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Hypersurfaces of Constant Curvature in Hyperbolic Space I

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Abstract We investigate the problem of finding, in hyperbolic space, a complete strictly convex hypersurface which has a prescribed asymptotic boundary at infinity and which has some fixed curvature function being constant. Our results apply to a very general class of curvature functions.

Keywords Hypersurfaces of constant curvature · Hyperbolic space · Asymptotic boundary · Fully nonlinear elliptic equations

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1 Introduction

In this paper we study Weingarten hypersurfaces of constant curvature in hyperbolic space $\mathbb{H}^{n+1}$ with a prescribed asymptotic boundary at infinity. More precisely, given a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed embedded $(n-1)$ dimensional submanifolds of $\partial_\infty \mathbb{H}^{n+1}$, the ideal boundary of $\mathbb{H}^{n+1}$ at infinity, and a smooth symmetric function f of n variables, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1.1}$$

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where $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$ denotes the hyperbolic principal curvatures of Σ and σ is a constant, with the asymptotic boundary

$$\partial \Sigma = \Gamma. \tag{1.2}$$

We will use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}. \tag{1.3}$$

Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect the Euclidean metric in $\mathbb{R}^n$.

The function f is assumed to satisfy the fundamental structure conditions:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \quad \text{in } K,\ 1 \le i \le n, \tag{1.4}$$

$$f \text{ is a concave function in } K, \tag{1.5}$$

and

$$f > 0 \quad \text{in } K, \qquad f = 0 \quad \text{on } \partial K \tag{1.6}$$

where $K \subset \mathbb{R}^n$ is an open symmetric convex cone such that

$$K_n^+ := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subset K. \tag{1.7}$$

In addition, we shall assume that f is normalized

$$f(1, \dots, 1) = 1 \tag{1.8}$$

and satisfies the following more technical assumptions

$$f \text{ is homogeneous of degree one} \tag{1.9}$$

and

$$\lim_{R \to +\infty} f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \ge 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.10}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$.

All these assumptions are satisfied by $f = (\sigma_k/\sigma_l)^{\frac{1}{k-l}}$, $0 \le l < k \le n$, defined in K_k where σ_k is the normalized k-th elementary symmetric polynomial ($\sigma_0 = 1$) and

$$K_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0,\ \forall 1 \le j \le k\}.$$

See [3] for proof of (1.4) and (1.5). For (1.10) one easily computes that

$$\lim_{R\to+\infty} f(\lambda_1,\ldots,\lambda_{n-1},\lambda_n+R)=\left(\frac{k}{l}\right)^{\frac{1}{k-l}}.$$

Since f is symmetric, by (1.5), (1.8) and (1.9) we have

$$f(\lambda)\le f(\mathbf{1})+\sum f_i(\mathbf{1})(\lambda_i-1)=\sum f_i(\mathbf{1})\lambda_i=\frac{1}{n}\sum\lambda_i \quad \text{in } K\subset K_1 \tag{1.11}$$

and

$$\sum f_i(\lambda)=f(\lambda)+\sum f_i(\lambda)(1-\lambda_i)\ge f(\mathbf{1})=1 \quad \text{in } K. \tag{1.12}$$

Moreover, (1.4) and $f(0)=0$ imply that

$$f>0 \quad \text{in } K_n^+. \tag{1.13}$$

In this paper we shall focus on the case of finding complete hypersurfaces satisfying (1.1)–(1.2) with positive hyperbolic principal curvatures everywhere; for convenience we shall call such hypersurfaces *(hyperbolically) locally strictly convex*. In Part II [8] we will allow f satisfying (1.4)–(1.10) and general cones K.

Before we state our first result we need to explain the orientation of hypersurfaces under consideration. In this paper all hypersurfaces in $\mathbb{H}^{n+1}$ we consider are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+\setminus\Sigma$, and the (both hyperbolic and Euclidean) principal curvatures of Σ are calculated with respect to this normal vector field.

Theorem 1.1 *Let Σ be a complete locally strictly convex C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Then Σ is the (vertical) graph of a function $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$, $u>0$ in Ω and $u=0$ on $\overline{\Omega}$, for some domain $\Omega\subset\mathbb{R}^n$:*

$$\Sigma=\left\{(x,u(x))\in\mathbb{R}^{n+1}_+ : x\in\Omega\right\}$$

such that

$$\{\delta_{ij}+u_iu_j+uu_{ij}\}>0 \quad \text{in } \Omega. \tag{1.14}$$

That is, the function $u^2+|x|^2$ is strictly convex. Moreover,

$$e^u\sqrt{1+|Du|^2}\le\max\left\{\max_{\overline{\Omega}} e^u,\ \max_{\partial\Omega}\sqrt{1+|Du|^2}\right\} \quad \text{in } \Omega. \tag{1.15}$$

According to Theorem 1.1, problem (1.1)–(1.2) for complete locally strictly convex hypersurfaces reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u,Du,u)=\frac{\sigma}{u},\quad u>0 \text{ in } \Omega\subset\mathbb{R}^n \tag{1.16}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.17}$$

In particular, the asymptotic boundary Γ must be the boundary of some bounded domain Ω in $\mathbb{R}^n$. The exact formula of G will be given in Sect. 2 (see (2.9)).

We seek solutions of (1.16) satisfying (1.14). Following the literature we call such solutions *admissible*. By [3] condition (1.4) implies that (1.16) is elliptic for admissible solutions. Our goal is to show that the Dirichlet problem (1.16)–(1.17) admits smooth admissible solutions for all $0 < \sigma < 1$ which is also a necessary condition by the comparison principle under conditions (1.8) and (1.9), as we shall see in Sect. 3.

Our main result of the paper may be stated as follows.

Theorem 1.2 *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that $\sigma \in (0, 1)$ satisfies $\sigma^2 > \frac{1}{8}$. Under conditions (1.4)–(1.10) with $K = K_n^+$, there exists a complete locally strictly convex hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)–(1.2) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma]| \le C \quad \text{on } \Sigma. \tag{1.18}$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.16)–(1.17). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\begin{aligned} &\sqrt{1+|Du|^2} \le \frac{1}{\sigma}, \qquad u|D^2u| \le C \quad \text{in } \Omega, \\ &\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.19}$$

Theorem (1.2) holds for a large family of $f = \frac{1}{N}\sum_{l=1}^{N} f_l$ where each f_l consisting of sums and "concave products" (that is of the form $(f_1 \cdots f_{N_l})^{\frac{1}{N_l}}$) where each f_l satisfies (1.4)–(1.10). For Gauss curvature, $f(\lambda) = (\Pi\lambda_i)^{\frac{1}{n}}$, Theorem 1.2 was proved by Rosenberg and Spruck [9] who in fact allowed all $\sigma \in (0, 1)$.

As we shall see in Sect. 2, (1.16) is singular where $u = 0$. It is therefore natural to approximate the boundary condition (1.17) by

$$u = \epsilon > 0 \quad \text{on } \partial\Omega. \tag{1.20}$$

When ϵ is sufficiently small, the Dirichlet problem (1.16), (1.20) is solvable for all $\sigma \in (0, 1)$.

Theorem 1.3 *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $\sigma \in (0, 1)$. Suppose f satisfies (1.4)–(1.10) with $K = K_n^+$. Then for any $\epsilon > 0$ sufficiently small, there exists an admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.16), (1.20). Moreover, u^ϵ satisfies the a priori estimates*

$$\sqrt{1+|Du^\epsilon|^2} \le \frac{1}{\sigma} + \epsilon C \quad \text{in } \Omega \tag{1.21}$$

and

$$u^\epsilon |D^2 u^\epsilon| \le \frac{C}{\epsilon^2} \quad in\ \Omega \tag{1.22}$$

where C is independent of ϵ.

The organization of the paper is as follows. Section 2 summarizes the basic information about vertical and radial graphs that we will need in the sequel. In Sect. 3 we prove global gradient bounds and some sharp estimates on the vertical component of the upward normal near the boundary. These are essential for the boundary second derivative estimates which we derive in Sect. 4. Here we make essential use of the exact form of the linearized operator to derive the mixed normal-tangential estimates and assumption (1.10) for the pure normal second derivative estimate. In Sect. 5 we prove a maximum principle for the maximum hyperbolic principle curvature. Our approach uses radial graphs and is new and rather delicate. It is here that we have had to restrict the allowable $\sigma \in (0, 1)$ to $\sigma^2 > \frac{1}{8}$. Otherwise our approach is completely general and we expect Theorem 1.2 is valid for all $\sigma \in (0, 1)$. Because the linearized operator is not necessarily invertible we cannot prove Theorem 1.2 by the continuity method directly. To overcome this difficulty we construct an iterative procedure which is carried out in Sect. 6. Because of this, we have derived all our estimates for a fairly general class of hypersurfaces of prescribed curvature as a function of position.

