equations to the boundary, as well as to interior hypersurfaces. These results imply corresponding results for approximate eigenfunctions of second-order elliptic operators, by considering the special case of time-independent operators and time-harmonic solutions. In 2007 Burq, Gérard, and Tzvetkov [6] proved estimates for general submanifolds and Laplacian eigenfunctions. Their estimates are sharp for subsequences of spherical harmonics. For high p these estimates are optimized by eigenfunctions concentrating at a point. For low p the optimizing examples are eigenfunctions concentrating in a small tube around a stable periodic geodesic. Like Tataru, Burq, Gérard, and Tzvetkov [6] were able to obtain better estimates for small p in dimension two when the submanifold is a curve with positive geodesic curvature. Hu [12] obtained a similar result for hypersurfaces in n dimensions where the hypersurface has positive curvature. In the special case of a flat two- or three-dimensional torus Bourgain and Rudnick obtain an improved nonconcentration result for curved hypersurfaces [4].

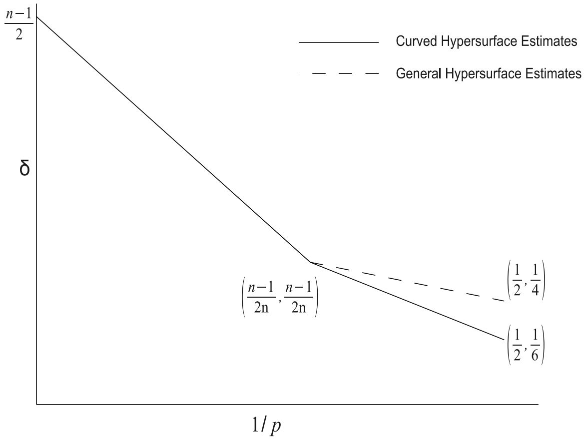

Fig. 1 $\delta(p)$ plotted against $1/p$ for a general hypersurface and for a hypersurface curved with respect to the flow

In 2010 Tacy [17] extended results of Tataru and Burq–Gérard–Tzvetkov on Laplacian eigenfunctions to quasimodes of semiclassical operators. This extension uses the semiclassical framework set up in Burq–Gérard–Tzvetkov [5] and Koch–Tataru–Zworski [13]. The main result of [17] is the following, where we refer to Definitions 2.1 and 2.5 for the precise definitions of localization and admissibility.

Theorem 1.1 *Let (M, g) be a smooth manifold without boundary and let H be a smooth embedded hypersurface. Let $u(h)$ be a family of L^2 normalized functions that satisfy $P(h)u(h) = O_{L^2}(h)$ for $P(h)$ a semiclassical operator with symbol $p(x, \xi)$. Assume further that u satisfies the localization property and that the symbol $p(x, \xi)$ is admissible. Then*

$$\|u(h)\|_{L^p(H)} \lesssim h^{-\delta(n,p)},$$

$$\delta(n, p) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p}, & \frac{2n}{n-1} \leq p \leq \infty, \\ \frac{n-1}{4} - \frac{n-2}{2p}, & 2 \leq p \leq \frac{2n}{n-1}. \end{cases} \tag{1}$$

Remark 1.2 We have only given the results of [17] pertaining to hypersurfaces. Higher codimension submanifolds were also treated there.

The present paper extends the improved estimates of Tataru, Burq–Gérard–Tzvetkov and Hu for curved hypersurfaces (see Definition 2.6) to the semiclassical regime, framing the geometric conditions in terms of the classical (bicharacteristic) flow. To motivate the condition of curvature, recall that the classical flow defined by

$$\begin{cases} \dot{x} = \partial_\xi p(x, \xi) \\ \dot{\xi} = -\partial_x p(x, \xi) \end{cases} \tag{2}$$

describes the movement in phase space of a classical particle with classical Hamiltonian $p(x, \xi)$. For the model case of the Laplacian the flow defined by (2) is the geodesic flow. In the semiclassical regime we wish to find estimates that link the properties of this classical flow to concentrations of quasimodes. Intuitively we can think of highly localized packets moving on trajectories defined by the flow. The more time a packet spends near a hypersurface the move concentration we would expect to see there. In [13] and [17] it is shown that for a hypersurface T with boundary defining function[1] t, if at some point (x_0, ξ_0) we have $\dot{t} \neq 0$ (hence locally T is a 'time slice' for the flow), where the dot indicates derivative with respect to bicharacteristic flow, and if u is a quasimode sufficiently localized near (x_0, ξ_0), then u does not concentrate on T. That is, if $\chi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a cutoff function with small enough support around (x_0, ξ_0) then

$$\|\chi(x, hD)u(h)\|_{L^2(T)} \lesssim \|u(h)\|_{L^2(M)}. \tag{3}$$

However, for a general hypersurface H, with boundary defining function r, a bicharacteristic may stay inside H, allowing considerable concentration of an associated wave packet on H. As shown in [13] and [17], concentration (as measured by L^2 norm) could be as bad as $\sim h^{-1/2}$ assuming just the localization condition and assumption (A1) below, while additionally assuming (A2) introduces dispersion effects which reduces the concentration to $\sim h^{-1/4}$. To improve on this, we need to rule out bicharacteristics that stay inside H. A natural assumption to make is that the projections of bicharacteristics are only simply tangent to H. In local coordinates this is the same as saying that whenever a bicharacteristic is tangent to H, i.e., $\dot{r}(x_0, \xi_0)$ vanishes, $x_0 \in H$, then the normal acceleration $\ddot{r}(x_0, \xi_0)$ is nonzero. We phrase this by saying that H is *curved* with respect to the bicharacteristic flow.

