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On Supnorm Estimates for $\bar{\partial}$ on Infinite Type Convex Domains in $\mathbb{C}^2$

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Abstract In this paper, we study the $\bar{\partial}$ equation on some convex domains of infinite type in $\mathbb{C}^2$. In detail, we prove that supnorm estimates hold for infinite exponential type domains, provided the exponent is less than 1.

Keywords Supnorm estimate · Infinite D'Angelo type · Pseudoconvex

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{C}^n$. Given a smooth $\bar{\partial}$ closed $(0,1)$-form f, one fundamental question is to study the supnorm estimates for the solutions of $\bar{\partial}u = f$. A positive answer is well-known when Ω is strictly convex or strongly pseudoconvex, by constructing integral formulas for $\bar{\partial}$ equations. Supnorm and Hölder estimates have been established by Grauert and Lieb [3], Henkin [4, 5], Kerzman [8], etc. Indeed, supnorm and Hölder estimates with order $\frac{1}{m}$ still hold if $\Omega \subset \mathbb{C}^n$ is convex of finite type with type m (see [1, 2]). However, for infinite type convex domains, even supnorm estimates are still unknown. We discuss in this paper

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some examples of convex domains of infinite type in $\mathbb{C}^2$. In particular, for the bidisc rounded off by an infinite exponential type hypersurface, we prove if the exponent is less than 1, the solutions given by the integral representation to the $\bar\partial$ equations have supnorm estimates. However, whether the exponent 1 is optimal for the supnorm estimates is still unclear to us.

1.1 Infinite Type Convex Domains in $\mathbb{C}^2$

Problem Assume that Ω is a bounded convex domain with smooth boundary in $\mathbb{C}^2$. Can one solve the $\bar\partial$ equation with supnorm estimates on Ω? More precisely, does there exist a constant C depending only on Ω so that for any $f=\sum_{i=1}^2 f_i(z)d\bar z_i \in C^1_{(0,1)}(\bar\Omega)$ and $\bar\partial f=0$ on Ω, then there is a $u(z)$ on Ω so that $\bar\partial u=f$ and $\|u\|_\infty \le C\|f\|_\infty$. Here $\|u\|_\infty=\sup\{|u(z)|: z\in\Omega\}$.

We note that if we moreover would like to prove that the constant $C(\Omega)$ only depends on the diameter of Ω, which is the case in the corresponding problem for L^2 estimates [7], then it suffices to study the case when the domain has smooth boundary and is strongly convex.

There is a natural integral kernel to solve the $\bar\partial$ problem, namely the Henkin Kernel [5]:

$$\begin{aligned}4\pi^2 u(z) &= \int_{\zeta\in\partial\Omega} \frac{\rho_{\zeta_1}(\bar\zeta_2-\bar z_2)-\rho_{\zeta_2}(\bar\zeta_1-\bar z_1)}{[\rho_{\zeta_1}(z_1-\zeta_1)+\rho_{\zeta_2}(z_2-\zeta_2)]|\zeta-z|^2} f\wedge\omega(\zeta)\\ &\quad+\int_{\zeta\in\Omega}\frac{f_1(\bar\zeta_1-\bar z_1)+f_2(\bar\zeta_2-\bar z_2)}{|\zeta-z|^4}\omega(\bar\zeta)\wedge\omega(\zeta)=Hf+Kf \qquad (1.1)\end{aligned}$$

where ρ is a C^1 defining function of Ω and $\omega(\zeta)=d\zeta_1\wedge d\zeta_2$.

For smooth finite type convex domains in $\mathbb{C}^2$, the result has been known for some time (see [9]). In that paper, Range used Skoda's technique to construct a different Cauchy-Fantappié kernel and carried out the Hölder estimates for $\bar\partial$ equations.

In [6], Henkin proved the conclusion in the Problem holds in the case of the bidisc. In this case, the integral formula is still valid, although the domain does not have a smooth boundary.

The methods in the two cases are different. In the finite type case one can directly estimate the kernel H and show it is uniformly in L^1 norm for points $z\in\Omega$. In other words, one can estimate the integral of the absolute value. For the bidisc case, this approach fails. Instead, Henkin uses an argument involving integration by parts which essentially takes the integral over the flat piece $\{|\zeta_1|=1\}$ to an integral over $\{|\zeta_2|=1\}$ and vice versa. For these new integrals one can integrate the absolute values of the integrands to prove uniform L^1 estimates. This idea has been carried over to more general polyhedra, see [6] even in higher dimension. It is not known whether one can solve for infinite type smooth convex domains.

In this paper we will investigate the case of some convex domains which have a relatively open part of the boundary which is Levi flat and another part which is strongly convex. For such domains one can split the boundary into three pieces: one which consists of the flat points, where one can use the method from Henkin's bidisc result, one which is a compact uniformly convex part, and a third one which contains strongly convex points near the flat part. The main difficulty is to deal with the third part.

This difficulty is already apparent in the case of smooth domains which are strongly convex except for one infinitely flat point. We will discuss this case and show that there is a critical exponent which decides absolute integrability of the Henkin kernel. This information motivates how to find a rounded-off polydisc where one can solve $\bar\partial$ with supnorm estimates.

Our main case is the following:

Example 1 Let $\chi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ be a smooth function such that $\chi''(t) \geq 0$ everywhere, $\chi''(t) > 0$ for all $t \in (1, 1+a)$, and

$$\chi(t) = \begin{cases} 1, & t \in [0,1]; \\ 1 + \exp(-\frac{1}{(t-1)^{\frac{\alpha}{2}}}), & t \in (1, 1+\epsilon); \\ t - \eta, & t \geq 1 + a, \end{cases}$$

where $a > \epsilon > 0$, $a > \eta > 0$ are small numbers such that $\chi', \chi'', \chi''' > 0$ on $\mathbb{R}^+ \cup \{0\}$.

Let us define a domain $\Omega \subset \mathbb{C}^2$ as follows:

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \chi(|z_1|^2) + |z_2|^2 < 4\}.$$

We show that we can solve $\bar\partial$ with supnorm estimates on this domain.

The paper is organized as follows. In the next section we recall the proof of Henkin's bidisc theorem and the strongly convex case. In Sect. 3 we discuss domains with totally real flat parts. In Sect. 4 we discuss the example above. We remark that as an application of this we get supnorm estimates for some rounded-off bidisc.

2 Background Results

2.1 Henkin's Integral Formula on a Bidisc

Proposition 1 *Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. There is a constant $C > 0$, so that for any $f = f_1 d\bar z_1 + f_2 d\bar z_2 \in C^1_{(0,1)}(\bar\Omega)$ with $\bar\partial f = 0$ on Ω and $\|f\|_\infty < \infty$. Then*

$$\begin{aligned} u(z) &= \frac{3}{4\pi^2} \int_\Omega \frac{f_1(\bar\zeta_1 - \bar z_1) + f_2(\bar\zeta_2 - \bar z_2)}{|\zeta - z|^4} d\bar\zeta_1 \wedge d\bar\zeta_2 \wedge d\zeta_1 \wedge d\zeta_2 \\ &+ \frac{1}{4\pi^2} \int_{|\zeta_2|=1, |\zeta_1|<1} \frac{f_1}{\zeta_1 - z_1} \frac{\bar\zeta_2 - \bar z_2}{|\zeta - z|^2} d\bar\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \\ &+ \frac{i}{2\pi} \int_{|\zeta_2|<1} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - z_2} d\bar\zeta_2 \wedge d\zeta_2 \end{aligned}$$

$$- \frac{1}{4\pi^2} \int_{|\zeta_1|=1, |\zeta_2|<1} \frac{f_2}{\zeta_2 - z_2} \frac{\bar{\zeta}_1 - \bar{z}_1}{|\zeta - z|^2} d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2$$
$$+ \frac{i}{2\pi} \int_{|\zeta_1|<1} \frac{f_1(\zeta_1, z_2)}{\zeta_1 - z_1} d\bar{\zeta}_1 \wedge d\zeta_1 \tag{2.1}$$

gives a solution for $\bar{\partial} u = f$ *on* Ω *with* $\|u\|_\infty \le C\|f\|_\infty$.

