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On Some Properties of the Quaternionic Functional Calculus

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Abstract In some recent works we have developed a new functional calculus for bounded and unbounded quaternionic operators acting on a quaternionic Banach space. That functional calculus is based on the theory of slice regular functions and on a Cauchy formula which holds for particular domains where the admissible functions have power series expansions. In this paper, we use a new version of the Cauchy formula with slice regular kernel to extend the validity of the quaternionic functional calculus to functions defined on more general domains. Moreover, we show some of the algebraic properties of the quaternionic functional calculus such as the S-spectral radius theorem and the S-spectral mapping theorem. Our functional calculus is also a natural tool to define the semigroup e^{tA} when A is a linear quaternionic operator.

Keywords Slice regular functions · Functional calculus · Spectral theory · Algebraic properties · S-spectral radius theorem · S-spectral mapping theorem · Semigroup of a linear quaternionic operator

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1 Introduction

In the paper [13] a new theory of quaternionic regular functions, called slice regular, has been introduced. The version of the Cauchy formula in [13] allows us to define a functional calculus for quaternionic operators $T = T_0 + T_1 i + T_2 j + T_3 k$, using slice

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regular functions defined on particular open sets containing the S-spectrum of T (see [5, 6, 10]).

In the paper [11], the theory of quaternionic slice regular functions has been generalized to the Clifford algebras setting. Such a theory is a natural tool to define a functional calculus for n-tuples of noncommuting operators, see [8] and [2, 3]. Even though the algebraic properties of quaternions are different from those of the Clifford numbers, we find very deep and unexpected analogies in the two functional calculi. We can say that slice hyperholomorphy (that is slice regularity for quaternions and slice monogenicity for the Clifford setting) offers a unified vision for the functional calculus of a quaternionic operator and for n-tuples of (noncommuting) operators. In this paper we treat the case of quaternionic operators and we generalize the results in [5, 6, 10] to slice regular functions defined on open sets more general than the union of balls and spherical shells with center at real points, as it was required in [5, 6, 10].

Our starting point to generalize the quaternionic functional calculus is the new version of the Cauchy formula for slice regular functions proved in [7]. This generalization of the quaternionic functional calculus allows us to prove some properties such as the S-spectral radius theorem and the S-spectral mapping theorem. We also provide the quaternionic evolution operator as an example.

Let us start by recalling the definition of slice regular (s-regular, for short) functions (see [13]).

Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$, where $\mathbb{S}$ is the sphere of purely imaginary unit quaternions. Let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on L_I. We say that f is a (left) *s-regular function* if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0.$$

In the sequel we will denote by $\mathcal{R}(U)$ the set of (left) s-regular functions on the open set U. We say that f is right s-regular function if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x+Iy) + \frac{\partial}{\partial y} f_I(x+Iy)I\right) = 0.$$

If T is a linear bounded quaternionic operator, its S-spectrum consists of all those quaternions $s = s_0 + s_1 i + s_2 j + s_3 k$ such that the operator $T^2 - 2s_0 T + |s|^2\mathcal{I}$ is not invertible, where $\mathcal{I}$ denotes the identity operator and $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$. To define a functional calculus, in [5, 10] we used the following notion of functions s-regular on the S-spectrum of operator T:

let T be a quaternionic linear bounded operator acting on a quaternionic Banach space V. Let $U \subset \mathbb{H}$ be an open set that contains the S-spectrum of T, and such that

(a) $\partial(U \cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$,
(b) $\sigma_S(T)$ is contained in a finite union of open balls $B_i \subseteq U$ with center in real points and of spherical shells $A_j = \{q \in \mathbb{H} \mid r_j < |q - \alpha_j| < R_j,\ r_j, R_j \in \mathbb{R}^+\} \subseteq U$ with center in real points α_j, and whose boundaries do not intersect $\sigma_S(T)$.

We say that a function f is (B, A)-locally s-regular on $\sigma_S(T)$ if there exists an open set $U \subset \mathbb{H}$, as above, such that $\overline{U}$ is contained on an open set on which f is s-regular.

We will denote by $\mathcal{R}^{B,A}_{\sigma_S(T)}$ the set of (B, A)-locally s-regular functions on $\sigma_S(T)$.