Under this additional assumption, which we label (A3) in Definition 2.6, we show that the concentration is at most $\sim h^{-1/6}$:

Theorem 1.3 *Let M, H, $P(h)$, and $u(h)$ be as in Theorem 1.1. If H is curved with respect to the flow given by $p(x, \xi)$, i.e., satisfies assumption* (A3) *in Definition 2.6, then the estimate (1) for $p = 2$ can be improved from $\delta = 1/4$ to $\tilde{\delta} = 1/6$. By interpolation with the result for $p = 2n/(n-1)$, we obtain*

$$\begin{aligned} \|u(h)\|_{L^p(H)} &\lesssim h^{-\tilde{\delta}(n,p)}, \quad 2 \leq p \leq \frac{2n}{n-1}, \\ \tilde{\delta}(n, p) &= \frac{n-1}{3} - \frac{2n-3}{3p}, \end{aligned} \tag{4}$$

under assumption (A3).

Remark 1.4 For $p \geq 2n/(n-1)$ there is no improvement in the curved case. In this case the $\|\cdot\|_{L^p(H)}$ norm is maximized by functions that concentrate at points so we would not expect the geometry of the hypersurface to affect such estimates.

[1]We say that the real function t is a boundary defining function for T if $T = \{t = 0\}$ and if t vanishes simply at T, i.e., $dt \neq 0$ at H'.

2 Semiclassical Analysis

We work with semiclassical pseudodifferential operators (for a full introduction see [5, 8], or [13]). Such operators are defined by their symbol $p(x, \xi, h)$ and a quantization procedure

$$P(h)u(h) = p(x, hD, h)u(h) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} p(x, \xi, h)u(y, h)d\xi dy$$

where h is a small parameter. Because we will assume that $u(h)$ is localized (Definition 2.1), it is harmless to assume that p is a C_c^∞ function of (x, ξ), and for simplicity we take it to be smooth in $h \in [0, h_0]$. By abuse of notation we denote the principal symbol $p(x, \xi, 0)$ by $p(x, \xi)$, and we will write $p(x, hD)$ for $p(x, hD, h)$.

Following [13], we assume that our family of quasimodes $p(x, hD)u(h) = O_{L^2}(h)$ is semiclassically localized:

Definition 2.1 A function $u(h)$ depending parametrically on h is localized if there exists $\chi \in C_c^\infty(T^\star M)$ such that

$$u(h) = \chi(x, hD)u(h) + O_{C^\infty}(h^\infty).$$

Remark 2.2 On a non-compact manifold we can still define a localized function $u(h)$ by replacing C^∞ with the space of Schwartz functions $\mathcal{S}$.

Remark 2.3 For the rest of this paper we assume that all quasimodes $u = u(h)$ have a parametric dependence on h and for ease of notation suppress the h from our expressions.

Localization is compatible with the assumption that $p(x, hD)u = O_{L^2}(h)$: that is, if $\chi \in C_c^\infty(T^\star M)$ then

$$p(x, hD)u = O_{L^2}(h) \Rightarrow p(x, hD)(\chi(x, hD)u) = O_{L^2}(h).$$

Using this localization assumption we are able to turn the global problem into a local problem on small patches in $T^\star M$. If $\chi \in C_c(T^\star M)$ is such that

$$u = \chi(x, hD)u + O_{C^\infty}(h^\infty)$$

then, using compactness of the support of χ, we can write

$$\chi(x, \xi) = \sum_{i=1}^{N} \chi_i(x, \xi)$$

for some $N < \infty$ where each χ_i has arbitrarily small support. In this fashion we reduce estimating $\|\chi(x, hD)u\|_{L^p(H)}$ to estimates on $\|\chi_i(x, hD)u\|_{L^p(H)}$ (the error term $O_{C^\infty}(h^\infty)$ is of course trivial to estimate). Due to this localization we can replace M with $\mathbb{R}^n$, H with $\mathbb{R}^{n-1}$ and $T^\star M$ with $\mathbb{R}^n \times \mathbb{R}^n$. We work in local Fermi coordinates (y, r) where $H = \{(y, r) \mid r = 0\}$.

Still following [13], we further reduce this problem to localizing around points (x_0, ξ_0) where $p(x_0, \xi_0) = 0$. To achieve this we use Lemma 2.1 of [13] which shows that if $|p(x, \xi)| \geq 1/C$ on a local patch then we can invert $p(x, hD)$ up to order h^∞. That is, choosing $\chi(x, \xi)$ supported on this patch, we can find some $q(x, hD)$ such that

$$q(x, hD)p(x, hD)\chi(x, hD) = \chi(x, hD) + O_{L^2 \to L^2}(h^\infty)$$

and

$$p(x, hD)q(x, hD)\chi(x, hD) = \chi(x, hD) + O_{L^2 \to L^2}(h^\infty).$$

So if $p(x, hD)u = O_{L^2}(h)$ and $|p(x, \xi)| > 1/C$ we can invert $p(x, hD)$ to get

$$\chi(x, hD)u = O_{L^2}(h).$$

We can combine this estimate with the following 'semiclassical Sobolev inequality' (see [5, 8], or [13] for proof) to obtain hypersurface restriction estimates.

