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On the Reconstruction of Conductivity of a Bordered Two-dimensional Surface in $\mathbb{R}^3$ from Electrical Current Measurements on Its Boundary

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Abstract An electrical potential U on a bordered real surface X in $\mathbb{R}^3$ with isotropic conductivity function $\sigma > 0$ satisfies the equation $d(\sigma d^c U)|_X = 0$, where $d^c = i(\bar\partial - \partial)$, $d = \bar\partial + \partial$ are real operators associated with a complex (conformal) structure on X induced by the Euclidean metric of $\mathbb{R}^3$. This paper gives an exact reconstruction of the conductivity function σ on X from the Dirichlet-to-Neumann mapping $U|_{bX} \to \sigma d^c U|_{bX}$. This paper extends to the case of Riemann surfaces the reconstruction schemes of R. Novikov (Funkt. Anal. Prilozh. 22(4):11–22, 1988) and of A. Bukhgeim (J. Inv. Ill-posed Probl. 16:19–34, 2008), given for the case $X \subset \mathbb{R}^2$. The paper extends and corrects the statements of Henkin and Michel (J. Geom. Anal. 18:1033–1052, 2008), where the inverse boundary value problem on the Riemann surfaces was first considered.

Keywords Riemann surface · Electrical current · Inverse conductivity problem · $\bar\partial$-method

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0 Introduction

0.1 Reduction of the Inverse Boundary Value Problem on a Surface in $\mathbb{R}^3$ to the Corresponding Problem on an Affine Algebraic Riemann Surface in $\mathbb{C}^3$

Let X be a bordered oriented two-dimensional manifold in $\mathbb{R}^3$. The manifold X is equipped with the complex (conformal) structure induced by the Euclidean metric of $\mathbb{R}^3$. We say that X possesses an isotropic conductivity function $\sigma > 0$ if any electrical potential u on bX generates an electrical potential U on X, solving the Dirichlet problem:

$$U|_{bX} = u \quad \text{and} \quad d\sigma d^c U|_X = 0, \tag{0.1}$$

where $d^c = i(\bar\partial - \partial)$, $d = \bar\partial + \partial$, and the Cauchy–Riemann operator $\bar\partial$ corresponds to the complex (conformal) structure on X. The inverse conductivity problem consists of the reconstruction of $\sigma|_X$ from the mapping potential $U|_{bX} \to$ current $j = \sigma d^c U|_{bX}$ for solutions of (0.1). This mapping is called the Dirichlet-to-Neumann mapping.

This problem is the special case of the following more general inverse boundary value problem, going back to I.M. Gelfand [13] and A. Calderón [6]: to find the potential (2-form) q on X in the equation

$$dd^c\psi = q\psi \tag{0.2}$$

from knowledge of the Dirichlet-to-Neumann mapping $\psi|_{bX} \to d^c\psi|_{bX}$ for solutions of (0.2). In some contexts, (0.2) is called the stationary Schrödinger equation, in other contexts the monochromatic acoustic equation, etc. Equation (0.1) can be reduced to (0.2) with $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ by the substitution $\psi = \sqrt{\sigma}U$.

Let the restriction of the Euclidean metric of $\mathbb{R}^3$ on X have (in local coordinates) the form

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2 = Adz^2 + 2Bdzd\bar z + \bar A d\bar z^2,$$

where $z = x + iy$, $B = \frac{E+G}{4}$, and $A = \frac{E-G-2iF}{4}$. Put $\mu = \frac{\bar A}{B+\sqrt{B^2-|A|^2}}$. By classical results (going back to Gauss and Riemann) one can construct a holomorphic embedding $\varphi : X \to \mathbb{C}^3$, using some solution of the Beltrami equation: $\bar\partial\varphi = \mu\partial\varphi$ on X. Moreover, the embedding φ can be chosen in such a way that $\varphi(X)$ belongs to a smooth algebraic curve V in $\mathbb{C}^3$. Using the existence of the embedding φ, we can further identify X with $\varphi(X)$.

0.2 Reconstruction Schemes for the Case $X \subset \mathbb{R}^2 \simeq \mathbb{C}$

For the case $X = \Omega \subset \mathbb{R}^2$, the exact reconstruction scheme for formulated inverse problems was given in [27, 28] under some restriction (a smallness assumption) for σ or q (see Corollary 2 of [27]). For the case of the inverse conductivity problem (see (0.1), (0.2)), when $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, the restriction on σ in this scheme was eliminated by A. Nachman [25] by the reduction to the equivalent question for the first-order system studied by R. Beals and R. Coifman [2]. Recently, A. Bukhgeim [5] has found a new original reconstruction scheme for the inverse boundary value problem (see (0.2)), without a smallness assumption on q.

In a particular case, the scheme of [27] for the inverse conductivity problem consists of the following. Let $\sigma(x) > 0$ for $x \in \bar{\Omega}$ and $\sigma \in C^{(2)}(\bar{\Omega})$. Put $\sigma(x) = 1$ for $x \in \mathbb{R}^2 \backslash \bar{\Omega}$.

Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$.

From a result of L. Faddeev [10], it follows that $\exists$ a compact set $E \subset \mathbb{C}$ such that for each $\lambda \in \mathbb{C}\backslash E$ there exists a unique solution $\psi(z,\lambda)$ of the equation $dd^c\psi = q\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$, with asymptotics

$$\psi(z,\lambda)e^{-\lambda z} \stackrel{\text{def}}{=} \mu(z,\lambda) = 1 + o(1), \quad z \to \infty.$$

Such a solution can be found from the integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2}\int_{\xi\in\Omega} g(z-\xi,\lambda)\frac{\mu(\xi,\lambda)dd^c\sqrt{\sigma}}{\sqrt{\sigma}}, \tag{0.3}$$

where the function

$$g(z,\lambda) = \frac{i}{(2\pi)^2}\int_{w\in\mathbb{C}} \frac{e^{\lambda w - \bar{\lambda}\bar{w}}dw\wedge d\bar{w}}{(w+z)\bar{w}} = \frac{i}{(2\pi)^2}\int_{w\in\mathbb{C}} \frac{e^{i(w\bar{z}+\bar{w}z)}dw\wedge d\bar{w}}{w(\bar{w}-i\lambda)}$$

is called the Faddeev–Green function for the operator

$$\mu \mapsto \bar{\partial}(\partial + \lambda dz)\mu.$$

From [27] it follows that $\forall\lambda \in \mathbb{C}\backslash E$ the function $\psi|_{b\Omega}$ can be found from the Dirichlet-to-Neumann mapping via the integral equation

$$\psi(z,\lambda)\big|_{b\Omega} = e^{\lambda z} + \int_{\xi\in b\Omega} e^{\lambda(z-\xi)}g(z-\xi,\lambda)(\hat{\Phi}\psi(\xi,\lambda) - \hat{\Phi}_0\psi(\xi,\lambda)), \tag{0.4}$$

where $\hat{\Phi}\psi = \bar{\partial}\psi|_{b\Omega}$, $\hat{\Phi}_0\psi = \bar{\partial}\psi_0|_{b\Omega}$, $\psi_0|_{b\Omega} = \psi|_{b\Omega}$, and $\partial\bar{\partial}\psi_0|_{\Omega} = 0$.

By results of [1, 15], and [27] it follows that $\psi(z,\lambda)$ satisfies the $\bar{\partial}$-equation of Bers–Vekua-type with respect to $\lambda \in \mathbb{C}\backslash E$:

$$\frac{\partial\psi}{\partial\bar{\lambda}} = b(\lambda)\bar{\psi}, \quad \text{where} \tag{0.5}$$

$$\bar{\lambda}b(\lambda) = -\frac{1}{2\pi i}\int_{z\in b\Omega} e^{\lambda z - \bar{\lambda}\bar{z}}\bar{\partial}_z\mu(z,\lambda) = \frac{1}{4\pi}\int_{\Omega} e^{\lambda z - \bar{\lambda}\bar{z}}q\mu, \tag{0.6}$$

$$\psi(z,\lambda)e^{-\lambda z} = \mu(z,\lambda) \to 1, \quad \lambda \to \infty, \ \forall z \in \mathbb{C}. \tag{0.7}$$

From [2] and [25], it follows that for $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, $\sigma > 0$, and $\sigma \in C^{(2)}(\bar{\Omega})$ the exceptional set $E = \{\emptyset\}$ and the function $\lambda \mapsto b(\lambda)$ belong to $L^{2+\varepsilon}(\mathbb{C}) \cap L^{2-\varepsilon}(\mathbb{C})$ for some $\varepsilon > 0$. As a consequence, the function $\mu = e^{-\lambda z}\psi$ is a unique solution of the Fredholm integral equation

$$\mu(z,\lambda) + \frac{1}{2\pi i}\int_{\lambda'\in\mathbb{C}} b(\lambda')e^{\bar{\lambda}'\bar{z}-\lambda' z}\overline{\mu(z,\lambda')}\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda} = 1. \tag{0.8}$$

Integral equations (0.4) and (0.8) permit us, starting from the Dirichlet-to-Neumann mapping, to find first the boundary values $\psi|_{b\Omega}$, second "$\bar{\partial}$-scattering data" $b(\lambda)$, and third the function $\psi|_{\Omega}$. From the equality $dd^c\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$ on X, we finally find $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ on X.

The scheme of the Bukhgeim-type [5] can be presented in the following way. Let $q = Qdd^c|z|^2$, where $Q \in C^{(1)}(\bar{\Omega})$, but the potential Q is not necessarily of the conductivity form $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. By a variation of the Faddeev statement and proof, we obtain that $\forall a \in \mathbb{C}\ \exists$ a compact set $E \subset \mathbb{C}$ such that $\forall \lambda \in \mathbb{C}\backslash E$ there exists a unique solution $\psi_a(z,\lambda)$ of the equation $dd^c\psi = q\psi$ with asymptotics

$$\psi_a(z,\lambda)e^{-\lambda(z-a)^2} = \mu_a(z,\lambda) = 1 + o(1), \quad z \to \infty.$$

Such a solution can be found from the integral equation (0.3), where the kernel $g(z-\zeta,\lambda)$ is replaced by the kernel

$$g_a(z,\zeta,\lambda) = \frac{ie^{\lambda a^2-\bar{\lambda}\bar{a}^2}}{2\pi^2}\int_{\mathbb{C}} \frac{e^{-\lambda(\zeta-\eta+a)^2+\bar{\lambda}(\bar{\zeta}-\bar{\eta}+\bar{a})^2}}{(\eta-z)(\bar{\zeta}-\bar{\eta})}d\eta\wedge d\bar{\eta}.$$

The kernel $g_a(z,\zeta,\lambda)$ can be called the Faddeev-type Green function for the operator $\mu \to \bar{\partial}(\partial + \lambda d(z-a)^2)\mu$. The equation $\bar{\partial}(\partial + \lambda d(z-a)^2)\mu = \frac{i}{2}q\mu$ and the Green formula imply

$$\int_{b\Omega} e^{\lambda(z-a)^2-\bar{\lambda}(\bar{z}-\bar{a})^2}\bar{\partial}\mu = \int_{\Omega} e^{\lambda(z-a)^2-\bar{\lambda}(\bar{z}-\bar{a})^2}\frac{q\mu}{2i}. \tag{0.9}$$

The stationary phase method, applied to the integral in the right-hand side of (0.9), gives for $\tau \to \infty$, $\tau \in \mathbb{R}$, the equality

$$\lim_{\substack{\tau\to\infty\\ \tau\in\mathbb{R}}} \frac{4\tau}{\pi i}\int_{z\in b\Omega} e^{i\tau[(z-a)^2+(\bar{z}-\bar{a})^2]}\bar{\partial}_z\mu_a(z,i\tau) = Q(a). \tag{0.10}$$

Formula (0.10) means that the values of the potential Q in an arbitrary point a of Ω can be reconstructed from the Dirichlet-to-Neumann mapping $\mu_a|_{b\Omega} \mapsto \bar{\partial}_z\mu_a|_{b\Omega}$ for a family of functions $\mu_a(z,\lambda)$ depending on the parameter λ, $|\lambda| > const$, where we assume that $\mu_a|_{b\Omega}$ is found using an analog of (0.4) for $\psi_a|_{b\Omega}$.

Bukhgeim's scheme works well at least $\forall Q \in C^{(1)}(\bar{\Omega})$.

The more constructive scheme of [27] works quite well only in the absence of an exceptional set E in the λ-plane for Faddeev-type functions. The papers [4, 32], and [28] constructed modified Faddeev–Green functions that permits solving the inverse boundary problem (0.2), at least on $\mathbb{R}^2 = \mathbb{C}$, under some smallness assumptions on the potential Q.

Let us note that the first uniqueness results in the two-dimensional inverse boundary value or scattering problems for (0.1) or (0.2) go back to A. Calderon [6], V. Druskin [7], R. Kohn, M. Vogelius [24], J. Sylvester and G. Uhlmann [31], and R. Novikov [26].

Note in this connection that the first seminal results on reconstruction of the two-dimensional Schrödinger operator H on the torus from the data "extracted" from the family of eigenfunctions (Bloch–Floquet) of the single energy level $H\psi = E\psi$ were obtained in a series of papers starting from B. Dubrovin, I. Krichever, and S.P. Novikov [8], and S.P. Novikov and A. Veselov [29]. These results were obtained in connection with (2+1)-dimensional evolution equations.

This paper extends to the case of Riemann surfaces the reconstruction procedures of [27] and of [5]. The paper extends (and also corrects) the recent paper [21] where the inverse boundary value problem on a Riemann surface was first considered. Earlier, in [20], it was proved that if $X \subset \mathbb{R}^3$ possesses constant conductivity, then X with complex structure can be effectively reconstructed by at most three generic potential $\to$ current measurements on bX.

Very recently, motivated by [5, 20], and [21], C. Guillarmou and L. Tzou [16] have obtained a general identifiability result (without reconstruction procedure): if for all $W^{1,2}(X)$ solutions of equations $dd^c u + q_j u = 0$, $q_j \in C^{(2)}(X)$, $j = 1, 2$, the Cauchy datas $u|_{bX}$, $d^c u|_{bX}$ coincide, then $q_1 = q_2$ on X.

1 Preliminaries and Main Results

Let $\mathbb{C}P^3$ be a complex projective space with homogeneous coordinates $w = (w_0 : w_1 : w_2 : w_3)$. Let $\mathbb{C}P^2_\infty = \{w \in \mathbb{C}P^3 : w_0 = 0\}$. Then $\mathbb{C}P^3 \backslash \mathbb{C}P^2_\infty$ can be considered as the complex affine space with coordinates $z_k = w_k/w_0$, $k = 1, 2, 3$. By a classical result of G. Halphen (see R. Hartshorne [17], Chap. IV, §6), any compact Riemann surface of genus g can be embedded in $\mathbb{C}P^3$ as a projective algebraic curve $\tilde{V}$, which intersects $\mathbb{C}P^2_\infty$ transversally in $d > g$ points, where $d \geq 1$ if $g = 0$, $d \geq 3$ if $g = 1$, and $d \geq g + 3$ if $g \geq 2$. Without loss of generality one can suppose that

(i) $V = \tilde{V} \backslash \mathbb{C}P^2_\infty$ is a connected affine algebraic curve in $\mathbb{C}^3$ defined by the polynomial equations $V = \{z \in \mathbb{C}^3 : p_1(z) = p_2(z) = p_3(z) = 0\}$ such that the rank of the matrix $[\frac{\partial p_1}{\partial z}(z), \frac{\partial p_2}{\partial z}(z), \frac{\partial p_3}{\partial z}(z)] \equiv 2$ $\forall z \in V$.

(ii) $\tilde{V} \cap \mathbb{C}P^2_\infty = \{\beta_1, \ldots, \beta_d\}$, where

$$\beta_l = (0 : \beta_l^1 : \beta_l^2 : \beta_l^3), \quad \left(\frac{\beta_l^2}{\beta_l^1}, \frac{\beta_l^3}{\beta_l^1}\right) \in \mathbb{C}^2, \quad l = 1, 2, \ldots, d.$$

(iii) For $r_0 > 0$ large enough,

$$\det \begin{vmatrix} \frac{\partial p_\alpha}{\partial z_2} & \frac{\partial p_\alpha}{\partial z_3} \\ \frac{\partial p_\beta}{\partial z_2} & \frac{\partial p_\beta}{\partial z_3} \end{vmatrix} \neq 0 \quad \text{for } z \in V : |z_1| \geq r_0 \text{ and } \alpha \neq \beta.$$

(iv) For $|z|$ large enough,

$$\frac{dz_2}{dz_1}\bigg|_{V_l} = \gamma_l + \frac{\gamma_l^0}{z_1^2} + O\left(\frac{1}{z_1^3}\right), \qquad \frac{dz_3}{dz_1}\bigg|_{V_l} = \tilde{\gamma}_l + \frac{\tilde{\gamma}_l^0}{z_1^2} + O\left(\frac{1}{z_1^3}\right),$$

where $\gamma_l, \tilde{\gamma}_l, \gamma_l^0, \tilde{\gamma}_l^0 \neq 0$, for $l = 1, \dots, d$, $d \geq 2$.

Let $V_0 = \{z \in V : |z_1| \leq r_0\}$ and $V \backslash V_0 = \bigcup_{l=1}^d V_l$, where $\{V_l\}$ are connected components of $V \backslash V_0$. Let us equip V with the Euclidean volume form $dd^c|z|^2$. Let $\tilde{W}^{1,\tilde{p}}(V) = \{F \in L^\infty(V) : \bar{\partial} F \in L^{\tilde{p}}_{0,1}(V)\}$, $\tilde{W}^{1,\tilde{p}}_{1,0}(V) = \{f \in L^\infty_{1,0}(V) : \bar{\partial} f \in L^{\tilde{p}}_{1,1}(V)\}$, $\tilde{p} > 2$. Let $H_{0,1}(V)$ denote the space of antiholomorphic $(0,1)$-forms on V. Let $H^p_{0,1}(V) = H_{0,1}(V) \cap L^p_{0,1}(V)$, $1 < p < 2$.

Let $W^{1,p}(V) = \{F \in L^p(V) : \bar{\partial} F \in L^p_{0,1}(V)\}$.

From the Hodge–Riemann decomposition theorem (see [14, 22]) $\forall \Phi_0 \in W^{1,p}_{0,1}(\tilde{V})$ we have $\Phi_0 = \bar{\partial}(\bar{\partial}^* G \Phi_0) + \mathcal{H}\Phi_0$, where $\mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V})$, and G is the Hodge–Green operator for the Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $\tilde{V}$ with the properties $G(H_{0,1}(\tilde{V})) = 0$, $\bar{\partial} G = G\bar{\partial}$, and $\bar{\partial}^* G = G\bar{\partial}^*$.