Many of the techniques developed in this paper and its subsequent extension to general curvature functions [8] have application to many other problems.

2 Formulas for Hyperbolic Principal Curvatures

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$ and g the induced metric on Σ from $\mathbb{H}^{n+1}$. For convenience we call g the hyperbolic metric, while the Euclidean metric on Σ means the induced metric from $\mathbb{R}^{n+1}$. We use X and u to denote the position vector and *height function*, defined as

$$u = X \cdot \mathbf{e},$$

of Σ in $\mathbb{R}^{n+1}$, respectively. Here and throughout this paper, $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '$\cdot$' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. We assume Σ is orientable and let $\mathbf{n}$ be a fixed global unit normal vector field to Σ with respect to the hyperbolic metric. This also determines an Euclidean unit normal ν to Σ by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote $\nu^{n+1} = \mathbf{e} \cdot \nu$.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a local orthnormal frame of vector fields on (Σ, g). The second fundamental form of Σ is locally given by

$$h_{ij} := \langle \nabla_{\mathbf{e}_i} \mathbf{e}_j, \mathbf{n} \rangle$$

where $\langle\cdot,\cdot\rangle$ and ∇ denote the metric and Levi-Civita connection of $\mathbb{H}^{n+1}$ respectively. The (hyperbolic) principal curvatures of Σ, denoted as $\kappa[\Sigma]=(\kappa_1,\dots,\kappa_n)$, are the eigenvalues of the second fundamental form. The relation between $\kappa[\Sigma]$ and the Euclidean principal curvatures $\kappa^E[\Sigma]=(\kappa_1^E,\dots,\kappa_n^E)$ is given by

$$\kappa_i = u\kappa_i^E + \nu^{n+1}, \quad 1\le i\le n. \tag{2.1}$$

We shall derive equations for Σ based on this formula when Σ satisfies (1.1).

2.1 Vertical Graphs

Suppose Σ is locally represented as the graph of a function $u\in C^2(\Omega)$, $u>0$, in a domain $\Omega\subset\mathbb{R}^n$:

$$\Sigma=\{(x,u(x))\in\mathbb{R}^{n+1}: x\in\Omega\}.$$

In this case we take ν to be the upward (Euclidean) unit normal vector field to Σ:

$$\nu=\left(\frac{-Du}{w},\frac{1}{w}\right),\quad w=\sqrt{1+|Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$g_{ij}^E=\delta_{ij}+u_iu_j,$$

and

$$h_{ij}^E=\frac{u_{ij}}{w}.$$

According to [4], the Euclidean principal curvatures $\kappa^E[\Sigma]$ are the eigenvalues of the symmetric matrix $A^E[u]=[a_{ij}^E]$:

$$a_{ij}^E := \frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}, \tag{2.2}$$

where

$$\gamma^{ij}=\delta_{ij}-\frac{u_iu_j}{w(1+w)}.$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible with inverse

$$\gamma_{ij}=\delta_{ij}+\frac{u_iu_j}{1+w}$$

which is the square root of $\{g_{ij}^E\}$, i.e., $\gamma_{ik}\gamma_{kj}=g_{ij}^E$. From (2.1) we see that the hyperbolic principal curvatures $\kappa[u]$ of Σ are the eigenvalues of the matrix $A^{\mathbf{v}}[u]=\{a_{ij}^{\mathbf{v}}[u]\}$:

$$a_{ij}^{\mathbf{v}}[u] := \frac{1}{w}\left(\delta_{ij}+u\gamma^{ik}u_{kl}\gamma^{lj}\right). \tag{2.3}$$

For K as in Sect. 1, let $\mathcal{S}$ be the vector space of $n \times n$ symmetric matrices and

$$\mathcal{S}_K = \{A \in \mathcal{S} : \lambda(A) \in K\},$$

where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of A. Define a function F by

$$F(A) = f(\lambda(A)), \quad A \in \mathcal{S}_K. \tag{2.4}$$

We recall some properties of F. Throughout the paper we denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \qquad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \tag{2.5}$$

The matrix $\{F^{ij}(A)\}$, which is symmetric, has eigenvalues $f_1, \dots, f_n$, and therefore is positive definite for $A \in \mathcal{S}_K$ if f satisfies (1.4), while (1.5) implies that F is concave for $A \in \mathcal{S}_K$ (see [3]), that is

$$F^{ij,kl}(A)\xi_{ij}\xi_{kl} \le 0, \quad \forall \{\xi_{ij}\} \in \mathcal{S},\ A \in \mathcal{S}_K. \tag{2.6}$$

We have

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i, \tag{2.7}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{2.8}$$

Finally, the function G in (1.16) is determined by

$$G(D^2u, Du, u) = \frac{1}{u} F(A^{\mathbf{v}}[u]), \tag{2.9}$$

where $A^{\mathbf{v}}[u] = \{a^{\mathbf{v}}_{ij}[u]\}$ is given by (2.3). Note that

$$\begin{aligned} G^{st}[u] := \frac{\partial G}{\partial u_{st}} &= \frac{1}{w} F^{ij} \gamma^{is} \gamma^{tj}, \\ G^{st}[u]u_{st} &= \frac{1}{u}\left(F^{ij}a_{ij} - \frac{1}{w}\sum F^{ii}\right), \\ G_u &= -\frac{1}{u^2 w}\sum F^{ii} \end{aligned} \tag{2.10}$$

and

$$G^{pq,st}[u] := \frac{\partial^2 G}{\partial u_{pq} \partial u_{st}} = \frac{u}{w^2} F^{ij,kl} \gamma^{is} \gamma^{tj} \gamma^{kp} \gamma^{ql} \tag{2.11}$$

where $F^{ij} = F^{ij}(A^{\mathbf{v}}[u])$, etc. It follows that, under condition (1.4), (1.16) is elliptic for u if $A^{\mathbf{v}}[u] \in \mathcal{S}_K$, while (1.5) implies that $G(D^2u, Du, u)$ is concave with respect to D^2u.

For later use in Sect. 6 note that if u is a solution of

$$\tilde{G}(D^2u, Du, u) = G(D^2u, Du, u) - \psi(x, u) = 0, \tag{2.12}$$

then from (2.10),

$$\tilde{G}_u = \frac{1}{u^2}\left(\Psi - u\Psi_u - \frac{1}{w}\sum f_i\right) \tag{2.13}$$

where $\Psi(x,u) = u\psi(x,u)$. Since $\sum f_i \geq 1$, we obtain from (2.13)

$$\tilde{G}_u \leq \frac{1}{u^2}\left(\Psi - u\Psi_u - \frac{1}{w}\right). \tag{2.14}$$

2.2 Radial Graphs

Let ∇ denote the covariant derivative on the standard unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and $y = \mathbf{e}\cdot\mathbf{z}$ for $\mathbf{z}\in\mathbb{S}^n\subset\mathbb{R}^{n+1}$. Let $\tau_1,\ldots,\tau_n$ be an orthnormal local frame of smooth vector fields on $\mathbb{S}^n$ so that $\tau_i\cdot\tau_j = \delta_{ij}$. For a function v on $\mathbb{S}^n$, we denote $v_i = \nabla_i v = \nabla_{\tau_i} v$, $v_{ij} = \nabla_j\nabla_i v$, etc.