Let T be a quaternionic linear bounded operator and let $f \in \mathcal{R}^{B,A}_{\sigma_S(T)}$. Let $U \subset \mathbb{H}$ be an open set as above. In [10] we have defined the functional calculus for T as

$$f(T) = -\frac{1}{2\pi} \int_{\partial(U \cap L_I)} (T^2 - 2s_0]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I})\, ds_I\, f(s). \tag{1}$$

The restrictions on the open set U were imposed by the need of using the power series expansion of the s-regular function f to show that the integral (1) does not depend on the open set U and on the choice of the imaginary unit $I \in \mathbb{S}$. In this paper we will show that formula (1) still holds when the hypothesis on the open set U are weakened. With the new formulation of the quaternionic functional calculus it is possible to prove that most of the properties that hold for the functional calculus in the complex case, still hold in the quaternionic setting.

We conclude by recalling that among the possible approaches to a functional calculus, there is one which makes use of functions with hypercomplex values and the function theory that plays an important role to treat the case of n-tuples of operators is the one of monogenic functions (see the classical book [1] for the theory of one variable and [4] for the several variables case). Since the literature in this setting is very wide we mention, without claim of completeness, the papers [14–17, 19, 21, 22] and the literature therein.

2 The Quaternionic Functional Calculus

We begin with some preliminary considerations and definitions that will be useful in the sequel. In the following we will denote by $\mathcal{I}$ the identity operator. The composition of operators, and in particular the powers T^n of a quaternionic operator, are defined in the usual way. An operator T is said to be invertible if there exists S such that $TS = ST = \mathcal{I}$ and we will write $S = T^{-1}$. We recall the definition of right linear operators.

Definition 2.1 Let V be a right vector space on $\mathbb{H}$. A map $T : V \to V$ is said to be a right linear operator if

$$T(u + v) = T(u) + T(v),$$
$$T(us) = T(u)s$$

for all $s \in \mathbb{H}$ and all $u, v \in V$.

In the sequel, we will consider only two sided vector spaces V, otherwise the set of right linear operators is not a (left or right) vector space. With this assumption, the set $\mathrm{End}(V)$ of right linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations

$$(aT)(v) := aT(v), \qquad (Ta)(v) := T(av).$$

Definition 2.2 Let V be a bilateral quaternionic Banach space. We will denote by $\mathcal{B}(V)$ the bilateral Banach space of all right linear bounded operators $T : V \to V$.

It is easy to verify that $\mathcal{B}(V)$ is a Banach space endowed with its natural norm. It is obvious that the set of all invertible elements in $\mathcal{B}(V)$ is a group with respect to the composition of operators defined in $\mathcal{B}(V)$. The notion of left spectrum of T related to the resolvent $(s\mathcal{I} - T)^{-1}$, that is $\sigma_L(T) = \{s \in \mathbb{H} : s\mathcal{I} - T \text{ is not invertible}\}$, is not the right tool to define our functional calculus, and similarly for the right spectrum of T. This is due to the fact that the resolvent operator used in the complex case to define the functional calculus, i.e. $(s\mathcal{I} - T)^{-1}$, here has to be replaced by a different resolvent operator that, in the sequel, will be called S-resolvent operator. Our quaternionic functional calculus is based on the notion of S-spectrum defined below. For the proofs of the following results on the S-spectrum of T, we refer the reader to [10].

Definition 2.3 (The S-spectrum and the S-resolvent sets) Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I} \text{ is not invertible}\},$$

where $\mathrm{Re}[s]$, $|s|$ denote the real part and the module of the quaternion s, respectively. The S-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

The notion of S-spectrum of a quaternionic operator T is suggested by the definition of S-resolvent operator that is the kernel for the quaternionic functional calculus.

Definition 2.4 (The S-resolvent operator) Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. When $s \in \rho_S(T)$ we define the S-resolvent operator as

$$S^{-1}(s, T) := -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{2}$$

where $\overline{s}$ is the conjugate of $s \in \mathbb{H}$.

The S-resolvent operator (2) admits a power series expansion formula which is the analogue of the series expansion $(\lambda e - B)^{-1} = \sum_{n\geq 0} B^n \lambda^{-1-n}$ of the classical resolvent operator in the complex case, where B is a complex bounded linear operator, e is the identity operator and the series converges in the uniform topology in the space of all bounded linear operators for $\|B\| < |\lambda|$. In the quaternionic case we have the following.

Theorem 2.5 (Power series expansion of the S-resolvent operator) *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. If $s \in \rho_S(T)$, then the following equality*:

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}) \tag{3}$$

holds for $\|T\| < |s|$.