Straight generalization of Proposition 1 from [18] gives the explicit operators: $R_1 : L^p_{0,1}(V) \to L^{\tilde{p}}(V)$, $R_0 : L^p_{0,1}(V) \to \tilde{W}^{1,\tilde{p}}(V)$, and $\mathcal{H} : L^p_{0,1}(V) \to H^p_{0,1}(V)$, $1 < p < 2$, $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$, such that $\forall \Phi \in L^p_{0,1}(V)$ we have a decomposition of the Hodge–Riemann-type:

$$\Phi = \bar{\partial} R\Phi + \mathcal{H}\Phi, \quad \text{where } R = R_1 + R_0,$$

$$R_1\Phi(z) = \frac{1}{2\pi i}\int_{\xi \in V} \Phi(\xi) \wedge (dp_\alpha \wedge dp_\beta) \rfloor d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \times \det\left[\frac{\partial p_\alpha(\xi)}{\partial \xi}, \frac{\partial p_\beta(\xi)}{\partial \xi}, \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}\right],$$

$$R_0\Phi(z) = (\bar{\partial}^* G(\bar{\partial} R_1\Phi - \Phi))(z) - (\bar{\partial}^* G(\bar{\partial} R_1 \Phi - \Phi))(\beta_1),$$

$$(\bar{\partial} R_1 \Phi - \Phi) \in W^{1,p}_{0,1}(\tilde{V}),$$

where G is the Hodge–Green operator for the Laplacian $\bar{\partial}\bar{\partial}^*$ for $(0,1)$-forms on $\tilde{V}$, the $(1,1)$-form under the integral sign does not depend on the choice of indexes $\alpha, \beta = 1, 2, 3$, $\alpha \neq \beta$,

$$\mathcal{H}\Phi = \sum_{j=1}^{g}\left(\int_V \Phi \wedge \omega_j\right)\bar{\omega}_j,$$

$\{\omega_j\}$ is an orthonormal basis of holomorphic $(1,0)$-forms on $\tilde{V}$, i.e.,

$$\int_V \omega_j \wedge \bar{\omega}_k = \delta_{jk}, \quad j, k = 1, 2, \dots, g.$$

Note that as a corollary of the construction of R we have that $\lim_{\substack{z \in V_1 \\ z \to \infty}} R\Phi(z) = R\Phi(\beta_1) = 0$.

Remark 1.1 If $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$ is an algebraic curve in $\mathbb{C}^2$, then the formula for the operator R_1 is reduced to the following:

$$R_1\Phi(z) = \frac{1}{2\pi i}\int_{\xi\in V}\Phi(\xi)\frac{d\xi_1}{\frac{\partial P}{\partial \xi_2}}\det\left[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi}-\bar{z}}{|\xi - z|^2}\right].$$

Remark 1.2 Based on [19], one can construct an explicit formula not only for the main part R_1 of the R-operator, but for the whole operator $R = R_1 + R_0$.

Let $\varphi\in L^1_{1,1}(V)\cap L^\infty_{1,1}(V)$, $f \in \tilde{W}^{1,\tilde{p}}_{1,0}(V)$, $\lambda \in \mathbb{C}$, $\theta \in \mathbb{C}$.

Let

$$\hat{R}_\theta\varphi = R((dz_1 + \theta dz_2)\rfloor\varphi)(dz_1 + \theta dz_2),$$
$$R_{\lambda,\theta} f = e_{-\lambda,\theta}\overline{R(\overline{e_{\lambda,\theta} f})}, \quad \text{where } e_{\lambda,\theta}(z) = e^{\lambda(z_1+\theta z_2)-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}.$$

By a straight generalization of Propositions 2 and 3 from [18], the form $f = \hat{R}_\theta\varphi$ is a solution of $\bar{\partial} f = \varphi$ on V, the function $u = R_{\lambda,\theta} f$ is a solution of

$$(\partial + \lambda(dz_1 + \theta dz_2))u = f - \mathcal{H}_{\lambda,\theta} f, \quad \text{where}$$
$$\mathcal{H}_{\lambda,\theta} f \stackrel{\text{def}}{=} e_{-\lambda,\theta}\overline{\mathcal{H}(\overline{e_{\lambda,\theta} f})}, \quad u \in W^{1,\tilde{p}}(V),\ \tilde{p} > 2.$$

In addition, by a straight generalization of Proposition 4 from [18], we have that

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))u = \varphi + \bar{\lambda}(d\bar{z}_1 + \bar{\theta} d\bar{z}_2) \wedge \mathcal{H}_{\lambda,\theta}(\hat{R}_\theta\varphi) \quad \text{on } V.$$

Definition 1.1 The kernel $g_{\lambda,\theta}(z,\xi)$, $z,\xi \in V$, $\lambda \in \mathbb{C}$, of the integral operator $R_{\lambda,\theta} \circ \hat{R}_\theta$ is called in [18] the Faddeev-type Green function for the operator $\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))$.

Definition 1.2 Let $g = genus\ \tilde{V}$. Let $\{\omega_j\}$, $j = 1,\ldots,g$, be an orthonormal basis of holomorphic forms on $\tilde{V}$. Let $\{a_1,\ldots,a_g\}$ be different points (or effective divisor) on $V\backslash V_0$. Let

$$\Delta_\theta(\lambda) = \det\left[\int_{\xi\in V}\hat{R}_\theta(\delta(\xi,a_j)) \wedge \bar{\omega}_k(\xi)e_{\lambda,\theta}(\xi),\ j,k = 1,\ldots,g\right],$$

where $\delta(\xi,a_j)$ is the Dirac (1, 1)-form concentrated in $\{a_j\}$.

Let $E_\theta = \{\lambda \in \mathbb{C} : \Delta_\theta(\lambda) = 0\}$.

Definition 1.3 The parameter $\theta \in \mathbb{C}$ will be called generic if $\theta \notin \{\theta_1,\ldots,\theta_d\}$, where $\theta_l = -1/\gamma_l$. Divisor $\{a_1,\ldots,a_g\}$ on $V\backslash V_0$ will be called generic if

$$\det\left[\frac{\omega_j}{dz_1}(a_k)\right]_{j,k=1,\ldots,g} \neq 0.$$

Proposition 1.1 *Let the parameter $\theta \in \mathbb{C}$ and the divisor $\{a_1, \dots, a_g\}$ on $V \backslash V_0$ be generic, where $V_0 = \{z \in V : |z_1| \le r_0\}$, $g \ge 1$. Then for r_0 large enough we have the inequalities*

$$\overline{\lim_{\lambda\to\infty}} |\lambda^g \Delta_\theta(\lambda)| < \infty \quad \textit{and}$$
$$\forall\varepsilon > 0 \quad \underline{\lim_{\lambda\to\infty}} |\lambda^g \Delta_\theta(\lambda)|_\varepsilon > 0, \quad \textit{where}$$
$$|\lambda^g \Delta_\theta(\lambda)|_\varepsilon = \sup_{\{\lambda' : |\lambda'-\lambda|<\varepsilon\}} |(\lambda')^g \cdot \Delta_\theta(\lambda')|.$$

Besides, the set E_θ is a closed nowhere-dense subset of $\mathbb{C}$.

Let X be a domain containing V_0 and relatively compact on V. Let $\sigma \in C^{(3)}(V)$, $\sigma > 0$, on V, $\sigma = 1$ on $V \backslash X$. Let Y be a domain containing $\bar{X}$ and relatively compact on V. Let the divisor $\{a_1, \dots, a_g\}$ on $Y \backslash X$ and the parameter $\theta \in \mathbb{C}$ be generic.

Definition 1.4 The functions $\psi_\theta(z, \lambda) = \sqrt{\sigma} F_\theta(z, \lambda) = \mu_\theta(z, \lambda) e^{\lambda(z_1 + \theta z_2)}$, $z \in V$, $\theta \in \mathbb{C}\backslash\{\theta_1, \dots, \theta_d\}$, $\lambda \in \mathbb{C}\backslash E_\theta$, will be called the Faddeev-type functions associated with σ, θ, and $\{a_1, \dots, a_g\}$ if ψ_θ, F_θ, μ_θ satisfy the corresponding properties:

$$d\sigma d^c F_\theta = 2\sqrt{\sigma} e^{\lambda(z_1+\theta z_2)} \sum_{j=1}^{g} C_{j,\theta}(\lambda)\delta(z, a_j),$$
$$dd^c \psi_\theta = q\psi_\theta + 2e^{\lambda(z_1+\theta z_2)} \sum_{j=1}^{g} C_{j,\theta}(\lambda)\delta(z, a_j), \tag{1.1}$$
$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu_\theta = \frac{i}{2} q\mu_\theta + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)\delta(z, a_j),$$

and the normalization condition

$$\lim_{\substack{z\in V_1 \\ z\to\infty}} \mu_\theta(z, \lambda) = 1, \tag{1.2}$$

where $\mu_\theta|_Y \in L^{\tilde{p}}(Y)$, $\mu_\theta|_{V\backslash Y} \in L^\infty(V\backslash Y)$, $\tilde{p} > 2$, $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, $\{C_{j,\theta}\}$ are some functions of $\lambda \in \mathbb{C}\backslash E_\theta$.

Theorem 1.1 *Using the aforementioned notation and conditions, $\forall$ generic $\theta \in \mathbb{C}$, $\forall$ generic divisor $\{a_1, \dots, a_g\} \subset V\backslash X$ and $\forall\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| > const(V, \{a_j\}, \theta, \sigma)$ there exists a unique Faddeev-type function*

$$\psi_\theta(z, \lambda) = \sqrt{\sigma} F_\theta(z, \lambda) = e^{\lambda(z_1+\theta z_2)} \mu_\theta(z, \lambda),$$

associated with the conductivity function σ and the divisor $\{a_1, \dots, a_g\}$. Moreover,

(A) *The function* $z \to \psi_\theta(z,\lambda)$ *and parameters* $\{C_{j,\theta}(\lambda)\}$ *can be found from the following equations, depending on parameters* $\theta \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash E_\theta$, $|\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$:

$$\psi_\theta(z,\lambda) - \frac{i}{2}\int_{\xi \in X} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi_\theta(\xi,\lambda)$$
$$= e^{\lambda(z_1+\theta z_2)} + i\sum_{j=1}^{g} C_{j,\theta}(\lambda) g_{\lambda,\theta}(z,a_j) e^{\lambda(z_1+\theta z_2)}, \tag{1.3}$$

$$2\sum_{j=1}^{g} C_{j,\theta}(\lambda) e_{\lambda,\theta}(a_j)\frac{\bar{\omega}_k}{d\bar{z}_1+\bar{\theta}d\bar{z}_2}(a_j)$$
$$= -\int_{z\in V} e^{-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi_\theta(z,\lambda)\frac{\bar{\omega}_k}{d\bar{z}_1+\bar{\theta}d\bar{z}_2}(z), \tag{1.4}$$

where $k = 1, 2, \ldots, g$, *and* $\{\omega_j\}$ *is an orthonormal basis of holomorphic forms on* $\tilde{V}$;

(B) *The functions* $z \to \psi_\theta(z,\lambda)$ *and the parameters* $\{C_{j,\theta}(\lambda)\}$ *satisfy the following properties for* $\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$:

$$\exists \lim_{\substack{z\to\infty,\ z\in V_l \\ l=1,2,\ldots,d}} \frac{\bar{z}_1+\bar{\theta}\bar{z}_2}{\bar{\lambda}} e^{-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\left(\frac{\partial\psi_\theta}{\partial\bar{z}_1}+\bar{\theta}\frac{\partial\psi_\theta}{\partial\bar{z}_2}\right)$$
$$= \lim_{\substack{z\to\infty \\ z\in V_l}} \psi_\theta e^{-\lambda(z_1+\theta z_2)} b_\theta(\lambda), \tag{1.5}$$

$$iC_{j,\theta}(\lambda) = (2\pi i) Res_{a_j} e^{-\lambda(z_1+\theta z_2)}\partial\psi_\theta$$
$$\stackrel{\text{def}}{=} 2\pi i \lim_{\varepsilon\to 0}\int_{|z-a_j|=\varepsilon} e^{-\lambda(z_1+\theta z_2)}\partial\psi_\theta, \tag{1.6}$$

$$\frac{\partial\psi_\theta(z,\lambda)}{\partial\bar{\lambda}} = b_\theta(\lambda)\overline{\psi_\theta(z,\lambda)}, \tag{1.7}$$

$$\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar{\lambda}} e^{\lambda(a_{j,1}+\theta a_{j,2})} = b_\theta(\lambda)\overline{C_{j,\theta}(\lambda)} e^{\bar{\lambda}(\bar{a}_{j,1}+\bar{\theta}\bar{a}_{j,2})}. \tag{1.8}$$

Besides,

$$\bar{\lambda} b_\theta(\lambda) d = -\frac{1}{2\pi i}\int_{z\in bX} e_{\lambda,\theta}(z)\bar{\partial}\mu(z) + i\sum_{j=1}^{g} C_{j,\theta} e_{\lambda,\theta}(a_j),$$
$$|\lambda|\cdot|b_\theta(\lambda)| \leq const(V, \{a_j\}, \sigma)\frac{1}{(|\lambda|+1)^{1/3}}\frac{1}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}, \tag{1.9}$$
$$|C_{j,\theta}(\lambda)| \leq const(V, \{a_j\}, \sigma)\frac{1}{(|\lambda|+1)^{1/3}}\frac{1}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}.$$

Remark 1.3 If $\| \ln \sqrt{\sigma} \|_{C^{(2)}(X)} \le const(V, \{a_j\}, \theta)$, then the condition $\lambda \in \mathbb{C} \backslash E_\theta$: $|\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$ in Theorem 1.1 can be replaced by the condition $\lambda \in \mathbb{C} \backslash E_\theta$. The dependence of $const(V, \{a_j\}, \theta, \sigma)$ on σ means it depends only on $\| \ln \sqrt{\sigma} \|_{C^{(2)}(X)}$.

Definition 1.5 The functions $b_\theta(\lambda)$ and $\{C_{j,\theta}\}$ will be called "scattering" data for the potential q.

Let $\hat{\Phi}(\psi|_{bX}) = \bar{\partial}\psi|_{bX}$ for all sufficiently regular solutions ψ of (0.2) in $\bar{X}$, where $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. The operator Φ is equivalent to the Dirichlet-to-Neumann operator for (0.1). Let $\hat{\Phi}_0$ denote $\hat{\Phi}$ for $q \equiv 0$ on $\bar{X}$. Note that for the solutions ψ of (0.2) we have the property $\partial\psi|_{bX} = \overline{\hat{\Phi}(\bar{\psi}|_{bX})}$.

Theorem 1.2 *Under the conditions of Proposition 1.1 and Theorem 1.1, the following statements are valid*:

(A) $\forall \lambda \in \mathbb{C} \backslash E_\theta : |\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$ *the restriction of* $\psi_\theta(z, \lambda)$ *on* bX *and data* $\{C_{j,\theta}(\lambda)\}$ *can be reconstructed from the Dirichlet-to-Neumann data as the unique solution of the Fredholm integral equation*

$$\psi_\theta(z,\lambda)|_{bX} + \int_{\xi\in bX} e^{\lambda[(z_1-\xi_1)+\theta(z_2-\xi_2)]} g_{\lambda,\theta}(z,\xi)(\hat{\Phi}-\hat{\Phi}_0)\psi_\theta(\xi,\lambda)$$
$$= e^{\lambda(z_1+\theta z_2)} + i\sum_{j=1}^{g} C_{j,\theta}(\lambda) g_{\lambda,\theta}(z,a_j) e^{\lambda(z_1+\theta z_2)}, \quad \textit{where} \tag{1.10}$$

$$\int_{z\in bX} (z_1+\theta z_2)^{-k} e^{-\lambda(z_1+\theta z_2)} \overline{\hat{\Phi}\bar{\psi}_\theta} = -\sum_{j=1}^{g} (a_{j,1}+\theta a_{j,2})^{-k} C_{j,\theta}(\lambda), \tag{1.11}$$

$k = 2, \dots, g+1$, *and for the coordinates of the points* $\{a_j\}$ *the values* $\{a_{j,1} + \theta a_{j,2}\}$ *are supposed to be mutually different*;

(B) *Under the additional assumption that* $\sigma \in C^{(3)}(V)$, *the function* $\sigma(w)$, $w \in X$, *can be reconstructed from the Dirichlet-to-Neumann data*

$$\psi_\theta|_{bX} \overset{\text{def}}{=} \mu_\theta|_{bX} e^{\lambda(z_1+\theta z_2)} \to \bar{\partial}\psi_\theta|_{bX}$$

by explicit formulas, *where we assume that* $\psi_\theta|_{bX}$ *is found using* (1.10), (1.11).

For the case $V = \{z \in \mathbb{C}^2 : P(z) = 0\}$, *where* P *is a polynomial of degree* N, *this formula has the following form. Let* $\{w_m\}$ *be points of* V, *where* $(dz_1 + \theta dz_2)|_V(w_m) = 0$, $m = 1, \dots, M$. *Then for all* $\theta \in \mathbb{C}$, *except for a finite number of* θ, *the values* $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}|_V(w_m)$ *can be found from the following linear system*:

$$\tau(1+o(1))\frac{d^k}{d\tau^k}\left(\int_{z\in bX} e_{i\tau,\theta}(z)\bar{\partial}\mu_\theta(z,i\tau)\right)$$
$$= \sum_{m=1}^{M} \frac{i\pi(1+|\theta|^2)}{2} \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\bigg|_V (w_m)$$

$$\times \frac{|\frac{\partial P}{\partial z_1}(w)|^3 \frac{d^k}{d\tau^k} \exp i\tau[(w_{m,1} + \theta w_{m,2}) + (\bar{w}_{m,n} + \bar{\theta}\bar{w}_{m,2})]}{|\frac{\partial^2 P}{\partial z_1^2}(\frac{\partial P}{\partial z_2})^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}(\frac{\partial P}{\partial z_2})(\frac{\partial P}{\partial z_1}) + \frac{\partial^2 P}{\partial z_2^2}(\frac{\partial P}{\partial z_1})^2|(w_m)}, \quad (1.12)$$

where $m, k = 1, \ldots, M$; $M = N(N-1)$, $\tau \in \mathbb{R}$, $\tau \to \infty$, $|\tau|^g |\Delta_\theta(i\tau)| \ge \varepsilon > 0$, *and* ε *is small enough. The determinant of system* (1.12) *is proportional to the determinant of Vandermonde.*

Note that $\forall w \in V\ \exists! \theta \in \mathbb{C} : (dz_1 + \theta dz_2)|_V(w) = 0$.

(C) *If* $g = 0$ *and if* $\theta = \theta(\lambda) = \lambda^{-2}$, *then* $\forall z \in X$ *and* $\forall \lambda \in \mathbb{C}$ *the function* $\mu_\theta(z, \lambda) = \psi_\theta(z, \lambda) e^{-\lambda(z_1 + \theta z_2}$ *is the unique solution of the Fredholm integral equation*

$$\mu_{\theta(\lambda)}(z, \lambda) + \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} b_{\theta(\xi)}(\xi) e^{\bar{\xi}(\bar{z}_1 + \bar{\theta}(\xi)\bar{z}_2) - \xi(z_1 + \theta(\xi) z_2)} \overline{\mu_{\theta(\xi)}(z, \xi)} \frac{d\xi \wedge d\bar{\xi}}{\xi - \lambda} = 1,$$

$$\textit{where } |b_{\theta(\xi)}(\xi)| \le \frac{const(V)}{(1 + |\xi|)^2},$$

and the function $z \to \sigma(z)$, $z \in X$, *can be found from the equality*

$$dd^c \psi_{\theta(\lambda)}(z, \lambda) = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}}(z) \psi_{\theta(\lambda)}(z, \lambda), \quad z \in X.$$

Remark 1.4 The statement and proof of Theorem 1.2B are still valid if we replace in the formulations the form $q = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}}$ with the arbitrary real form (potential) $q \in C^{(1)}_{1,1}(V)$, $q|_{V \setminus X} = 0$.

Remark 1.5 Using the Faddeev-type Green function constructed in [18], in [21] there were obtained natural analogues of the main steps of the reconstruction scheme of [27] on the Riemann surface V. In particular, under a smallness assumption on $\partial \log \sqrt{\sigma}$ the existence (and uniqueness) of the solution $\mu_\theta(z, \lambda)$ of the Faddeev-type integral equation

$$\mu_\theta(z, \lambda) = 1 + \frac{i}{2} \int_{\xi \in V} g_{\lambda,\theta}(z, \xi) \frac{\mu_\theta(\xi, \lambda) dd^c \sqrt{\sigma}}{\sqrt{\sigma}} + i \sum_{j=1}^{g} C_j g_{\lambda,\theta}(z, a_j),$$

$$z \in V,\ \lambda \in \mathbb{C},$$

holds for any a priori fixed constants $C_1, \ldots, C_g$. However (and this fact was overlooked in [21]), for $\lambda \in \mathbb{C} \setminus E_\theta$ there exists a unique choice of constants $C_{j,\theta}(\lambda, \sigma)$ for which the integral equation above is equivalent to the differential equation

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu = \frac{i}{2}\left(\frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}} \mu\right) + i \sum_{j=1}^{g} C_j \delta(z, a_j),$$

where $\delta(z, a_j)$ are Dirac measures concentrated in the points a_j.