From now on, we also denote $\|f\|_\infty \le C\|g\|_\infty$ for some positive constant C by $\|f\|_\infty \lesssim \|g\|_\infty$, for simplification. In [6], the details of the proof were not given. For the completeness of our result, we give the detailed computation here. At the end of the section, we will also show the supnorm estimates of the $\bar{\partial}$ equation.

Proof The usual Henkin's integral formula on a domain $\Omega = \{\rho < 0\} \subset \mathbb{C}^2$ is as follows:

$$4\pi^2 u(z) = \int_\Omega \frac{f_1(\bar{\zeta}_1 - \bar{z}_1) + f_2(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2$$
$$+ \int_{\partial\Omega} \frac{\rho_{\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \rho_{\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{[\rho_{\zeta_1}(z_1 - \zeta_1) + \rho_{\zeta_2}(z_2 - \zeta_2)]|\zeta - z|^2} f \wedge d\zeta_1 \wedge d\zeta_2. \tag{2.2}$$

Applying this formula on the bidisc, we get

$$4\pi^2 u(z) = \int_\Omega \cdots + \int_{\partial\Omega} \cdots = \int_\Omega \cdots + \int_{|\zeta_1|=1, |\zeta_2|<1} \cdots + \int_{|\zeta_2|=1, |\zeta_1|<1} \cdots$$
$$= A_1 + A_2 + A_3,$$

where

$$A_1 = \int_\Omega \frac{f_1(\bar{\zeta}_1 - \bar{z}_1) + f_2(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2,$$
$$A_2 = \int_{|\zeta_1|=1, |\zeta_2|<1} \frac{\bar{\zeta}_1(\bar{\zeta}_2 - \bar{z}_2)}{\bar{\zeta}_1(z_1 - \zeta_1)|\zeta - z|^2} f \wedge \omega(\zeta)$$
$$= \int_{|\zeta_1|=1, |\zeta_2|<1} \frac{(\bar{\zeta}_2 - \bar{z}_2)}{(z_1 - \zeta_1)|\zeta - z|^2} f_2 d\bar{\zeta}_2 \wedge \omega(\zeta),$$
$$A_3 = \int_{|\zeta_2|=1, |\zeta_1|<1} \frac{\bar{\zeta}_2(\bar{\zeta}_1 - \bar{z}_1)}{\bar{\zeta}_2(z_2 - \zeta_2)|\zeta - z|^2} f \wedge \omega(\zeta)$$
$$= \int_{|\zeta_2|=1, |\zeta_1|<1} \frac{(\bar{\zeta}_1 - \bar{z}_1)}{(z_2 - \zeta_2)|\zeta - z|^2} f_1 d\bar{\zeta}_1 \wedge \omega(\zeta)$$

and $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$.

Now let us look at A_2. We may pick a small $\epsilon > 0$ such that $B(z_1, \epsilon) = \{\zeta : |\zeta - z_1| < \epsilon\} \subset D_1 = \{\zeta_1 : |\zeta_1| < 1\}$. We then apply Stokes' Theorem on $D_1 \setminus B(z_1, \epsilon)$ for the boundary integral on $|\zeta_1| = 1$:

$$\begin{aligned}
A_2 &= \int_{|\zeta_1|=1,|\zeta_2|<1} \frac{(\bar\zeta_2 - \bar z_2)}{(z_1-\zeta_1)|\zeta - z|^2} f_2 d\bar\zeta_2 \wedge \omega(\zeta) \\
&= \int_{|\zeta_1 - z_1|=\epsilon,|\zeta_2|<1} \frac{(\bar\zeta_2 - \bar z_2)}{(z_1-\zeta_1)|\zeta - z|^2} f_2 d\bar\zeta_2 \wedge \omega(\zeta) \\
&\quad + \int_{D_1\setminus B(z_1,\epsilon),|\zeta_2|<1} \frac{\partial}{\partial\bar\zeta_1}(\cdots)\omega(\bar\zeta)\wedge\omega(\zeta) \\
&= \int_{|\zeta_1 - z_1|=\epsilon,|\zeta_2|<1} \frac{(\bar\zeta_2 - \bar z_2)}{(z_1-\zeta_1)|\zeta - z|^2} f_2 d\bar\zeta_2 \wedge \omega(\zeta) \\
&\quad - \int_{D_1\setminus B(z_1,\epsilon),|\zeta_2|<1} \frac{\bar\zeta_2 - \bar z_2}{(z_1-\zeta_1)} \cdot \frac{\zeta_1 - z_1}{|\zeta - z|^4} f_2 \omega(\bar\zeta)\wedge\omega(\zeta) \\
&\quad + \int_{D_1\setminus B(z_1,\epsilon),|\zeta_2|<1} \frac{\bar\zeta_2 - \bar z_2}{(z_1-\zeta_1)} \cdot \frac{1}{|\zeta - z|^2} \frac{\partial f_2}{\partial\bar\zeta_1} \omega(\bar\zeta)\wedge\omega(\zeta) = B_1 + B_2 + B_3.
\end{aligned}$$

As $\epsilon \to 0$, we have

$$B_1 \longrightarrow 2\pi i \int_{|\zeta_2|<1} \frac{f_2(z_1,\zeta_2)}{\zeta_2 - z_2} d\bar\zeta_2 \wedge d\zeta_2,$$

$$B_2 \longrightarrow \int_\Omega \frac{f_2(\bar\zeta_2 - \bar z_2)}{|\zeta - z|^4}\omega(\bar\zeta)\wedge\omega(\zeta).$$

Applying $\bar\partial f = 0$ and integration by parts, we get

$$\begin{aligned}
B_3 &= \int_{D_1\setminus B(z_1,\epsilon),|\zeta_2|<1} \frac{\bar\zeta_2 - \bar z_2}{(z_1-\zeta_1)} \cdot \frac{1}{|\zeta - z|^2} \frac{\partial f_2}{\partial\bar\zeta_1} \omega(\bar\zeta)\wedge\omega(\zeta) \\
&= \int_{D_1\setminus B(z_1,\epsilon),|\zeta_2|<1} \frac{\bar\zeta_2 - \bar z_2}{(z_1-\zeta_1)} \cdot \frac{1}{|\zeta - z|^2} \frac{\partial f_1}{\partial\bar\zeta_2} \omega(\bar\zeta)\wedge\omega(\zeta) \\
&= \int_{|\zeta_2|=1,D_1\setminus B(z_1,\epsilon)} \frac{f_1}{\zeta_1 - z_1} \frac{\bar\zeta_2 - \bar z_2}{|\zeta - z|^2} d\bar\zeta_1 \wedge \omega(\zeta) \\
&\quad + \int_{|\zeta_2|<1,D_1\setminus B(z_1,\epsilon)} \frac{f_1(\bar\zeta_1 - \bar z_1)}{|\zeta - z|^4}\omega(\bar\zeta)\wedge\omega(\zeta) \\
&\longrightarrow \int_{|\zeta_2|=1,|\zeta_1|<1} \frac{f_1}{\zeta_1 - z_1} \frac{\bar\zeta_2 - \bar z_2}{|\zeta - z|^2} d\bar\zeta_1 \wedge \omega(\zeta) + \int_\Omega \frac{f_1(\bar\zeta_1 - \bar z_1)}{|\zeta - z|^4}\omega(\bar\zeta)\wedge\omega(\zeta).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
A_2 &= B_1 + B_2 + B_3 \\
&= 2\pi i \int_{|\zeta_2|<1} \frac{f_2(z_1,\zeta_2)}{\zeta_2 - z_2} d\bar\zeta_2 \wedge d\zeta_2 + \int_\Omega \frac{f_2(\bar\zeta_2 - \bar z_2)}{|\zeta - z|^4}\omega(\bar\zeta)\wedge\omega(\zeta)
\end{aligned}$$