Suppose that locally Σ is a radial graph over the upper hemisphere $\mathbb{S}^n_+\subset\mathbb{R}^{n+1}$, i.e., it is locally represented as

$$X = e^v\mathbf{z}, \quad \mathbf{z}\in\mathbb{S}^n_+\subset\mathbb{R}^{n+1}. \tag{2.15}$$

The Euclidean metric, outward unit normal vector, and second fundamental from of Σ are

$$\begin{aligned} g^E_{ij} &= e^{2v}(\delta_{ij} + v_i v_j),\\ \nu &= \frac{\mathbf{z} - \nabla v}{w}, \quad w = (1 + |\nabla v|^2)^{1/2}, \end{aligned}$$

and

$$h^E_{ij} = \frac{1}{w}e^v(v_{ij} - v_i v_j - \delta_{ij}),$$

respectively. Therefore the Euclidean principal curvatures are the eigenvalues of the matrix

$$a^E_{ij} = \frac{1}{w}e^{-v}\gamma^{ik}(v_{kl} - v_k v_l - \delta_{kl})\gamma^{lj} = \frac{1}{w}e^{-v}(\gamma^{ik}v_{kl}\gamma^{lj} - \delta_{ij}) \tag{2.16}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{v_i v_j}{w(1+w)}.$$

Note that the height function $u = ye^v$. We see that the hyperbolic principal curvatures are the eigenvalues of the matrix $A^{\mathbf{s}}[v] = \{a^{\mathbf{s}}_{ij}[v]\}$:

$$a^{\mathbf{s}}_{ij}[v] := \frac{1}{w}(y\gamma^{ik}v_{kl}\gamma^{lj} - \mathbf{e}\cdot\nabla v\,\delta_{ij}). \tag{2.17}$$

In this case (1.1) takes the form

$$F(A^{\mathbf{s}}[v]) = \sigma. \tag{2.18}$$

3 Locally Convex Hypersurfaces. C^1 Estimates

3.1 Equidistance Spheres

There are two important facts which will be used repeatedly. One is the invariance of (1.1) under scaling $X \mapsto \lambda X$ in $\mathbb{R}^{n+1}$, as it is an isometry of $\mathbb{H}^{n+1}$. The other is that the Euclidean spheres, known as equidistance spheres, have constant hyperbolic principal curvatures. Let $B_R(a)$ be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ where $\sigma \in (0, 1)$ and $S = \partial B_R(a) \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S] = \sigma$ for all $1 \leq i \leq n$ with respect to its outward normal. These spheres may serve as barriers in many situations. Especially, we have the following estimates which were first derived in [6] for hypersurfaces of constant mean curvature.

Lemma 3.1 *Suppose f satisfies* (1.4), (1.8) *and* (1.9). *Let Σ be a hypersurface in $\mathbb{H}^{n+1}$ with $\kappa[\Sigma] \in K$ and*

$$\sigma_1 \leq f(\kappa[\Sigma]) \leq \sigma_2$$

where $0 \leq \sigma_1 \leq \sigma_2 \leq 1$ are constants, and $\partial\Sigma \subset P(\epsilon) \equiv \{x_{n+1} = \epsilon\}$, $\epsilon \geq 0$. Let Ω be the region in $\mathbb{R}^n$ bounded by the projection of $\partial\Sigma$ to $\mathbb{R}^n = \{x_{n+1} = 0\}$ (such that $\mathbb{R}^n \setminus \Omega$ contains an unbounded component), and u denote the height function of Σ.

(i) *For any point $(x, u) \in \Sigma$,*

$$\frac{\epsilon\sigma_2}{1+\sigma_2} + d(x)\sqrt{\frac{1-\sigma_2}{1+\sigma_2}} \leq u \leq \frac{L}{2}\sqrt{\frac{1-\sigma_1}{1+\sigma_1}} + \epsilon, \tag{3.1}$$

where $d(x)$ and L denote the distance from $x \in \mathbb{R}^n$ to $\partial\Omega$ and the (Euclidean) diameter of Ω, respectively.

(ii) *Assume that $\partial\Sigma \in C^2$. For $\epsilon > 0$ sufficiently small,*

$$\begin{aligned}\sigma_1 - \frac{\epsilon\sqrt{1-\sigma_1^2}}{r_1} - \frac{\epsilon^2(1+\sigma_1)}{r_1^2} \\ < \nu^{n+1} < \sigma_2 + \frac{\epsilon\sqrt{1-\sigma_2^2}}{r_2} + \frac{\epsilon^2(1-\sigma_2)}{r_2^2} \quad \text{on } \partial\Sigma\end{aligned} \tag{3.2}$$

where r_1 and r_2 are the maximal radii of exterior and interior spheres to $\partial\Omega$, respectively. In particular, if $\sigma_1 = \sigma_2 = \sigma$ then $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\epsilon \to 0$.

While Lemma 3.1 was proved in [6] only for the mean curvature case, the proof remains valid for more general symmetric functions of principal curvatures with minor modifications. So we omit the proof here.

Another important class of hypersurfaces of constant principal curvatures are the horospheres $P(\epsilon) \equiv \{x_{n+1} = \epsilon\}$, $\epsilon > 0$. Indeed, from (2.1) we see that $\kappa[P(\epsilon)] = \mathbf{1}$. By the comparison principle we immediately obtain the following necessary condition for the solvability of problem (1.1)–(1.2).

Lemma 3.2 *Suppose that f satisfies* (1.4) *and* (1.8), *and that there is a hypersurface Σ in $\mathbb{H}^{n+1}$ which satisfies* (1.1) *and* (1.2) *with $\kappa[\Sigma] \in K$. Then $\sigma < 1$.*

3.2 Locally Strictly Convex Hypersurfaces

We now consider hypersurfaces of positive principal curvatures in $\mathbb{H}^{n+1}$; we call such hypersurfaces *locally strictly convex.*

Lemma 3.3 *Let $\Sigma \subset \{x_{n+1} \geq c\}$ be a locally strictly convex hypersurface of class C^2 in $\mathbb{H}^{n+1}$ with compact* (*asymptotic*) *boundary $\partial\Sigma \subset \{x_{n+1} = c\}$ for some constant $c \geq 0$. Then Σ is a vertical graph. In particular, $\partial\Sigma$ must be the boundary of a bounded domain in $\{x_{n+1} = c\}$.*

Proof Let T be the set of $t \geq c$ such that $\Sigma_t := \Sigma \cap \{x_{n+1} \geq t\}$ is a vertical graph and let t_0 be the minimum of T which is clearly nonempty. Suppose $t_0 > c$. Then there must be a point $p \in \partial\Sigma_{t_0}$ with $\nu^{n+1}(p) = 0$, that is, the normal vector to Σ at p is horizontal. It follows from (2.1) that $\kappa_i^E = \kappa_i/t_0 > 0$ for all $1 \leq i \leq n$ at p. On the other hand, the curve $\Sigma \cap P$ (near p) clearly has nonpositive curvature at p (with respect to the normal $\nu(p)$), where P is the plane through p spanned by $\mathbf{e}$ and $\nu(p)$. This is a contradiction, proving that $t_0 = c$. $\square$

By the formula (2.3) the graph of a function u is locally strictly convex if and only if the function $U = |x|^2 + u^2$ is (locally) strictly convex, i.e., its Hessian D^2U is positive definite. We define the class of *admissible* functions in a domain $\Omega \subset \mathbb{R}^n$ as

$$\mathcal{A}(\Omega) = \left\{u \in C^2(\Omega) : u > 0, |x|^2 + u^2 \text{ is locally strictly convex in } \Omega\right\}. \tag{3.3}$$

By the convexity of $|x|^2 + u^2$ we immediately have

$$|Du| \leq \frac{1}{u}\Big(L + \max_{\partial\Omega} u|Du|\Big) \tag{3.4}$$

where L is the diameter of Ω. The following gradient estimate, which improves (3.4) in the sense that it is independent of the (positive) lower bound of u, will be crucial to our results.

Lemma 3.4 *Let $u \in \mathcal{A}(\Omega)$. Then*

$$e^u\sqrt{1 + |Du|^2} \leq \max\Big\{\sup_{\Omega} e^u, \max_{\partial\Omega} e^u\sqrt{1 + |Du|^2}\Big\}. \tag{3.5}$$

Proof If $e^u\sqrt{1 + |Du|^2}$ attains its maximum at an interior point $x_0 \in \Omega$ then at x_0,

$$\sum_j u_j(\delta_{ij} + u_i u_j + u_{ij}) = e^{-u}\frac{\partial}{\partial x_i}\Big(e^u\sqrt{1 + |Du|^2}\Big) = 0, \quad \forall 1 \leq i \leq n.$$

If follows that $Du(x_0) = 0$ as the matrix $\{\delta_{ij} + u_i u_j + u_{ij}\}$ is positive definite. $\square$

Lemma 3.5 *Let $u \in \mathcal{A}(\Omega)$ satisfy*

$$\begin{cases} \sigma_1 \le f(\kappa[u]) \le \sigma_2, & \text{in } \Omega, \\ u \ge \epsilon, & \text{in } \Omega, \\ u = \epsilon, & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $0 < \sigma_1 \le \sigma_2 < 1$, $\epsilon \ge 0$ and $\partial\Omega \in C^2$. Suppose f satisfies (1.4), (1.8) and (1.9). Then, for ϵ sufficiently small,

$$\frac{1}{\sqrt{1+|Du|^2}} \ge \sigma_1 - C\epsilon(\epsilon + \sqrt{1-\sigma_1^2}) \quad \text{in } \overline{\Omega}. \tag{3.7}$$