2 Faddeev-type Functions on Riemann Surfaces. Uniqueness

Let a projective algebraic curve $\tilde{V}$ be embedded in $\mathbb{C}P^3$ and intersect $\mathbb{C}P^2_\infty = \{w \in \mathbb{C}P^3 : w_0 = 0\}$ transversally in $d > g$ points. Let $V = \tilde{V}\backslash\mathbb{C}P^2_\infty$, $V_0 = \{z \in V : |z_1| \le r_0\}$, and properties (i)–(iv) from Sect. 1 be valid.

Proposition 2.1 *Let σ be a positive function belonging to $C^{(2)}(V)$ such that $\sigma \equiv const = 1$ on $V\backslash X \subset V\backslash V_0 = \bigcup_{l=1}^d V_l$, where $\{V_l\}$ are connected components of $V\backslash\bar{V}_0$. Put $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Let $\{a_1, \dots, a_g\}$ be a generic divisor with support in $Y\backslash\bar{X}$, $\bar{X} \subset Y \subset \bar{Y} \subset V$. For generic $\theta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$, let $|\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$ and the function $z \mapsto \mu = \mu_\theta(z, \lambda)$ be such that*:

$$\begin{aligned} &\mu|_Y \in L^{\tilde{p}}(Y), \quad \mu|_{V\backslash Y} \in L^\infty(V\backslash\bar{Y}), \\ &\bar{\partial}\mu|_Y \in L^p(Y), \quad \bar{\partial}\mu|_{V\backslash\bar{Y}} \in L^{\tilde{p}}(V\backslash Y), \quad 1 \le p < 2,\ \tilde{p} > 2, \end{aligned} \tag{2.1}$$

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2)\mu = \frac{i}{2}q\mu + i\sum_{j=1}^g C_j\delta(z, a_j) \quad \textit{with some } C_j = C_{j,\theta}(\lambda) \tag{2.2}$$

$$\textit{and} \quad \mu_\theta(z, \lambda) \to 0, \quad z \to \infty,\ z \in V_1. \tag{2.3}$$

Then $\mu_\theta(z, \lambda) \equiv 0$, $z \in V$.

Remark 2.1 Proposition 2.1 is a corrected version of Proposition 2.1 of [21]. For the case $V = \mathbb{C}$, the equivalent result goes back to [2].

Lemma 2.1 *Let $\psi = \sqrt{\sigma}F = e^{\lambda(z_1+\theta z_2)}\mu$, where μ satisfies (2.1), (2.2) and*

$$F_1 = \sqrt{\sigma}\partial F, \qquad F_2 = \sqrt{\sigma}\bar{\partial}F. \tag{2.4}$$

Then the forms F_1, F_2 satisfy the system of equations

$$\begin{aligned} &\bar{\partial}F_1 + F_2 \wedge \partial \ln\sqrt{\sigma} = ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^g C_j\delta(z, a_j), \\ &\partial F_2 + F_1 \wedge \bar{\partial}\ln\sqrt{\sigma} = -ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^g C_j\delta(z, a_j). \end{aligned} \tag{2.5}$$

Proof of Lemma 2.1 From the definition of F_1 and F_2, it follows that

$$\begin{aligned} d\sigma d^c F &= i[2\sigma\partial\bar{\partial}F - \bar{\partial}\sigma \wedge \partial F + \partial\sigma \wedge \bar{\partial}F] \\ &= 2i\sqrt{\sigma}(\partial F_2 + F_1 \wedge \bar{\partial}\ln\sqrt{\sigma}) = -2i\sqrt{\sigma}(\bar{\partial}F_1 + F_2 \wedge \partial\ln\sqrt{\sigma}). \end{aligned}$$

From (2.4) and (2.2), we deduce also that

$$d(\sigma d^c F) = \sqrt{\sigma}\left(dd^c\psi - \psi \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\right) = 2\sqrt{\sigma} e^{\lambda(z_1+\theta z_2)} \sum_{j=1}^{g} C_j \delta(z, a_j).$$

These equalities imply (2.5).

Lemma 2.1 is proved. □

Lemma 2.2 *Let $\{b_m\}$ be the points of X where $(dz_1 + \theta dz_2)|_X(b_m) = 0$. Let $B^0 = \bigcup_m \{b_m\}$ and $A^0 = \bigcup_j \{a_j\}$. Let $u_\pm = m_1 \pm e_{-\lambda,\theta}(z)\bar{m}_2$, where $m_1 = e^{-\lambda(z_1+\theta z_2)} f_1$, $m_2 = e^{-\lambda(z_1+\theta z_2)} f_2$, $f_1 = \sqrt{\sigma}\frac{\partial F}{\partial z_1}$, and $f_2 = \sqrt{\sigma}\frac{\partial F}{\partial \bar{z}_1}$. Also let $q_1 = \frac{\partial \ln\sqrt{\sigma}}{\partial z_1}$ and $\delta_0(z, a_j) = \frac{\delta(z,a_j)}{dz_1 \wedge d\bar{z}_1}$. Then in the conditions of Lemma 2.1*

$$\begin{aligned} &\sup_{z\in X} |\bar{\partial} u_\pm\big|_X(z)\cdot \operatorname{dist}^2(z, B^0)| = O\Big(\sup_{z\in X} |u_\pm \operatorname{dist}(z, B^0)|\Big) < \infty; \\ &u_\pm|_{V\setminus X} \in L^1(V\setminus X) \cap O(V\setminus(X\cup A^0)) \end{aligned} \tag{2.6}$$

and the system (2.5) is equivalent to the system

$$\frac{\partial u_\pm}{\partial \bar{z}_1} d\bar{z}_1 = \mp(e_{-\lambda,\theta}(z) q_1 \bar{u}_\pm) d\bar{z}_1 + i \sum_{j=1}^{g} (C_j \pm \bar{C}_j e_{-\lambda,\theta}(z))\delta_0(z, a_j) d\bar{z}_1. \tag{2.7}$$

Proof of Lemma 2.2 From (2.1), we deduce the property

$$u_\pm|_Y \in L^p(Y), \quad 1 \le p < 2, \qquad u_\pm|_{V\setminus Y} \in L^{\tilde{p}}(V\setminus Y) \oplus L^\infty(V\setminus Y), \quad \tilde{p} > 2.$$

System (2.5) is equivalent to the system of equations

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}_1} &= -f_2 q_1 + i e^{\lambda(z_1+\theta z_2)} \sum_{j=1}^{g} C_j \delta_0(z, a_j), \\ \frac{\partial f_2}{\partial z_1} &= -f_1 \bar{q}_1 + i e^{\lambda(z_1+\theta z_2)} \sum_{j=1}^{g} C_j \delta_0(z, a_j). \end{aligned}$$

This system and the definition of m_1, m_2 imply

$$\begin{aligned} \frac{\partial m_1}{\partial \bar{z}_1} &= -q_1 m_2 + i \sum_{j=1}^{g} C_j \delta_0(z, a_j), \\ \frac{\partial m_2}{\partial z_1} + \lambda m_2\left(1 + \theta \frac{\partial z_2}{\partial z_1}\right) &= -\bar{q}_1 m_1 + i \sum_{j=1}^{g} C_j \delta_0(z, a_j). \end{aligned}$$

From the last equalities and the definition of $u_\pm$ we deduce

$$\frac{\partial u_\pm}{\partial \bar z_1} = \frac{\partial m_1}{\partial \bar z_1} \pm e_{-\lambda,\theta}(z)\left(\frac{\partial \bar m_2}{\partial \bar z_1} + \bar\lambda\left(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\right)\bar m_2\right)$$

$$= -q_1 m_2 + i\sum_{j=1}^{g} C_j\delta_0(z,a_j) \pm e_{-\lambda,\theta}(z)\left(\bar\lambda\left(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\right)\bar m_2\right.$$

$$\left. - \bar\lambda\bar m_2\left(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\right) - q_1\bar m_1 + i\sum_{j=1}^{g}\bar C_j\delta_0(z,a_j)\right)$$

$$= \mp(e_{-\lambda,\theta}(z)q_1\bar u_\pm) + i\sum_{j=1}^{g}(C_j \pm \bar C_j e_{-\lambda,\theta}(z))\delta_0(z,a_j).$$

Property (2.7) is proved.

For proving (2.6), we will use a construction from Bers and Vekua (see [30, 33]). Let $\beta_\pm$ be continuous on Y solutions of $\bar\partial$ equations

$$\bar\partial\beta_\pm = \pm e_{-\lambda,\theta}(z)q_1\frac{\bar u_\pm}{u_\pm}d\bar z_1,$$

where the right-hand side belongs to $L^\infty_{0,1}(Y)$.

The functions $v_\pm = u_\pm e^{-\beta_\pm}$ belong to $\mathcal{O}(Y)$. Indeed, from (2.1) and (2.2) it follows that $\mu \in W^{1,p}(Y)\cap W^{1,\tilde p}_{loc}(Y\backslash(A^0\cup B^0))$. From this and from the definition of $v_\pm$, we deduce that $\bar\partial v_\pm = q_1\bar u_\pm d\bar z_1 e^{-\beta_\pm} - q_1 u_\pm\frac{\bar u_\pm}{u_\pm}e^{-\beta_\pm}d\bar z_1 = 0$ on $Y\backslash(A^0\cup B^0)$, and the following formula for $u_\pm$ is valid:

$$u_\pm(z) = v_\pm(z)e^{\beta_\pm(z)}. \tag{2.8}$$

From this, (2.7), and (2.8), we obtain (2.6).

Lemma 2.2 is proved. □

Lemma 2.3 *Let $u_\pm$ be the functions from Lemma 2.2 and let μ be the function from Lemma 2.1. Then*

$$u_\pm = \frac{\partial\mu}{\partial z_1} + \lambda\left(1+\theta\frac{\partial z_2}{\partial z_1}\right)\mu - q_1\mu \pm e_{-\lambda,\theta}(z)\left(\frac{\partial\bar\mu}{\partial z_1} - q_1\bar\mu\right).$$

Proof of Lemma 2.3 We have

$$u_\pm = e^{-\lambda(z_1+\theta z_2)}f_1 \pm e^{-\lambda(z_1+\theta z_2)}\bar f_2 = e^{-\lambda(z_1+\theta z_2)}(f_1 \pm \bar f_2),$$

where

$$f_1 = \sqrt{\sigma}\frac{\partial F}{\partial z_1} = \sqrt{\sigma}\frac{\partial}{\partial z_1}\left(\frac{1}{\sqrt{\sigma}}e^{\lambda(z_1+\theta z_2)}\mu\right)$$

$$= e^{\lambda(z_1+\theta z_2)}\left(\frac{\partial\mu}{\partial z_1} + \lambda\left(1+\theta\frac{\partial z_2}{\partial z_1}\right)\mu - q_1\mu\right),$$

$$\bar{f}_2 = \sqrt{\sigma}\frac{\partial\bar{F}}{\partial z_1} = \sqrt{\sigma}\frac{\partial}{\partial z_1}\left(\frac{1}{\sqrt{\sigma}}e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\bar{\mu}\right) = e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\left(\frac{\partial\bar{\mu}}{\partial z_1} - q_1\bar{\mu}\right).$$

This implies Lemma 2.3. □

Lemma 2.4 *Let $\omega_1,\dots,\omega_g$ be an orthonormal basis of holomorphic 1-forms on $\tilde{V}$. Let $\{a_1,\dots,a_g\}$ be a generic divisor on $Y\backslash\bar{X}$, where $V_0\subset\bar{X}\subset Y\subset V$. Put $\omega^0_{j,k} = \frac{\omega_k}{dz_1}(a_j)$. For some generic $\theta\in\mathbb{C}$ and $\lambda\in\mathbb{C}$, let the functions $u_\pm$ from Lemmas 2.2–2.3 satisfy (2.6) and (2.7), with some $C_j = C_{j,\theta}(\lambda)$. Then*

$$\sup_j |C_{j,\theta}(\lambda)| \le const(V,\{a_j\},\theta)\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)}(1+|\lambda|)^{-1/3}\|u_\pm\|_{L^\infty(X,B^0)},$$

where $\|u_\pm\|_{L^\infty(X,B^0)} \stackrel{\text{def}}{=} \sup_{z\in X}|u_\pm(z)\,\text{dist}(z,B^0)|.$

Proof of Lemma 2.4 From condition (iv) of Sect. 1, we deduce that $|\omega^0_{j,k}|<\infty$. From the definition of a generic divisor, we obtain $\det[\omega^0_{j,k}]\neq 0$. From (2.7) and from the definition of Dirac measure $\forall k = 1,\dots,g$, we deduce

$$\begin{aligned}
&\overline{\lim_{r\to\infty}}\left(\int_{\{z\in V:|z_1|=r\}} u_\pm\wedge\omega_k\right) \pm \int_X e_{-\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}\bar{u}_\pm d\bar{z}_1\wedge\omega_k\\
&= i\int_Y\sum_{j=1}^g (C_j\pm\bar{C}_j e_{-\lambda,\theta}(z))\delta_0(z,a_j)d\bar{z}_1\wedge\omega_k\\
&= i\sum_{j=1}^g (C_j\pm\bar{C}_j e_{-\lambda,\theta}(a_j))\omega^0_{j,k},
\end{aligned}\tag{2.9}$$

$j,k = 1,2,\dots,g.$

From the estimates $\overline{\lim}_{r_n\to\infty}\sup_{\{z\in V:|z_1|=r_n\}}|u_\pm(z)|<\infty$, for some sequence $r_n\to\infty$, and $|\frac{\omega_k}{dz_1}|\le O(|\frac{1}{z_1^2}|)$, $z\in V\backslash Y$, $k=1,\dots,g$, we obtain

$$\overline{\lim_{r\to\infty}}\left|\int_{\{z\in V:|z_1|=r\}} u_\pm\wedge\omega_k\right| = 0. \tag{2.10}$$

From (2.9), (2.10), and Cramer's formula, we obtain

$$\begin{aligned}
&i(C_j\pm\bar{C}_j e_{-\lambda,\theta}(a_j))\\
&\quad= \frac{\det[\omega^0_{1,k};\dots;\omega^0_{j-1,k};\int_X \pm e_{-\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}\bar{u}_\pm d\bar{z}_1\wedge\omega_k;\omega^0_{j+1,k};\dots;\omega^0_{g,k}]}{\det[\omega^0_{j,k}]},
\end{aligned}\tag{2.11}$$

where $j,k = 1,\dots,g$.

Let us prove the estimate

$$\left| \int_X e_{-\lambda,\theta}(z) \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} \bar{u}_\pm d\bar{z}_1 \wedge \omega_k \right|$$
$$\le const(X,\theta)(1+|\lambda|)^{-1/3} \| \ln \sqrt{\sigma}\|^2_{W^{2,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}. \qquad (2.12)$$

For $|\lambda| \le 1$, the estimate follows directly, using that $\ln \sqrt{\sigma} \in W^{1,\infty}(X)$.

Let $B^\varepsilon = \bigcup_{m=1}^M \{z \in X : |z - b_m| \le \varepsilon\}$.

Let $\chi_{\varepsilon,\nu}$, $\nu = 1,2$, be functions from $C^{(1)}(V)$ such that $\chi_{\varepsilon,1} + \chi_{\varepsilon,2} \equiv 1$ on V, $\operatorname{supp} \chi_{\varepsilon,1} \subset B^{2\varepsilon}$, $\operatorname{supp} \chi_{\varepsilon,2} \subset V \backslash B^\varepsilon$, $|d\chi_{\varepsilon,\nu}| = O(\frac{1}{\varepsilon})$, $\nu = 1,2$.

Put $J^\varepsilon_\nu u_\pm = \int_X \chi_{\varepsilon,\nu}(z) e_{-\lambda,\theta}(z) \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} \bar{u}_\pm d\bar{z}_1 \wedge \omega_k$, $\nu = 1,2$. We have directly:

$$|J^\varepsilon_1 u_\pm| \le const(X)\varepsilon \| \ln \sqrt{\sigma}\|_{W^{1,1}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}. \qquad (2.13)$$

For $J^\varepsilon_2 u_\pm$ we obtain from integration by parts:

$$J^\varepsilon_2 u_\pm = -\frac{1}{\lambda} \int_X \chi_{\varepsilon,2} \partial e_{-\lambda,\theta}(z) \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} \bar{u}_\pm d\bar{z}_1 \wedge \frac{\omega_k}{dz_1 + \theta dz_2}$$
$$= \frac{1}{\lambda} \int_X e_{-\lambda,\theta}(z) \partial \left(\chi_{\varepsilon,2} \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} \bar{u}_\pm d\bar{z}_1 \wedge \frac{\omega_k}{dz_1 + \theta dz_2} \right). \qquad (2.14)$$

To estimate (2.14), we use (2.6) and the following properties: $|\partial \chi_{\varepsilon,2}| = O(\frac{1}{\varepsilon})$, $\operatorname{supp}(\partial \chi_{\varepsilon,2}) \subset B^{2\varepsilon}$,

$$\left\| \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} d\bar{z}_1 \wedge \partial \chi_{\varepsilon,2} u_\pm \frac{\omega_k}{dz_1 + \theta dz_2} \right\|_{L^1_{0,1}(X)}$$
$$\le \frac{const(X,\theta)}{\varepsilon} \| \ln \sqrt{\sigma}\|_{W^{1,\infty}(X)} \|u_\pm\|_{L^\infty(X,B^0)},$$
$$\left\| \frac{\partial^2 \ln \sqrt{\sigma}}{\partial z_1^2} dz_1 \wedge d\bar{z}_1 \chi_{\varepsilon,2} u_\pm \frac{\omega_k}{dz_1 + \theta dz_2} \right\|_{L^1_{1,1}(X)}$$
$$\le |\ln \varepsilon| const(X,\theta) \| \ln \sqrt{\sigma}\|_{W^{2,\infty}(X)} \|u_\pm\|_{L^\infty(X,B^0)},$$
$$\left\| \frac{\partial \ln \sqrt{\sigma}}{\partial z_1} d\bar{z}_1 \chi_{\varepsilon,2} u_\pm \wedge \partial \Big(\frac{\omega_k}{dz_1 + \theta dz_2} \Big) \right\|_{L^1_{0,1}(X)}$$
$$\le \frac{const(X,\theta)}{\varepsilon} \| \ln \sqrt{\sigma}\|_{W^{1,\infty}(X)} \|u_\pm\|_{L^\infty(X,B^0)},$$
$$\partial \bar{u}_\pm \big|_X = \mp (e_{\lambda,\theta}(z) \bar{q}_1 \bar{u}_\pm) dz_1.$$

From (2.14), (2.6), and these properties, we obtain

$$|J^\varepsilon_2 u_\pm| \le |\ln \varepsilon| \frac{const(X,\theta)}{|\lambda|} \| \ln \sqrt{\sigma}\|_{W^{2,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}$$

$$+ \frac{const(X,\theta)}{\varepsilon|\lambda|}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}$$
$$+ \frac{const(X,\theta,\delta)}{\varepsilon^{1+\delta}|\lambda|}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}. \tag{2.15}$$

Putting in (2.13) and (2.15) $\varepsilon = \frac{1}{\sqrt{\lambda}}$ and $\delta = 1/3$ we obtain (2.12) for $|\lambda| \geq 1$.