$$+\int_{|\zeta_2|=1,|\zeta_1|<1}\frac{f_1}{\zeta_1-z_1}\frac{\bar\zeta_2-\bar z_2}{|\zeta-z|^2}d\bar\zeta_1\wedge\omega(\zeta)+\int_\Omega\frac{f_1(\bar\zeta_1-\bar z_1)}{|\zeta-z|^4}\omega(\bar\zeta)\wedge\omega(\zeta)$$

$$=A_1+\int_{|\zeta_2|=1,|\zeta_1|<1}\frac{f_1}{\zeta_1-z_1}\frac{\bar\zeta_2-\bar z_2}{|\zeta-z|^2}d\bar\zeta_1\wedge\omega(\zeta)$$

$$+2\pi i\int_{|\zeta_2|<1}\frac{f_2(z_1,\zeta_2)}{\zeta_2-z_2}d\bar\zeta_2\wedge d\zeta_2. \tag{2.3}$$

Calculation of A_3 is similar. Hence we get (2.1) as in the proposition.

We show next that the solution u in (2.1) is bounded if $|f|$ is bounded. It suffices to estimate the second and fourth expression on the right side of (2.1). Since they are symmetric, we only provide the estimate for the second one here.

Notice that $a+b\gtrsim a^{2/3}b^{1/3}$ if $a>0,\ b>0$. Hence we have $|z-\zeta|^2\gtrsim|z_1-\zeta_1|^{2/3}|z_2-\zeta_2|^{4/3}$. Therefore,

$$\frac{1}{|\zeta_1-z_1|}\frac{|\bar\zeta_2-\bar z_2|}{|\zeta-z|^2}\lesssim\frac{1}{|\zeta_1-z_1|}\frac{|\bar\zeta_2-\bar z_2|}{|\zeta_1-z_1|^{2/3}|\zeta_2-z_2|^{4/3}}$$

$$=\frac{1}{|\zeta_1-z_1|^{5/3}}\frac{1}{|\zeta_2-z_2|^{1/3}}.$$

Then the integrand in the second expression of (2.1) is integrable. □

2.2 Henkin's Integral Formula on a Strongly Convex Domain

The estimate of Henkin's integral formula is well-known for strongly pseudoconvex domains (see [5]). Here we include the proof for strongly convex domains first because the method we use is different from that in [5] and also because the estimates for other special domains in Sects. 3 and 4 are based on the estimate we show below. Hence it would not inconvenience the reader if we skip the estimate in Sects. 3 and 4 which are the same as in the case for strongly convex domains.

Proposition 2 *Let $\Omega=\{\rho<0\}\subset\mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary. For any $f\in C^1_{(0,1)}(\bar\Omega)$ and $\bar\partial f=0$, there is a u such that $\bar\partial u=f$ on Ω with $\|u\|_\infty\lesssim\|f\|_\infty$, where $\|u\|_\infty=\sup\{|u(z)|:z\in\bar\Omega\}$.*

Proof It suffices to show that the integral Hf in (1.1) is bounded. Recall the formula for Hf:

$$Hf(z)=\int_{\zeta\in\partial\Omega}\frac{\rho_{\zeta_1}(\bar\zeta_2-\bar z_2)-\rho_{\zeta_2}(\bar\zeta_1-\bar z_1)}{[\rho_{\zeta_1}(z_1-\zeta_1)+\rho_{\zeta_2}(z_2-\zeta_2)]|\zeta-z|^2}f\wedge\omega(\zeta).$$

Let $F(z,\zeta)=\rho_{\zeta_1}(z_1-\zeta_1)+\rho_{\zeta_2}(z_2-\zeta_2)$. Then we have

$$|Hf(z)|\lesssim\int_{\zeta\in\partial\Omega}\frac{|f_1|d\bar\zeta_1+|f_2|d\bar\zeta_2}{(|\mathrm{Im}\,F|+|\mathrm{Re}\,F|)|\zeta-z|}\wedge\omega(\zeta).$$

If we let $z = (x_1 + ix_2, x_3 + ix_4)$ and $\zeta = (t_1 + it_2, t_3 + it_4)$, then

$$\operatorname{Re} F(z,\zeta) = \frac{1}{2}\sum_{j=1}^{4} \frac{\partial\rho}{\partial t_j}(x_j - t_j).$$

For fixed $\zeta \in \partial\Omega$, the Taylor expansion of ρ at ζ evaluated at $z \in \bar{\Omega}$ is as follows:

$$\begin{aligned}\rho(z) &= 2\operatorname{Re} F(z,\zeta) + \frac{1}{2}\sum_{j,k=1}^{4} \frac{\partial^2\rho}{\partial t_j \partial t_k}(x_j - t_j)(x_k - t_k) + O(|z-\zeta|^3)\\ &\geq 2\operatorname{Re} F(z,\zeta) + C|z-\zeta|^2 + O(|z-\zeta|^3).\end{aligned}$$

Hence for $\delta > 0$ small enough, we have

$$|\operatorname{Re} F(z,\zeta)| \geq C'|z-\zeta|^2, \quad \text{for } z \in \bar{\Omega}, |z-\zeta| < \delta, \tag{2.4}$$

since $\operatorname{Re} F(z,\zeta) \leq 0$. We may assume $\|\nabla\rho(\zeta)\| = 1$ for all $\zeta \in \partial\Omega$. Then $|\operatorname{Re} F(z,\zeta)|$ is comparable to the square of the distance from $z \in \bar{\Omega}$ to the real tangent plane to $\partial\Omega$ at $\zeta \in \partial\Omega$. Hence if $z \in \bar{\Omega}$ and $|z-\zeta| > \delta$, then $|\operatorname{Re} F(z,\zeta)| \geq \inf\{|\operatorname{Re} F(z,\zeta)| : |z-\zeta| = \delta,\ z \in \bar{\Omega}\} \gtrsim \delta^2$. Therefore, we have

$$|Hf(z)| \lesssim \int_{\zeta\in\partial\Omega, |z-\zeta|\leq\delta} \cdots + \int_{\zeta\in\partial\Omega, |z-\zeta|>\delta} \cdots = I + II$$

and $II \lesssim \|f\|_\infty$. Therefore, it is sufficient to show $I \lesssim \|f\|_\infty$. Let $z \in \bar{\Omega}$ be fixed such that $S_{z,\delta} = \{\zeta \in \partial\Omega : |z-\zeta| \leq \delta\} \neq \emptyset$. We may assume $\delta > 0$ is small enough such that $\partial\rho/\partial t_3 \approx 1$ on $S_{z,\delta}$. Then we have $d\rho = \rho_{t_1}dt_1 + \rho_{t_2}dt_2 + \rho_{t_3}dt_3 + \rho_{t_4}dt_4$ and

$$dt_3 = \frac{-1}{\rho_{t_3}}(\rho_{t_1}dt_1 + \rho_{t_2}dt_2 + \rho_{t_4}dt_4), \quad \text{on } S_{z,\delta}.$$

Therefore, $|d\bar{\zeta}_1 \wedge \omega(\zeta)| \approx dt_1dt_2dt_4$ and $|d\bar{\zeta}_2 \wedge \omega(\zeta)| \approx dt_1dt_2dt_4$. So we have

$$I \lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2+(t_4-x_2)^2<\delta} \frac{1}{(|\operatorname{Im} F| + |\operatorname{Re} F|)|\zeta - z|} dt_1dt_2dt_4. \tag{2.5}$$