Proof By Lemma 3.1 we have, for ϵ sufficiently small,

$$\sqrt{1+|Du|^2} \ge \frac{2}{1+\sigma_2} \quad \text{on } \partial\Omega. \tag{3.8}$$

Fix $\lambda > 0$ (sufficiently large) such that

$$\frac{L}{2\lambda}\sqrt{\frac{1-\sigma_1}{1+\sigma_1}} \le \ln \frac{2}{1+\sigma_2} \tag{3.9}$$

where L is the diameter of Ω, and let

$$u^\lambda(x) = \frac{1}{\lambda} u(\lambda x), \quad x \in \Omega^\lambda$$

where $\Omega^\lambda = \frac{\Omega}{\lambda}$. Then $\kappa[u^\lambda](x) = \kappa[u](\lambda x)$ in Ω^λ. It follow that $u^\lambda \in \mathcal{A}(\Omega^\lambda)$ and

$$\begin{cases} \sigma_1 \le f(\kappa[u^\lambda]) \le \sigma_2, & \text{in } \Omega^\lambda, \\ u^\lambda \ge \frac{\epsilon}{\lambda}, & \text{in } \Omega^\lambda, \\ u^\lambda = \frac{\epsilon}{\lambda}, & \text{on } \partial\Omega^\lambda. \end{cases} \tag{3.10}$$

Applying Lemma 3.1 to u^λ, we have by (3.8) and (3.9),

$$u^\lambda - \frac{\epsilon}{\lambda} \le \frac{L}{2\lambda}\sqrt{\frac{1-\sigma_1}{1+\sigma_1}} \le \max_{\partial\Omega^\lambda} \ln \sqrt{1+|Du^\lambda|^2} \quad \text{in } \Omega^\lambda$$

or

$$\sup_{\Omega^\lambda} e^{u^\lambda} \le \max_{\partial\Omega^\lambda} e^{\frac{\varepsilon}{\lambda}} \sqrt{1+|Du^\lambda|^2}. \tag{3.11}$$

By (3.11), Lemma 3.4 and Lemma 3.1 (part ii, formula (3.2)),

$$\begin{aligned} \frac{1}{\sqrt{1+|Du^\lambda|^2}} &\ge e^{(u^\lambda - \frac{\epsilon}{\lambda})} \min_{\partial\Omega^\lambda} \frac{1}{\sqrt{1+|Du^\lambda|^2}} \\ &\ge \min_{\partial\Omega^\lambda} \frac{1}{\sqrt{1+|Du^\lambda|^2}} \\ &\ge \sigma_1 - C\epsilon\Big(\epsilon + \sqrt{1-\sigma_1^2}\Big). \end{aligned} \tag{3.12}$$

This proves (3.7). □

4 Boundary Estimates for Second Derivatives

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. In this section we establish boundary estimates for second derivatives of admissible solutions to the Dirichlet problem

$$\begin{cases} G(D^2u, Du, u) = \psi(x, u), & \text{in } \Omega, \\ u = \epsilon, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where G is defined in (2.9) and $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R}_+)$. We assume that ψ satisfies the following conditions:

$$0 < \psi(x, z) \le \frac{1-\epsilon_1}{z}, \qquad |D_x\psi(x, z)| + |\psi_z(x, z)| \le \frac{C}{z^2}, \quad \forall x \in \overline{\Omega_\delta},\ z \in (0, \epsilon_1) \tag{4.2}$$

and

$$\psi(x, z) = \frac{\sigma}{z}, \quad \forall x \in \partial\Omega,\ z \in (0, \epsilon_1), \tag{4.3}$$

where C is a large fixed constant, $\epsilon_1 > 0$ is a small fixed constant, $\delta = \frac{\epsilon}{C^2}$ and

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}.$$

Remark 4.1 We have in mind $\Psi := u\psi(x, u) = \sigma + M(u - v(x))$ where $M \in [0, \frac{1}{\epsilon}]$ and $v = \epsilon$ on $\partial\Omega$, $|\nabla v| \le C$ in Ω. We will need this generality because (see (2.14)) $\tilde{G}_u = G_u - \psi_u$ may be positive in Ω causing us some trouble when we try to prove Theorem 1.3 using the method of continuity directly. Note also that conditions (4.2), (4.3) imply $\operatorname{osc}_{\Omega_\delta} \Psi \le \frac{C}{\epsilon}\delta \le \frac{1}{C}$ which is used at the end of the proof when we need to appeal to Lemma 3.5.

Theorem 4.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^3$, and $u \in C^3(\bar{\Omega}) \cap \mathcal{A}(\Omega)$ a solution of problem* (4.1). *Suppose that f satisfies* (1.4)–(1.10) *and ψ satisfies* (4.2) *and* (4.3). *Then, if ϵ is sufficiently small,*

$$u|D^2u| \le C \quad \text{on } \partial\Omega \tag{4.4}$$

where C is independent of ϵ.

The notation of this section follows that of Sect. 2.1. We first consider the partial linearized operator of G at u:

$$L = G^{st}\partial_s\partial_t + G^s\partial_s,$$

where G^{st} is defined in (2.10) and

$$G^s := \frac{\partial G}{\partial u_s} = -\frac{u_s}{w^2u}F^{ij}a_{ij} - \frac{2}{wu}F^{ij}a_{ik}\left(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\right) + \frac{2}{w^2u}F^{ij}u_i\gamma^{sj} \tag{4.5}$$

by the formula (2.21) in [7]. Here $F^{ij} = F^{ij}(A^{\mathbf{v}}[u])$ and $a_{ij} = a^{\mathbf{v}}_{ij}[u]$. Since $\{F^{ij}\}$ and $\{a_{ij}\}$ are both positive definite and can be diagonalized simultaneously, we see that

$$F^{ij} a_{ik} \xi_k \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n. \tag{4.6}$$

Moreover, by the concavity of f we have the following inequality similar to Lemma 2.3 in [7],

$$\sum |G^s| \leq \frac{C}{u}\Big(1 + \sum F^{ii}\Big). \tag{4.7}$$

Since $\gamma^{sj} u_s = u_j/w$,

$$G^s u_s = \frac{1}{u}\left\{\left(\frac{1}{w^2} - 1\right) F^{ij} a_{ij} - \frac{2}{w^2} F^{ij} a_{ik} u_k u_j + \frac{2}{w^3} F^{ij} u_i u_j\right\}. \tag{4.8}$$

It follows from (2.10) and (4.8) that

$$Lu = \frac{1}{w^2 u} F^{ij} a_{ij} - \frac{1}{wu} \sum F^{ii} - \frac{2}{w^2 u} F^{ij} a_{ik} u_k u_j + \frac{2}{w^3 u} F^{ij} u_i u_j. \tag{4.9}$$

Lemma 4.3 *Suppose that f satisfies* (1.4), (1.5), (1.8) *and* (1.9). *Then*

$$L\left(1 - \frac{\epsilon}{u}\right) \leq -\frac{\epsilon(1 - \frac{u\psi}{w})}{2wu^3}\Big(1 + \sum F^{ii}\Big) \quad \text{in } \Omega. \tag{4.10}$$

Proof By (4.9), (4.6) and (1.9),

$$\begin{aligned} L\frac{1}{u} &= -\frac{1}{u^2} Lu + \frac{2}{u^3} G^{st} u_s u_t \\ &= -\frac{1}{u^2} Lu + \frac{2}{w^3 u^3} F^{ij} u_i u_j \\ &\geq \frac{1}{wu^3}\left(\sum F^{ii} - \frac{u\psi}{w}\right). \end{aligned} \tag{4.11}$$

Thus (4.10) follows from (1.12) and (4.2). □

Lemma 4.4 *For* $1 \leq i, j \leq n$,

$$L(x_i u_j - x_j u_i) = (\psi_u - G_u)(x_i u_j - x_j u_i) + x_i \psi_{x_j} - x_j \psi_{x_i} \tag{4.12}$$

where

$$G_u := \frac{\partial G}{\partial u} = -\frac{1}{wu^2} \sum_i F^{ii}. \tag{4.13}$$

Proof For $\theta \in \mathbb{R}$ let

$$y_i = x_i \cos\theta - x_j \sin\theta,$$

$$
\begin{aligned}
y_j &= x_i \sin\theta + x_j \cos\theta, \\
y_k &= x_k, \quad \forall k \neq i, j.
\end{aligned}
$$

Differentiate the equation

$$
G(D^2u(y), Du(y), u(y)) = \psi(y, u(y))
$$

with respect to θ and set $\theta = 0$ afterwards. We obtain

$$
\begin{aligned}
&L(x_i u_j - x_j u_i) + G_u(x_i u_j - x_j u_i) \\
&\quad = (L + G_u)\frac{\partial u}{\partial \theta}\bigg|_{\theta=0} = \frac{\partial}{\partial \theta}\psi(y, u(y))\bigg|_{\theta=0}
\end{aligned}
$$

which yields (4.12). □

Proof of Theorem 4.2 Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$
x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}). \tag{4.14}
$$

Since $u = \epsilon$ on $\partial\Omega$, we see that $u(x', \rho(x')) = \epsilon$ and

$$
u_{\alpha\beta}(0) = -u_n \rho_{\alpha\beta}, \quad \alpha, \beta < n. \tag{4.15}
$$

Consequently,

$$
|u_{\alpha\beta}(0)| \le C|Du(0)|, \quad \alpha, \beta < n \tag{4.16}
$$

where C depends only on the maximal (Euclidean principal) curvature of $\partial\Omega$.