Lemma 2.4 (semiclassical Sobolev estimates) *Suppose that a family $u = u(h)$ satisfies the localization condition. Then for $1 \leq q \leq p \leq \infty$*

$$\|u\|_{L^p} \lesssim h^{n(1/p - 1/q)} \|u\|_{L^q} + O(h^\infty).$$

To get the L^2 norm of the restriction of u to H we use Lemma 2.4 in only the r coordinates. This is justified as localization in $T^\star\mathbb{R}^n$ implies localization in $T^\star\mathbb{R}^{n-1}$ (see [17]). We have

$$\|u(y, 0)\|_{L^2_y} \lesssim \|u(y, r)\|_{L^\infty_r L^2_y} \lesssim h^{-\frac{1}{2}} \|u(y, z)\|_{L^2_z L^2_y}. \tag{5}$$

So, if $|p(x, \xi)| \geq 1/C$, and $Pu = O_{L^2}(h)$, the L^2 norm of u when restricted to a hypersurface H containing x_0 is $O(h^{\frac{1}{2}})$. This is significantly better than the L^2 estimate given by Theorem 1.3. Consequently we can ignore regions where $p(x, \xi)$ is bounded away from zero.

To get better estimates when $p(x_0, \xi_0) = 0$ than what can be obtained from Lemma 2.4 (which uses only localization), we need to make assumptions on the function p (to prevent p vanishing identically, for example, in which case the assumption $Pu = O(h)$ is vacuous!). Our first assumption (A1) is that p vanishes simply on each cotangent fiber:

(A1) for any point (x_0, ξ_0) such that $p(x_0, \xi_0) = 0$, we have $\partial_\xi p(x_0, \xi_0) \neq 0$.

Our second condition is a geometric condition on the characteristic variety. The condition eliminates examples such as $p(x, \xi) = \xi_1$, i.e., $P = hD_{x_1}$, for which we cannot estimate $\|u\|_{L^2(H)}$ by better than the $h^{-1/2}$ estimate given by Lemma 2.4 alone. Let us note that (A1) implies that the set

$$\{\xi \mid p(x_0, \xi) = 0\} \subset T^\star_{x_0} M \tag{6}$$

is a smooth hypersurface in $T^\star_{x_0} M$.

(A2) For each $x_0 \in M$, the second fundamental form of (6) is positive definite.

Definition 2.5 A symbol $p(x,\xi)$ is *admissible* if it satisfies condition (A1) and (A2).

In addition we make the geometric assumption of curvature with respect to the flow.

Definition 2.6 A hypersurface H of M is *curved* with respect to the flow if the projection of the bicharacteristic flow to M is at most simply tangent to H, or in other words, if for one (and hence any) boundary defining function r for H, we have

(A3) For any (x_0,ξ_0), $\dot r(x_0,\xi_0)=0$ implies that $\ddot r(x_0,\xi_0)\neq 0$.

Remark 2.7 In the case $P(h)=h^2\Delta-1$, where Δ is the Laplacian on M with respect to a Riemannian metric, assumptions (A1) and (A2) are satisfied, and (A3) is satisfied iff H has positive definite second fundamental form. Thus, in this case our result reduces to that of Burq–Gérard–Tzvetkov [6] ($n=2$) and Hu [12] ($n\geq 2$).

3 Evolution Equation

Using the argument in the previous section we can assume that $p(x_0,\xi_0)=0$. Assumption (A1) then tells us that $\partial_\xi p(x_0,\xi_0)\neq 0$. Let us choose coordinates $x=(y,r)$ where $y\in\mathbb{R}^{n-1}$ and $r\in\mathbb{R}$ is a boundary defining function for H. Let $\xi=(\eta,\nu)$ be the dual coordinates. If $\partial_\nu p(x_0,\xi_0)\neq 0$ then we have $\dot r\neq 0$ and, as mentioned in the Introduction (see (3)), u does not concentrate at H at all. So we may assume that $\partial_\nu p(x_0,\xi_0)=0$, in which case (A1) gives $\partial_\eta p(x_0,\xi_0)\neq 0$. By a linear change of y coordinates we can assume that $\partial_{\eta_1} p(x_0,\xi_0)\neq 0$ and $\partial_{\eta_j} p(x_0,\xi_0)=0$ for $j\geq 2$.

Now we apply the implicit function theorem and deduce that the characteristic variety $\{p=0\}$ implicitly defines ξ_1 as a smooth function of $(x,\xi_2,\dots,\xi_n)$:

$$p=0 \implies \xi_1=a(x,\xi_2,\dots,\xi_n). \tag{7}$$

We shall now write $x_1=t$ and think of it as a time variable. We write $x=(t,\bar x)$ and similarly, $\xi_1=\tau$ and $\xi=(\tau,\bar\xi)$. We also write $y=(t,y')$ and $\eta=(\tau,\eta')$. Thus $x=(t,y',r)$ and correspondingly $\xi=(\tau,\eta',\nu)$. We write T for the 'initial' hypersurface $\{t=0\}$, and recall that $H=\{r=0\}$. We assume that $t=0$ at (x_0,ξ_0) and write $(x_0,\xi_0)=((0,\bar x_0),\xi_0)=((0,y_0',0),(\tau_0,\eta_0',\nu_0))$.

As a consequence of (7), we have

$$p=e(x,\xi)\bigl(\tau-a(x,\bar\xi)\bigr)$$

near (x_0,ξ_0), where $e(x_0,\xi_0)\neq 0$. By localizing suitably we may assume that $e\neq 0$ on the support of our localizing function χ. The condition $Pu=O(h)$ in L^2 then implies that

$$e(x,hD_x)\bigl(hD_t-a(x,hD_{\bar x})\bigr)u=O_{L^2(M)}(h)$$

and using the local invertibility modulo $O(h^\infty)$ of $e(x, hD_x)$, we find that

$$\big(hD_t - a(x, hD_{\bar{x}})\big)u = hf(t, \bar{x}) \tag{8}$$

where $\|f\|_{L^2(M)} = O(1)$.