Theorem 2.6 (The S-resolvent equation) *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. If $s \in \rho_S(T)$, then the S-resolvent operator defined in (2) satisfies the equation*

$$S^{-1}(s,T)s - TS^{-1}(s,T) = \mathcal{I}.$$

The fact that the spectrum of bounded linear operators in the complex case is a non empty compact set still holds for the S-spectrum of a linear bounded quaternionic operator as the following result shows.

Theorem 2.7 (Compactness of S-spectrum) *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set contained in $\{s \in \mathbb{H} : |s| \le \|T\|\}$.*

We denote by $\mathbb{S}$ the set of unit purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k \ : \ x_1^2 + x_2^2 + x_3^2 = 1\}.$$

To each quaternion p it is possible to uniquely associate an element on the sphere $\mathbb{S}$:

$$I_p = \begin{cases} \frac{\mathrm{Im}[p]}{|\mathrm{Im}[p]|} & \text{if } \mathrm{Im}[p] \neq 0, \\ \text{any element of } \mathbb{S} \text{ otherwise.} \end{cases}$$

The imaginary unit I_p determines the complex plane L_{I_p} containing p.

Definition 2.8 Given $p \in \mathbb{H}$, $p = \mathrm{Re}[p] + I_p|\mathrm{Im}[p]|$ we denote by $[p]$ the set of all elements of the form $\mathrm{Re}[q] + J|\mathrm{Im}[p]|$ when J varies in $\mathbb{S}$. We say that $[p]$ is the 2-sphere defined by p.

Remark 2.9 The set $[p]$ is a 2-sphere which is reduced to the point p when $p \in \mathbb{R}$.

We can now describe the structure of the S-spectrum:

Theorem 2.10 (Structure of the S-spectrum) *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. If $p \in \mathbb{H}$ belongs to $\sigma_S(T)$, then all the elements of the 2-sphere $[p]$ are contained in $\sigma_S(T)$.*

We point out that in the Riesz-Dunford functional calculus (see for example [12] and [20]) a crucial tool is the resolvent equation

$$(\lambda e - B)^{-1} - (\mu e - B)^{-1} = -(\lambda - \mu)(\lambda e - B)^{-1}(\mu e - B)^{-1}, \tag{4}$$

where B is a complex linear operator acting on complex Banach space and λ and μ belong to the resolvent set of B. In the quaternionic case the analogue of the resolvent equation would be the following: take s and $p \in \rho_S(T)$ so from the S-resolvent equation we get

$$S^{-1}(s,T)s - TS^{-1}(s,T) = S^{-1}(p,T)p - TS^{-1}(p,T),$$

but the noncommutative setting does not allow us to get an analogue of the expression (4). This fact does not compromise the development of the quaternionic functional calculus. As we shall see, most of the results that are based on (4) for the Riesz-Dunford functional calculus, can be proved also in our case overcoming this difficulty.

We now state a structure formula for s-regular functions, see [7], which will be used in the sequel and which is the analogue of the structure formula for s-monogenic functions proved by the authors in [3].

Lemma 2.11 (The structure formula for s-regular functions) *Let $U \subseteq \mathbb{H}$ be a domain such that $U \cap \mathbb{R} \neq \emptyset$, $U \cap L_I$ is a domain for all $I \in \mathbb{S}$ and U contains the 2-sphere $[q]$ defined by q whenever $q \in U$. Let $f : U \to \mathbb{H}$ be an s-regular function. Set the positions*: $u = \mathrm{Re}[q]$, $v = |\mathrm{Im}[q]|$. *Then for all $q \in U$ and $I \in \mathbb{S}$ the following formula holds*:

$$f(q) = \frac{1}{2}\big[1 - I_q I\big] f(u + Iv) + \frac{1}{2}\big[1 + I_q I\big] f(u - Iv). \tag{5}$$

Remark 2.12 Define the functions

$$\eta_I : U \cap L_I \to \mathbb{H}, \qquad \eta_I(u, v) := \frac{1}{2}[f(u + Iv) + f(u - Iv)],$$

and

$$\theta_I : U \cap L_I \to \mathbb{H}, \qquad \theta_I(u, v) := \frac{1}{2} I[f(u - Iv) - f(u + Iv)].$$

We have the following identity:

$$\frac{1}{2}\big[1 - I_q I\big] f(u + Iv) + \frac{1}{2}\big[1 + I_q I\big] f(u - Iv) = \eta_I(u, v) + I_q \theta_I(u, v), \tag{6}$$

moreover, see Theorem 2.26 in [7], the quaternionic valued functions $\eta_I(u, v)$ and $\theta_I(u, v)$ are independent of $I \in \mathbb{S}$.