Next, following [2] we consider for fixed $\alpha < n$ the operator

$$
T = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta\partial_n - x_n\partial_\beta). \tag{4.17}
$$

We have

$$
\begin{aligned}
|Tu| &\le C, \quad \text{in } \Omega \cap B_\delta(0), \\
|Tu| &\le C|x|^2, \quad \text{on } \partial\Omega \cap B_\delta
\end{aligned} \tag{4.18}
$$

since $u = \epsilon$ on $\partial\Omega$. By Lemma 4.4 and (4.2), (4.13),

$$
\begin{aligned}
|L(Tu)| &= |TG(D^2u, Du, u) - G_u Tu| \\
&= |T\psi(x, u) - G_u Tu| \\
&\le \frac{C}{u^2}\Big(1 + \sum F^{ii}\Big).
\end{aligned} \tag{4.19}
$$

Let

$$\phi = A\left(1 - \frac{\epsilon}{u}\right) + B|x|^2 \pm Tu.$$

By (4.2), (4.7), (4.19) and Lemma 4.3,

$$L\phi \le \left(-\frac{\epsilon_1 \epsilon A}{2w} + Cu^2 B(u+\delta) + Cu\right)\frac{(1+\sum F^{ii})}{u^3} \quad \text{in } \Omega \cap B_\delta. \tag{4.20}$$

We first choose $B = \frac{C_1}{\delta^2}$ with $C_1 = C$ the constant in (4.18) so that $\phi \ge 0$ on $\partial(\Omega \cap B_\delta)$. Then choosing $A \gg C_1/\epsilon_1$ makes $L\phi \le 0$ in $\Omega \cap B_\delta$.

By the maximum principle $\phi \ge 0$ in $\Omega \cap B_\delta$. Since $\phi(0) = 0$, we have $\phi_n(0) \ge 0$ which gives

$$|u_{\alpha n}(0)| \le \frac{Au_n(0)}{u(0)}. \tag{4.21}$$

Finally to estimate $|u_{nn}(0)|$ we use our hypothesis (1.10). We may assume $[u_{\alpha\beta}(0)]$, $1 \le \alpha, \beta < n$, to be diagonal. Note that $u_\alpha(0) = 0$ for $\alpha < n$. We have at $x = 0$

$$A^{\mathrm{v}}[u] = \frac{1}{w}\begin{bmatrix} 1+uu_{11} & 0 & \dots & \frac{uu_{1n}}{w} \\ 0 & 1+uu_{22} & \dots & \frac{uu_{2n}}{w} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{uu_{n1}}{w} & \frac{uu_{n2}}{w} & \dots & 1+\frac{uu_{nn}}{w^2} \end{bmatrix}.$$

By Lemma 1.2 in [3], if $\epsilon u_{nn}(0)$ is very large, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A^{\mathrm{v}}[u]$ are asymptotically given by

$$\begin{aligned} \lambda_\alpha &= \frac{1}{w}(1 + \epsilon u_{\alpha\alpha}(0)) + o(1), \quad \alpha < n, \\ \lambda_n &= \frac{\epsilon u_{nn}(0)}{w^3}\left(1 + O\left(\frac{1}{\epsilon u_{nn}(0)}\right)\right). \end{aligned} \tag{4.22}$$

By (4.16) and assumptions (1.9)–(1.10), for all $\epsilon > 0$ sufficiently small,

$$\epsilon\psi(0,\epsilon) = \frac{1}{w}F(wA^{\mathrm{v}}[u](0)) \ge \frac{1}{w}\left(1 + \frac{\varepsilon_0}{2}\right)$$

if $\epsilon u_{nn}(0) \ge R$ where R is a uniform constant. By the hypothesis (4.3) and Lemma 3.5, however,

$$\sigma \ge \frac{1}{w}\left(1+\frac{\varepsilon_0}{2}\right) \ge (\sigma - C\epsilon)\left(1+\frac{\varepsilon_0}{2}\right) > \sigma$$

which is a contradiction. Therefore

$$|u_{nn}(0)| \le \frac{R}{\epsilon}.$$

The proof is complete. □

5 Global Estimates for Second Derivatives

In this section we prove a maximum principle for the largest hyperbolic principal curvature $\kappa_{\max}(x)$ of solutions of general curvature equations. For later applications we keep track of how the estimates depend on the right hand side of (4.1). We consider

$$M(x)=\frac{\kappa_{\max}(x)}{u^2(x)(\nu^{n+1}(x)-a)}. \tag{5.1}$$

Theorem 5.1 *Let $u\in C^4(\bar{\Omega})$ be a positive solution of $f(\kappa[u])=\Psi(x,u)$ where f satisfies (1.4)–(1.9), $A^{\mathbf{v}}[u]\in\mathcal{S}_K$, $\nu^{n+1}\geq 2a$ and $\Psi\geq\sigma_0>0$. Suppose M achieves its maximum at an interior point $x_0\in\Omega$. Then either $\kappa_{\max}(x_0)\leq 16(a+\frac{1}{a})$ or*

$$M^2(x_0)\leq C\frac{\{(|\Psi_x|+|\Psi_u|)^2+(|\Psi_{xx}|+|\Psi_{ux}|+|\Psi_{uu}|)\}(x_0)}{u^2(x_0)} \tag{5.2}$$

where C is a controlled constant.

If we assume, for example, that

$$\Psi\geq\sigma_0,\qquad |\Psi_x|+|\Psi_u|\leq\frac{L_1}{\epsilon},\qquad |\Psi_{xx}|+|\Psi_{ux}|+|\Psi_{uu}|\leq\frac{L_2}{\epsilon^2}\quad \text{in } \Omega \tag{5.3}$$

with L_1, L_2 independent of ϵ, we obtain using Theorem 4.2.

Theorem 5.2 *Let $u\in C^4(\bar{\Omega})$ be a solution of problem (4.1) with $A^{\mathbf{v}}[u]\in\mathcal{S}_K$. Suppose that f satisfies (1.4)–(1.9) and ψ satisfies (4.2), (5.3). Then if ϵ is sufficiently small,*

$$u|D^2u|\leq C\Big(1+\max_{\partial\Omega}u|D^2u|\Big)\frac{u^2}{\epsilon^2}\quad \text{in } \Omega \tag{5.4}$$

where C is independent of ϵ.

We begin the proof of Theorem 5.1 which is long and computational.

Let Σ be the graph of u. For $x\in\Omega$ let $\kappa_{\max}(x)$ be the largest principal curvature of Σ at the point $X=(x,u(x))\in\Sigma$. We consider

$$M_0=\max_{x\in\overline{\Omega}}\frac{\kappa_{\max}(x)}{\phi(\eta,u)} \tag{5.5}$$

where $\eta=\mathbf{e}\cdot\nu$, ν is the upward (Euclidean) unit normal to Σ, and ϕ a smooth positive function to be chosen later. Suppose that M_0 is attained at an interior point $x_0\in\Omega$ and let $X_0=(x_0,u(x_0))$.

After a horizontal translation of the origin in $\mathbb{R}^{n+1}$, we may write Σ locally near X_0 as a radial graph

$$X=e^v\mathbf{z},\quad \mathbf{z}\in\mathbb{S}^n_+\subset\mathbb{R}^{n+1} \tag{5.6}$$

with $X_0=e^{v(\mathbf{z}_0)}\mathbf{z}_0$, $\mathbf{z}_0\in\mathbb{S}^n_+$, such that $\nu(X_0)=\mathbf{z}_0$.