We view (8) as an evolution equation for u, which determines u given the 'initial data' $u(0, \bar{x})$ and the inhomogeneous term $f(t, \bar{x})$. This determines a family of solution operators $U_s(t)$, such that $U_s(t)$ is the solution operator for the evolution equation

$$\big(hD_t - a(s+t, \bar{x}, hD_{\bar{x}})\big)u = 0, \qquad u(0, \bar{x}) = u(\bar{x}).$$

Using Duhamel's principle we write

$$u(t, \bar{x}) = U_0(t)u(0, \bar{x}) + i\int_0^t U_s(t-s)f(s, \bar{x})ds.$$

Now let R_H be the operation of restriction to the hypersurface H, and let $W_s(t) = R_H \circ U_s(t)$. Also, let $u_0 = u(0, \bar{x})$ be the restriction of u to the initial hypersurface $T = \{t = 0\}$. We then have

$$u(t, y', 0) = W_0(t)u_0 + i\int_0^t W_s(t-s)f(s, \bar{x})ds.$$

Using Minkowski's inequality we have

$$\begin{aligned}\|u\|_{L^2(H)} \lesssim & \left(\int \|W_0(t)u_0\|^2_{L^2_{y'}}\, dt\right)^{1/2} \\ & + \int_{\mathbb{R}} \left(\int \|W_s(t-s)f(s, \bar{x})\|^2_{L^2_{y'}}\, dt\right)^{1/2} ds.\end{aligned} \tag{9}$$

We recall from (3) (with $H' = T$) that $\|u_0\|_{L^2(T)} \lesssim \|u\|_{L^2(M)}$. Therefore, to prove Theorem 1.3, i.e., obtain a L^2 bound of

$$\|u\|_{L^2(H)} \lesssim h^{-1/6}\|u\|_{L^2(M)}$$

it suffices to obtain

$$\left(\int \|W_0(t)u_0\|^2_{L^2_{y'}}\, dt\right)^{1/2} \lesssim h^{-1/6}\|u_0\|_{L^2(T)} \tag{10}$$

and an estimate, uniform in s, of the form

$$\left(\int \|W_s(t-s)f\|^2_{L^2_{y'}}\, dt\right)^{1/2} \lesssim h^{-1/6}\|f\|_{L^2(T)}. \tag{11}$$

For each s we will show that (11), and therefore (10), holds with a constant that depends only on the seminorms of $a(x, \xi)$. In fact, the estimates are uniform given uniform bounds on a finite number of derivatives of a, and given uniform lower bounds on the nondegeneracies involved in the computation in Sect. 5—see Remark 5.5. Such uniform bounds hold provided that the patch size is chosen sufficiently small. Therefore we only address the estimate for $W_0(t)$, which we denote by $W(t)$ from here on. To obtain this estimate we view $W(t)$, thought of as a single operator from $L^2(T)$ to $L^2(H)$ instead of as a family parameterized by t, as a Fourier integral operator.

4 Fourier Integral Representation

We need to express the solution operator for the evolution equation

$$hD_t - a(t, \bar{x}, hD_{\bar{x}}) = 0 \tag{12}$$

as a Fourier integral operator. We will then be able to transfer properties of the flow to properties of the phase function defining the operator $U(t)$.

Proposition 4.1 *Suppose $U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ satisfies*

$$hD_t U(t) - A(t)U(t) = 0, \qquad U(0) = \mathrm{Id}$$

where $A(t)$ is a semiclassical pseudodifferential operator such that the symbol $a(t, \bar{x}, \eta)$ of $A(t)$ is real and is smooth in h. Then there exists some $t_0 > 0$ independent of h such that for $0 \le t \le t_0$

$$U(t)u(\bar{x}) = \frac{1}{(2\pi h)^d} \int\int e^{\frac{i}{h}(\phi(t,\bar{x},\bar{\xi}) - \bar{w}\cdot\bar{\xi})} b(t, \bar{x}, \bar{\xi}, h) u(\bar{w}) d\bar{w} d\bar{\xi} + E(t)u(\bar{x})$$

where

$$\partial_t \phi(t, \bar{x}, \bar{\xi}) - a(t, \bar{x}, \partial_{\bar{x}} \phi(t, \bar{x}, \bar{\xi})) = 0, \qquad \phi(0, \bar{x}, \bar{\xi}) = \bar{x} \cdot \bar{\xi},$$

$$b(t, \bar{x}, \bar{\xi}, h) \in C_c^\infty(\mathbb{R} \times T^\star\mathbb{R}^d \times \mathbb{R}) \quad \textit{and}$$

$$E(t) = O(h^\infty) : \mathcal{S}' \to \mathcal{S}.$$

Proof This is in fact the normal parametrix construction yielding the eikonal equation for the phase function. See [8] Sect. 10.2. □

Recall that $W(t) = R_H \circ U(t)$ so we have

$$W(t)f(y') = \frac{1}{(2\pi h)^{n-1}} \iint e^{\frac{i}{h}(\phi(t,(y',0),\bar{\xi}) - \bar{w}\cdot\bar{\xi})} b(t, y', \eta, h) f(\bar{w}) d\bar{w} d\bar{\xi}.$$