We now state the Cauchy formula with s-regular kernel proved in [7].

Theorem 2.13 (The Cauchy formula with s-regular kernel) *Let $f \in \mathcal{R}(W)$ where W is an open set in $\mathbb{H}$. Let $U \subset \mathbb{H}$ be a domain such that*: $\overline{U} \subset W$, $[q] \subset U$ *for every* $q \in U$, $U \cap \mathbb{R} \neq \emptyset$, $U \cap L_I$ *is a domain for every* $I \in \mathbb{S}$, *and* $\partial(U \cap L_I)$ *is union of a finite number of rectifiable Jordan curves for every* $I \in \mathbb{S}$. *Set* $ds_I = ds/I$. *Then, if* $q \in U$, *we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, q)\, ds_I\, f(s), \tag{7}$$

where $S^{-1}(s, q)$ is defined by

$$S^{-1}(s, q) = -(q^2 - 2q\mathrm{Re}[s] + |s|^2)^{-1}(q - \overline{s})$$

and the integral (7) *does not depend on the choice of the imaginary unit* $I \in \mathbb{S}$ *and on* U.

If f *is a right s-regular function on* W *and* U *is as above, then we have the following integral formula*:

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} f(s) ds_I \widetilde{S}^{-1}(s,q) = -\frac{1}{2\pi} \int_{\partial(U \cap L_I)} f(s) ds_I S^{-1}(q,s), \quad (8)$$

where

$$\widetilde{S}^{-1}(s,q) := (s^2 - 2s\mathrm{Re}[q] + |q|^2)^{-1}(s - \overline{q}) = -S^{-1}(q,s)$$

and the integral (8) *does not depend on the choice of the imaginary unit* $I \in \mathbb{S}$ *and on* U.

Remark 2.14 In our Cauchy formula for s-regular quaternionic functions (7) it is always possible to replace, at least formally, the variable q by a quaternionic operator $T = T_0 + T_1 i + T_2 j + T_3 k$. This substitution is not always possible in other function theories. For example, if one considers the Fueter's notion of regular functions for quaternions, the substitution $q \to T$ in the Cauchy-Fueter kernel is not allowed, unless the component of the quaternionic operator T_ℓ, $\ell = 0, \dots, 3$ commute. This is the reason for which the definition of the "Fueter functional calculus" has some obstructions (see [9]).

Remark 2.15 In the Cauchy formula (7) the kernel $-(q^2 - 2q\mathrm{Re}[s] + |s|^2)^{-1}(q - \overline{s})$ has been obtained by summing the Cauchy kernel series $\sum_{n\geq 0} q^n s^{-1-n}$ when the series converges. Thanks to Theorem 2.5 the sum of the series $\sum_{n\geq 0} T^n s^{-1-n}$ is formally obtained by substituting the quaternion q by T, also when the components of T do not commute.

This is the reason for which our functional calculus can be developed in a natural way starting from the Cauchy formula (7).

We are now in the position to introduce the admissible functions to define the quaternionic functional calculus.

Definition 2.16 Let V be a quaternionic Banach space, $T \in \mathcal{B}(V)$ and let $U \subset \mathbb{H}$ be a domain that contains the S-spectrum $\sigma_S(T)$ and such:

(i) $[q] \subset U$ for every $q \in U$,
(ii) $U \cap \mathbb{R} \neq \emptyset$,
(iii) $U \cap L_I$ is a domain for every $I \in \mathbb{S}$,
(iv) $\partial(U \cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

A function $f \in \mathcal{R}(W)$, where W is an open set in $\mathbb{H}$, is said to be locally s-regular on $\sigma_S(T)$ if there exists a domain $U \subset \mathbb{H}$, satisfying (i)–(iv) and such that $\overline{U} \subset W$, on which f is s-regular. We will denote by $\mathcal{R}_{\sigma_S(T)}$ the set of locally s-regular functions on $\sigma_S(T)$.