In the rest of this section we shall follow the notation in Sect. 2.2 and rewrite the equation in (4.1) in the form

$$F(A^{\mathbf{s}}[v]) = \Psi \equiv u\psi \tag{5.7}$$

where $A^{\mathbf{s}}[v]$ is given in (2.17); henceforth we write $A[v] = A^{\mathbf{s}}[v]$ and $a_{ij} = a^{\mathbf{s}}_{ij}[v]$.

We choose an orthnormal local frame $\tau_1, \ldots, \tau_n$ around $\mathbf{z}_0$ on $\mathbb{S}^n_+$ such that $v_{ij}(\mathbf{z}_0)$ is diagonal. Since $\nu(X_0) = \mathbf{z}_0$, $\nabla v(\mathbf{z}_0) = 0$ and, by (2.17), at $\mathbf{z}_0$,

$$a_{ij} = y v_{ij} = \kappa_i \delta_{ij} \tag{5.8}$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at X_0. We may assume

$$\kappa_1 = \kappa_{\max}(X_0). \tag{5.9}$$

The function $\frac{a_{11}}{\phi}$, which is defined locally near $\mathbf{z}_0$, then achieves its maximum at $\mathbf{z}_0$ where, therefore

$$\left(\frac{a_{11}}{\phi}\right)_i = 0, \quad 1 \le i \le n \tag{5.10}$$

and

$$F^{ii}\left(\frac{a_{11}}{\phi}\right)_{ii} = \frac{1}{\phi} F^{ii} a_{11,ii} - \frac{\kappa_1}{\phi^2} F^{ii} \phi_{ii} \le 0. \tag{5.11}$$

Proposition 5.3 *At* $\mathbf{z}_0$,

$$\begin{aligned}
& y^2 \phi F^{ii} a_{11,ii} - y^2 \kappa_1 F^{ii} \phi_{ii} \\
&\quad = y^2 \phi F^{ii} a_{ii,11} + y(\phi_\eta \kappa_1 - \phi) F^{ii} y_j a_{ii,j} - 2y\phi F^{ii} y_1 a_{ii,1} \\
&\qquad + (y\phi_\eta - \phi)\kappa_1 \sum f_i \kappa_i^2 - \phi_{\eta\eta}\kappa_1 \sum f_i (y - \kappa_i)^2 y_i^2 \\
&\qquad + (\phi(\kappa_1^2 + 2y_1^2 + 1) - y^2 e^v \phi_u \kappa_1 - (1 + y^2)\phi_\eta \kappa_1) \sum f_i \kappa_i \\
&\qquad + (y\phi_\eta + y e^v \phi_u - (1 + 2y_1^2)\phi)\kappa_1 \sum f_i \\
&\qquad + (2y\phi_\eta + 2y e^v \phi_u - y^2 e^{2v} \phi_{uu} - 2y^2 e^v \phi_{u\eta} - 2\phi)\kappa_1 \sum f_i y_i^2 \\
&\qquad + 2(\phi - \phi_\eta \kappa_1 + y e^v \phi_{u\eta} \kappa_1) \sum f_i \kappa_i y_i^2.
\end{aligned} \tag{5.12}$$

Proof In what follows all calculations are evaluated at $\mathbf{z}_0$. Since $\nabla v = 0$, we have

$$w = 1, \qquad w_i = 0, \qquad w_{ij} = v_{ki} v_{kj}. \tag{5.13}$$

(Recall $w = \sqrt{1 + |\nabla v|^2}$.) Straightforward calculations show that

$$a_{ij,k} = y v_{ijk} + y_k v_{ij} - (\mathbf{e} \cdot \nabla v)_k \delta_{ij}, \tag{5.14}$$

$$a_{kk,ii} = y v_{kkii} + 2y_i v_{kki} + y_{ii} v_{kk} - (\mathbf{e} \cdot \nabla v)_{ii} - y v_{kk} v_{ii}^2 - 2y v_{kk}^3 \delta_{ki}. \tag{5.15}$$

Therefore,

$$\begin{aligned} a_{11,ii} - a_{ii,11} &= y(v_{11ii} - v_{ii11} - v_{11}v_{ii}^2 + v_{ii}v_{11}^2) \\ &\quad + y_{ii}v_{11} - y_{11}v_{ii} + 2(y_i v_{11i} - y_1 v_{ii1}) \\ &\quad + (\mathbf{e}\cdot\nabla v)_{11} - (\mathbf{e}\cdot\nabla v)_{ii}. \end{aligned} \tag{5.16}$$

We recall the following formulas

$$v_{ijk} = v_{ikj} = v_{kij}, \tag{5.17}$$

$$v_{kkii} = v_{iikk} + 2(v_{kk} - v_{ii}) \tag{5.18}$$

(where we have used the fact that $\nabla v = 0$) and, from [6],

$$\sum y_j^2 = 1 - y^2, \tag{5.19}$$

$$y_{ij} = -y\delta_{ij}, \tag{5.20}$$

$$(\mathbf{e}\cdot\nabla v)_i = y_i v_{ii}, \tag{5.21}$$

and

$$\begin{aligned} (\mathbf{e}\cdot\nabla v)_{ij} &= \mathbf{e}\cdot\tau_k v_{kij} - 2yv_{ij} - \mathbf{e}\cdot\tau_j v_i \\ &= y_k v_{ijk} - 2yv_{ij} - y_j v_i. \end{aligned} \tag{5.22}$$

By (5.14) and (5.8),

$$v_{iij} = \frac{a_{ii,j}}{y} - \frac{y_j\kappa_i}{y^2} + \frac{y_j\kappa_j}{y^2}, \tag{5.23}$$

$$(\mathbf{e}\cdot\nabla v)_{ii} = \frac{y_j a_{ii,j}}{y} - \left(1 + \frac{1}{y^2}\right)\kappa_i + \frac{1}{y^2}\sum_j \kappa_j y_j^2. \tag{5.24}$$

Plug these formulas into (5.16) and note that $F^{ij} = f_i\delta_{ij}$. We obtain

$$\begin{aligned} y^2F^{ii}a_{11,ii} &= y^2F^{ii}a_{ii,11} - yF^{ii}(y_j a_{ii,j} + 2y_1 a_{ii,1}) \\ &\quad + yF^{ii}(y_j a_{11,j} + 2y_i a_{11,i}) - \kappa_1\sum f_i\kappa_i^2 \\ &\quad + (\kappa_1^2 + 2y_1^2 + 1)\sum f_i\kappa_i - (1 + 2y_1^2)\kappa_1\sum f_i \\ &\quad + 2\sum f_i\kappa_i y_i^2 - 2\kappa_1\sum f_i y_i^2. \end{aligned} \tag{5.25}$$

Next, recall that $u = ye^v$ and $\eta := \mathbf{e}\cdot\nu = \frac{y - \mathbf{e}\cdot\nabla v}{w}$. At $\mathbf{z}_0$ we have $\eta = y$,

$$\eta_i = y_i - (\mathbf{e}\cdot\nabla v)_i = y_i(1 - v_{ii}), \tag{5.26}$$

$$\eta_{ii} = y_{ii} - (\mathbf{e}\cdot\nabla v)_{ii} - yv_{ii}^2$$

$$= -y - \frac{\kappa_i^2}{y} + \left(1 + \frac{1}{y^2}\right)\kappa_i$$

$$- \frac{1}{y^2} \sum_j (y y_j a_{ii,j} + \kappa_j y_j^2), \tag{5.27}$$

and

$$u_i = e^v y_i, \qquad u_{ii} = e^v(\kappa_i - y). \tag{5.28}$$

It follows that

$$\begin{aligned} y^2 F^{ii} \phi_{ii} &= y^2 F^{ii} (\phi_\eta \eta_{ii} + \phi_{\eta\eta} \eta_i^2 + 2\phi_{u\eta} u_i \eta_i + \phi_{uu} u_i^2 + \phi_u u_{ii}) \\ &= -y\phi_\eta \sum f_i \kappa_i^2 + (y^2 e^v \phi_u + (1 + y^2)\phi_\eta) \sum f_i \kappa_i \\ &\quad - (y^3 \phi_\eta + y^3 e^v \phi_u + \kappa_j y_j^2 \phi_\eta) \sum f_i + \phi_{\eta\eta} \sum f_i y_i^2 (y - \kappa_i)^2 \\ &\quad + (y^2 e^{2v} \phi_{uu} + 2y^2 e^v \phi_{u\eta}) \sum f_i y_i^2 - 2y e^v \phi_{u\eta} \sum f_i \kappa_i y_i^2 \\ &\quad - y\phi_\eta \sum y y_j F^{ii} a_{ii,j}. \end{aligned} \tag{5.29}$$