Inequalities (2.11) and (2.12) imply the estimate

$$|C_j \pm \bar{C}_j e_{-\lambda,\theta}(a_j)|$$
$$\leq const(X,\{a_j\},\theta)(1+|\lambda|)^{-1/3}\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}.$$

We have obtained the statement of Lemma 2.4. □

Lemma 2.5 *Let the functions $u_\pm$ satisfy* (2.6), (2.7), *and R, the operator from Sect.* 1. *Then*

$$\|R[e_{-\lambda,\theta}q_1\bar{u}_\pm d\bar{\xi}_1\|_{L^\infty(X,B^0)}$$
$$\leq const(X,\theta)(1+|\lambda|)^{-1/5}\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)} \cdot \|u_\pm\|_{L^\infty(X,B^0)}.$$

Proof of Lemma 2.5 Let $\chi_{\varepsilon,\nu}$, $\nu = 1,2$, be the partition of unity from Lemma 2.4. Put $S^\varepsilon_\nu u_\pm = R[\chi_{\varepsilon,\nu}q_1\bar{u}_\pm d\bar{\xi}_1]$, $\nu = 1,2$. Using (2.6) and the formula for the operator R, we deduce the estimate

$$\|S^\varepsilon_1 u_\pm\|_{L^\infty(X,B^0)} = O(\varepsilon)\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\|u_\pm\|_{L^\infty(X,B^0)}. \tag{2.16}$$

Let $R_{1,0}(\xi,z)$ be the kernel of the operator R. This means, in particular, that $\bar{\partial}_\xi R_{1,0}(\xi,z) = -\delta(\xi,z)$, where $\delta(\xi,z)$ is the Dirac $(1,1)$-measure, concentrated in the point $\xi = z$. We have

$$S^\varepsilon_2 u_\pm = \int_X \chi_{\varepsilon,2}e_{-\lambda,\theta}q_1\bar{u}_\pm d\bar{\xi}_1 \wedge R_{1,0}(\xi,z). \tag{2.17}$$

Integration by parts in (2.17) gives the following:

$$S^\varepsilon_2 u_\pm = \frac{1}{\bar{\lambda}}\int_X \bar{\partial}e_{-\lambda,\theta}(\xi)\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}\chi_{\varepsilon,2}(\xi)q_1(\xi)\bar{u}_\pm(\xi) \wedge R_{1,0}(\xi,z)$$
$$= -\frac{1}{\bar{\lambda}}\int_X e_{-\lambda,\theta}(\xi)\bar{\partial}\left(\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}\chi_{\varepsilon,2}(\xi)q_1(\xi)\bar{u}_\pm(\xi)\right) \wedge R_{1,0}(\xi,z)$$
$$+ \frac{1}{\bar{\lambda}}e_{-\lambda,\theta}(z)\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}(z)\chi_{\varepsilon,2}(z)q_1(z)\bar{u}_\pm(z). \tag{2.18}$$

To estimate (2.18), we use (2.6), the properties of the partition of unity $\{\chi_{\varepsilon,\nu}\}$, and the inequalities

$$\begin{aligned}\left|\frac{d\bar{\xi}_1}{d\bar{\xi}_1+\bar{\theta}d\bar{\xi}_2}(\xi)\right| &= O\left(\frac{1}{\operatorname{dist}(\xi,B^0)}\right),\\ \left|\bar{\partial}\frac{d\bar{\xi}_1}{d\bar{\xi}_1+\bar{\theta}d\bar{\xi}_2}(\xi)\right| &= O\left(\frac{1}{(\operatorname{dist}(\xi,B^0))^2}\right),\\ |q_1(\xi)| &= O\left(\frac{1}{\operatorname{dist}(\xi,B^0)}\right),\\ |\bar{\partial}q_1(\xi)| &= O\left(\frac{1}{(\operatorname{dist}(\xi,B^0))^2}\right), \quad \xi\in X.\end{aligned} \tag{2.19}$$

From (2.19), (2.8), and the formula for the operator R, we deduce the estimate

$$\|S_2^{\varepsilon}u_{\pm}\|_{L^\infty(X)} = O\left(\frac{1}{\varepsilon^4|\lambda|}\right)\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)}\|u_{\pm}\|_{L^\infty(X,B^0)}. \tag{2.20}$$

Putting in (2.16) and (2.20) $\varepsilon=\frac{1}{|\lambda|^{1/5}}$, we obtain the statement of Lemma 2.5. □

Proof of Proposition 2.1 Let the function μ satisfy conditions (2.1)–(2.3), and let $u_{\pm}$ be the functions defined in Lemma 2.2. Then by Lemma 2.3, we have

$$\begin{aligned}\lim_{\substack{z\to\infty\\ z\in V_1}} u_{\pm}(z,\lambda) &= \lim_{\substack{z\to\infty\\ z\in V_1}}(m_1\pm e_{-\lambda,\theta}(z)\bar{m}_2)\\ &= \lim_{\substack{z\to\infty\\ z\in V_1}}\left[\lambda\left(1+\theta\frac{dz_2}{dz_1}\right)\mu+\frac{\partial\mu}{\partial z_1}\pm e_{-\lambda,\theta}(z)\frac{\partial\bar{\mu}}{\partial z_1}\right]\to 0.\end{aligned} \tag{2.21}$$

Let

$$h_{\pm}=u_{\pm}\pm R\left[(e_{-\lambda,\theta}(z)q_1\bar{u}_{\pm})d\bar{z}_1 - i\sum_{j=1}^{g}(C_j\pm\bar{C}_je_{-\lambda,\theta}(z))\delta_0(z,a_j)d\bar{z}_1\right], \tag{2.22}$$

where R is the operator from Sect. 1.

By Lemmas 2.2–2.5 and the properties of the operator R, we have $h_{\pm}\in\mathcal{O}(V)\cap L^\infty(V)$ and $h_{\pm}(z,\lambda)\to 0$, $z\to\infty$, $z\in V_1$. By Liouville's theorem, $h_{\pm}(z,\lambda)\equiv 0$ on V, $\lambda\in\mathbb{C}$. Then from (2.22) with $h_{\pm}(z,\lambda)\equiv 0$, and Lemmas 2.4 and 2.5, it follows that $u_{\pm}(z,\lambda)\equiv 0$, $z\in V$, if $\lambda\in\mathbb{C}\backslash E_\theta : |\lambda|\geq const(V,\{a_j\},\theta)\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)}$. Property $u_{\pm}(z,\lambda)\equiv 0$, $z\in V$, implies by Lemma 2.3 the equality $\frac{\partial\mu}{\partial\bar{z}_1}-\bar{q}_1\mu=0$, $z\in V$, where $\mu(z)\to\infty$ if $z\in V_1$, $z\to\infty$. The Liouville-type theorem for generalized holomorphic functions ([30], Theorem 7.1) implies $\mu\equiv 0$. Proposition 2.1 is proved. □

3 Faddeev-type Functions on a Riemann Surface. Existence. Proof of Theorem 1.1A

Proposition 3.1 *Let the conductivity σ and the divisor $\{a_1,\ldots,a_g\}$ satisfy the conditions of Proposition 2.1. Then $\forall$ generic $\theta\in\mathbb{C}$ and $\forall\lambda\in\mathbb{C}\backslash E_\theta:|\lambda|\geq const(V,\{a_j\},\theta,\sigma)$ there exists a unique Faddeev-type function*

$$\psi \overset{\text{def}}{=} \sqrt{\sigma}F \overset{\text{def}}{=} e^{\lambda(z_1+\theta z_2)}\mu,\quad \textit{where}$$
$$\psi=\psi_\theta(z,\lambda),\ F=F_\theta(z,\lambda),\ \mu=\mu_\theta(z,\lambda), \tag{3.1}$$

associated with σ and the divisor $\{a_1,\ldots,a_g\}$, i.e.,

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu=\frac{i}{2}q\mu+\sum_{j=1}^{g}C_j\delta(z,a_j),$$
$$\textit{for some } C_j=C_{j,\theta}(\lambda),\quad \textit{where}$$
$$q=\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}},\ \mu|_Y\in L^{\tilde p}(Y),\ \mu|_{V\backslash\bar Y}\in L^\infty(V\backslash\bar Y),\ \lim_{\substack{z\to\infty\\ z\in V_1}}\mu_\theta(z,\lambda)=1. \tag{3.1a}$$

In addition,

$$\|\mu_\theta(z,\lambda)-\mu_\theta(\infty_l,\lambda)\|_{L^{\tilde p}(V)}\leq\frac{const(V,\{a_j\},\theta,\sigma,\tilde p,\varepsilon)}{|\Delta_\theta(\lambda)|\cdot(1+|\lambda|)^{g+1-\varepsilon}},$$
$$\textit{where } \mu_\theta(\infty_l,\lambda)\overset{\text{def}}{=}\lim_{\substack{z\to\infty\\ z\in V_l}}\mu_\theta(z,\lambda),\ l=1,\ldots,d, \tag{3.1b}$$

$$\|\partial\mu\|_{L^p_{1,0}(Y)}+\|\partial\mu\|_{L^{\tilde p}_{1,0}(V\backslash Y)}\leq\frac{const(V,\{a_j\},\theta,\sigma,p,\tilde p,\varepsilon)}{|\Delta_\theta(\lambda)|\cdot(1+|\lambda|)^{g-\varepsilon}},\quad p<2,\ \tilde p>2,$$
$$\forall \textit{ generic } \theta\in\mathbb{C} \textit{ and } \lambda\in\mathbb{C}\backslash E_\theta:|\lambda|\geq const(V,\{a_j\},\theta,\sigma),$$
$$\left.\frac{\partial\mu}{\partial\bar\lambda}\right|_Y\in W^{1,p}(Y),\ \left.\frac{\partial\mu}{\partial\bar\lambda}\right|_{V_l\backslash Y}\in L^\infty(V_l\backslash Y)\cup W^{1,\tilde p}(V_l\backslash Y), \tag{3.1c}$$

where $\{V_l\}$ are connected components of $V\backslash V_0$, $l=1,\ldots,d$,

$$e_{\lambda,\theta}(z)=e^{\lambda(z_1+\theta z_2)-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}.$$

Remark 3.1 Proposition 3.1 is a corrected version of Proposition 2.2 from [21]. For the case $V=\mathbb{C}$, the results of such a type go back to [10, 11].

Lemma 3.1 *Under the conditions of Proposition 3.1, $\forall\lambda\in\mathbb{C}\backslash E_\theta$ the function $z\to\mu_\theta(z,\lambda)$ belonging to $L^{\tilde p}(Y)$ on Y and to $L^\infty(V\backslash Y)$ on $V\backslash Y$ satisfies (3.1a) iff there exists $C_j=C_{j,\theta}(\lambda)$, $j=1,\ldots,g$, such that*

$$\mu_\theta(z,\lambda)=1+\frac{i}{2}\int_{\xi\in X}g_{\lambda,\theta}(z,\xi)q(\xi)\mu_\theta(\xi,\lambda)+i\sum_{j=1}^{g}C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j) \tag{3.2}$$

and one of two equivalent conditions is valid:

$$\mathcal{H}_{\lambda,\theta}\left(\hat{R}_\theta\left(\frac{i}{2}q\mu\right)\right)+i\sum_{j=1}^{g}C_{j,\theta}(\lambda)\mathcal{H}_{\lambda,\theta}(\hat{R}_\theta(\delta(z,a_j)))=0 \quad or$$
$$(\partial+\lambda(dz_1+\theta dz_2))\mu_\theta(z,\lambda)\in H_{1,0}\left(V\Big\backslash\left(X\bigcup_{j=1}^{g}\{a_j\}\right)\right)\cap L^1_{1,0}(Y\backslash X), \tag{3.3}$$

where $g_{\lambda,\theta}$ is a Faddeev-type Green function, $\hat{R}_\theta$, $\mathcal{H}_{\lambda,\theta}$ are the operators defined in Sect. 1.

Proof of Lemma 3.1 From Proposition 4 in [18] and from the definition of the Green function $g_{\lambda,\theta}(z,\xi)$ we deduce that the integral equation (3.2) is equivalent to the following differential equation:

$$\begin{aligned}\bar{\partial}(\partial+\lambda(dz_1+\theta dz_2))\mu=&\frac{i}{2}q\mu+i\sum_{j=1}^{g}C_{j,\theta}\delta(z,a_j)\\&+\bar{\lambda}(d\bar{z}_1+\bar{\theta}d\bar{z}_2)\wedge\left[\mathcal{H}_{\lambda,\theta}\left(\hat{R}_\theta\left(\frac{i}{2}q\mu\right)\right)\right.\\&\left.+i\sum_{j=1}^{g}C_{j,\theta}\mathcal{H}_{\lambda,\theta}(\hat{R}_\theta(\delta(z,a_j)))\right].\end{aligned} \tag{3.4}$$

Equation (3.4) is equivalent to (3.1a) if one of the two equivalent conditions (3.3) are valid.

Lemma 3.1 is proved. □

Lemma 3.2 *Let $\{a_1,\dots,a_g\}$ be a generic divisor in $Y\backslash\bar{X}$. Then for any generic $\theta\in\mathbb{C}$ and $\forall\lambda\in\mathbb{C}\backslash E_\theta:|\lambda|\geq const(V,\{a_j\},\theta,\sigma)$, integral equations* (3.2), (3.3) *are a uniquely solvable Fredholm integral equation in the space $\tilde{W}^{1,\tilde{p}}(V)$.*

Proof of Lemma 3.2 Let $\theta\in\mathbb{C}$ and $\lambda\in\mathbb{C}\backslash E_\theta:|\lambda|\geq const(V,\{a_j\},\theta,\sigma)$. From (3.2), (3.3) we obtain an integral equation for $\tilde{\mu}_\theta=\mu_\theta-1$ and $\tilde{C}_{j,\theta}$:

$$\begin{aligned}&\tilde{\mu}_\theta(z,\lambda)-\frac{i}{2}\int_{\xi\in V}g_{\lambda,\theta}(z,\xi)q(\xi)\tilde{\mu}_\theta(\xi,\lambda)-i\sum_{j=1}^{g}\tilde{C}_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j)\\&\quad=\frac{i}{2}\int_{\xi\in V}g_{\lambda,\theta}(z,\xi)q(\xi)+i\sum_{j=1}^{g}C^0_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j).\end{aligned} \tag{3.5}$$

The parameters $\tilde{C}_j = \tilde{C}_{j,\theta}(\lambda)$, $j = 1, \dots, g$, are defined by the equations

$$-i \sum_{j=1}^{g} \tilde{C}_j \int_{\xi \in V} \hat{R}_\theta(\delta(\xi, a_j)) \bar{\omega}_k(\xi) e_{\lambda,\theta}(\xi) = \int_{\xi \in V} e_{\lambda,\theta}(\xi) \hat{R}_\theta\left(\frac{i}{2} q \tilde{\mu}\right) \bar{\omega}_k(\xi), \quad k = 1, 2, \dots, g. \tag{3.6}$$

Recall that the determinant of system (3.6) is exactly $\Delta_\theta(\lambda)$.

The parameters $C^0_{j,\theta}$ are defined by (3.6) with $C^0_{j,\theta}$ in place of $\tilde{C}_{j,\theta}$ and 1 in place of $\tilde{\mu}$. One can see also that $C^0_{j,\theta}(\lambda) = C_{j,\theta}(\lambda) - \tilde{C}_{j,\theta}(\lambda)$.

Let us prove that (3.5), (3.6) determine a Fredholm integral equation in the space $\tilde{W}^{1,\tilde{p}}(V)$, $\tilde{p} > 2$.

Propositions 2, 3 of [18] imply that the correspondence

$$\tilde{\mu} \mapsto R_{\lambda,\theta} \circ \left(\hat{R}_\theta\left(\frac{i}{2} q \tilde{\mu}\right) + i \sum_{j=1}^{g} \tilde{C}_{j,\theta} \hat{R}_\theta(\delta(z, a_j)) \right)$$

defines a linear continuous mapping of $\tilde{W}^{1,\tilde{p}}(V)$ into itself. This mapping is compact because the mapping $\tilde{\mu} \to q\tilde{\mu}$, $\operatorname{supp} q \subset X$, from $\tilde{W}^{1,\tilde{p}}(V)$ into $L^{\tilde{p}}_{1,1}(X)$ is compact, the operator $\hat{R}_\theta : L^{\tilde{p}}_{1,1}(X) \to \tilde{W}^{1,\tilde{p}}_{1,0}(V)$ and the operator $R_{\lambda,\theta} : \tilde{W}^{1,\tilde{p}}_{1,0}(V) \to \tilde{W}^{1,\tilde{p}}(V)$ are bounded.

If for fixed $\lambda \notin E_\theta$ the Fredholm equations (3.5), (3.6) are not solvable, then the corresponding homogeneous equation, when the right-hand side of (3.5) is replaced by zero, admits a nontrivial solution $\tilde{\mu}^* = \mu^* - 1$.

By Lemma 3.1, the function $\tilde{\mu}^*$ satisfies the differential equation (2.2) with C_j replaced by $\tilde{C}_j$ and with property $\tilde{\mu}^*(z) \to 0$, $z \to \infty$, $z \in V_1$.

By Proposition 2.1, $\tilde{\mu}^* \equiv 0$ if $\lambda \in \mathbb{C} \backslash E_\theta : |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$.

This means (3.2), (3.3) are a uniquely solvable Fredholm integral equation for any $\lambda \in \mathbb{C} \backslash E_\theta : |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$.

Lemma 3.2 is proved. □

Lemma 3.3 *Let $\{a_1, \dots, a_g\}$ be a generic divisor on $Y \backslash X$. Let $\lambda \in \mathbb{C} \backslash E_\theta$. Let μ be the solution of integral equations* (3.2), (3.3). *Then the relations* (3.3) *determining parameters $C_j = C_{j,\theta}(\lambda)$ are reduced to the following explicit formulas*:

$$2i \sum_{j=1}^{g} C_j e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{z}_1}(a_j) = \int_{z \in X} e_{\lambda,\theta}(z) \left(i \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial} \ln \sqrt{\sigma} \wedge \partial \ln \sqrt{\sigma} \right) \mu \frac{\bar{\omega}_k}{d\bar{z}_1}(z). \tag{3.7}$$

Proof of Lemma 3.3 By Lemma 3.1, (3.2), (3.3) are equivalent to the equation

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu=\frac{i}{2}q\mu+i\sum_{j=1}^{g}C_{j,\theta}\delta(z,a_j), \tag{3.8}$$

where $\mu=\mu_\theta(z,\lambda)\to 1$, $z\in V_1$, $z\to\infty$.

System (2.7) implies the following relation:

$$\overline{\lim_{R\to\infty}}\int_{|z_1|=R}\bar u_\pm\wedge\bar\omega_k+i\int_{z\in V\setminus X}\sum_{j=1}^{g}(\bar C_{j,\theta}\mp C_{j,\theta}e_{\lambda,\theta}(z))\frac{\delta(z,a_j)}{d\bar z_1}\bar\omega_k$$
$$=\mp\int_{z\in X}e^{\lambda(z_1+\theta z_2)-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\bar q_1u_\pm dz_1\wedge\bar\omega_k, \tag{3.9}$$

where $\bar q_1=\frac{\partial\ln\sqrt{\sigma}}{\partial\bar z_1}$.

To obtain (3.9) we multiply both sides of (2.7) by $\wedge\omega_k$, integrate on V, and take conjugation.