It is easy to check that $\partial\operatorname{Im} F/\partial t_4 \neq 0$ near $S_{z,\delta}$. Hence the Jacobian of the coordinate change mapping $\Phi(t) = (t_1, t_2, t_4) \to (t_1, t_2, \operatorname{Im} F)$ does not vanish on $S_{z,\delta}$. We have

$$\begin{aligned} I &\lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2+(t_4-x_2)^2<\delta} \frac{1}{(|t_4| + |\operatorname{Re} F|)|\zeta_1 - z_1|} dt_1dt_2dt_4\\ &\lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2<\delta} \frac{|\ln(|\operatorname{Re} F| + \sqrt{\delta}) - \ln|\operatorname{Re} F||}{|\zeta_1 - z_1|} dt_1dt_2\\ &\lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2<\delta} \frac{|\ln|\operatorname{Re} F||}{|\zeta_1 - z_1|} dt_1dt_2. \end{aligned} \tag{2.6}$$

By (2.4), we get

$$I \lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2<\delta} \frac{|\ln|\mathrm{Re}\,F||}{|\zeta_1 - z_1|} dt_1 dt_2$$
$$\lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2<\delta} \frac{|\ln(|z_1-\zeta_1|^2+|z_2-\zeta_2|^2)|}{|\zeta_1 - z_1|} dt_1 dt_2$$
$$\lesssim \|f\|_\infty \int_{(t_1-x_1)^2+(t_2-x_2)^2<\delta} \frac{|\ln|z_1-\zeta_1||}{|\zeta_1 - z_1|} dt_1 dt_2. \qquad (2.7)$$

Let us use polar coordinates for (t_1, t_2) such that $|z_1 - \zeta_1| = r$ and $t_1 = x_1 + r\cos\theta$. Then we get

$$I \lesssim \|f\|_\infty \int_{r<\delta} \frac{|\ln r|}{r} r dr \lesssim \|f\|_\infty. \qquad \square$$

3 Some Convex Domains with Totally Real Flat Parts

In this section we shall study bounded convex domains with smooth boundaries in $\mathbb{C}^2$ which are strongly convex except on some totally real flat boundary pieces. We assume those flat parts to be of exponentially infinite type.

As one can see in the proof of Proposition 2, to discuss the integrability of the Henkin kernel H in (1.1), it is enough to consider the case when z is close to the boundary and the integral on a small boundary piece close to z. For the domains we consider in this section, if z is close to the strongly convex boundary point, and if one can choose a small neighborhood around z such that the boundary piece in that neighborhood is strongly convex, then we can use the same estimates as in (2.6) and (2.7), and the integral is finite on that boundary piece. Therefore, it is enough to consider the case when z is close to the exponentially flat points, and evaluate the integral on $S_{z,\delta} = \{\zeta \in \partial\Omega : |z-\zeta| < \delta\}$.

Lemma 3 *If a C^2-smooth function $\phi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ satisfies $\phi'(t) \geq 0$ and $\phi''(t) \geq 0$ for all $t \in \mathbb{R}^+ \cup \{0\}$, then $\psi(x) := \phi(x_1^2 + \cdots + x_n^2)$ is a convex function on $\mathbb{R}^n$.*

The proof is straightforward and is omitted here.

Lemma 4 *Let ϕ be a smooth function on $\mathbb{R}^+ \cup \{0\}$ such that $\phi(0) = \phi'(0) = 0$, $\phi''(t) \geq 0$, and $\phi'''(t) \geq 0$, for all $t \in \mathbb{R}^+ \cup \{0\}$. Let $p, q \in \mathbb{R}^+ \cup \{0\}$. Then we have*

$$\phi(q) - \phi(p) - \phi'(p)(q-p) \geq \phi(q-p), \quad \text{if } 0 \leq p \leq q,$$

and

$$\phi(q) - \phi(p) - \phi'(p)(q-p) \geq \phi''\left(\frac{p+q}{2}\right)\left(\frac{p-q}{2}\right)^2, \quad \text{if } 0 \leq q < p.$$

Proof Let us assume $0 \le p \le q$ and let $q = p + s$. We want to show the following:

$$g(p,s) = \phi(p+s) - (\phi(p) + s\phi'(p)) - \phi(s) \ge 0, \quad \forall p, s \ge 0.$$

Obviously $g(0,s) = 0$. Moreover $\partial g/\partial p = \phi'(p+s) - \phi'(p) - s\phi''(p) \ge 0$ since ϕ' is a convex function on $\mathbb{R}^+ \cup \{0\}$. Therefore $g(p,s) \ge 0$ for all $p, s \ge 0$.

Now assume $0 \le q < p$ and let $p = q + s$, $s > 0$. Let $h(q,s) = \phi(q+s) - \phi'(q+s)s$. Then we have $h(q,0) = \phi(q)$ and $\partial h/\partial s = -\phi''(q+s)s \le 0$. Hence we get $h(q,0) - h(q,s) \ge h(q,s/2) - h(q,s)$. Note that $\partial^2 h/\partial s^2 = -\phi'''(q+s) \le 0$. Therefore we have

$$\frac{h(q,s/2) - h(q,s)}{-s/2} \le \frac{\partial h}{\partial s}(q, s/2).$$

Hence

$$\begin{aligned}\phi(q) - \phi(p) - \phi'(p)(q-p) = h(q,0) - h(q,s) &\ge \frac{s^2}{4}\phi''\left(q + \frac{s}{2}\right)\\ &= \phi''\left(\frac{p+q}{2}\right)\left(\frac{p-q}{2}\right)^2.\end{aligned}$$ □

We construct two bounded convex domains with smooth boundaries $\Omega_1, \Omega_2 \subset \mathbb{C}^2$ as follows. Locally, we study $\partial\Omega_1 \cap B(0,\epsilon) \subset \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|z_1|^\alpha) = 0\}$ and $\partial\Omega_2 \cap B(0,\epsilon) \subset \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|\operatorname{Re} z_1|^\alpha) = 0\}$. Here ϵ is small enough so that $\exp(-1/|z_1|^\alpha)$ and $\exp(-1/|\operatorname{Re} z_1|^\alpha)$ are convex if $|z_1| < \epsilon$. As one can see, the boundaries of Ω_1 and Ω_2 are strongly convex everywhere except along the imaginary z_2 axis in Ω_1 and along both imaginary z_1, z_2 axes in Ω_2. Moreover, in the origin in Ω_1 and in the imaginary z_1 axis in Ω_2, the boundaries are of exponentially infinite type. To get bounded convex domains in $\mathbb{C}^2$, we need the following patching proposition:

Proposition 5 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary and $0 \in \partial\Omega$. Then there exists a domain $\tilde{\Omega} \Subset \mathbb{C}^n$ such that $\partial\tilde{\Omega}$ is smooth and $\partial\tilde{\Omega} \cap B(0,\epsilon) = \partial\Omega \cap B(0,\epsilon)$ for some small $\epsilon > 0$ and that $\partial\tilde{\Omega}$ is strongly convex except possibly on $\partial\tilde{\Omega} \cap B(0,2\epsilon)$. Moreover, if Ω is convex, then $\tilde{\Omega}$ can be chosen to be bounded convex.*

Proof We use Lemma 3 with the following function:

$$\psi(t) = \begin{cases} 0, & t \in [0, \epsilon^2); \\ e^{-1/(t-\epsilon^2)}(t^2 - \epsilon^4), & t \ge \epsilon^2. \end{cases} \tag{3.1}$$