By (5.10),

$$a_{11,i}\phi = \kappa_1 \phi_i = \kappa_1(\phi_\eta \eta_i + e^v \phi_u y_i) = \kappa_1 \phi_\eta \left(1 - \frac{\kappa_i}{y}\right) y_i + e^v \phi_u \kappa_1 y_i. \tag{5.30}$$

Using (5.30) we have

$$\begin{aligned} &y\phi F^{ii}(y_j a_{11,j} + 2y_i a_{11,i}) \\ &\quad = 2(y\phi_\eta + y e^v \phi_u)\kappa_1 \sum f_i y_i^2 - 2\phi_\eta \kappa_1 \sum f_i \kappa_i y_i^2 \\ &\qquad + \left(y(1 - y^2)\phi_\eta + y(1 - y^2) e^v \phi_u - \phi_\eta \sum \kappa_j y_j^2\right) \kappa_1 \sum f_i. \end{aligned} \tag{5.31}$$

Combining (5.11), (5.25), (5.29) and (5.31), we obtain (5.12). □

Lemma 5.4

$$\begin{aligned} &y^2 F^{ii} a_{ii,11} - 2y F^{ii} y_1 a_{ii,1} + y\left(\frac{\phi_\eta}{\phi}\kappa_1 - 1\right) F^{ii} y_j a_{ii,j} \\ &\quad \geq -y^2 F^{ij,kl} a_{ij,1} a_{kl,1} \\ &\qquad - C\{u\kappa_1(|\Psi_x| + |\Psi_u|) + u^2(|\Psi_{xx}| + |\Psi_{ux}| + |\Psi_{uu}|)\} \end{aligned} \tag{5.32}$$

where C depends on an upper bound for $|\frac{\phi_\eta}{\phi}|$.

Proof Since $X = e^v \mathbf{z}$, $x = e^v(\mathbf{z} - y\mathbf{e})$ and $u = e^v y$. Hence,

$$\nabla_{\tau_j} x = e^v(\tau_j - y_j \mathbf{e}) + v_j x,$$

$$
\begin{aligned}
\nabla_{\tau_j}\Psi(x,u) &= e^v(\Psi_x\cdot(\tau_j - y_j\mathbf{e}) + \Psi_u(yv_j + y_j)) + (x\cdot\Psi_x)v_j,\\
\nabla_{\tau_1\tau_1}\Psi(x,u) &= e^{2v}(D_x^2\Psi(\tau_1 - y_1\mathbf{e})(\tau_1 - y_1\mathbf{e})\\
&\quad - 2y_1\Psi_{ux}\cdot(\tau_1 - y_1\mathbf{e}) + \Psi_{uu}y_1^2)\\
&\quad + e^v(\Psi_x\cdot(\nabla_{\tau_1}\tau_1 - y_{11}\mathbf{e}) + \Psi_u(\kappa_1 + y_{11})) + (x\cdot\Psi_x)v_{11}.
\end{aligned}
\tag{5.33}
$$

Using (5.33) and differentiating equation (5.7) twice gives

$$
\begin{aligned}
yF^{ii}y_ja_{ii,j} &= u(\Psi_x\cdot(y_j\tau_j - (1-y^2)\mathbf{e}) + \Psi_u(1-y^2)),\\
yF^{ii}y_1a_{ii,1} &= u(\Psi_x\cdot(y_1\tau_1 - y_1^2\mathbf{e}) + \Psi_u y_1^2),\\
y^2F^{ii}a_{ii,11} &= u^2(D_x^2\Psi(\tau_1 - y_1\mathbf{e})\cdot(\tau_1 - y_1\mathbf{e})\\
&\quad + 2y_1\Psi_{ux}\cdot(\tau_1 - y_1\mathbf{e}) + \Psi_{uu}y_1^2)\\
&\quad + yu(\Psi_x\cdot(\nabla_{\tau_1}\tau_1 + y\mathbf{e}) + \Psi_u(\kappa_1 - y))\\
&\quad + (x\cdot\Psi_x)y\kappa_1 - y^2F^{ij,kl}a_{ij,1}a_{kl,1}.
\end{aligned}
\tag{5.34}
$$

Formula (5.32) follows immediately from (5.34). □

We now make the choice $\phi(\eta,u) = (\eta - a)u^2$ where $0 < a \le \eta/2$. We have

$$
(y-a)\phi_\eta = \phi, \qquad \phi_{\eta\eta} = 0, \qquad u\phi_u = u^2\phi_{uu} = 2\phi, \qquad u(y-a)\phi_{u\eta} = 2\phi.
$$

By Proposition 5.3 and Lemma 5.4, for $\kappa_1 \ge 16(a + \frac{1}{a})$

$$
\begin{aligned}
&-y^2(y-a)F^{ij,kl}a_{ij,1}a_{kl,1} + a\kappa_1\sum f_i\kappa_i^2\\
&\quad + \frac{a}{2}\sigma_0\kappa_1^2 + (a + 2y^2(y-a))\kappa_1\sum f_i\\
&\quad + 2(\kappa_1 + y - a)\sum f_i(\kappa_i - y)y_i^2 + 2y(y-a)\sum f_i y_i^2\\
&\le C\{u\kappa_1(|\Psi_x| + |\Psi_u|) + u^2(|\Psi_{xx}| + |\Psi_{ux}| + |\Psi_{uu}|)\}.
\end{aligned}
\tag{5.35}
$$

Let $0 < \theta < 1$ (to be chosen in a moment) and set

$$
\begin{aligned}
I &= \{i : \kappa_i \le -\theta\kappa_1\},\\
J &= \{i : -\theta\kappa_1 < \kappa_i \le y,\ f_i < \theta^{-1}f_1\},\\
K &= \{i : -\theta\kappa_1 < \kappa_i \le y,\ f_i \ge \theta^{-1}f_1\},\\
L &= \{i : \kappa_i > y\}.
\end{aligned}
$$

Note that for $i \in L$, all the terms on the left hand side of (5.35)are nonnegative.

We have (provided that $\theta\kappa_1 \ge 1$),

$$
\sum_{i\in I} f_i\kappa_i^2 \ge \frac{1}{2}\sum_{i\in I} f_i(\kappa_i^2 + \theta^2\kappa_1^2)
$$

$$
\begin{aligned}
&\geq \frac{\theta\kappa_1}{2}\sum_{i\in I} f_i(|\kappa_i| + \theta\kappa_1)\\
&\geq \frac{\theta\kappa_1}{2}\sum_{i\in I} f_i(|\kappa_i| + y)y_i^2, \qquad (5.36)
\end{aligned}
$$

and

$$
\sum_{i\in J} f_i(\kappa_i - y)y_i^2 \geq -2\kappa_1 f_1. \qquad (5.37)
$$

According to Andrews [1] and Gerhardt [5] (see also [11], Lemma 3.1 and [10]),

$$
-F^{ij,kl}a_{ij,1}a_{kl,1} \geq \sum_{i\neq j}\frac{f_i - f_j}{\kappa_j - \kappa_i}a_{ij,1}^2 \geq 2\sum_{i=2}^{n}\frac{f_i - f_1}{\kappa_1 - \kappa_i}a_{i1,1}^2. \qquad (5.38)
$$

By (5.14) and (5.30),

$$
ya_{i1,1} = ya_{11,i} + \kappa_i y_i - \kappa_1 y_i = \left(\kappa_i + \kappa_1 + \frac{y - \kappa_i}{y - a}\kappa_1\right)y_i.
$$

Therefore,

$$
y^2 a_{i1,1}^2 \geq \frac{2(1-\theta)\kappa_1^2}{y - a}(y - \kappa_i)y_i^2, \quad \forall\, i \in K.
$$

Note that

$$
\frac{f_i - f_1}{\kappa_1 - \kappa_i} \geq \frac{f_i - \theta f_i}{\kappa_1 + \theta\kappa_1} = \frac{(1-\theta)f_i}{(1+\theta)\kappa_1}, \quad \forall i \in K. \qquad (5.39)
$$

It follows that

$$
-y^2(y-a)F^{ij,kl}a_{ij,1}a_{kl,1} \geq \frac{4(1-\theta)^2}{1+\theta}\kappa_1\sum_{i\in K} f_i(\kappa_i - y)y_i^2. \qquad (5.40)
$$

We now fix θ such that

$$
\frac{4(1-\theta)^2}{1+\theta} \geq 2 + \theta.
$$

For example, we can choose $\theta = \frac{1}{6}$. From (5.36), (5.37) and (5.40) we obtain

$$
y^2(y-a)F^{ij,kl}a_{ij,1}a_{kl,1} + a\kappa_1\sum f_i\kappa_i^2 + 2(\kappa_1 + y - a)\sum f_i(\kappa_i - y)y_i^2 \geq 0 \quad (5.41)
$$

provided that $\kappa_1 \geq 16(a + \frac{1}{a})$. Consequently,

$$
\frac{a\sigma_0}{2}\kappa_1^2 \leq C\{(|\Psi_x| + |\Psi_u|)u\kappa_1 + u^2(|\Psi_{xx}| + |\Psi_{ux}| + |\Psi_{uu}|)\}. \qquad (5.42)
$$

Formula (5.2) follows easily from (5.42) completing the proof of Theorem 5.1.