From Lemmas 2.3 and 3.2, it follows that

$$u_\pm(z)\to\lambda(1+\theta\gamma_l)\cdot\lim_{\substack{z\to\infty\\ z\in V_l}}\mu_\theta(z,\lambda),\quad z\to\infty,\ z\in V_l,$$

$$\text{where }\gamma_l=\lim_{\substack{z\to\infty\\ z\in V_l}}\frac{\partial z_2}{\partial z_1},\quad \lim_{\substack{z\to\infty\\ z\in V_1}}\mu_\theta(z,\lambda)=1.$$

The existence of $\lim_{\substack{z\to\infty\\ z\in V_l}}\mu_\theta(z,\lambda)$ follows from Lemma 4.1, below. This implies that

$$\overline{\lim_{R\to\infty}}\left|\int_{|z_1|=R}\bar u_\pm\wedge\bar\omega_k\right|=\overline{\lim_{R\to\infty}}\left|\int_{|z_1|=R}\bar\lambda(1+\bar\theta\bar\gamma_l)\bar\omega_k\right|=\lim_{R\to\infty}|\lambda|O\left(\frac{1}{R}\right)=0. \tag{3.10}$$

From (3.9), (3.10), and the definition of $u_\pm$, we obtain

$$2i\sum_{j=1}^{g}\int_{V\setminus X}C_je_{\lambda,\theta}(z)\frac{\delta(z,a_j)}{d\bar z_1}\wedge\bar\omega_k$$
$$=\int_{z\in X}e_{\lambda,\theta}(z)\bar q_1(u_++u_-)dz_1\wedge\bar\omega_k$$
$$=2\int_{z\in X}e^{-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\bar q_1f_1dz_1\wedge\bar\omega_k,\quad\text{where }f_1=\sqrt{\sigma}\frac{\partial F}{\partial z_1}.$$

By Lemma 2.3, we have

$$2\int_{z\in X}e^{-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\bar q_1f_1dz_1\wedge\bar\omega_k$$
$$=2\int_{z\in X}e_{\lambda,\theta}(z)\bar q_1\left(\frac{\partial\mu}{\partial z_1}+\lambda\mu+\lambda\theta\frac{\partial z_2}{\partial z_1}\mu-q_1\mu\right)dz_1\wedge\bar\omega_k. \tag{3.11}$$

From the definition of $\delta(z, a_j)$, we have

$$2i \sum_{j=1}^{g} \int_{z \in V \setminus X} C_j e_{\lambda,\theta}(z) \frac{\delta(z, a_j)}{d\bar{z}_1} \wedge \bar{\omega}_k = -2i \sum_{j=1}^{g} C_j e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{z}_1}(a_j). \quad (3.12)$$

From integration by parts, we have

$$\begin{aligned} 2 \int_{z \in X} & e_{\lambda,\theta}(z) \bar{q}_1 \left(\frac{\partial \mu}{\partial z_1} + \lambda \mu \right) dz_1 \wedge \bar{\omega}_k \\ &= 2 \int_X e_{\lambda,\theta}(z) \frac{\partial \ln \sqrt{\sigma}}{\partial \bar{z}_1} \lambda \mu dz_1 \wedge \bar{\omega}_k \\ &\quad - 2 \int_X e_{\lambda,\theta}(z) \frac{\partial \ln \sqrt{\sigma}}{\partial \bar{z}_1} \left(\lambda \mu + \lambda \theta \frac{\partial z_2}{\partial z_1} \mu \right) dz_1 \wedge \bar{\omega}_k \\ &\quad - 2 \int_X e_{\lambda,\theta}(z) \frac{\partial^2 \ln \sqrt{\sigma}}{\partial z_1 \partial \bar{z}_1} \mu dz_1 \wedge \bar{\omega}_k \\ &= -2 \int_X e_{\lambda,\theta}(z) \left(\frac{\partial^2 \ln \sqrt{\sigma}}{\partial z_1 \partial \bar{z}_1} + \frac{\partial \ln \sqrt{\sigma}}{\partial \bar{z}_1} \lambda \theta \frac{\partial z_2}{\partial z_1} \right) \mu dz_1 \wedge \bar{\omega}_k. \end{aligned} \quad (3.13)$$

Using (3.11), (3.12), and (3.13), we obtain

$$i \sum_{j=1}^{g} C_{j,\theta} e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{z}_1}(a_j) = \int_{z \in X} e_{\lambda,\theta}(z) \left(\frac{\partial^2 \ln \sqrt{\sigma}}{\partial z_1 \partial \bar{z}_1} + \left| \frac{\partial \ln \sqrt{\sigma}}{\partial \bar{z}_1} \right|^2 \right) \mu dz_1 \wedge \bar{\omega}_k.$$

Lemma 3.3 is proved. □

Proof of Proposition 3.1 (a) By Lemmas 3.1–3.3, the statement (3.1a) of the proposition is valid, i.e., there exists a function $z \to \mu_\theta(z, \lambda)$, $z \in V$ with the property (3.1a) $\forall \lambda \in \mathbb{C} \setminus E_\theta : |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$.

(b) Put $f_0 = \hat{R}_\theta(\frac{i}{2} q \mu)$, $f_1 = \hat{R}_\theta(i \sum_{j=1}^{g} C_{j,\theta} \delta(z, a_j))$, and $f = f_0 + f_1$. By (3.2) we have $\mu - 1 = R_{\lambda,\theta} f = R_{\lambda,\theta} f_0 + R_{\lambda,\theta} f_1$.

Put

$$L_{0,q}^{p,\tilde{p}}(V) = \{u : u|_Y \in L_{0,q}^{p}(Y),\ u|_{V \setminus \bar{Y}} \in L_{0,q}^{\tilde{p}}(V \setminus Y)\}, \quad 1 \leq p < 2,\ \tilde{p} > 2,\ q = 0, 1.$$

By Proposition 3(ii′) from [18], we obtain

$$\begin{aligned} &\|\mu - \mu_\theta(\infty_l, \lambda)\|_{L^{\tilde{p}}(V_l \setminus Y)} \\ &\quad \leq const(V, \tilde{p}, \theta) \cdot \min(|\lambda|^{-1/2}, |\lambda|^{-1}) \left(\|f_0\|_{\tilde{W}_{1,0}^{1,\tilde{p}}(V)} + \sum_{j=1}^{g} |C_{j,\theta}| \right), \\ &\|\partial \mu\|_{L_{1,0}^{p,\tilde{p}}(V)} \leq const(V, \tilde{p}, \theta) \left(\|f_0\|_{\tilde{W}_{1,0}^{1,\tilde{p}}(V)} + \sum_{j=1}^{g} |C_{j,\theta}| \right). \end{aligned} \quad (3.14)$$

For proving the estimates (3.1b) we need to estimate $\{C^0_{j,\theta}\}$.

In order to estimate $\{C^0_{j,\theta}\}$ we must use (3.6), where the parameters $\{\tilde{C}_{j,\theta}\}$ are replaced by $\{C^0_{j,\theta}\}$ and the function $\tilde{\mu}$ is replaced by 1. For modified equations (3.6), we apply Cramer's formula for the solution of a linear system and integration by parts in all integrals of this system, using $e_{\lambda,\theta}(z)(d\bar{z}_1+\bar{\theta}d\bar{z}_2)=\frac{1}{\bar{\lambda}}\bar{\partial}e_{\lambda,\theta}(z)$. In addition, we use: formula (1.2) for $\Delta_\theta(\lambda)$, the formula $\bar{\partial}\hat{R}_\theta\left(\frac{i}{2}q\mu\right)=\frac{i}{2}q\mu$, and an estimate of the singular integral containing $\bar{\partial}\left(\frac{\bar{\omega}_k}{d\bar{z}_1+\bar{\theta}d\bar{z}_2}\right)$. This gives the inequality

$$\sum_j |C^0_{j,\theta}(\lambda)| \le \frac{const(V,\{a_j\},\theta,\sigma)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}.$$

Further using (3.5), together with the obtained inequality for $\sum |C^0_{j,\theta}(\lambda)|$ and the inequality for the Faddeev-type Green function $|g_{\lambda,\theta}(z,\xi)|=O\left(\frac{1}{|\lambda|^{1-\varepsilon}}\right)$, we obtain estimates for $\sum|\tilde{C}_{j,\theta}(\lambda)|$ and $|\mu_\theta(\lambda)|$:

$$|\lambda|^{-\varepsilon}\|\mu\|_{\tilde{W}^{1,\tilde{p}}(V)}+\sum_j|\tilde{C}_{j,\theta}(\lambda)|\le\frac{const(V,\{a_j\},\theta,\sigma,\tilde{p},\varepsilon)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}\quad\text{and}$$

$$\|\mu-\mu(\infty_l,\cdot)\|_{L^{\tilde{p}}(V_l\backslash Y)}\le\frac{const(V,\{a_j\},\theta,\sigma,\tilde{p},\varepsilon)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^{g+1-\varepsilon}},$$
$$\text{where }\lambda\in\mathbb{C}\backslash E_\theta:|\lambda|\ge const(V,\{a_j\},\theta,\sigma,\varepsilon),\ l=1,\ldots,d,$$
$$\mu_\theta(\infty_1,\lambda)=1. \tag{3.15}$$

Estimates (3.14), (3.15) imply estimates (3.1b).

(c) Differentiation of (3.2) with respect to $\bar{\lambda}$ gives the equality

$$\frac{\partial\mu}{\partial\bar{\lambda}}-R_{\lambda,\theta}\circ\left(\hat{R}_\theta\left(\frac{i}{2}q\frac{\partial\mu}{\partial\bar{\lambda}}+i\sum_{j=1}^g\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar{\lambda}}\delta(z,a_j)\right)\right)$$
$$=(\bar{z}_1+\bar{\theta}\bar{z}_2)(\mu-1)-R_{\lambda,\theta}\left((\bar{\xi}_1+\bar{\theta}\bar{\xi}_2)\hat{R}_\theta\left(\frac{i}{2}q\mu+i\sum_{j=1}^g C_{j,\theta}\delta(z,a_j)\right)\right). \tag{3.16}$$

Equality (3.16) can be rewritten in the following form:

$$\frac{\partial\mu}{\partial\bar{\lambda}}=\left(I-R_{\lambda,\theta}\circ\hat{R}_\theta\left(\frac{i}{2}q\cdot\right)\right)^{-1}\Bigg[(\bar{z}_1+\bar{\theta}\bar{z}_2)(\mu-1)$$
$$+R_{\lambda,\theta}\circ\hat{R}_\theta\left(i\sum_{j=1}^g\frac{\partial C_{j,\theta}}{\partial\bar{\lambda}}\delta(z,a_j)\right)$$
$$-R_{\lambda,\theta}\left((\bar{\xi}_1+\bar{\theta}\bar{\xi}_2)\hat{R}_\theta\left(\frac{i}{2}q\mu+i\sum_{j=1}^g C_{j,\theta}(\lambda)\delta(z,a_j)\right)\right)\Bigg]. \tag{3.17}$$

Using Propositions 2, 3 from [18], and the estimates from Part (b) of this proof, we obtain from (3.17)

$$e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial\bar\lambda}\bigg|_Y \in W^{1,p}(Y),$$

$$e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial\bar\lambda}\bigg|_{V_l} \in W^{1,\tilde p}(V_l\backslash Y)\cup L^\infty(V_l\backslash Y).$$

Statement (3.1c) is proved.
Proposition 3.1 is proved. □

4 Equation $\frac{\partial\mu_\theta(z,\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(z)\overline{\mu_\theta(z,\lambda)}$. Proof of Theorem 1.1B

Proposition 4.1 *Let the conductivity σ, the divisor $\{a_1,\dots,a_g\}$, and θ satisfy the conditions of Proposition 2.1. Let the function $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$ be the Faddeev-type function associated with σ, θ, and the divisor $\{a_1,\dots,a_g\}$. Then for $\lambda\in\mathbb{C}\backslash E_\theta : |\lambda|\ge const(V,\{a_j\},\theta,\sigma)$.*

(i) *The following $\bar\partial$-equations take place*:

$$\frac{\partial\mu_\theta(z,\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(z)\overline{\mu_\theta(z,\lambda)}, \quad \text{if } z\in V\backslash\{a_1,\dots,a_g\}, \tag{4.1}$$

$$\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(a_j)\overline{C_{j,\theta}(\lambda)}, \quad j=1,\dots,g. \tag{4.2}$$

(ii) *The function $b_\theta(\lambda)$ satisfies the following equations*:

$$\begin{aligned} b_\theta(\lambda)\lim_{\substack{z\to\infty\\ z\in V_l}}\overline{\mu_\theta(z,\lambda)} &= \lim_{\substack{z\to\infty\\ z\in V_l}}\frac{\bar z_1+\bar\theta\bar z_2}{\bar\lambda}e_{\lambda,\theta}(z)\frac{\partial\mu_\theta(z,\lambda)}{\partial(\bar z_1+\bar\theta\bar z_2)},\\ \bar\lambda b_\theta(\lambda)d &= -\frac{1}{2\pi i}\int_{z\in bX} e_{\lambda,\theta}(z)\bar\partial\mu_\theta(z,\lambda) + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)e_{\lambda,\theta}(a_j),\\ l &= 1,\dots,d \end{aligned} \tag{4.3}$$

and the inequality

$$|\lambda|(1+|\lambda|)^g|\Delta_\theta(\lambda)|\cdot|b_\theta(\lambda)| \le const(V,\{a_j\},\theta,\sigma)\frac{1}{(|\lambda|+1)^{1/3}}. \tag{4.4}$$

Remark 4.1 For the case $V=\mathbb{C}$, this statement is obtained in [15, 27], and [28]. Proposition 4.1 is a corrected version of Proposition 3.2 of [21].

Lemma 4.1

(i) *Let the function* $\mu = \mu_\theta(z,\lambda)$, $z \in V\backslash Y$, $\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$ *satisfy the equation*

$$\bar\partial(\partial + \lambda(dz_1 + \theta dz_2))\mu = 0 \quad on\ V\backslash Y \tag{4.5}$$

and the property

$$[\mu - \mu_\theta(\infty_l,\lambda)]|_{V_l\backslash Y} \in W^{1,\tilde p}(V_l\backslash \bar Y), \quad where\ \tilde p > 2,$$
$$\mu_\theta(\infty_l,\lambda) \stackrel{\text{def}}{=} \lim_{\substack{z\to\infty\\ z\in V_l}} \mu_\theta(z,\lambda),\ l = 1,\dots,d.$$

Then

$$A \stackrel{\text{def}}{=} \frac{\partial\mu}{\partial(z_1+\theta z_2)} + \lambda\mu \in \mathcal{O}(\tilde V\backslash \bar Y) \quad and$$
$$A|_{V_l\backslash Y} = \lambda\mu(\infty_l) + \sum_{k=1}^{\infty} A_{k,l}\frac{1}{(z_1+\theta z_2)^k},$$

$$\begin{aligned}\bar B &\stackrel{\text{def}}{=} e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)} \in \overline{\mathcal{O}(\tilde V\backslash \bar Y)} \quad and\\ \bar B|_{V_l\backslash Y} &= \sum_{k=1}^{\infty} B_{k,l}\frac{1}{(\bar z_1+\bar\theta\bar z_2)^k}, \quad l=1,\dots,d,\end{aligned} \tag{4.6}$$

where $\mathcal{O}(\tilde V\backslash \bar Y)$ *is the space of holomorphic functions on* $(\tilde V\backslash \bar Y)$.

(ii) *Let*

$$M|_{V_l} = \mu_\theta(\infty_l,\lambda) + \sum_{k=1}^{\infty}\frac{a_{k,l}(\lambda)}{(z_1+\theta z_2)^k} \quad and \quad \bar N|_{V_l} = \sum_{k=1}^{\infty}\frac{b_{k,l}(\lambda)}{(\bar z_1+\bar\theta\bar z_2)^k}$$

be formal series with coefficients determined by the relations

$$\lambda a_{k,l} - (k-1)a_{k-1,l} = A_{k,l}, \quad \bar\lambda b_{k,l} - (k-1)b_{k-1,l} = B_{k,l},$$
$$l = 1,\dots,d,\ k = 1,2,\dots.$$

Let

$$M_\nu|_{V_l} = \mu_\theta(\infty_l,\lambda) + \sum_{k=1}^{\nu}\frac{a_{k,l}}{(z_1+\theta z_2)^k}, \qquad \bar N_\nu|_{V_l} = \sum_{k=1}^{\nu}\frac{b_{k,l}}{(\bar z_1+\bar\theta\bar z_2)^k}. \tag{4.7}$$

Then the function μ *has the asymptotic decomposition*

$$\mu|_{V_l} = M|_{V_l} + e_{-\lambda,\theta}(z)\bar N|_{V_l}, \quad z_1\to\infty, \quad i.e.$$
$$\mu|_{V_l} = M|_{V_l} + e_{-\lambda,\theta}(z)\bar N_\nu|_{V_l} + O\left(\frac{1}{|z_1|^{\nu+1}}\right).$$

Proof of Lemma 4.1 (i) From (4.5), it follows that

$$\partial\bar{\partial}(e^{\lambda(z_1+\theta z_2)}\mu(z,\lambda))|_{V\backslash\bar{Y}}=0.$$

Thus $\bar{\partial}(e^{\lambda(z_1+\theta z_2)}\mu(z,\lambda)) = e^{\lambda(z_1+\theta z_2)}\bar{\partial}\mu$ is an antiholomorphic form on $V\backslash\bar{Y}$, and $\partial\mu+\lambda\mu(dz_1+\theta dz_2)$ is a holomorphic form on $V\backslash\bar{Y}$. From this, the condition $\bar{\partial}\mu\in L^{\tilde{p}}_{0,1}(V\backslash\bar{Y})$, and the Cauchy theorem, it follows that

$$e^{\lambda(z_1+\theta z_2)}\bar{\partial}\mu|_{V_l\backslash\bar{Y}} = e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\bar{B}(d\bar{z}_1+\bar{\theta}d\bar{z}_2)|_{V_l\backslash\bar{Y}}$$

$$= e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\sum_{k=1}^{\infty}\frac{B_{k,l}}{(\bar{z}_1+\bar{\theta}\bar{z}_2)^k}(d\bar{z}_1+\bar{\theta}d\bar{z}_2)\big|_{V_l} \quad \text{and}$$

$$(\partial\mu+\lambda\mu(dz_1+\theta dz_2))\big|_{V_l\backslash\bar{Y}} = A(dz_1+\theta dz_2)\big|_{V_l\backslash\bar{Y}}$$

$$= \left(\lambda\mu(\infty_l)+\sum_{k=1}^{\infty}\frac{A_{k,l}}{(z_1+\theta z_2)^k}\right)(dz_1+\theta dz_2)|_{V_l\backslash\bar{Y}}.$$

This gives (4.6).

(ii) From (4.6), (4.7) we obtain, first, that

$$\bar{\partial}\mu|_{V_l} = e^{-\lambda(z_1+\theta z_2)}\bar{\partial}\big(e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\bar{N}_\nu\big)\big|_{V_l}+O\left(\frac{1}{|\bar{z}_1|^{\nu+1}}\right)$$

$$\text{then } \mu|_{V_l} = M_\nu|_{V_l}+e_{-\lambda,\theta}(z)\bar{N}_\nu|_{V_l}+\tilde{O}\left(\frac{1}{|\bar{z}_1|^{\nu}}\right). \tag{4.8}$$

Comparison of the last equality for different indexes ν and $\nu+1$ implies that $\tilde{O}(\frac{1}{|\bar{z}_1|^\nu}) = O(\frac{1}{|\bar{z}_1|^{\nu+1}})$.