Remark 2.17 Let W is an open set in $\mathbb{R}^{n+1}$ and let $f \in \mathcal{R}(W)$. In the Cauchy formula (7) the open set $U \subset W$ need not to be necessarily connected. Indeed formula (7) obviously holds when $U = \bigcup_{i=1}^{r} U_i$, $\overline{U}_i \cap \overline{U}_j = \emptyset$ when $i \neq j$ where U_i are as in Definition 2.16 for all $i = 1, \dots, r$ and the boundaries of $U_i \cap L_I$ consists of a finite number of continuously differentiable Jordan curves for $I \in \mathbb{S}$ for all $i = 1, \dots, r$. So when we choose $f \in \mathcal{R}_{\sigma_S(T)}$ the related open set U need not to be connected. In the sequel we will state our results relating them to a domain U but our results obviously hold for open sets $U = \bigcup_{i=1}^{r} U_i$ as above.

The Hahn–Banach theorem holds for right (or left) vector spaces on $\mathbb{H}$. The proof is very similar to the one for the complex case. We recall the result and its proof below, for sake of completeness. We will use it for the proof of an important theorem that allows us to define the quaternionic functional calculus.

Theorem 2.18 (The quaternionic version of the Hahn-Banach theorem) *Let V_0 be a right subspace of a right vector space V on $\mathbb{H}$. Suppose that p is a seminorm on V and let ϕ be a linear and continuous functional on V_0 such that*

$$|\langle \phi, v \rangle| \leq p(v), \quad \forall v \in V_0. \tag{9}$$

Then it is possible to extend ϕ to a linear and continuous functional on V satisfying the estimate (9) for all $v \in V$.

Proof Note that, for any quaternion q we have $q = q_0 + q_1 i + q_2 j + q_3 k = z_1(q) + z_2(q) j$, where $z_1, z_2 \in \mathbb{C} = \mathbb{R} + \mathbb{R}i$ and $qj = -z_2(q) + z_1(q) j$, so $q = z_1(q) - z_1(qj) j$. The functional ϕ can be written as $\phi = \phi_0 + \phi_1 i + \phi_2 j + \phi_3 k = \psi_1(\phi) + \psi_2(\phi) j$, with $\psi_1(\phi) = \phi_0 + \phi_1 i$ and $\psi_2(\phi) = \phi_2 + \phi_3 i$ which are complex functionals. It is immediate that

$$\langle \phi, v \rangle = \langle \psi_1, v \rangle - \langle \psi_1, vj \rangle j, \qquad \forall v \in V_0,$$

where ψ_1 is a $\mathbb{C}$-linear functional. So we can apply the complex version of the Hahn–Banach theorem to deduce the existence of a functional $\tilde{\psi}_1$ that extends ψ_1 to the whole of V (as a complex vector space). The functional Ψ given by

$$\langle \Psi, v \rangle = \langle \tilde{\psi}_1, v \rangle - \langle \tilde{\psi}_1, vj \rangle j$$

is defined on V and it is the extension that satisfies estimate (9) for all $v \in V$. □

The following result is an immediate consequence of the quaternionic version of the Hahn-Banach theorem. Its proof mimics the analogous proof in the complex case.

Corollary 2.19 *Let V be a right vector space on $\mathbb{H}$ and let $v \in V$. If $\langle \phi, v \rangle = 0$ for every linear and continuous functional ϕ in V', then $v = 0$.*

Theorem 2.20 *Let V be a quaternionic Banach space, $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. Let $U \subset \mathbb{H}$ be a domain as in Definition 2.16. Set $ds_I = ds/I$. Then the integral*

$$\frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s), \tag{10}$$

where $S^{-1}(s,T)$ is the S-resolvent operator, does not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on U.

Proof We first observe that the function $S^{-1}(s,q)$ is right s-regular in the variable s in its domain of definition. In fact, by a direct computation, setting $s = x + Iy$ we have

$$\frac{\partial}{\partial x}S^{-1}(s,q) = (q^2 - 2qx + x^2 + y^2)^{-2}(-2q + 2x)(q - x + Iy) + (q^2 - 2qx + x^2 + y^2)^{-1},$$

$$\frac{\partial}{\partial y}S^{-1}(s,q) = (q^2 - 2qx + x^2 + y^2)^{-2}2v(q - x + Iy) - (q^2 - 2qx + x^2 + y^2)^{-1}I.$$

Easy calculations show that

$$\frac{\partial}{\partial x}S^{-1}(s,q) + \frac{\partial}{\partial y}S^{-1}(s,q)I = 0, \quad \forall I \in \mathbb{S}$$

which proves the assertion.