Then $\psi : [0,\infty) \to [0,\infty)$ is a smooth convex function and we have $\psi'(t) > 0$, $\psi''(t) > 0$ if $t > \epsilon^2$. Moreover, there exists $\beta > 0$ such that $\psi''(t) > \beta$ if $t > 4\epsilon^2$. Let ρ be a smooth local defining function of Ω near 0, i.e., $\Omega \cap U = \{\rho < 0\} \cap U$ for some neighborhood of U. We consider $\tilde{\Omega} = \{z \in \mathbb{C}^n : \rho + M\psi(|z|^2) < 0\}$ with some appropriate constant $M > 0$. Since $\psi(t) \to \infty$ as $t \to \infty$, clearly $\tilde{\Omega} \Subset \mathbb{C}^n$. Also $\tilde{\Omega} \cap B(0,\epsilon) = \Omega \cap B(0,\epsilon)$, where $\epsilon > 0$ is chosen small enough that $B(0,\epsilon) \subset U$. We may choose M large enough that $\tilde{\Omega}$ is strongly convex on $\partial\tilde{\Omega} \setminus B(0,2\epsilon)$. □

Using Proposition 5, we now have constructed two bounded convex domains with smooth boundaries $\Omega_1, \Omega_2 \subset \mathbb{C}^2$ such that

$$\begin{aligned} \Omega_1 &= \left\{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|z_1|^\alpha) + M\psi(|z|^2) < 0\right\} \quad \text{and} \\ \Omega_2 &= \left\{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|\operatorname{Re} z_1|^\alpha) + M\psi(|z|^2) < 0\right\}, \end{aligned} \tag{3.2}$$

where ψ is given as in (3.1).

Theorem 6 *Let $\alpha < 1$ on both of the two cases in (3.2), then there is a solution to the $\bar\partial$-equation $\bar\partial u = f$, for any $f = f_1 d\bar z_1 + f_2 d\bar z_2$, $f \in C^1_{(0,1)}(\bar\Omega_j)$ and $\bar\partial f = 0$, that satisfies $\|u\|_\infty \lesssim \|f\|_\infty$.*

Proof We show that the Henkin integral is bounded on Ω_1 and Ω_2. We will split the boundary into three types of pieces. For strongly convex boundary pieces, we use the same method as in Proposition 2. For pieces in $\mathbf{B}(0,\epsilon)$, the estimate (2.6) still holds for z and ζ close to 0. By the following Proposition 7 and Proposition 8, we will show the estimate is valid near 0. For the pieces on $\partial\mathbf{B}(0,\epsilon) \cap \Omega$ where the defining equation is not strictly convex, the same method as in Proposition 7 and Proposition 8 can still be applied. Indeed, in the proof of those propositions we only used the estimates in Lemma 4. The estimates of $|\operatorname{Re} F|$ continue to be valid. We leave this part to the reader.

We follow the same method as in Proposition 2 and get (2.6). It comes down to estimating $|\operatorname{Re} F|$ near 0 for each of these two special cases, where

$$F(z,\zeta) = \rho_{\zeta_1}(z_1 - \zeta_1) + \rho_{\zeta_2}(z_2 - \zeta_2).$$

Proposition 7 *Let $\Omega \cap B(0,\delta) = \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|z_1|^\alpha) < 0\}$. Then, for sufficiently small $\delta > 0$, we have*

$$\int_{\sqrt{t_1^2+t_2^2}<\delta} \frac{|\ln|\operatorname{Re} F(z,\zeta)||}{|z_1 - \zeta_1|} dt_1 dt_2 < \infty,$$

for $z \in \bar\Omega \cap B(0,\delta)$, if $\alpha < 1$.

Proof Let $\phi(t) = \exp(-1/t^{\alpha/2})$. Then we have $\rho(z) = \operatorname{Re} z_2 + \phi(|z_1|^2)$. Since $\zeta \in \partial\Omega$, we have $\operatorname{Re}\zeta_2 = -\phi(|\zeta_1|^2)$. Hence we get

$$\begin{aligned} \operatorname{Re} F &= \operatorname{Re} \phi'(|\zeta_1|^2)\bar\zeta_1(z_1 - \zeta_1) + \frac{1}{2}\operatorname{Re}(z_2 - \zeta_2) \\ &= \phi'(|\zeta_1|^2)\operatorname{Re}\bar\zeta_1(z_1 - \zeta_1) + \frac{1}{2}\left(\operatorname{Re} z_2 + \phi(|\zeta_1|^2)\right). \end{aligned}$$

Since $\operatorname{Re} F \le 0$ and $\operatorname{Re} z_2 + \phi(|z_1|^2) \le 0$, we have

$$|\operatorname{Re} F| = -\operatorname{Re} F = \phi'(|\zeta_1|^2)\left[|\zeta_1|^2 - \operatorname{Re}\bar\zeta_1 z_1\right] - \frac{1}{2}\phi(|\zeta_1|^2) - \frac{1}{2}\operatorname{Re} z_2$$

$$\geq \phi'(|\zeta_1|^2)\Big[|\zeta_1|^2 - \operatorname{Re}\bar\zeta_1 z_1\Big] - \frac{1}{2}\phi(|\zeta_1|^2) + \frac{1}{2}\phi(|z_1|^2)$$
$$\geq \frac{1}{2}\Big[\phi'(|\zeta_1|^2)(|\zeta_1|^2 - |z_1|^2) - \phi(|\zeta_1|^2) + \phi(|z_1|)^2\Big].$$

Apply Lemma 4 with $p = |\zeta_1|^2$ and $q = |z_1|^2$. Then we get

$$|\operatorname{Re} F| \geq \frac{1}{2}\phi(|z_1|^2 - |\zeta_1|^2), \quad \text{if } |z_1| \geq |\zeta_1| \quad \text{and}$$
$$|\operatorname{Re} F| \geq \frac{1}{2}\phi''\left(\frac{|z_1|^2 + |\zeta_1|^2}{2}\right)\left(\frac{|\zeta_1|^2 - |z_1|^2}{2}\right)^2 \quad \text{if } |z_1| < |\zeta_1|.$$

Hence we have

$$\begin{aligned}
&\int_{|t|<\delta} \frac{|\ln|\operatorname{Re} F(z,\zeta)||}{|z_1 - \zeta_1|} dt_1 dt_2 \\
&\lesssim \int_{|t|<\delta, |z_1|\geq|\zeta_1|} \frac{|\ln\phi(|z_1|^2 - |\zeta_1|^2)|}{|z_1 - \zeta_1|} dt_1 dt_2 \\
&\quad + \int_{|t|<\delta, |z_1|<|\zeta_1|} \frac{|\ln(\phi''((|z_1|^2 + |\zeta_1|^2)/2)(|\zeta_1|^2 - |z_1|^2)^2/4)|}{|z_1 - \zeta_1|} dt_1 dt_2 \\
&\lesssim \int_{|t|<\delta} \frac{|\ln\phi(|z_1|^2 - |\zeta_1|^2)|}{|z_1 - \zeta_1|} dt_1 dt_2 \\
&\quad + \int_{|t|<\delta} \frac{|\ln(\phi''((|z_1|^2 + |\zeta_1|^2)/2)(|\zeta_1|^2 - |z_1|^2)^2/4)|}{|z_1 - \zeta_1|} dt_1 dt_2 \\
&= I + II.
\end{aligned}$$

We shall show that $I < \infty$ and $II < \infty$. Let us first consider I:

$$I \lesssim \int \frac{|\ln\phi(|z_1|^2 - |\zeta_1|^2)|}{|z_1 - \zeta_1|} dt_1 dt_2 \lesssim \int \frac{1}{||z_1|^2 - |\zeta_1|^2|^{\alpha/2}|z_1 - \zeta_1|} dt_1 dt_2. \tag{3.3}$$

Let us use the polar coordinate centered at z_1 and let $\zeta_1 = z_1 + re^{i\theta}$. Then we get

$$\begin{aligned}
I &\lesssim \int_0^{2\pi}\int_0^1 \left[\frac{1}{r(r + (z_1 e^{-i\theta} + \bar z_1 e^{i\theta}))}\right]^{\alpha/2} dr d\theta \\
&\lesssim \int_0^{2\pi}\int_0^1 \left[\frac{1}{r^2} + \frac{1}{(r + (z_1 e^{-i\theta} + \bar z_1 e^{i\theta}))^2}\right]^{\alpha/2} dr d\theta \\
&\lesssim \int_0^{2\pi}\int_0^1 \left(\frac{1}{r}\right)^{\alpha} + \left[\frac{1}{r + (z_1 e^{-i\theta} + \bar z_1 e^{i\theta})}\right]^{\alpha} dr d\theta < \infty,
\end{aligned}$$

if $\alpha < 1$.