This gives statement of Lemma 4.1. □

Lemma 4.2

(i) *The functions* M_ν *and* N_ν (*conjugated to* $\bar{N}_\nu$) *from the decomposition* (4.8) *have the following properties*:

$$\forall z\in\tilde{V}\backslash Y \quad \exists\lim_{\nu\to\infty}\left(\frac{\partial M_\nu}{\partial(z_1+\theta z_2)}+\lambda M_\nu\right)\stackrel{\text{def}}{=}\frac{\partial M}{\partial(z_1+\theta z_2)}+\lambda M \quad \textit{and}$$

$$\exists\lim_{\nu\to\infty}\left(\frac{\partial N_\nu}{\partial(z_1+\theta z_2)}+\lambda N_\nu\right)\stackrel{\text{def}}{=}\frac{\partial N}{\partial(z_1+\theta z_2)}+\lambda N.$$

(ii) *The functions* $\frac{\partial M}{\partial(z_1+\theta z_2)}+\lambda M$ *and* $\frac{\partial N}{\partial(z_1+\theta z_2)}+\lambda N$ *belong to* $\mathcal{O}(\tilde{V}\backslash Y)$ *and*

$$\begin{aligned}\frac{\partial\mu}{\partial(\bar{z}_1+\bar{\theta}\bar{z}_2)} &= e_{-\lambda,\theta}(z)\left(\frac{\partial\bar{N}}{\partial(\bar{z}_1+\bar{\theta}\bar{z}_2)}+\bar{\lambda}\bar{N}\right),\\ \frac{\partial\mu}{\partial(z_1+\theta z_2)}+\lambda\mu &= \frac{\partial M}{\partial(z_1+\theta z_2)}+\lambda M,\end{aligned} \tag{4.9}$$

$$\frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N \to 0, \quad z_1 \to \infty. \tag{4.10}$$

Proof of Lemma 4.2 Part (i) and the equalities (4.9), (4.10) from Part (ii) follow directly from (4.8).

Properties (4.8), (4.9), (4.10), property $\bar\partial\mu \in L^{p,\tilde p}_{0,1}$ (Proposition 3.1b), and the extension property of bounded holomorphic functions through isolated singularities imply that

$$\frac{\partial M}{\partial(z_1+\theta z_2)} + \lambda M \quad \text{and} \quad \frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N$$

belong to $\mathcal{O}(\tilde V \backslash Y)$.

Lemma 4.2 is proved. □

Lemma 4.3 *Let $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$ be the Faddeev-type function on V associated with the potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ and the divisor $\{a_1,\dots,a_g\}$ on $Y\backslash \bar X$. Then $\forall \lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \ge const(V,\{a_j\},\theta,\sigma)$*

$$\begin{aligned}
&e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)}\bigg|_{V_l\backslash\bar Y} = \sum_{k=1}^{\infty} B_{k,l}(\bar z_1+\bar\theta\bar z_2)^{-k}, \quad \textit{where}\\
&B_{1,l} = -\frac{1}{2\pi i}\int_{\{z\in V_l:|z_1|=r_1\}} e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)}(d\bar z_1+\bar\theta d\bar z_2)\\
&\forall r_1 : Y \subset \{z\in V : |z_1| < r_1\}.
\end{aligned} \tag{4.11}$$

Proof of Lemma 4.3 The estimate of $\partial\mu$ from (3.1b) and the Cauchy theorem, applied to the antiholomorphic function $e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)}|_{V_l\backslash\bar Y}$ imply (4.11). □

Proof of Proposition 4.1 Since ψ, μ are Faddeev-type functions, we have the equations

$$\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu + i\sum_{j=1}^{\infty} C_{j,\theta}(\lambda)\delta(z,a_j),$$

$$dd^c\psi = q\psi + 2\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)}C_{j,\theta}(\lambda)\delta(z,a_j).$$

Put $\psi_{\bar\lambda} = \frac{\partial\psi}{\partial\bar\lambda}$ and $\mu_{\bar\lambda} = \frac{\partial\mu}{\partial\bar\lambda}$.

We obtain

$$dd^c\psi_{\bar\lambda} = q\psi_{\bar\lambda} + 2\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)}\frac{\partial C_{j,\theta}}{\partial\bar\lambda}(\lambda)\delta(z,a_j).$$

From Lemma 4.1, we deduce

$$\begin{aligned}\left.\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)}\right|_{V_l\setminus\bar Y} &= e_{-\lambda,\theta}(z)\frac{B_{1,l}(\lambda)}{\bar z_1+\bar\theta\bar z_2}+O\left(\frac{1}{|z_1|^2}\right), \quad \text{and}\\ \left.\left(\frac{\partial\mu}{\partial(z_1+\theta z_2)}+\lambda\mu\right)\right|_{V_l\setminus\bar Y} &= \lambda\mu(\infty_l)+\frac{A_{1,l}(\lambda)}{z_1+\theta z_2}+O\left(\frac{1}{|z_1|^2}\right).\end{aligned} \tag{4.12}$$

From (4.6), (4.7), and (4.8), we deduce

$$\mu|_{V_l\setminus\bar Y}=\mu(\infty_l,\lambda)+\frac{a_l(\lambda)}{z_1+\theta z_2}+e_{-\lambda,\theta}(z)\frac{b_l(\lambda)}{\bar z_1+\bar\theta\bar z_2}+O\left(\frac{1}{|z_1|^2}\right), \quad z_1\to\infty, \tag{4.13}$$

$$\text{where } \bar\lambda b_l(\lambda)\stackrel{\text{def}}{=}\bar\lambda b_{1,l}(\lambda)=B_{1,l},\ \lambda a_l(\lambda)\stackrel{\text{def}}{=}\lambda a_{1,l}(\lambda)=A_{1,l},\ l=1,\dots,d. \tag{4.14}$$

From (4.13) and (3.1c), we obtain for $l=1,\dots,d$

$$\begin{aligned}\psi|_{V_l\setminus Y} &= e^{\lambda(z_1+\theta z_2)}\mu\\ &= e^{\lambda(z_1+\theta z_2)}\left(\mu(\infty_l,\lambda)+\frac{a_l(\lambda)}{z_1+\theta z_2}+e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)-\lambda(z_1+\theta z_2)}\frac{b_l(\lambda)}{\bar z_1+\bar\theta\bar z_2}\right.\\ &\quad\left.+O\left(\frac{1}{|z_1|^2}\right)\right),\end{aligned}$$

$$\begin{aligned}\psi_{\bar\lambda}|_{V_l\setminus Y} = \left.\frac{\partial\psi}{\partial\bar\lambda}\right|_{V_l\setminus Y} &= e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\left[(\bar z_1+\bar\theta\bar z_2)\frac{b_l(\lambda)+e_{\lambda,\theta}(z)\frac{\partial\mu(\infty_l,\lambda)}{\partial\bar\lambda}}{\bar z_1+\bar\theta\bar z_2}+O\left(\frac{1}{|z_1|}\right)\right]\\ &= e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\left(b_l(\lambda)+e_{\lambda,\theta}(z)\frac{\partial\mu(\infty_l,\lambda)}{\partial\bar\lambda}+O\left(\frac{1}{|z_1|}\right)\right).\end{aligned}$$

For the function $\mu_{\bar\lambda}=e^{-\lambda(z_1+\theta z_2)}\psi_{\bar\lambda}$ we obtain

$$\begin{aligned}&\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu_{\bar\lambda}=\frac{i}{2}q\mu_{\bar\lambda}+i\sum_{j=1}^{g}\frac{\partial C_{j,\theta}}{\partial\bar\lambda}\delta(z,a_j) \quad \text{and}\\ &\mu_{\bar\lambda}=e_{-\lambda,\theta}(z)\left(b_l(\lambda)+e_{\lambda,\theta}(z)\frac{\partial\mu(\infty_l,\lambda)}{\partial\bar\lambda}+O\left(\frac{1}{|z_1|}\right)\right), \quad z\in V_l.\end{aligned}$$

For z_1 large enough, the function $e_{-\lambda,\theta}(z)\bar\mu_{\bar\lambda}\stackrel{\text{def}}{=}\varphi(z,\lambda)$ satisfies the equation $\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\varphi=0$. From this, Lemma 4.1, and the property $\overline{\lim}_{z\to\infty}|\varphi(z,\lambda)|_V<\infty$, we deduce that $\varphi|_{V_l}(z,\lambda)\to const_l(\lambda)\stackrel{\text{def}}{=}\varphi(\infty_l,\lambda)$, if $z\in V_l, z\to\infty, l=1,\dots,d$. So in the relations above we have $e_{\lambda,\theta}(z)\mu_{\bar\lambda}(\infty_l,\lambda)\equiv 0$, $l=1,\dots,d$. The functions $e_{-\lambda,\theta}(z)\bar\mu_{\bar\lambda}$ and μ both satisfy the equation $\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu=\frac{i}{2}q\mu$ on $V\setminus\{a_1,\dots,a_g\}$. Besides, $\overline{\mu|_{V_l}(z,\lambda)}\to\overline{\mu(\infty_l,\lambda)}$ and

$e_{\lambda,\theta}(z)\mu_{\bar{\lambda}}(z,\lambda) \to b_l(\lambda)$, if $z \in V_l$, $z \to \infty$. Applying Proposition 2.1, we obtain

$$e_{\lambda,\theta}(z)\mu_{\bar{\lambda}} = b_l(\lambda)\overline{\mu(z,\lambda)}(\overline{\mu(\infty_l,\lambda)})^{-1}, \quad l = 1,\ldots,d.$$

This implies the equalities (4.1) and (4.2), where

$$b_\theta(\lambda) = \frac{b_l(\lambda)}{\overline{\mu(\infty_l,\lambda)}}, \quad l = 1,\ldots,d. \tag{4.15}$$

The asymptotic formula (4.3) follows from (4.11), (4.14), and (4.15). These formulas and the Cauchy–Green formula imply also the following important expression for $b_\theta(\lambda)$:

$$\begin{aligned}\bar{\lambda} b_\theta(\lambda) d &= -\frac{1}{2\pi i}\int_{z\in bY} e_{\lambda,\theta}(z)\bar{\partial}\mu \\ &= -\frac{1}{2\pi i}\int_{z\in bX} e_{\lambda,\theta}(z)\bar{\partial}\mu + i\sum_{j=1}^{g} C_{j,\theta} e_{\lambda,\theta}(a_j),\end{aligned} \tag{4.16}$$

where

$$\int_{z\in bX} e_{\lambda,\theta}(z)\bar{\partial}\mu = \int_X \frac{1}{2i} e_{\lambda,\theta}(z) q\mu. \tag{4.17}$$

The equality (4.3) follows from (4.16). This equality, together with the estimate of $\{C_j\}$ from Lemma 2.4 and an estimate through integration by parts of $\int_X e_{\lambda,\theta} q\mu$, imply (4.4).

Proposition 4.1 is proved. □

5 Reconstruction of the Function $\psi_\theta|_{bX}$ from the Dirichlet-to-Neumann Data on bX. Proof of Theorem 1.2A

Let X be a domain containing V_0, relatively compact in V with smooth (of class $C^{(2)}$) boundary. Let $\sigma \in C^{(2)}(V)$, $\sigma > 0$ on V, $\sigma = 1$ on $V\backslash X$. Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Let $u \in C(bX)$ and $\tilde{u} \in W^{1,\tilde{p}}(X)$, $\tilde{p} > 2$, be a solution of the Dirichlet problem $d\sigma d^c \tilde{u}|_X = 0$, $\tilde{u}|_{bX} = u$, where $d^c = i(\bar{\partial} - \partial)$, $d = \bar{\partial} + \partial$. Let $\tilde{\psi} = \sqrt{\sigma}\tilde{u}$ and $\psi = \sqrt{\sigma} u$. Then

$$dd^c\tilde{\psi} = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\tilde{\psi} = q\tilde{\psi} \quad \text{on } X,\ \tilde{\psi}\big|_{bX} = \psi. \tag{5.1}$$

Let ψ_0 be a solution of the Dirichlet problem

$$dd^c\psi_0\big|_X = 0, \qquad \psi_0\big|_{bX} = \psi\big|_{bX}.$$

Let

$$\hat{\Phi}\psi = \bar{\partial}\tilde{\psi}\big|_{bX} \quad \text{and} \quad \hat{\Phi}_0\psi = \bar{\partial}\tilde{\psi}_0\big|_{bX}. \tag{5.2}$$

The operator $\psi|_{bX} \mapsto \bar{\partial}\tilde{\psi}|_{bX}$ is equivalent to the Dirichlet-to-Neumann operator $u|_{bX} \mapsto \sigma d^c \tilde{u}|_{bX}$.

Proposition 5.1 *Let $\psi = e^{\lambda(z_1+\theta z_2)}\mu$ be the Faddeev-type function associated with the potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ (see Definition 1.4), the generic divisor $\{a_1, \ldots, a_g\}$ with support in $V\backslash\bar{X}$ and generic $\theta \in \mathbb{C}$. Then $\forall\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \geq const(V, \{a_j\}\theta, \sigma)$, the restriction $\psi|_{bX}$ of ψ on bX can be found from the Dirichlet-to-Neumann operator $\psi|_{bX} \to \sigma d^c\psi|_{bX}$ through the uniquely solvable Fredholm integral equation*

$$\begin{aligned}
&\mu_\theta(z,\lambda)\big|_{bX} + \int_{\zeta\in bX} g_{\lambda,\theta}(z,\zeta)m_{-\lambda}(\hat{\Phi} - \hat{\Phi}_0)m_\lambda\mu_\theta(\zeta,\lambda)\\
&\quad = 1 + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j),\\
&i\sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k}C_{j,\theta}(\lambda) + \int_{z\in bX}(z_1+\theta z_2)^{-k}e^{-\lambda(z_1+\theta z_2)}\overline{\hat{\Phi}\bar{\psi}_\theta} = 0,\\
&\quad k = 2,\ldots,g+1,
\end{aligned} \tag{5.3}$$

where $g_{\lambda,\theta}(z,\xi)$ is the kernel of the operator $R_{\lambda,\theta} \circ \hat{R}_\theta$,

$$\begin{aligned}
&m_{-\lambda}(\hat{\Phi} - \hat{\Phi}_0)m_\lambda\mu_\theta(\zeta,\lambda)\\
&\quad = \int_{w\in bX} e^{-\lambda(\zeta_1+\theta\zeta_2)}(\Phi(\zeta,w) - \Phi_0(\zeta,w))e^{\lambda(w_1+\theta w_2)}\mu_\theta(w,\lambda),
\end{aligned} \tag{5.4}$$

$\Phi(\zeta,w)$, $\Phi_0(\zeta,w)$ are the kernels of the operators $\hat{\Phi}$ and $\hat{\Phi}_0$, $m_{\pm\lambda}$ denote the multiplication operators by $e^{\pm\lambda(z_1+\theta z_2)}$, and for the coordinates of the points $\{a_j\}$ the values $\{a_{j,1} + \theta a_{j,2}\}$ are supposed to be mutually different.

This proposition for the case $V = \mathbb{C}$ is equivalent to the second part of Theorem 1 from [27].

Lemma 5.1 *Let $\psi = e^{\lambda(z_1+\theta z_2)}\mu$ be a Faddeev-type function of Proposition 5.1. Then $\forall z \in V\backslash X$ and $\forall\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$ we have the equalities*

$$\begin{aligned}
\mu_\theta(z,\lambda) = 1 &- \int_{\xi\in bX} g_{\lambda,\theta}(z,\xi)\bar{\partial}\mu_\theta(\xi,\lambda)\\
&- \int_{\xi\in bX}\mu_\theta(z,\xi)e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}(z,\xi)\big)\\
&+ i\sum_{j=1}^{g} C_{j,\theta}(\lambda)g_{j,\theta}(z,a_j)
\end{aligned} \tag{5.5}$$

and

$$-\int_{z\in bX}(z_1+\theta z_2)^{-k}(\partial+\lambda(dz_1+\theta dz_2))\mu_\theta(z,\lambda)$$
$$=\sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k} iC_{j,\theta}(\lambda), \quad k=2,\ldots. \tag{5.6}$$

Proof of Lemma 5.1 The equation

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu=\frac{i}{2}q\mu+i\sum_{j=1}^{g}C_{j,\theta}(\lambda)\delta(z,a_j), \tag{5.7}$$

where $\operatorname{supp} q\subseteq X$ implies that the $(1,0)$-form $f=(\partial+\lambda(dz_1+\theta dz_2))\mu$ is holomorphic on $(V\backslash(X\bigcup_{j=1}^{g}\{a_j\})$ and $Res_{a_j}(\partial+\lambda(dz_1+\theta dz_2))\mu=\frac{iC_j}{2\pi i}$. This and the property (4.12) imply that $\forall\lambda\in\mathbb{C}\backslash E_\theta$ and $\forall k\ge 2$, the form $(z_1+\theta z_2)^{-k}f$ is holomorphic in the neighborhood of $(\tilde V\backslash V)$. By the residue theorem applied to the form $(z_1+\theta z_2)^{-k}f$ on $\tilde V\backslash X$, we obtain

$$\int_{z\in bX}(z_1+\theta z_2)^{-k}f(z,\lambda)=-2\pi i\sum_{j=1}^{g}Res_{a_j}(z_1+\theta z_2)^{-k}f(z,\lambda)$$
$$=-\sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k}(iC_{j,\theta}(\lambda)),$$

$k=2,3,\ldots$. The equalities (5.6) are proved.

Let us now prove (5.5). The differential equation (5.7), where $\mu|_Y\in L^{\tilde p}(Y)$, $\mu|_{V\backslash\bar Y}\in L^\infty(V\backslash\bar Y)$, $\mu(z)\to 1$, $z\to\infty$, $z\in V_1$, is equivalent by Lemma 3.1 to the system of equations

$$\mu(z,\lambda)=1+R_{\lambda,\theta}\circ\hat R_\theta\left(\frac{i}{2}q\mu+i\sum_{j=1}^{g}C_j\delta(\cdot,a_j)\right),\quad z\in V,\quad \text{and} \tag{5.8}$$

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu=0,\quad z\in V\backslash\left(X\bigcup_{j=1}^{g}\{a_j\}\right). \tag{5.9}$$

Besides, we have the equality

$$\int_{\xi\in X}g_{\lambda,\theta}(z,\xi)\frac{i}{2}q(\xi)\mu(\xi,\lambda)=\int_{\xi\in X}g_{\lambda,\theta}(z,\xi)\bar\partial(\partial+\lambda(d\xi_1+\theta d\xi_2))\mu(\xi,\lambda).$$

Using the Green–Riemann formula, we obtain

$$\int_{\xi\in X}e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\partial\bar\partial\psi$$

$$= \int_{\xi\in X} \psi\partial\bar\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\big) + \int_{\xi\in bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\bar\partial\psi + \int_{\xi\in bX} \psi\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\big).$$

For $z\in V\backslash X$ we have $\partial\bar\partial(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi))=0$. Then

$$-\int_{\xi\in X} g_{\lambda,\theta}(z,\xi)\left(\frac{i}{2}q\mu\right) = \int_{\xi\in bX} g_{\lambda,\theta}\bar\partial\mu + \int_{\xi\in bX} e^{\lambda(\xi_1+\theta\xi_2)}\mu\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}(z,\xi)\big). \tag{5.10}$$

From (5.8) and (5.10), we deduce statement (5.5) of Lemma 5.1. □

Proof of Proposition 5.1 Let $\psi_0 : \bar\partial\partial\psi_0|_X = 0$ and $\psi_0|_{bX} = \psi$. By the Green–Riemann formula, $\forall z\in V\backslash X$ we have

$$\int_{\xi\in bX} \psi\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\big) + \int_{\xi\in bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\bar\partial\psi_0 = 0. \tag{5.11}$$

Formulas (5.11), (5.5), and (5.6) imply

$$\psi(z,\lambda) = e^{\lambda(z_1+\theta z_2)} - \int_{\xi\in bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)(\bar\partial\psi(\xi)-\bar\partial\psi_0(\xi)) + i\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)}C_j g_{\lambda,\theta}(z,a_j). \tag{5.12}$$

Formulas (5.12) and (5.6) are equivalent to (5.3). Integral equation (5.3) is the Fredholm equation in $C(bX)$, because the operator $(\hat\Phi-\hat\Phi_0)$ is a compact operator in $C(bX)$. The existence for $\lambda\in\mathbb{C}\backslash E_\theta : |\lambda|\ge const(V,\{a_j\},\theta,\sigma)$ of a unique Faddeev-type function $\psi = e^{\lambda(z_1+\theta z_2)}\mu$, associated with q and divisor $\{a_1,\dots,a_g\}$, implies the existence for such λ of a solution of (5.3) with residue data $iC_j = Res_{a_j}(\partial+\lambda(dz_1+\theta dz_2))\mu$, $j=1,\dots,g$. Let us prove uniqueness for $\lambda\in\mathbb{C}\backslash E_\theta$: $|\lambda|\ge const(V,\{a_j\},\theta,\sigma)$ of the solution (5.3) in $C(bX)$ with residue data $\{C_j\}$. Suppose that $\mu\in C(bX)$ solves (5.3), (5.6). Consider this μ as Dirichlet data for the equation $\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu$ on X, the solution of which well defines μ on $\bar X$.