Now observe that we can replace q with an operator $T \in \mathcal{B}(V)$ in the Cauchy formula (7), thanks to Theorem 2.5. For every linear and continuous functional $\phi \in V'$, consider the duality $\langle \phi, S^{-1}(s,T)v\rangle$, for $v \in V$ and define the function

$$g(s) := \langle \phi, S^{-1}(s,T)v\rangle, \quad \text{for } v \in V,\ \phi \in V'.$$

The function g remains right s-regular in the variable s on the complement of $\sigma_S(T)$ and since $g(s) \to 0$ as $s \to \infty$ we have that g is s-regular also at infinity. Suppose that U is as in Definition 2.16 so that $\partial(U \cap L_I)$ does not cross the S-spectrum of T for every $I \in \mathbb{S}$. The fact that, for fixed $I \in \mathbb{S}$, the integral

$$\frac{1}{2\pi}\int_{\partial(U\cap L_I)} g(s)ds_I f(s) \tag{11}$$

does not depend on U follows from the Cauchy theorem. By Corollary 2.19 also the integral (10) does not depend on U. We now prove that the integral (11) does not depend on $I \in \mathbb{S}$. Since g is a right s-regular function on the complement of the S-spectrum of T, we can consider an open set U' such that $\overline{U}' \subset \rho_S(T)$ and $[q] \subset U'$ whenever $q \in U'$. Suppose that $\partial U \subset U'$ where U is as above so, in particular, it contains $[s]$ whenever $s \in U$. Choose $J \in \mathbb{S}$, $J \neq I$ and represent $g(s)$ by the Cauchy integral formula (8) as

$$g(s) = -\frac{1}{2\pi}\int_{\partial(U'\cap L_J)^-} g(t)\,dt_J S^{-1}(s,t), \tag{12}$$

where the boundary $\partial(U' \cap L_J)^-$ is oriented clockwise to include the points $[s] \in \partial(U \cap L_J)$ (recalling that the singularities of $S^{-1}(s,t)$ correspond to the 2-sphere $[s]$)

and to exclude the points belonging to the S-spectrum of T. Taking into account the orientation of $\partial(U' \cap L_J)^-$ we can rewrite the integral (12) as

$$g(s) = \frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J\, S^{-1}(s,t). \tag{13}$$

Let us now plug the expression of $g(s)$ in (13) into the integral (11) to obtain

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial(U \cap L_I)} g(s)\, ds_I\, f(s) \\
&\quad = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} \left[\frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J\, S^{-1}(s,t) \right] ds_I\, f(s) \\
&\quad = \frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J \left[\frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s,t)\, ds_I\, f(s) \right],
\end{aligned} \tag{14}$$

where we have used the Fubini theorem. Now observe that, in general, $\partial(U' \cap L_J)$ consists of a finite number of Jordan curves inside and possibly outside $U \cap L_J$, but the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s,t)\, ds_I\, f(s)$$

equals $f(t)$ for those $t \in \partial(U' \cap L_J)$ belonging to $U \cap L_J$. Thus we obtain:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J \left[\frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s,t)\, ds_I\, f(s) \right] \\
&\quad = \frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J\, f(t).
\end{aligned} \tag{15}$$

So from (14) and (15) we can write

$$\frac{1}{2\pi} \int_{\partial(U \cap L_I)} g(s)\, ds_I\, f(s) = \frac{1}{2\pi} \int_{\partial(U' \cap L_J)} g(t)\, dt_J\, f(t). \tag{16}$$

Now observe that $\partial(U' \cap L_J)$ is positively oriented and surrounds the S-spectrum of T. By the first part of the statement in (16) we can substitute $\partial(U' \cap L_J)$ by $\partial(U \cap L_J)$ because of the independence of the integral on the open set U and we get:

$$\frac{1}{2\pi} \int_{\partial(U \cap L_I)} g(s)\, ds_I\, f(s) = \frac{1}{2\pi} \int_{\partial(U \cap L_J)} g(t)\, dt_J\, f(t),$$

that is

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial(U \cap L_I)} \langle \phi, S^{-1}(s,T)v \rangle ds_I\, f(s) \\
&\quad = \frac{1}{2\pi} \int_{\partial(U \cap L_J)} \langle \phi, S^{-1}(t,T)v \rangle\, dt_J\, f(t), \quad \text{for every } v \in V,\ \phi \in V',\ I, J \in \mathbb{S}.
\end{aligned}$$

Thus by Corollary 2.19 the integral (10) does not depend on $I \in \mathbb{S}$. □

Thanks to Theorem 2.20 the following definition of the quaternionic functional calculus is well posed.