In what follows we will write $\phi(t, y', \eta', \nu)$ for $\phi(t, (y', 0), \bar{\xi})$ (recall that $\bar{\xi} = (\eta', \nu)$). We want to estimate the operator norm of $W(t)$ regarded as a single operator acting from $L^2(T)$ to $L^2(H)$. Note that $W(t) = Z \circ \mathcal{F}_h$ where $\mathcal{F}_h$ is the semiclassical Fourier transform:

$$\mathcal{F}_h f(\bar{\xi}) = \frac{1}{(2\pi h)^{\frac{n-1}{2}}} \int e^{-\frac{i}{h}\bar{\xi}\cdot\bar{v}} f(\bar{v}) d\bar{v}$$

and the operator Z is given by

$$Zg(t, y') = \frac{1}{(2\pi h)^{\frac{n-1}{2}}} \iint e^{\frac{i}{h}\phi(t,y',\eta',\nu)} b(t, y', \eta', \nu, h) g(\eta', \nu)\, d\eta'\, d\nu.$$

As $\|\mathcal{F}_h f\|_{L^2} = \|f\|_{L^2}$ it is enough to estimate $L^2 \to L^2$ operator norm of Z. To estimate the operator norm of Z we view it as a semiclassical Fourier integral operator and analyze its canonical relation.

5 Canonical Relation

To prove Theorem 1.3 we need to show that the operator norm of Z is bounded by $Ch^{-1/6}$. To do this we use the following theorem of Pan and Sogge [15] which is the analogue for oscillatory integral operators of Melrose and Taylor's [14] theorem on Fourier integral operators with folding canonical relations.

Theorem 5.1 *Let the oscillatory integral operator T_λ be defined by*

$$T_\lambda f(x) = \int_{\mathbb{R}^d} e^{i\lambda\psi(x,y)} \beta(x, y) f(y) dy$$

where $\beta \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the phase function $\psi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is real. If the left and right projections from the associated canonical relation

$$\mathcal{C}_\psi = \{(x, \psi'_x(x, y), y, -\psi'_y(x, y))\}$$

are at most folding singularities then

$$\|T_\lambda f\|_{L^2(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d}{2}+1/6} \|f\|_{L^2(\mathbb{R}^d)}.$$

Let us recall (see, for example, [10]) that a smooth map $F : \mathbb{R}^d \to \mathbb{R}^d$ has a folding singularity at $x \in \mathbb{R}^d$ if

(i) $dF(x)$ is rank $d - 1$,
(ii) the function $\det dF$ vanishes simply at x, implying in particular that locally near x, the set of $y \in \mathbb{R}^d$ such that $dF(y)$ has rank $d - 1$ is a smooth hypersurface S containing x, and
(iii) the kernel of $dF(x)$ is not contained in the tangent space to S:

$$T_x S + \ker dF(x) = T_x\mathbb{R}^d.$$

Given (i) an equivalent condition to (ii) and (iii) is that, if v is a nonzero element of $\ker dF(x)$, then

$$D_v(\det dF(x)) \neq 0. \tag{13}$$

The operator Z is a Fourier integral operator with canonical relation

$$\mathcal{C} = \left\{(t, y', \partial_t\phi, \partial_{y'}\phi, \eta', \nu, -\partial_{\eta'}\phi, -\partial_\nu\phi)\right\}.$$

The left and right projections on $\mathcal{C}$ are represented in local coordinates by

$$\pi_L : (t, y', \eta', \nu) \mapsto (t, y', \partial_t \phi, \partial_{y'} \phi)$$

and

$$\pi_R : (t, y', \eta', \nu) \mapsto (\eta', \nu, \partial_{\eta'} \phi, \partial_\nu \phi)$$

(where we removed the irrelevant minus signs from π_R for notational convenience).

The matrix $d\pi_L$ takes the form

$$d\pi_L = \begin{pmatrix} \mathrm{Id} & 0 \\ * & B \end{pmatrix}$$

where

$$B = \begin{pmatrix} \partial^2_{t\eta'}\phi & \partial^2_{t\nu}\phi \\ \partial^2_{y'\eta'}\phi & \partial^2_{y'\nu}\phi \end{pmatrix}.$$

At (x_0, ξ_0) we have $\partial^2_{y'\eta'}\phi = \mathrm{Id}$, $\partial^2_{t\eta'}\phi = \partial_{\eta'} a = 0$, $\partial^2_{y'\nu}\phi = 0$ and $\partial^2_{t\nu}\phi = \partial_\nu a = 0$, so we get

$$B = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix} \quad \text{at } (x_0, \xi_0).$$

It is clear that the vector field ∂_ν is in the kernel of $d\pi_L(x_0, \xi_0)$. Moreover, $\det d\pi_L$ is given by $\partial^2_{t\nu}\phi \cdot \det(\partial^2_{y'\eta'}\phi)$ plus terms vanishing to second-order at (x_0, ξ_0). To show that π_L has a fold at (x_0, ξ_0) we need by (13) to show that $\partial_\nu(\det d\pi_L)$ is nonzero at (x_0, ξ_0). Due to the vanishing of both 'off-diagonal' terms $\partial^2_{t\eta'}\phi$ and $\partial^2_{y'\nu}\phi$, the nonvanishing of $\partial_\nu(\det d\pi_L)$ at (x_0, ξ_0) is equivalent to the nonvanishing of $\partial_\nu(\partial^2_{t\nu}\phi) = \partial^3_{t\nu\nu}\phi$.