Let us also define μ on $V\backslash\bar X$ by (5.5). The function $\mu(z,\lambda)$ defined in such a way on V belongs to $C(V\backslash\bigcup_{j=1}^{g}\{a_j\})$.

Let us show that μ satisfies (5.7). By the Sohotsky–Plemelj jump formula, $\forall z^* \in bX$ we have

$$\frac{i}{2}\mu(z^*) = \lim_{\substack{z\to z^*\\ z\in X}} \left(\int_{bX} g_{\lambda,\theta}\bar\partial\mu + \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\left(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\right)\right) - \lim_{\substack{z\to z^*\\ z\in V\setminus X}} \left(\int_{bX} g_{\lambda,\theta}\bar\partial\mu + \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\left(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\right)\right). \quad (5.13)$$

From (5.5) and (5.13), we deduce the equality

$$\mu - \frac{i}{2}\mu = 1 - \int_{\xi\in bX} g_{\lambda,\theta}\bar\partial\mu - \int_{\xi\in bX} \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\left(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\right) + i\sum_{j=1}^{g} C_j g_{\lambda,\theta}(z,a_j), \quad z\in X. \quad (5.14)$$

By the Green–Riemann formula, we have also

$$\begin{aligned} &-\int_{bX} g_{\lambda,\theta}\bar\partial\mu - \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\left(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\right) + i\sum_{j=1}^{g} C_{j,\theta} g_{\lambda,\theta}(z,a_j)\\ &= -\int_X \mu(\bar\partial(\partial + \lambda(d\xi_1+\theta d\xi_2))g_{\lambda,\theta} + \int_X g_{\lambda,\theta}\bar\partial(\partial+\lambda(d\xi_1+\theta d\xi_2))\mu\\ &\quad + i\sum_{j=1}^{g} C_{j,\theta} g_{\lambda,\theta}(z,a_j)\\ &= \begin{cases} \frac{\mu}{2i} + \int_{\xi\in X} g_{\lambda,\theta}\bar\partial(\partial+\lambda(d\xi_1+\theta d\xi_2))\mu + i\sum_{j=1}^{g} C_{j,\theta} g_{\lambda,\theta}(z,a_j),\\ \quad z\in X,\\ \int_{\xi\in X} g_{\lambda,\theta}\bar\partial(\partial+\lambda(d\xi_1+\theta d\xi_2))\mu + i\sum_{j=1}^{g} C_{j,\theta} g_{\lambda,\theta}(z,a_j),\\ \quad z\in V\setminus(X\bigcup_{j=1}^{g}\{a_j\}). \end{cases} \end{aligned} \quad (5.15)$$

The equalities (5.5), (5.6), (5.14), and (5.15) imply (3.3) and

$$\begin{aligned} \mu(z) &= 1 + \int_{\xi\in V} g_{\lambda,\theta}\bar\partial(\partial+\lambda(d\xi_1+\theta d\xi_2))\mu + i\sum_{j=1}^{g} C_{j,\theta} g_{\lambda,\theta}(z,a_j)\\ &= 1 + R_{\lambda,\theta}\circ\hat R_\theta\left(\frac{i}{2}q\mu + i\sum_{j=1}^{g} C_{j,\theta}\delta(\cdot,a_j)\right), \quad z\in V. \end{aligned}$$

By Lemma 3.1, the function $\mu_\theta(z,\lambda)$ is the Faddeev-type function associated with q and the divisor $\{a_1,\dots,a_g\}$. The uniqueness of the solution of (5.3) in $C(bX)$ with residue data $\{C_{j,\theta}\}$ follows now from the uniqueness of the Faddeev-type function for $\lambda\in\mathbb{C}\setminus E_\theta : |\lambda| \ge const(V,\{a_j\},\theta,\sigma)$. □

6 Reconstruction of the Conductivity Function from the Dirichlet-to-Neumann Data. Proof of Theorem 1.2B

We will obtain here exact formulas for the reconstruction of the conductivity function $\sigma \in C^{(3)}(V)$, $\sigma > 0$, $\sigma \equiv 1$ on $V \backslash X$, from the Dirichlet-to-Neumann data

$$\psi_\theta\big|_{bX} \to \bar\partial\psi_\theta\big|_{bX}$$

for Faddeev-type functions $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$, $\theta \in \mathbb{C}\backslash\{\theta_1,\theta_d\}$, $\lambda \in \mathbb{C}\backslash E_\theta : |\lambda| \geq const(V,\{a_j\},\theta,\sigma)$, $\{a_1,\dots,a_g\} \subset Y\backslash X$.

For simplicity of presentation we consider in detail only the case of regular algebraic curves in $\mathbb{C}^2 \subset \mathbb{C}P^2$.

Let $\tilde V = \{\tilde z = (\tilde z_0 : \tilde z_1 : \tilde z_2) \in \mathbb{C}P^2 : \tilde P(\tilde z) = 0\}$, where $\tilde P(\tilde z)$ is a homogeneous polynomial of degree N. Let $\mathbb{C}P^1_\infty = \{\tilde z : \mathbb{C}P^2 : \tilde z_0 = 0\}$. Put

$$\begin{aligned} &\mathbb{C}^2 = \{\tilde z \in \mathbb{C}P^2 : \tilde z_0 \neq 0\}, \qquad z_1 = \frac{\tilde z_1}{\tilde z_0}, \qquad z_2 = \frac{\tilde z_2}{\tilde z_0}, \\ &P(z) = \tilde P(1, z_1, z_2), \qquad V = \{z \in \mathbb{C}^2 : P(z) = 0\} = \tilde V \cap \mathbb{C}^2. \end{aligned} \tag{6.1}$$

Without loss of generality we suppose that $\tilde V$ is a (regular) curve of degree $N \geq 2$ with the following property:

$$\begin{aligned} &\tilde V \cap \mathbb{C}P^1_\infty = \{\beta_1,\dots,\beta_d\}, \quad \text{where } \beta_1,\dots,\beta_d \text{ are different points of } \mathbb{C}P^1_\infty, \\ &\beta_l = (0 : \beta_l^1 : \beta_l^2), \quad \frac{\beta_l^2}{\beta_l^1} \in \mathbb{C}, \quad l = 1,\dots,N, \\ &\frac{\partial P}{\partial z_2}(z) \neq 0, \quad \text{if } z \in V : |z_1| \geq r_0 = const(V). \end{aligned} \tag{6.2}$$

For $\theta \in \mathbb{C}$ let $\{w_m\}$ be points of V, where $(dz_1 + \theta dz_2)|_V(w_m) = 0$. Then for all $\theta \in \mathbb{C}$, except for a finite number of θ, the following relations are valid:

$$\begin{aligned} &\theta = \frac{\partial P}{\partial z_2}(w_m) \Big/ \frac{\partial P}{\partial z_1}(w_m), \qquad \frac{\partial P}{\partial z_1}(w_m) \neq 0, \\ &\left[\frac{\partial^2 P}{\partial z_1^2}\left(\frac{\partial P}{\partial z_2}\right)^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}\left(\frac{\partial P}{\partial z_2}\right)\left(\frac{\partial P}{\partial z_1}\right) + \frac{\partial^2 P}{\partial z_2^2}\left(\frac{\partial P}{\partial z_1}\right)^2\right](w_m) \neq 0. \end{aligned}$$

Without loss of generality it is sufficient to give proof under the condition that $\theta = 0$, i.e., for points $w_m = (w_{m,1}, w_{m,2}) \in V$ such that

$$\frac{\partial P}{\partial z_1}(w_m) \neq 0, \qquad \frac{\partial P}{\partial z_2}(w_m) = 0, \qquad \frac{\partial^2 P}{\partial z_2^2}(w_m) \neq 0 \tag{6.3}$$

and also such that $\forall m$ the line $\{z \in \mathbb{C}^2 : z_1 = w_{m,1}\}$ has tangency with X only in the single point w_m, $m = 1,\dots,M$. By the Hurwitz–Riemann formula $M = N(N-1)$.

In the neighborhood of the point $w_m \in V$, the curve V can be represented in the form

$$V = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = w_{m,1}\} + \left(\frac{\partial P}{\partial z_1}(w_m)\right)^{-1}\left[-\frac{1}{2}\frac{\partial^2 P}{\partial z_2^2}(w_m)(z_2 - w_{m,2})^2 + O((z_2 - w_{m,2})^3)\right]. \quad (6.4)$$

The reconstruction formula for $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(w_m)$, $m = 1, \dots, M$, will be obtained here by the stationary phase method, using formula (4.17).

Let μ be a Faddeev-type function (3.1) with properties (3.1a)–(3.1c), and with $\theta = 0$.

In this section we will write $\hat{R}_0$, $R_{\lambda,0}$, $e_{\lambda,0}$, μ_0, ψ_0, Δ_0, E_0, and $C_{j,0}$ as $\hat{R}$, R_λ, e_λ, μ, ψ, Δ, E, and C_j, respectively.

Let

$$f_0 = F_0 dz_1 = \frac{i}{2}\hat{R}(q\mu), \qquad f_1 = F_1 dz_1 = i\sum_{j=1}^{g} C_j(\lambda)\hat{R}(\delta(\cdot, a_j)),$$

where $\mu = \mu(z, \lambda)$, $z \in V$, $\lambda \in \mathbb{C}\backslash E : |\lambda| \geq const(V, \{a_j\}, \sigma)$.

Lemma 6.1 *For $u_0 = R_\lambda f_0$ the following estimate holds*:

$$\left\| u_0(\cdot, \lambda) - \frac{F_0(\cdot, \lambda)}{\lambda} \right\|_{L^{9/4}(X)} \leq \frac{const(V, \tilde{p})}{|\lambda|^{7/5}} \| f_0(\cdot, \lambda)\|_{\tilde{W}^{2,\tilde{p}}_{1,0}(V)}.$$

Proof of Lemma 6.1 By Lemma 2.1 and Proposition 2 from [18], we have $f_0 \in \tilde{W}^{2,\tilde{p}}_{1,0}(V)$, $F_0 \in \tilde{W}^{1,p}(V)$. Using the equality $\partial_z e_\lambda(z) = \lambda e_\lambda(z) dz_1$ and integration by parts, formula $u_0 = R_\lambda f_0 = e_{-\lambda}(z)\overline{R(\overline{e_\lambda f_0})}$ can be transformed into the following:

$$\begin{aligned} u_0(z) &= e_{-\lambda}(z)\overline{R_1(\overline{e_\lambda f_0})} + e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda f_0})} \\ &= -\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int_V \frac{e_\lambda(\xi)\partial F_0 \wedge \overline{d\xi_1} \det[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi) \cdot |\xi - z|^2} \\ &\quad - \frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int_V e_\lambda(\xi) F_0 \partial\left(\frac{\det[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z] \wedge d\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi) \cdot |\xi - z|^2}\right) + e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda f_0})}, \end{aligned} \quad (6.5)$$

where R_1 and R_0 are the operators defined in Sect. 1 (see Remark 1.1). From (6.5), using Corollary 1.2 from [18], we deduce

$$\lambda u_0 - F_0 = -e_{-\lambda}(z)\overline{R_1(\overline{e_\lambda(\xi)\partial F_0})} - e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda(\xi)\partial F_0})} \stackrel{\text{def}}{=} J_1(z) + J_0(z). \quad (6.6)$$

We will estimate further only the term $J_1(z)$. The estimate for $J_0(z)$ is similar.

For $J_1(z)$ we have $J_1(z) = J_1^+(z) + J_1^-(z)$, where

$$J_1^{\pm}(z) = \frac{e_{-\lambda}(z)}{2\pi i} \int_V \frac{e_\lambda(\xi)\chi_\rho^{\pm}(\xi)\partial F_0(\xi) \wedge \overline{d\xi_2}\, \det[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_1}(\xi) \cdot |\xi - z|^2},$$

$$\chi_\rho^{\pm} \text{ are smooth functions such that } \chi_\rho^+ + \chi_\rho^- \equiv 1,$$

$$\chi_\rho^+ = 1, \text{ if } \left|\frac{d\xi_1}{d\xi_2}\right| \le \rho, \text{ supp}\, \chi_\rho^+ \subset \left\{\xi : \left|\frac{d\xi_1}{d\xi_2}\right| \le 2\rho\right\} \quad \text{and}$$

$$|d\chi_\rho^{\pm}| = O\left(\frac{1}{\rho}\right). \tag{6.7}$$

Let $B_0 = \{z \in V : d\xi_1|_V(z) = 0\}$. The property $\bar{\partial} F_0 = dz_1 \rfloor \frac{i}{2} q\mu$ implies the estimate $\bar{\partial} F_0 = O(\frac{1}{\mathrm{dist}(z, B_0)}) dz_2$. From this, the formula for $J_1^+(z)$, and Lemma 3.1 of [18], we obtain an estimate for

$$J_1^+ : \|J_1^+\|_{L^{9/4}(X)} = O(\rho^{2/3}) \|f_0\|_{\tilde{W}^{2,\tilde{p}}_{1,0}(V)}. \tag{6.8}$$

In order to estimate $J_1^-(z)$, we integrate by parts in the formula for J_1^-, using $\partial_z e_\lambda(z) = \lambda e_\lambda(z) dz_1$. Then the inequalities

$$|\bar{\partial} F_0(z)| = O\left(\frac{1}{\mathrm{dist}(z, B_0)}\right), \quad |\bar{\partial}\partial F_0(z)| = O\left(\frac{1}{\mathrm{dist}(z, B_0)}\right)^2, \quad z \in X \backslash B_0$$

and the inequality

$$\left|\int_{\rho \le |\xi_2| \le 1} \frac{d\xi_2 \wedge d\bar{\xi}_2}{|\xi_2|^2 (\bar{\xi}_2 - \bar{z}_2)}\right| + \left|\int_{\rho \le |\xi_2| \le 1} \frac{d\xi_2 \wedge d\bar{\xi}_2}{|\xi_2| (\bar{\xi}_2 - \bar{z}_2)^2}\right| = O\left(\frac{1}{\rho}\right)$$

imply the estimate

$$\|J_1^-\|_{L^\infty(X)} = O\left(\frac{1}{|\lambda|\rho}\right) \|f_0\|_{\tilde{W}^{2,\tilde{p}}_{1,0}(V)}. \tag{6.9}$$

From (6.6), (6.8), and (6.9) with $\rho = |\lambda|^{-3/5}$, we obtain the statement of Lemma 6.1. □

Lemma 6.2 *Let* $q \in C^{(1)}_{1,1}(V)$, $\mathrm{supp}\, q \subseteq X$, $f_0 = \frac{i}{2} \hat{R}(q\mu)$, $u_0 = R_\lambda f_0$. *Then the following asymptotic estimate is valid*:

$$\left|\int_X e_\lambda(z) q(z) u_0(z)\right| = o\left(\frac{1}{|\lambda|}\right), \quad \textit{for } \lambda \in \mathbb{C}: \ |\lambda| \ge const(V, \{a_j\}, \sigma),$$

$$|\Delta(\lambda)(1 + |\lambda|)^g| \ge \delta > 0, \quad \textit{for some sufficiently small } \delta.$$

Proof of Lemma 6.2 From Lemma 6.1, using the estimate of μ from (3.1b), we obtain an asymptotic relation in the space $L^{\tilde{p}}(V)$, $2<\tilde{p}<9/4$:

$$\begin{aligned} u_0(z,\lambda) &= \frac{F_0(z,\lambda)}{\lambda} + O\left(\frac{1}{|\lambda|^{7/5}}\right) \\ &= \frac{dz_1 \rfloor \frac{i}{2}\hat{R}(q)}{\lambda} + O\left(\frac{1}{|\lambda|^{7/5}}\right) \quad \text{if } |\Delta_\theta(\lambda)(1+|\lambda|)^g| \ge \delta > 0. \end{aligned}$$

Putting this relation into $\int_X e_\lambda(z)q(z)u_0(z)$, we obtain

$$\int_X e_\lambda(z)q(z)u_0(z) = \frac{i}{2\lambda}\int_X e_\lambda(z)q(z)(dz_1 \rfloor \hat{R}(q)) + O\left(\frac{1}{|\lambda|^{7/5}}\right).$$

By a Riemann–Lebesgue-type theorem,

$$\int_X e_\lambda(z)q(z)(dz_1 \rfloor \hat{R}(q)) = o(1) \quad \text{if } \lambda \to \infty, \qquad |\Delta(\lambda)|(1+|\lambda|)^g \ge \delta > 0.$$

This implies the statement of Lemma 6.2. □

Lemma 6.3 *Let* $q \in C^{(1)}_{1,1}(V)$, $\operatorname{supp} q \subset X$. *Let* $w_1,\dots,w_M$ *be the points where* $dz_1|_V(w_m)=0$. *Then the following consequence of the stationary phase method is valid*:

$$\int_V e^{i\tau(z_1+\bar{z}_1)}q(z) = \sum_m (1+o(1))\sum_{m=1}^{M} -\frac{\pi}{r}\frac{|\frac{\partial P}{\partial z_1}(w_m)|Q_2(w_m)}{|\frac{\partial^2 P}{\partial z_2^2}(w_m)|}e^{i\tau(w_{m,1}+\bar{w}_{m,1})}, \tag{6.10}$$

where $Q_2(w_m) = \frac{q}{2i dz_2\wedge d\bar{z}_2}(w_m)$.