Now let us consider II. We have

$$\begin{aligned}\phi'(t) &= \exp\left(-1/t^{\alpha/2}\right)\frac{\alpha}{2}\frac{1}{t^{1+\alpha/2}};\\ \phi''(t) &= \exp\left(-1/t^{\alpha/2}\right)\frac{1}{t^{2+\alpha}}\frac{\alpha}{2}\left[\frac{\alpha}{2}-\left(1+\frac{\alpha}{2}\right)t^{\alpha/2}\right]\\ &\geq C(\alpha)\exp\left(-1/t^{\alpha/2}\right)\frac{1}{t^{2+\alpha}},\end{aligned} \tag{3.4}$$

if we choose $\delta > 0$ sufficiently small such that $\phi''(t) \geq 0$ for all $t < \delta^2$. Therefore we get

$$\begin{aligned} II &\lesssim \int_{|t|<\delta}\left(\frac{1}{(|z_1|^2+|\zeta_1|^2)^{\alpha/2}} + \left|\ln||z_1|^2 - |\zeta_1|^2|\right|\right)\frac{1}{|z_1-\zeta_1|}dt_1dt_2\\ &\lesssim \int_{|t|<\delta}\left(\frac{1}{|z_1-\zeta_1|^{\alpha}}\right)\frac{1}{|z_1-\zeta_1|}dt_1dt_2 < \infty,\end{aligned}$$

if $\alpha < 1$. □

Proposition 8 *Let* $\Omega \cap B(0,\delta) = \{\rho(z) = \mathrm{Re}\, z_2 + \exp(-1/|\mathrm{Re}\, z_1|^\alpha) < 0\}$. *Then we have*

$$\int_{\sqrt{t_1^2+t_2^2}<\delta}\frac{|\ln|\mathrm{Re}\, F(z,\zeta)||}{|z_1-\zeta_1|}dt_1dt_2 < \infty,$$

for $z \in \bar{\Omega}\cap B(0,\delta)$, *if* $\alpha < 1$.

Proof Let $\phi(t) = \exp(-1/t^{\alpha/2})$. Then we have $\rho(z) = \mathrm{Re}\, z_2 + \phi(|\mathrm{Re}\, z_1|^2)$. Since $\zeta \in \partial\Omega$, we have $-\mathrm{Re}\,\zeta_2 = \phi(|\mathrm{Re}\,\zeta_1|^2)$. Hence we get

$$\begin{aligned} 2\mathrm{Re}\, F &= \phi'(t_1^2)2t_1(x_1-t_1) + (\mathrm{Re}\, z_2 - \mathrm{Re}\,\zeta_2)\\ &= \phi'(t_1^2)2t_1(x_1-t_1) + \left(\mathrm{Re}\, z_2 + \phi(t_1^2)\right).\end{aligned}$$

Since $\mathrm{Re}\, F \leq 0$ and $\mathrm{Re}\, z_2 + \phi(|z_1|^2) \leq 0$, we have

$$\begin{aligned} |2\mathrm{Re}\, F| = -2\mathrm{Re}\, F &= 2\phi'(t_1^2)(t_1^2 - t_1x_1) - (\mathrm{Re}\, z_2 + \phi(t_1^2))\\ &\geq \phi'(t_1^2)\left(t_1^2 - x_1^2\right) - \phi(t_1^2) + \phi(x_1^2).\end{aligned}$$

Apply Lemma 4 with $p = t_1^2$ and $q = x_1^2$. Then we get

$$|\mathrm{Re}\, F| \geq \frac{1}{2}\phi(x_1^2 - t_1^2) \quad \text{if } x_1^2 \geq t_1^2 \quad \text{and}$$

$$|\mathrm{Re}\, F| \geq \frac{1}{2}\phi''\left(\frac{x_1^2+t_1^2}{2}\right)\left(\frac{t_1^2-x_1^2}{2}\right)^2 \quad \text{if } x_1^2 < t_1^2.$$

Hence we have

$$\begin{aligned}
&\int_{|t|<\delta} \frac{|\ln|\mathrm{Re}\, F(z,\zeta)||}{|z_1-\zeta_1|} dt_1 dt_2 \\
&\quad \lesssim \int_{|t|<\delta, |x_1|\geq |t_1|} \frac{|\ln \phi(x_1^2 - t_1^2)|}{|z_1-\zeta_1|} dt_1 dt_2 \\
&\qquad + \int_{|t|<\delta, |x_1|<|t_1|} \frac{|\ln(\phi''((x_1^2+t_1^2)/2)(t_1^2-x_1^2)^2/4)|}{|z_1-\zeta_1|} dt_1 dt_2 \\
&\quad \lesssim \int_{|t|<\delta} \frac{|\ln \phi(|x_1^2-t_1^2|)|}{|z_1-\zeta_1|} dt_1 dt_2 \\
&\qquad + \int_{|t|<\delta} \frac{|\ln(\phi''((x_1^2+t_1^2)/2)(t_1^2-x_1^2)^2/4)|}{|z_1-\zeta_1|} dt_1 dt_2 = I + II.
\end{aligned}$$

We shall show that $I, II < \infty$. First let us consider I. If we assume $\alpha < 1$, then we can find $\epsilon > 0$ such that $\alpha + 2\epsilon < 1$ and we have

$$\begin{aligned}
I &\lesssim \int \frac{1}{|x_1^2 - t_1^2|^{\alpha/2}|z_1-\zeta_1|} dt_1 dt_2 \\
&= \int \frac{1}{|x_1+t_1|^{\alpha/2}|x_1-t_1|^{\alpha/2}|x_1-t_1|^{\epsilon}|x_2-t_2|^{1-\epsilon}} dt_1 dt_2 \\
&\lesssim \left(\int_{|t_1|<1} \frac{1}{|x_1+t_1|^{\alpha}} dt_1\right)^{1/2} \left(\int_{|t_1|<1} \frac{1}{|x_1-t_1|^{\alpha+2\epsilon}} dt_1\right)^{1/2} < \infty.
\end{aligned}$$

Let us consider II. From (3.4), we get

$$\begin{aligned}
II &\lesssim \int_{|t|<\delta} \left(\frac{1}{(x_1^2+t_1^2)^{\alpha/2}} + |\ln|x_1^2 - t_1^2||\right) \frac{1}{|z_1-\zeta_1|} dt_1 dt_2 \\
&\lesssim \int_{|t|<\delta} \left(\frac{1}{|x_1-t_1|^{\alpha}}\right) \frac{1}{|x_1-t_1|^{\epsilon}|x_2-t_2|^{1-\epsilon}} dt_1 dt_2 < \infty,
\end{aligned}$$

if we choose $\epsilon > 0$ such that $\alpha + \epsilon < 1$. □

The proof of Theorem 6 is thus complete. □

4 Rounding off a Ball Cut by a Cylinder

Let us recall Example 1:

Let $\chi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ be a smooth function such that $\chi''(t) \geq 0$ everywhere, $\chi''(t) > 0$ for all $t \in (1, 1+a)$, and

$$\chi(t) = \begin{cases} 1, & t \in [0,1]; \\ 1 + \exp(-\frac{1}{(t-1)^{\frac{\alpha}{2}}}), & t \in (1, 1+\epsilon); \\ t - \eta, & t \geq 1 + a, \end{cases}$$

where $a > \epsilon > 0$ are small numbers, $0 < \eta < a$. Let us define a domain $\Omega \subset \mathbb{C}^2$ as follows:

$$\Omega = \left\{ \rho(z_1, z_2) = \chi(|z_1|^2) + |z_2|^2 < 4 \right\}.$$

Proposition 9 *If $\alpha < 1$ in the domain Ω above, then there is a solution u to $\bar{\partial} u = f$ with $f \in C^1_{(0,1)}(\bar{\Omega})$ and $\bar{\partial} f = 0$ with supnorm estimate $\|u\|_\infty \lesssim \|f\|_\infty$.*

Proof Note that

$$\begin{aligned} \partial\Omega = & \left\{ |z_1| \le 1, |z_2|^2 = 3 \right\} \\ & \cup \left\{ 1 < |z_1|^2 < 1 + a,\ \chi(|z_1|^2) + |z_2|^2 = 4 \right\} \\ & \cup \left\{ |z_1|^2 > 1 + a,\ |z_1|^2 + |z_2|^2 = 4 + \eta \right\} = P_1 \cup P_2 \cup P_3. \end{aligned} \tag{4.1}$$

Solve $\bar{\partial} u = f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$ on Ω using Henkin's integral:

$$\begin{aligned} 4\pi^2 u(z) = & \int_{\partial\Omega} \frac{\rho_{\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \rho_{\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{F(z,\zeta)|\zeta - z|^2} f \wedge d\zeta_1 \wedge d\zeta_2 \\ & + \int_\Omega \frac{f_1(\bar{\zeta}_1 - \bar{z}_1) + f_2(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\ = & \int_{P_1} \cdots + \int_{P_2} \cdots + \int_{P_3} \cdots + \int_\Omega \cdots = u_1 + u_2 + u_3 + u_4, \end{aligned} \tag{4.2}$$

where

$$F(z,\zeta) = \rho_{\zeta_1}(z_1 - \zeta_2) + \rho_{\zeta_2}(z_2 - \zeta_2).$$

We shall show that

$$\left\| u_j \right\|_\infty \lesssim \|f\|_\infty, \quad j = 1, 2, 3, 4.$$

The proof of $\|u_4\|_\infty \lesssim \|f\|_\infty$ has been included in the literature. The estimates on u_2 and u_3 are also omitted here, as they are exactly the same as in Proposition 1 and Proposition 2.

For the estimate of $\|u_2\|_\infty$, we first compute

$$\begin{aligned} F(z,\zeta) &= \frac{\partial\rho}{\partial\zeta_1}(\zeta)(z_1 - \zeta_1) + \frac{\partial\rho}{\partial\zeta_2}(\zeta)(z_2 - \zeta_2) \\ &= e^{-\frac{1}{(|\zeta_1|^2 - 1)^{\frac{\alpha}{2}}}} \frac{\alpha\bar{\zeta}_1(z_1 - \zeta_1)}{2(|\zeta_1|^2 - 1)^{\frac{\alpha}{2}+1}} + \bar{\zeta}_2(z_2 - \zeta_2). \end{aligned}$$

Notice that on P_2, it is strictly convex except at those points (ζ_1, ζ_2) so that $(|\zeta_1|, |\zeta_2|) = (1, \sqrt{3})$. The integrand in u_2 becomes most singular when (z_1, z_2) is near those boundary points. From now on we assume $(|z_1|, |z_2|) \in \bar{\Omega}$, $(|\zeta_1|, |\zeta_2|) \in$

$\partial\Omega$ are both close to $(1, \sqrt{3})$. We can also assume that $f = f_1 d\bar{z}_1$. Indeed, since $\rho = 0$, $\rho_{\zeta_1} d\zeta_1 + \rho_{\bar\zeta_1} d\bar\zeta_1 + \rho_{\zeta_2} d\zeta_2 + \rho_{\bar\zeta_2} d\bar\zeta_2 = 0$ on $\partial\Omega$. Therefore $d\bar\zeta_2 = d_1 d\zeta_1 + d_2 d\bar\zeta_1 + d_3 d\zeta_2$, where $d_1, d_2 = o(1)$. Then $d\bar\zeta_2 \wedge d\zeta_1 \wedge d\zeta_2 = o(1) d\bar\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2$.

$$
\begin{aligned}
|A| \le \|f\|_\infty &\int_{1<|\zeta_1|<1+\epsilon,\chi(|\zeta_1|^2)+|\zeta_2|^2=4} \frac{1}{|F||z_1-\zeta_1|} d\bar\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \\
&+ \|f\|_\infty \int_{1+\epsilon<|\zeta_1|<1+a,\chi(|\zeta_1|^2)+|\zeta_2|^2=4} \frac{1}{|F||z_1-\zeta_1|} d\bar\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 := I + II, \\
I \le \|f\|_\infty &\int_{1<|\zeta_1|<1+\epsilon,\chi(|\zeta_1|^2)+|\zeta_2|^2=4} \frac{1}{|F||z_1-\zeta_1|} d\bar\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \\
= \|f\|_\infty &\int_{1<|\zeta_1|<1+\epsilon,\chi(|\zeta_1|^2)+|\zeta_2|^2=4} \frac{1}{|z_1-\zeta_1|(|\operatorname{Re} F| + |\operatorname{Im} F|)} d\zeta_2 \wedge d\bar\zeta_1 \wedge d\zeta_1, \\
II \lesssim \|f\|_\infty&.
\end{aligned}
$$

Since $\frac{\partial \operatorname{Im} F}{\partial \zeta_2} \approx 1$ as $\zeta_2 \approx \sqrt{3}$, by taking a change of coordinates, we can assume the last integration is actually over $d\operatorname{Im} F \wedge d\bar\zeta_1 \wedge d\zeta_1$. Therefore

$$
I \le \|f\|_\infty \int_{1<|\zeta_1|<1+\epsilon} \frac{\ln(|\operatorname{Re} F|)}{|z_1-\zeta_1|} d\bar\zeta_1 \wedge d\zeta_1.
$$

Here,

$$
\operatorname{Re} F = e^{-\frac{1}{(|\zeta_1|^2-1)^{\frac{\alpha}{2}}}} \frac{\alpha \operatorname{Re}(\bar\zeta_1(z_1-\zeta_1))}{2(|\zeta_1|^2-1)^{\frac{\alpha}{2}+1}} + \operatorname{Re}(\bar\zeta_2 z_2) - 3 + e^{-\frac{1}{(|\zeta_1|^2-1)^{\frac{\alpha}{2}}}}. \tag{4.3}
$$

We need the following lemma:

Lemma 10 *Let* $\operatorname{Re} F$ *be given by* (4.3) *and let* $\phi(t) = e^{-\frac{1}{t^{\frac{\alpha}{2}}}}$ *for* $t > 0$, $\phi(0) = 0$. *Denote* $y = \operatorname{Re} \bar\zeta_1 z_1 - 1$ *and* $x = |\zeta_1|^2 - 1$. *Then*

$$
|\operatorname{Re} F| \ge \begin{cases} \phi(y-x), & \text{if } y \ge x \ge 0; \\ \phi''(\frac{x-y}{2})(\frac{x-y}{2})^2, & \text{if } 0 \le y \le x; \\ \phi''(\frac{x}{2})(\frac{x}{2})^2, & \text{if } y \le 0. \end{cases} \tag{4.4}
$$

Proof Since $\zeta \in P_2$, $z \in \Omega$, we have

$$
\begin{aligned}
\phi(|\zeta_1|^2 - 1) + |\zeta_2|^2 - 3 = 0, \\
\phi(||z_1|^2 - 1|) + |z_2|^2 - 3 < 0.
\end{aligned}
$$