The matrix $d\pi_R$ takes the form

$$d\pi_R = \begin{pmatrix} 0 & \mathrm{Id} \\ D & * \end{pmatrix}$$

$$D = \begin{pmatrix} \partial^2_{\eta' t}\phi & \partial^2_{y'\eta'}\phi \\ \partial^2_{\nu t}\phi & \partial^2_{\nu y'}\phi \end{pmatrix}$$

and we see that ∂_t is in the kernel of $d\pi_R(x_0, \xi_0)$. To show that π_R has a fold at (x_0, ξ_0) we need by (13) to show that $\partial_t(\det d\pi_L)$ is nonzero at (x_0, ξ_0). As above, due to the vanishing of the 'off-diagonal' terms $\partial^2_{t\eta'}\phi$ and $\partial^2_{y'\nu}\phi$, the nonvanishing of $\partial_t(\det d\pi_L)$ at (x_0, ξ_0) is equivalent to the nonvanishing of $\partial_t(\partial^2_{t\nu}\phi) = \partial^3_{tt\nu}\phi$.

The proof of Theorem 1.3 is therefore completed by the following lemma:

Lemma 5.2 *Under assumptions* (A1), (A2), *and* (A3), *we have*

$$\partial^3_{t\nu\nu}\phi(x_0, \xi_0) \neq 0, \quad \textit{and} \quad \partial^3_{tt\nu}\phi(x_0, \xi_0) \neq 0.$$

Remark 5.3 To simplify notation we write (x_0, ξ_0) for the argument of ϕ corresponding to this point, although $(0, y'_0, 0, \tau_0, \eta'_0, \nu_0)$ would be more accurate.

Proof We use the Hamilton–Jacobi equation

$$\partial_t\phi(t, \bar{x}, \eta', \nu) = a\big(t, \bar{x}, \partial_{\bar{x}}\phi(t, \bar{x}, \eta', \nu)\big). \tag{14}$$

Since at $t = 0$ we have $\phi(0, \bar{x}, \eta', \nu) = y' \cdot \eta' + r\nu$ (recall that $\bar{x} = (y', r)$), we have

$$\partial^3_{t\nu\nu}\phi(0, \bar{x}, \eta', \nu) = \partial^2_{\nu\nu}a(0, \bar{x}, \eta', \nu).$$

Now we apply assumption (A2): it says that the second fundamental form of the submanifold $\{\tau = a(x, \bar{\xi}_0)\} \subset T_{x_0}M$ is positive definite. Since $\partial_{\bar{\xi}}a = 0$ at $(x_0, \bar{\xi}_0)$, the second fundamental form of this submanifold at (x_0, ξ_0) is given by the matrix of second derivatives of a:

$$h_{ij}(\xi_0) = \partial^2_{\xi_i\xi_j}a(x_0, \bar{\xi}_0), \quad 2 \leq i, j \leq n.$$

Therefore, $\partial^2_{\nu\nu}a \neq 0$ at $(x_0, \bar{\xi}_0)$, showing that π_L has a fold singularity at (x_0, ξ_0).

To treat the term $\partial^3_{tt\nu}\phi(x_0, \xi_0)$, we differentiate (14) in t, obtaining

$$\partial^2_{tt}\phi = \partial_t a + \partial_{\bar{\xi}}a \cdot \partial^2_{\bar{x}t}\phi.$$

Using (14) again on the term $\partial^2_{\bar{x}t}\phi$ we obtain

$$\partial^2_{tt}\phi = \partial_t a + \partial_{\bar{\xi}}a \cdot \big(\partial_{\bar{x}}a + \partial_{\bar{\xi}}a \cdot \partial^2_{\bar{x}\bar{x}}\phi\big).$$

We evaluate this at $t = 0$ since the next derivative to be applied, namely ∂_ν, is tangent to $\{t = 0\}$. At $t = 0$, we have $\partial^2_{\bar{x}\bar{x}}\phi = 0$, so we get

$$\partial^2_{tt}\phi\Big|_{t=0} = \partial_t a + \partial_{\bar{\xi}}a \cdot \partial_{\bar{x}}a.$$

Now when we differentiate in ν, we get

$$\partial^3_{tt\nu}\phi(x_0,\xi_0) = \partial^2_{t\nu}a(x_0,\bar{\xi}_0) + \partial^2_{\bar{\xi}\nu}a(x_0,\bar{\xi}_0)\cdot\partial_{\bar{x}}a(x_0,\bar{\xi}_0) \tag{15}$$

since $\partial_{\bar{\xi}}a(x_0,\bar{\xi}_0) = 0$.

At this point we remind the reader that we have chosen coordinates (t, y', r) and $(\tau, \eta', \nu) = (\tau, \bar{\xi})$ such that

$$\partial_{\bar{\xi}}\, p(x_0,\xi_0) = 0$$

and

$$\tau_0 - a(t_0,\bar{x}_0,\bar{\xi}_0) = 0.$$

It follows that

$$\begin{cases} \partial_{\bar{\xi}}a(x_0,\bar{\xi}_0) = 0 \\ \partial_\tau p(x_0.\xi_0) = e(x_0,\xi_0) \\ \partial_{\bar{x}}p(x_0,\xi_0) = -e(x_0,\xi_0)\partial_{\bar{x}}a(x_0,\bar{\xi}_0) \\ \partial_t p(x_0,\xi_0) = -e(x_0,\xi_0)\partial_t a(x_0,\xi_0). \end{cases} \tag{16}$$

Now we apply assumption (A3), which says that $\ddot{r} \neq 0$. We express $\ddot{r}$ in terms of a. We have

$$\dot{r} = \partial_\nu p = \partial_\nu\big(e(\tau - a)\big).$$

Differentiating a second time and using the flow identities

$$\dot{x} = \partial_\xi p(x,\xi), \qquad \dot{\xi} = -\partial_x p(x,\xi),$$

we have

$$\ddot{r} = \Big(\partial_\tau p\partial_t + \partial_{\bar{\xi}}p\partial_{\bar{x}} - \partial_t p\partial_\tau - \partial_{\bar{x}}p\partial_{\bar{\xi}}\Big)\Big((\tau - a)\partial_\nu e - e\partial_\nu a\Big).$$