6 Existence: Proof of Theorems 1.3 and 1.2

In order to prove Theorem 1.3 we will construct a monotone sequence $\{u_k\}$ of admissible functions satisfying (1.2) in Ω starting from $u_0 \equiv \epsilon$. Having found

$$u_0 \le u_1 \le \cdots \le u_k,$$

$u = u_{k+1}$ is a solution of the Dirichlet problem

$$\begin{aligned} G(D^2u, Du, u) &= \frac{1}{u}\left(\sigma + \frac{1}{\epsilon}(u - u_k)\right) \equiv \psi(x, u) \quad \text{in } \Omega, \\ u &= \epsilon \quad \text{on } \partial\Omega. \end{aligned} \tag{6.1}$$

In order to solve (6.1) we use the continuity method for $u = u^t$, $0 \le t \le 1$:

$$\begin{aligned} G(D^2u, Du, u) &= \frac{1}{u}\left(\sigma + \frac{1}{\epsilon}(u - (tu_k + (1-t)u_{k-1}))\right) \quad \text{in } \Omega,\ k \ge 1, \\ G(D^2u, Du, u) &= \frac{1}{u}\left(t(\sigma - 1) + \frac{1}{\epsilon}u\right) \quad \text{in } \Omega,\ k = 0, \\ u &= \epsilon \quad \text{on } \partial\Omega \end{aligned} \tag{6.2}$$

where $u \in \mathcal{A}_k = \{u \ge u_k$ and u admissible$\}$ and $u^0 = u_k$. Since u is admissible we have from Sect. 3 that $|u|_{C^1\Omega} \le C$ for a uniform constant C. Now according to (2.14),

$$\begin{aligned} G_u - \psi_u &\le \frac{1}{u^2}\left(\sigma - \frac{1}{w} - \frac{1}{\epsilon}(tu_k + (1-t)u_{k-1})\right) \\ &\le -\frac{1 - \sigma + \frac{1}{C}}{u^2}, \quad k \ge 1, \\ G_u - \psi_u &\le \frac{1}{u^2}\left(t(\sigma - 1) - \frac{1}{w}\right) \le -\frac{1}{Cu^2}, \quad k = 0. \end{aligned} \tag{6.3}$$

Hence for $\Omega \in C^{2+\alpha}$, the linearized operator for u^t is invertible and the set of t for which (6.2) is solvable is open. In particular, (6.2) is solvable for $0 \le t \le 2t_0$. Using standard regularity theory for concave fully nonlinear operators, to show the closedness of this set, it suffice to show $|u|_{C^2(\Omega)} \le C$ for a uniform constant C for $t_0 \le t \le 1$. Observe that $\Psi(x, u) = u\psi(x, u) = \sigma + \frac{1}{\epsilon}(u - (tu_k + (1-t)u_{k-1}))$ satisfies the conditions (4.2), (5.3) of Theorem 5.2. Hence we obtain an estimate

$$\sup_{\Omega} |D^2u| \le \frac{C_k}{\epsilon^3}$$

where C_k depends on k but is independent of t. Therefore (6.2) is solvable for all $0 \le t \le 1$ and so we have found a monotone increasing sequence of solutions to (6.1).

It remains to show that the sequence $\{u_k\}$ converges to a solution of (1.16). For this we need second derivative estimates independent of k. Define

$$M_k(x) = \frac{\kappa_{\max}(x)}{u_k^2(x)(\nu_k^{n+1}(x) - a)}.$$

If $M_k(x)$ achieves its maximum on $\partial\Omega$, then according to Theorem 4.2, $M_k(x) \le \frac{C}{\epsilon^2}$ where C is independent of k and ϵ (see Remark 4.1). Otherwise applying Theorem 5.1 with $\Psi(x,u) = \sigma + \frac{1}{\epsilon}(u - u_k)$ we obtain

$$M_{k+1}^2 \le \frac{C}{\epsilon^4} + \frac{C}{\epsilon^2} M_k \le \frac{C}{\epsilon^4} + \frac{1}{2} M_k^2 \tag{6.4}$$

where C is independent of k and ϵ. Iterating (6.4) gives

$$M_k^2 \le \frac{2C}{\epsilon^4} + \frac{1}{2} M_1^2 \le \frac{C}{\epsilon^4}.$$

It follows that the sequence u_k converges uniformly in $C^{2+\alpha}(\overline{\Omega})$. The proof of Theorem 1.3 is complete.

To finish the proof of Theorem 1.2 we need to show that for $\sigma^2 > \frac{1}{8}$, we can obtain an estimate for $\sup_\Omega \kappa_{\max}$ which is independent of ϵ as ϵ tends to zero.

As in Sect. 5 we define

$$M_0 = \max_{x \in \overline{\Omega}} \frac{\kappa_{\max}(x)}{\phi(\eta, u)}.$$

We now choose $\phi = \eta - a$, where $\inf \eta > a$. If M_0 is achieved on $\partial\Omega$, then we obtain a uniform bound by Theorem 5.1. Otherwise at an interior maximum, Proposition 5.3 and Lemma 5.4 give

$$\sigma(y-a)\kappa_1^2 + (a - 2(1-y^2)(y-a))\kappa_1 \sum f_i \le 4\sigma\kappa_1 \tag{6.5}$$

where we have dropped some positive terms from the left hand side of (6.5) and used $\sum f_i \kappa_i y_i^2 \le \sigma$. From (6.5) we see that we must find the minimum of the function

$$\gamma(y) = a - 2(1-y^2)(y-a) = 2y^3 - 2ay^2 - 2y + 3a \quad \text{on } [a, 1]. \tag{6.6}$$

We have

$$\begin{aligned} \gamma'(y) &= 2(3y^2 - 2ay - 1), \\ \gamma''(y) &= 4(3y - a). \end{aligned} \tag{6.7}$$

The unique critical point of $\gamma(y)$ in $(a,1)$ is $y^* = \frac{a + \sqrt{a^2+3}}{3}$ and some computation shows that

$$\gamma(y^*) = \frac{7}{3}a - \frac{4}{27}a^3 - \frac{4}{27}(a^2+3)^{\frac{3}{2}}.$$

It is also not difficult to see that $\gamma(y^*) < a = \gamma(a) = \gamma(1)$.

We claim $\gamma(y^*) > 0$ if $a^2 > \frac{1}{8}$. This is equivalent to showing

$$4(a^2+3)^{\frac{3}{2}} < a(63 - 4a^2)$$

which after squaring both sides is in turn equivalent to

$$a^4 - \frac{131}{24}a^2 + \frac{2}{3} = \left(a^2 - \frac{1}{8}\right)\left(a^2 - \frac{16}{3}\right) < 0.$$

Thus our claim follows.

Now suppose $2\varepsilon_0 = \sigma^2 - \frac{1}{8} > 0$ and set $a^2 = \frac{1}{8} + \varepsilon_0$. Then

$$\sigma - a = \frac{\varepsilon_0}{\sigma + a} > \frac{\varepsilon_0}{2\sigma}.$$

According to Lemma 3.5 (see formula (3.7)), $\eta \geq \sigma - C\epsilon$ for a uniform constant C if ϵ is sufficiently small. Hence if $C\epsilon \leq \frac{\varepsilon_0}{4\sigma}$,

$$\eta - a \geq (\sigma - a) - C\varepsilon \geq \frac{\varepsilon_0}{2\sigma} - C\varepsilon > \frac{\varepsilon_0}{4\sigma}.$$

Returning to formula (6.5) we find

$$\frac{\varepsilon_0}{4}\kappa_1^2 \leq 4\sigma\kappa_1$$

or

$$\kappa_1 \leq \frac{16\sigma}{\varepsilon_0} = \frac{16\sigma}{\frac{1}{8} - \sigma^2}.$$

The proof of Theorem 1.2 is complete.

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