Notice $\operatorname{Re} \bar\zeta_1 z_1 \le \frac{1}{2}|\zeta_1|^2 + \frac{1}{2}|z_1|^2$, $\operatorname{Re} \bar\zeta_2 z_2 \le \frac{1}{2}|\zeta_2|^2 + \frac{1}{2}|z_2|^2$. Moreover, ϕ is increasing and convex when t is close to 0^+. So

$$
\phi(|y|) \le \phi\left(\frac{1}{2}(|\zeta_1|^2 - 1) + \frac{1}{2}||z_1|^2 - 1)|\right) \le \frac{1}{2}\phi(|\zeta_1|^2 - 1) + \frac{1}{2}\phi(||z_1|^2 - 1|)
$$

$$
\begin{aligned}
&\leq \frac{1}{2}(3-|\zeta_2|^2)+\frac{1}{2}(3-|z_2|^2)\\
&=3-\left(\frac{1}{2}|\zeta_2|^2+\frac{1}{2}|z_2|^2\right)\\
&\leq 3-\operatorname{Re}\bar{\zeta}_2 z_2.
\end{aligned}
$$

Therefore if $y \geq 0$,

$$
\begin{aligned}
\operatorname{Re} F &= \phi'(x)(y-x)+\operatorname{Re}(\zeta_2 z_2)-3+\phi(x)\\
&\leq \phi'(x)(y-x)-\phi(y)+\phi(x).
\end{aligned}
$$

Applying Lemma 4 to the above, we have the estimates in (4.4).

If $y \leq 0$,

$$
\begin{aligned}
\operatorname{Re} F &= \phi'(x)(y-x)+\operatorname{Re}(\zeta_2 z_2)-3+\phi(x)\\
&= \phi'(x)(0-x)-\phi(0)+\phi(x)\\
&\leq -\phi''\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^2.
\end{aligned}
$$

□

We then have

$$
\begin{aligned}
I &\lesssim \|f\|_\infty\left\{\int_{1<|\zeta_1|<1+\epsilon,\, y\geq x}\frac{1}{|\operatorname{Re}(\bar{\zeta}_1 z_1-|\zeta_1|^2)|^{\frac{\alpha}{2}}|z_1-\zeta_1|}d\bar{\zeta}_1\wedge d\zeta_1\right.\\
&\quad\times\int_{1<|\zeta_1|<1+\epsilon,\, 0\leq y\leq x}\frac{|\ln(|\zeta_1|^2-\operatorname{Re}(\bar{\zeta}_1 z_1))|}{(|\zeta_1|^2-\operatorname{Re}(\bar{\zeta}_1 z_1))^{\frac{\alpha}{2}}|z_1-\zeta_1|}d\bar{\zeta}_1\wedge d\zeta_1\\
&\quad\left.+\int_{1<|\zeta_1|<1+\epsilon,\, y\leq 0}\frac{|\ln(|\zeta_1|^2-1)||}{(|\zeta_1|^2-1)^{\frac{\alpha}{2}}|z_1-\zeta_1|}d\bar{\zeta}_1\wedge d\zeta_1\right\}\\
&:= \|f\|_\infty(A_1+A_2+A_3).
\end{aligned}
$$

Here,

$$
\begin{aligned}
A_1 &\leq \int_{1<|\zeta_1|<1+\epsilon,\, y\geq x}\left[\operatorname{Re}\left(\frac{1}{\bar{\zeta}_1}\frac{1}{z_1-\zeta_1}\right)\right]^{\frac{\alpha}{2}}\left|\frac{1}{z_1-\zeta_1}\right|d\bar{\zeta}_1\wedge d\zeta_1\\
&\leq \int_{1<|\zeta_1|<1+\epsilon}\left[\frac{1}{|\bar{\zeta}_1|}\frac{1}{|z_1-\zeta_1|}\right]^{\frac{\alpha}{2}}\left|\frac{1}{z_1-\zeta_1}\right|d\bar{\zeta}_1\wedge d\zeta_1\\
&\lesssim \int_{1<|\zeta_1|<1+\epsilon}\left|\frac{1}{z_1-\zeta_1}\right|^{1+\frac{\alpha}{2}}d\bar{\zeta}_1\wedge d\zeta_1<\infty,\\
A_2 &\leq \int_{1<|\zeta_1|<1+\epsilon}\frac{|\ln|\zeta_1-z_1||}{|\zeta_1-z_1|^{1+\frac{\alpha}{2}}}d\bar{\zeta}_1\wedge d\zeta_1<\infty,
\end{aligned}
$$

when $\alpha < 1$.

For A_3, we need to use the inequality $a^p + b^q \gtrsim ab$ for $a, b \geq 0$ and $1/p + 1/q = 1$, $p, q > 1$. Now since $\alpha < 1$, we can choose $p_0 > 2$ so that $\frac{p_0\alpha}{2} < 1$ and q_0 satisfy $1/p_0 + 1/q_0 = 1$. Then $q_0 < 2$.

$$
\begin{aligned}
A_3 &\leq \int_{1<|\zeta_1|<1+\epsilon} \frac{|\ln(|\zeta_1|^2-1)|^{p_0}}{(|\zeta_1|^2-1)^{\frac{p_0\alpha}{2}}} d\bar{\zeta}_1 \wedge d\zeta_1 + \int_{1<|\zeta_1|<1+\epsilon} \frac{1}{|z_1-\zeta_1|^{q_0}} d\bar{\zeta}_1 \wedge d\zeta_1 \\
&\leq \int_{1<|\zeta_1|<1+\epsilon} \frac{|\ln(|\zeta_1|^2-1)|^{p_0}}{(|\zeta_1|^2-1)^{\frac{p_0\alpha}{2}}} d\bar{\zeta}_1 \wedge d\zeta_1 + const.
\end{aligned}
$$

In the first integral, take the polar coordinates $\zeta = re^{i\theta}$. Then

$$
\begin{aligned}
A_3 &\lesssim \int_{1<r<1+\epsilon} \frac{|\ln(r^2-1)|^{p_0}}{(r^2-1)^{\frac{p_0\alpha}{2}}} dr + const \\
&\lesssim \int_{1<r<1+\epsilon} \frac{|\ln(r-1)|^{p_0}}{(r-1)^{\frac{p_0\alpha}{2}}} dr + const < \infty,
\end{aligned}
$$

by our choices of p_0, q_0. □

One can similarly consider the domain obtained by rounding off the corners of a bidisc. See the following example:

Example 2 Let

$$
\chi(t) = \begin{cases} 1-a, & t \leq 1-a; \\ k\exp\left(-\frac{1}{(t^2-(1-a)^2)^{\frac{\alpha}{2}}}\right) + 1 - a, & t > 1-a, \end{cases}
$$

where $a > 0$ is a small constant such that $\chi(t)$ is convex on [0, 1] and k is a constant chosen such that $\chi(1) = 1$, i.e.,

$$
k = a \cdot \exp\left(\frac{1}{(2a-a^2)^{\frac{\alpha}{2}}}\right).
$$

Let

$$
\Omega = \{\rho(z) = \chi(|z_1|) + \chi(|z_2|) - 2 + a < 0\} \subset \mathbb{C}^2.
$$

As we can see, the boundary of Ω consists of the following sets:

$$
\begin{aligned}
\partial\Omega &= \{|z_1| \leq 1-a,\ |z_2| = 1\} \cup \{|z_1| = 1,\ |z_2| \leq 1-a\} \\
&\quad \cup \{|z_1|, |z_2| > 1-a,\ \rho(z_1, z_2) = 0\} = P_1 \cup P_2 \cup P_3. \qquad (4.5)
\end{aligned}
$$

Integrals on flat pieces P_1 and P_2 fall into the Henkin's bidisc situation. The supnorm estimates of the integral over rounded off piece P_3 can be carried out the same way as we discussed in Example 1 when $\alpha < 1$.

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