At (x_0,ξ_0) using the identities given in (16) we can simplify this to

$$\ddot{r}(x_0,\xi_0) = -e\Big(\partial^2_{\nu t}a(x_0,\bar{\xi}_0) + \partial_{\bar{x}}a(x_0,\bar{\xi}_0)\cdot\partial^2_{\nu\bar{\xi}}a(x_0,\bar{\xi}_0)\Big). \tag{17}$$

Therefore, combining (15) and (17), we find

$$\partial^3_{tt\nu}\phi(x_0,\xi_0) = -\frac{\ddot{r}(x_0,\xi_0)}{e(x_0,\xi_0)} \neq 0.$$

This shows that π_R has a fold singularity at (x_0,ξ_0) and completes the proof. □

Remark 5.4 It is easy to see from the calculations above that assumption (A3) is equivalent to the statement that π_R has a folding singularity. Similarly, assumption (A2) is equivalent to the statement that π_L has a folding singularity for every hypersurface H whose tangent space $T_{x_0}H$ at x_0 contains $\partial_\xi p(x_0,\xi_0)\partial_x$, i.e., the tangent vector of the projected bicharacteristic through (x_0,ξ_0).

Remark 5.5 According to [7], Theorem 2.2, one obtains uniform bounds of the form $Ch^{-1/6}$ on the norms of the operators W_s given by (11) provided that there are uniform bounds on a finite number of derivatives of the symbol of W_s, and uniform lower bounds on the determinant of $\partial^2_{y'\eta'}\phi$, $\partial_\nu(\partial^2_{t\nu}\phi)$, and $\partial_t(\partial^2_{t\nu}\phi)$. These lower bounds are achieved simply by shrinking the patch size sufficiently and using continuity. Thus we obtain a bound as in (11) uniformly in s, as desired.

6 Optimality of Theorem 1.3

All the estimates given by Theorem 1.3 are sharp. We study a simple local model around (0, 0) for a hypersurface curved with respect to the flow. Let $H = \{x \mid x_n = 0\}$ and $p(x, \xi)$ be given by

$$p(x,\xi) = \xi_1 - x_n - \sum_{i=2}^{n} \xi_i^2.$$

Note that

$$\begin{array}{lll} \dot{t} = 1 & \dot{y}' = -2\eta' & \dot{r} = -2\nu \\ \dot{\tau} = 0 & \dot{\eta}' = 0 & \dot{\nu} = 1. \end{array}$$

Therefore the flow $(x(s), \xi(s))$ with initial point (0, 0) is given by

$$\begin{array}{lll} t(s) = s & y'(s) = 0 & r(s) = -s^2 \\ \tau(s) = 0 & \eta'(s) = 0 & \nu(s) = s. \end{array}$$

So we have that condition (A3) is clearly satisfied as $\ddot{r}(0) = -2$. We have

$$p(x, hD) = hD_t - r - h^2 D_r^2 - \sum_{i=1}^{n-2} h^2 D_{y'}^2.$$

It is easier to develop a solution in Fourier space. Note that

$$\mathcal{F}_h \circ p(x, hD) \circ \mathcal{F}_h^{-1} = \tau - hD_\nu - \nu^2 - \eta' \cdot \eta'.$$

As the semiclassical Fourier transform preserves L^2 norms, if

$$\left\| (\tau - hD_\nu - \nu^2 - \eta' \cdot \eta') f \right\|_{L^2} = O_{L^2}(h)$$

and $u = \mathcal{F}_h^{-1} f$, then

$$\|p(x, hD)u\|_{L^2} = O_{L^2}(h).$$

We therefore seek a solution for

$$(\tau - hD_\nu - \nu^2 - \eta' \cdot \eta') f = 0; \tag{18}$$

it is obvious that

$$g(\tau, \eta', \nu) = e^{\frac{i}{h}(\frac{1}{3}\nu^3 + \nu(\tau - \eta' \cdot \eta'))}$$

is a solution to (18). The natural scaling $\nu \to h^{-1/3}\nu$ induces a scaling of $\tau \to h^{-2/3}\tau$ and $\eta' \to h^{-1/3}\eta'$ and accordingly we place cutoff functions appropriate to that scale. Let $\chi \in C_c^\infty(\mathbb{R})$ satisfy $\chi(0) = 1$, $\chi \geq 0$, $\operatorname{supp} \chi \subset [-1, 1]$, and let

$$f(\tau, \eta', \nu) = h^{-\frac{n-2}{6} - \frac{1}{3}} \chi(|\nu|) \chi\left(\frac{\tau}{\epsilon h^{2/3}}\right) \chi\left(\frac{|\eta'|}{\epsilon h^{1/3}}\right) e^{\frac{i}{h}\psi(\tau, \eta', \nu)}$$

where

$$\psi(\tau, \eta', \nu) = \frac{1}{3}\nu^3 + \nu(\tau - \eta' \cdot \eta').$$

Now $\|f\|_{L^2} = O_{L^2}(1)$ and f satisfies (18) up to an $O(h)$ error coming from the D_ν hitting the cutoff function $\chi(|\nu|)$. We define the function u as

$$u = \chi(|x|)\mathcal{F}_h^{-1} f.$$

Now $R_H u$ is given by

$$R_H u(y) = \frac{h^{-\frac{n-2}{6} - \frac{1}{3}} \chi(|y|)}{(2\pi h)^{\frac{n}{2}}} \int e^{\frac{i}{h}(t\tau + y' \cdot \eta' + \psi(\tau, \eta', \nu))} \chi(\nu) \chi\left(\frac{\tau}{\epsilon h^{2/3}}\right)$$
$$\times \chi\left(\frac{|\eta'|}{\epsilon h^{1/3}}\right) d\tau d\nu d\eta'.$$