Proof of Lemma 6.3 See [12], Theorem 2.1, [23]. □

Lemma 6.4 *Let* $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} \in C^{(1)}_{1,1}(X)$, $\operatorname{supp} q \subset X$, $f_1 = i\sum_{j=1}^{g} C_j(\lambda)\hat{R}(\delta(\cdot,a_j))$, $u_1 = R_\lambda f_1$. *Then the following asymptotic estimate is valid*

$$\left|\int_X e_\lambda(z)q(z)u_1(z)\right| = O\left(\frac{1}{|\lambda|^{3/2-\varepsilon}}\right), \quad \textit{for } \lambda\in\mathbb{C}: |\lambda| \ge const(V,\{a_j\},\sigma,\varepsilon),$$
$$|\Delta(\lambda)(1+|\lambda|)^g| \ge \delta > 0, \quad \delta \textit{ for some sufficiently small } \delta.$$

Proof of Lemma 6.4 Using that $\{a_1,\dots,a_g\}$ is a generic divisor, from the estimate (3.7) (Lemma 3.3) we obtain the inequality

$$\begin{aligned} \sup_{j,\lambda} |C_j(\lambda)| &\le const(V,\{a_j\}) \\ &\quad \times \sup_k \left| \int_X e_\lambda(z)\left(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma}\wedge\partial\ln\sqrt{\sigma}\right)\mu\frac{\bar{\omega}_k}{d\bar{z}_1}(z)\right|. \end{aligned}$$

Let $\varepsilon > 0$ be small enough and $B_\varepsilon = \{z \in X : |\frac{dz_1}{dz_2}| < \varepsilon\}$. Then

$$\left|\frac{\bar{\omega}_k(z)}{d\bar{z}_1}\right|_X = O\left(\sum_{m=1}^{M} \frac{1}{|z_2 - w_{m,2}|}\right), \quad z \in X.$$

Let $\chi_\rho^\pm \in C^{(1)}(X)$ be functions with the properties (6.7). Using that $\sigma \in C^{(3)}(X)$, $\mu \in \tilde{W}^{1,\tilde{p}}(X)$, $\partial_z e_\lambda(z) = \lambda e_\lambda(z) dz_1$, from integration by parts we obtain

$$\left|\int_X \chi_\rho^-(z) e_\lambda(z)\left(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma} \wedge \partial\ln\sqrt{\sigma}\right)\mu(z,\lambda)\frac{\bar{\omega}_k}{d\bar{z}_1}(z)\right| \\ \leq \frac{const(V,\sigma)}{\rho\lambda}.$$

We have also directly

$$\left|\int_X \chi_\rho^+(z) e_\lambda(z)\left(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma} \wedge \partial\ln\sqrt{\sigma}\right)\mu(z,\lambda)\frac{\bar{\omega}_k}{d\bar{z}_1}(z)\right| \\ \leq const(V,\sigma)\rho.$$

These estimates with $\rho = \frac{1}{\sqrt{|\lambda|}}$ and estimates for the Faddeev-type Green function $|R_\lambda \circ \hat{R}(\delta(\cdot, a_j)| = O(\frac{1}{|\lambda|^{1-\varepsilon}})$ from Theorem 4 of [18] imply the statement of Lemma 6.4. □

Proposition 6.1 *Under the conditions* (6.1)–(6.4), *for* $\lambda = i\tau$, $\tau \in \mathbb{R} : |\tau^g \Delta(i\tau)| \geq \delta > 0$, δ *is small enough, the following asymptotic equality is valid*:

$$\begin{aligned}&\int_{z\in bX} e_{i\tau}(z)\bar{\partial}_z\mu(z,i\tau) \\ &\quad= \int_{z\in X} e_{i\tau}(z)\frac{q\mu}{2i} \\ &\quad= \frac{1+o(1)}{\tau}\sum_{m=1}^{M}\frac{\pi i}{2}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\bigg|_V(w_m)e^{i\tau(w_{m,1}+\bar{w}_{m,1})}\left|\frac{\partial^2 P}{\partial z_2^2}(w_m)\right|^{-1}\frac{\partial P}{\partial z_1}(w_m),\end{aligned} \tag{6.11}$$

$o(1) \to 0$, *if* $|\tau| \to \infty$.

*Proof of Proposition 6.1 and Theorem 1.2*B From Lemma 3.1 we have the equality

$$\mu = 1 + R_\lambda \circ \hat{R}\left(\frac{i}{2}q\mu\right) + R_\lambda \circ \hat{R}\left(i\sum_{j=1}^{g} C_j\delta(z,a_j)\right) = 1 + u_0 + u_1. \tag{6.12}$$

Let $\delta > 0$ be small enough. The estimates of Lemmas 6.2, 6.4 and (6.12) give the asymptotic equality

$$\mu = 1 + o\left(\frac{1}{\lambda}\right) \tag{6.13}$$

under the conditions $\lambda \in \mathbb{C} : |\lambda| \geq const(V, \{a_j\}, \sigma)$, $|\Delta(\lambda)(1 + |\lambda|)^g| \geq \delta > 0$.

By Proposition 1.1, $\forall \varepsilon > 0$ we have the inequality

$$\varliminf_{\lambda\to\infty} |\lambda^g \Delta(\lambda)|_\varepsilon = \delta(\varepsilon) > 0, \quad \text{where } |\lambda^g \Delta(\lambda)|_\varepsilon = \sup_{\{\lambda' : |\lambda'-\lambda| \leq \varepsilon\}} |(\lambda')^g \Delta(\lambda'|.$$

So for any $\varepsilon > 0$ and any positive $\delta < \delta(\varepsilon)$ there exists r such that the set $\{\lambda \in \mathbb{C} : |\Delta(\lambda)(1 + |\lambda|)^g| \geq \delta > 0\}$ intersects any disc $\{\lambda' : |\lambda - \lambda'| < \varepsilon\}$, with $|\lambda| \geq r$. This property, Lemma 6.3, and the property (6.13) imply Proposition 6.1.

Theorem 1.2B follows from Proposition 1.1. Indeed, the stationary phase method permits the differentiation of (6.11) with respect to τ, keeping (in our case) terms of order $\frac{1}{\tau}$. Differentiation of the right-hand side of (6.11) gives for $\theta = 0$ the right-hand side of (1.12).

Theorem 1.2B is proved. □

Remark 6.1 To obtain a version of Proposition 6.1 with arbitrary generic θ from Proposition 6.1 with $\theta = 0$, it is sufficient to change the coordinate system: $\tilde{z}_1 = z_1 + \theta z_2$, $\tilde{z}_2 = z_2$.

Remark 6.2 Proposition 6.1 can be reformulated also as a formula for the reconstruction of a conductivity function from scattering data $b_\theta(i\tau)$ and $C_{j,\theta}(i\tau)$. Indeed, by formula (4.16), we have

$$\int_{bX} e_{i\tau,\theta}(z)\bar{\partial}\mu(z, i\tau) = -2\pi\left[\tau b_\theta(i\tau)d + \sum_{j=1}^{g} C_{j,\tau}(i\tau)e_{i\tau,\theta}(a_j)\right],$$

where d is defined in Sect. 1.

7 Proof of Proposition 1.1

For simplicity of presentation we give proof only for the case when V is an algebraic curve in $\mathbb{C}^2$. Proposition 1.1 will be obtained here as a corollary of the following statement.

Proposition 7.1 *Let* $\theta \in \mathbb{C}\backslash\{\theta_1, \ldots, \theta_N\}$, $\delta = \delta(\theta) = \inf_l |\theta - \theta_l|$, $V_0 = \{z \in V : |z_1| \leq r_0(\delta)\}$, $r_0(\delta) = \frac{const(V)}{\sqrt{\delta}}$. *Let* $\{b_m\}$ *be the points of* V *where* $(dz_1 + \theta dz_2)|_V(b_m) = 0$, $m = 1, \ldots, M$, *and let* $\{a_1, \ldots, a_g\}$ *be the points of a generic divisor in* $V\backslash\bar{V}_0$. *Then* $\forall j, k = 1, \ldots, g$ *and for* $\lambda \in \mathbb{C} : |\lambda|^g|\Delta_\theta(\lambda)| \geq \delta > 0$, *for* δ

small enough, the following asymptotic equality is valid:

$$\int_V \hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{\lambda,\theta}(\xi)$$
$$= -\frac{1}{\bar{\lambda}} e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{\xi}_1 + \bar{\theta} d\bar{\xi}_2}(a_j)$$
$$- \frac{\pi}{|\lambda|} \sum_{m=1}^{M} \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar{\lambda}(\bar{b}_{m,1} + \bar{\theta}\bar{b}_{m,2})] K_{j,k}(b_m, a_j)$$
$$+ O\left(\frac{1}{|\lambda|^2}\right),$$

where

$$K_{j,k}(b_m, a_j) = \frac{|\frac{\partial P}{\partial z_1}(b_m)|^3 \hat{R}_\theta(\delta(b_m, a_j)) \wedge \bar{\omega}_k(b_m)(1 + |\theta|^2)}{|\frac{\partial^2 P}{\partial z_1^2}(\frac{\partial P}{\partial z_2})^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}\frac{\partial P}{\partial z_2}\frac{\partial P}{\partial z_1} + \frac{\partial^2 P}{\partial z_2^2}(\frac{\partial P}{\partial z_1})^2 |dd^c|z|^2|_V(b_m)}. \tag{7.1}$$

Lemma 7.1 *Let* $V \backslash V_0 = \bigcup_{l=1}^{g} V_l$ *be a curve with the properties* (i)–(iv) *of Sect.* 1. *Then* $\forall \theta \neq \theta_1, \ldots, \theta_d$, *any point* w, *where* $(dz_1 + \theta dz_2)|_V(w) = 0$, *belongs to* $V_0 = \{z \in V : |z_1| \leq r_0(\delta)\}$, *where* $r_0(\delta) = const(V)/\sqrt{\delta}$, $\delta = \min_l |\theta - \theta_l|$.

Proof of Lemma 7.1 For any point $w \in V \backslash V_0$, where $(dz_1 + \theta dz_2)|_V(w) = 0$, the definition $\theta_l = -\frac{1}{\gamma_l}$, $l = 1, \ldots, d$, and property (iii) of Sect. 1 imply for some $l = l(w)$ the equality

$$0 = (dz_1 + \theta dz_2)\big|_V(w) = \left[1 + \theta\left(\gamma_l + \frac{\gamma_l^0}{w_1^2} + O\left(\frac{1}{w_1^3}\right)\right)\right] dz_1$$
$$= \left[1 + \theta\gamma_l + O\left(\frac{\theta}{w_1^2}\right)\right] dz_1 = \gamma_l\left[(\theta - \theta_l) + O\left(\frac{\theta}{w_1^2}\right)\right] dz_1.$$

This gives the equality $\theta(1 + O(\frac{1}{w_1^2})) = \theta_l$. This equality, together with inequality $|\theta - \theta_l| \geq \delta$, implies the inequality $|w_1| \leq \frac{const(V)}{\sqrt{\delta}} = r_0(\delta)$.

Lemma 7.1 is proved. □

Further let

$$A_{\varepsilon,j} = \{z \in V : |z - a_j| \leq \varepsilon\}, \qquad A_\varepsilon = \bigcup_{j=1}^{g} A_{\varepsilon,j},$$

$$B_{\varepsilon,m} = \{z \in V : |z - b_m| \leq \varepsilon\}, \qquad B_\varepsilon = \bigcup_{m=1}^{M} B_{\varepsilon,m}.$$

Lemma 7.2 *Let $r_0(\delta)$, $\delta=\delta(\theta)$ be as in Lemma 7.1. Let χ^{A_ε}, χ^{B_ε} be smooth functions with the following properties*:

$$\chi^{A_\varepsilon}\big|_{A_\varepsilon}=1,\qquad \chi^{A_\varepsilon}\big|_{V\setminus A_{2\varepsilon}}=0,\qquad |d\chi^{A_\varepsilon}|=O\left(\frac{1}{\varepsilon}\right),$$

$$\chi^{B_\varepsilon}\big|_{B_\varepsilon}=1,\qquad \chi^{B_\varepsilon}\big|_{V\setminus B_{2\varepsilon}}=0,\qquad |d\chi^{B_\varepsilon}|=O\left(\frac{1}{\varepsilon}\right).$$

Then for any $\varepsilon>0$ small enough we have $B_{2\varepsilon}\cap A_{2\varepsilon}=\{\emptyset\}$ and $\forall j,k=1,\dots,g$

$$\Delta^{j,k}_{\theta,\varepsilon}(\lambda)\overset{\text{def}}{=}\int_{\xi\in V}(1-\chi^{A_\varepsilon}-\chi^{B_\varepsilon})\hat{R}(\delta(\xi,a_j))\wedge\bar{\omega}_k(\xi)e_{\lambda,\theta}(\xi)=O\left(\frac{1}{\lambda^2}\right).$$

Proof of Lemma 7.2 By Lemma 7.1, any point b_m, where $(dz_1+\theta dz_2)|_V(b_m)=0$, belongs to $\{z\in V:|z_1|\le r_0\}$. Under the conditions of Lemma 7.2, any a_j from $\{a_1,\dots,a_g\}$ belongs to $\{z\in V:|z_1|>r_0(\delta)\}$, $\delta=\delta(\theta)$.

Then $B_{2\varepsilon}\cap A_{2\varepsilon}=\{\emptyset\}$, if ε is small enough. From the definition of $\Delta^{j,k}_{\theta,\varepsilon}$ and the equality $\bar\partial\hat{R}_\theta(\delta(\varepsilon,a_j))|_{V\setminus\{a_j\}}=0$ we obtain

$$\begin{aligned}\Delta^{j,k}_{\theta,\varepsilon}(\lambda)&=\frac{1}{\bar\lambda}\int_V(1-\chi^{A_\varepsilon}-\chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi,a_j))\wedge\frac{\bar\omega_k}{d\bar\xi_1+\bar\theta d\bar\xi_2}\bar\partial e_{\lambda,\theta}(\xi)\\&=-\frac{1}{\bar\lambda}\int_V(1-\chi^{A_\varepsilon}-\chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi,a_j))\wedge\bar\partial\left(\frac{\bar\omega_k}{d\bar\xi_1+\bar\theta d\bar\xi_2}\right)e_{\lambda,\theta}(\xi)\\&\quad-\frac{1}{\bar\lambda}\int_V\bar\partial(\chi^{A_\varepsilon}+\chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi,a_j))\wedge\frac{\bar\omega_k(\xi)}{d\bar\xi_1+\bar\theta d\bar\xi_2}e_{\lambda,\theta}(\xi)\\&\quad+\frac{1}{\bar\lambda}\lim_{r\to\infty}\int_{\{\xi\in V:|\xi_1|=r\}}\hat{R}_\theta(\delta(\xi,a_j))\wedge\frac{\bar\omega_k}{d\bar\xi_1+\bar\theta d\bar\xi_2}e_{\lambda,\theta}(\xi).\end{aligned}\tag{7.2}$$

From the asymptotic estimates $|\hat{R}_\theta(\delta(\xi,a_j))|=O(|d\xi_1|)$ and $|\bar\omega_k|=O(\frac{d\bar\xi_1}{\bar\xi_1^2})$, $\xi_1\to\infty$, and the property $\inf_l|\theta-\theta_l|>0$, we obtain the vanishing of the last term of the right-hand side of (7.2).

The property $(d\xi_1+\theta d\xi_2)|_{V\setminus B_\varepsilon}\neq 0$ lets us integrate other terms of the right-hand side of (7.2) by parts once more, and to obtain the statement of Lemma 7.2. □

Lemma 7.3 *For any $k,j\in\{1,\dots,g\}$, $\theta\notin\{\theta_1,\dots,\theta_d\}$ and any $\varepsilon>0$, we have the asymptotic equality*

$$\int_V\chi^{A_{\varepsilon,j}}\hat{R}_\theta(\delta(\xi,a_j))\wedge\bar\omega_k(\xi)e_{\lambda,\theta}(\xi)=-\frac{1}{\bar\lambda}e_{\lambda,\theta}(a_j)\frac{\bar\omega_k}{d\bar\xi_1+\bar\theta d\bar\xi_2}(a_j)+\left(\frac{1}{\lambda^2}\right).$$

Proof of Lemma 7.3 Integration by parts of the left-hand side, the equality $\bar\partial\hat{R}(\delta(\xi,a_j))=\delta(\xi,a_j)$, and the inequality $(d\xi_1+\theta d\xi_2)|_{A_{\varepsilon,j}}\neq 0$ imply the statement of Lemma 7.3. □

Lemma 7.4 *Under the conditions of Lemmas 7.1, 7.2, $\forall\delta > 0$, $\theta : \inf_l |\theta - \theta_l| > \delta$, $\forall j, k = 1, \dots, g$,*

$$\int_V \chi^{B_\varepsilon} \hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{i\tau,\theta}(\xi)$$
$$= -\frac{\pi}{|\lambda|} \sum_{m=1}^{M} \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar{\lambda}(\bar{b}_{m,1} + \bar{\theta}\bar{b}_{m,2})] K_{j,k}(b_m, a_j) + O\left(\frac{1}{|\lambda|^2}\right),$$

where $\theta = \theta(b_m)$, $m = 1, \dots, M$, and $K_{j,k}(b_m, a_j)$ are defined by (7.1).

Proof of Lemma 7.4 This statement is the consequence of the classical result of the stationary phase method [12, 23], applied to the left-hand side, taking into account the following equality for $e_{\lambda,\theta}(z)$ in the neighborhood of the stationary points $b_m \in V$, $m = 1, \dots, M$:

$$e_{\lambda,\theta}(z) = \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar{\lambda}(\bar{b}_{m,1} + \bar{\theta}\bar{b}_{m,2})] \times \exp[\lambda A(z_2 - b_{m,2})^2 - \bar{\lambda}\bar{A}(\bar{z}_2 - \bar{b}_{m,2})^2],$$

where

$$A = -\frac{(\frac{\partial^2 P}{\partial z_1^2}\theta^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}\theta + \frac{\partial^2 P}{\partial z_2^2})(b_m)(z_2 - b_{m,2})^2(1 + O(z_2 - b_{m,2}))}{2(\frac{\partial P}{\partial z_1})(b_m)}.$$

We use here $z_2, \bar{z}_2$ as the coordinates of integration.

Lemma 7.4 is proved. □

Proof of Proposition 7.1 Proposition 7.1 follows from Lemmas 7.2–7.4.

In the proof of Proposition 1.1 we will apply also the following statement about exponential polynomials discovered by L. Ehrenpreis [9] and reinforced by C. Berenstein and M. Dostal [3]. □

Proposition 7.2 ([3, 9]) *Let $Q(\xi)$ be an exponential polynomial*

$$Q(\xi) = \sum_{k=1}^{N} q_k(\xi) e^{\langle \alpha_k, \xi \rangle},$$

where $\{q_k\}$ are polynomials of $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, $\alpha_k = \{\alpha_{k,1}, \dots, \alpha_{k,n}\} \in \mathbb{C}^n$, $k = 1, \dots, N$.

Let $h(\xi) = \max_k Re\langle \alpha_k, \xi \rangle$. Then $\forall\varepsilon > 0\ \exists$ constant $C = C(\varepsilon, Q) > 0$ such that

$$|Q(\xi)|_\varepsilon \stackrel{\text{def}}{=} \sup_{\{\xi' \in \mathbb{C} : |\xi' - \xi| < \varepsilon\}} |Q(\xi')| \geq \frac{1}{C} e^{h(\xi)}.$$

The final part of the proof of Proposition 1.1 consists of the following.

Proposition 7.1 and the definition of $\Delta_\theta(\lambda)$ imply the asymptotic equality

$$|\lambda|^g \Delta_\theta(\lambda) = \det\Bigg(-\frac{\lambda}{\bar\lambda} e_{\lambda,\theta} \frac{\bar\omega_k}{d\bar\xi_1 + \bar\theta d\bar\xi_2}(a_j) - \pi \sum_{m=1}^{M} \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar\lambda(\bar b_{m,1} + \bar\theta \bar b_{m,2})] K_{j,k}(b_m, a_j)\Bigg) + O\left(\frac{1}{|\lambda|}\right), \tag{7.3}$$

where $j, k = 1, \dots, g$.

The determinant of the right-hand side of (7.3) is an exponential polynomial $Q(\lambda, \bar\lambda)$ of the form

$$Q(\lambda, \bar\lambda) = \sum_{k=1}^{N} q_k(\lambda, \bar\lambda) e^{\lambda\alpha_k - \bar\lambda\bar\alpha_k}, \tag{7.4}$$

where $\lambda \in \mathbb{C}$, $\alpha_k \in \mathbb{C}$, $k = 1, \dots, N$. The coefficient $q_k(\lambda, \bar\lambda)$ of the exponential polynomial $Q(\lambda, \bar\lambda)$ and complex frequencies $\{\alpha_k\}$ depend on V, $\{a_j\}$, θ, $\{b_m\}$. Applying Proposition 7.2 to the exponential polynomial (7.4) we obtain uniformly for $\lambda \in \mathbb{C}$ the estimate

$$|Q(\lambda, \bar\lambda)\big|_\varepsilon \geq \frac{1}{C(\varepsilon, Q)} e^{\max_k \mathrm{Re}(\lambda\alpha_k - \bar\lambda\bar\alpha_k)} = \frac{1}{C(\varepsilon, Q)}. \tag{7.5}$$

Both inequalities of Proposition 1.1 follow from (7.3)–(7.5).

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