For $|t| \leq h^{1/3}$, ϵ small, the factor $e^{\frac{i}{h}t\tau}$ does not oscillate significantly and can be ignored. Similarly for $|y'| \leq h^{2/3}$ the factor $e^{\frac{i}{h}y' \cdot \eta'}$ does not oscillate significantly and can also be ignored. On the other hand, there are oscillations in the ν variable. We claim that the ν integral

$$\int e^{\frac{i}{h}(t\tau + y' \cdot \eta' + \psi(\tau, \eta', \nu))} \chi(\nu) \chi\left(\frac{\tau}{\epsilon h^{2/3}}\right) \chi\left(\frac{|\eta'|}{\epsilon h^{1/3}}\right) d\nu \tag{19}$$

is bounded below by $ch^{1/3}$. To see this, we insert

$$1 = \chi\left(\frac{\nu}{Mh^{1/3}}\right) + (1 - \chi)\left(\frac{\nu}{Mh^{1/3}}\right),$$

and integrate by parts once to show that the $1 - \chi$ term is $O(M^{-1}h^{1/3})$. On the other hand, with the χ factor inserted into (19), then the $e^{i\nu(\tau - |\eta'|^2)/h}$ factor does not oscillate for ϵ sufficiently small (relative to M^{-1}), and can be ignored. We are thereby reduced to studying the integral with only the $e^{i\nu^3/h}$ oscillatory factor; Theorem 7.7.18 of [11] applies and shows that, with the χ factor inserted, there is a lower bound $ch^{1/3}$ on (19), compared to which the $1 - \chi$ contribution is negligible for large M. We conclude that

$$|R_H u(t, y')| \sim h^{-\frac{n-2}{6} - \frac{1}{3} - \frac{n}{2} + \frac{n-2}{3} + \frac{2}{3} + \frac{1}{3}} = h^{-\frac{n-1}{3}}$$

for $|t| \leq h^{1/3}$, $|y'| \leq h^{2/3}$. Thus on this set we get a lower bound on the L^p norm:

$$\|u\|_{L^p([0,h^{1/3}]_t \times B(0,h^{2/3})_{y'})} \sim h^{-\frac{n-1}{3}+\frac{1}{3p}+\frac{2(n-2)}{3p}} = h^{-(\frac{n-1}{3}-\frac{2n-3}{3p})}$$

which saturates the estimate of Theorem 1.3.

References

1. Anantharaman, N.: Entropy and the localization of eigenfunctions. Ann. Math. (2) **168**(2), 435–475 (2008)
2. Anantharaman, N., Nonnenmacher, S.: Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold. Ann. Inst. Fourier (Grenoble) **57**(7), 2465–2523 (2007). Festival Yves Colin de Verdière
3. Anantharaman, N., Koch, H., Nonnenmacher, S.: Entropy of eigenfunctions. arXiv:0704.1564 (2007)
4. Bourgain, J., Rudnick, Z.: Restriction of toral eigenfunctions to hypersurfaces. C. R. Math. **347**(21–22), 1249–1253 (2009)
5. Burq, N., Gérard, P., Tzvetkov, N.: The Cauchy problem for the nonlinear Schrödinger equation on compact manifolds, pp. 21–52. Pubbl. Cent. Ric. Mat. Ennio Giorgi. Scuola Norm. Sup., Pisa (2004)
6. Burq, N., Gérard, P., Tzvetkov, N.: Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Math. J. **138**(3), 445–486 (2007)
7. Comech, A.: Oscillatory integral operators in scattering theory. Commun. Partial Differ. Equ. **22**(5–6), 841–867 (1997)
8. Evans, L.C., Zworski, M.: Lectures on Semiclassical Analysis. Book in progress, http://math.berkeley.edu/~Zworski/semiclassical.pdf
9. Gérard, P., Leichtnam, É.: Ergodic properties of eigenfunctions for the Dirichlet problem. Duke Math. J. **71**(2), 559–607 (1993)
10. Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. Graduate Texts in Mathematics, vol. 14. Springer, Berlin (1973)
11. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. I. Springer, Berlin (1990)
12. Hu, R.: L^p norm estimates of eigenfunctions restricted to submanifolds. Forum Math. **21**(6), 1021–1052 (2009)
13. Koch, H., Tataru, D., Zworski, M.: Semiclassical L^p estimates. Ann. Henri Poincaré **8**(5), 885–916 (2007)
14. Melrose, R., Taylor, M.: Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. Adv. Math. **55**(3), 242–315 (1985)
15. Pan, Y., Sogge, C.: Oscillatory integrals associated to folding canonical relations. Colloq. Math. **60/61**(2), 413–419 (1990)
16. Sogge, C.: Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. J. Funct. Anal. **77**(1), 123–138 (1988)
17. Tacy, M.: Semiclassical L^p estimates of quasimodes on submanifolds. Commun. Partial Differ. Equ. **35**(8), 1538–1562 (2010)
18. Tataru, D.: On the regularity of boundary traces for the wave equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **26**(1), 185–206 (1998)
19. Zelditch, S.: Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke Math. J. **55**(4), 919–941 (1987)
20. Zelditch, S., Zworski, M.: Ergodicity of eigenfunctions for ergodic billiards. Commun. Math. Phys. **175**(3), 673–682 (1996)