Definition 2.21 Let V be a quaternionic Banach space, $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. Let $U \subset \mathbb{H}$ be a domain as in Definition 2.16 and set $I \in \mathbb{S}$. Set $ds_I = ds/I$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s). \tag{17}$$

From Definition 2.21 it easily follows that

Proposition 2.22 *Let V be a quaternionic Banach space, $T \in \mathcal{B}(V)$ and let f, $g \in \mathcal{R}_{\sigma_S(T)}$. Then*

$$(f+g)(T) = f(T) + g(T), \qquad (fp)(T) = f(T)p, \quad \textit{for all } p \in \mathbb{H}.$$

2.1 Some Comments

It is possible to give an easy proof of Theorem 2.20. This proof is of limited validity, but follows by a direct computation. It applies only in the case the functions we consider admit power series expansions on U. We recall this important fact: s-regular functions admit Taylor series expansions only on balls centered at real points and they admit Laurent series expansions only on spherical shells centered at real points.

Let us consider the case in which the domain U is contained in a ball $B(\alpha, r) \subset \mathbb{H}$ centered in a real point α and of radius $r > 0$ in which the s-regular function f admits a power series expansion.

Lemma 2.23 *Let V be a quaternionic Banach space, $T \in \mathcal{B}(V)$. Suppose that f is an s-regular function such that*

$$f(s) = \sum_{m \geq 0} (s-\alpha)^m a_m, \quad \forall s \in B(\alpha, r),\ \alpha \in \mathbb{R},\ a_m \in \mathbb{H},\ r > 0 \tag{18}$$

and assume that $\sigma_S(T) \subset U \subset B(\alpha, r)$ where U is as in Definition 2.16. Then

$$\frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s) \tag{19}$$

does not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on U.

Proof In $B(\alpha, r)$ the Taylor expansion of f has the form (18) where the elements a_m are fixed quaternions and do not depend on the particular plane L_I. Now observe that

$$f(s) = \sum_{m \geq 0} (s-\alpha)^m a_m = \sum_{m \geq 0} \sum_{j=0}^{m} \binom{m}{j} s^j (-\alpha)^{m-j} a_m.$$

Consider the integral (19) and replace the power series expansion for f. By the absolute and uniform convergence we get

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap L_I)} & S^{-1}(s,T)\,ds_I\,f(s)\\ &= \frac{1}{2\pi}\sum_{m\geq 0}\sum_{j=0}^{m}\binom{m}{j}\left(\int_{\partial(U\cap L_I)} S^{-1}(s,T)\,ds_I s^j\right)(-\alpha)^{m-j}a_m. \qquad (20)\end{aligned}$$

Now consider the integral

$$\int_{\partial(U\cap L_I)} S^{-1}(s,T)\,ds_I s^j$$

and observe that s^j is s-regular everywhere so we can deform the integration path in such a way that $S^{-1}(s,T)$ admits the power series expansion (3) in a suitable ball $B(0,r)$. We have:

$$\frac{1}{2\pi}\sum_{n\geq 0} T^n \int_{\partial(B(0,r)\cap L_I)} s^{-1-n+j}\,ds_I = T^j, \qquad (21)$$

since

$$\begin{aligned}&\int_{\partial(B(0,r)\cap L_I)} ds_I s^{-n-1+j} = 0 \quad \text{if } n\neq j,\\ &\int_{\partial(B(0,r)\cap L_I)} ds_I s^{-n-1+j} = 2\pi \quad \text{if } n = j.\end{aligned} \qquad (22)$$

The standard Cauchy theorem on the complex plane L_I shows that the above integral (21) is not affected if we replace $\partial(B(0,r)\cap L_I)$ by $\partial(U\cap L_I)$, so

$$\frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\,ds_I s^j = T^j.$$

We conclude that the integral (20) does not depend on U and on $I\in\mathbb{S}$ because the coefficient $(-\alpha)^{j-m}a_m$ are independent of $I\in\mathbb{S}$. □

As we did in [10], consider the open sets $U\subset\mathbb{H}$ that contain the S-spectrum of T, and such that

(a) $\partial(U\cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I\in\mathbb{S}$,
(b) $\sigma_S(T)$ is contained in a finite union of open balls $B_i\subseteq U$ with center in real points and of spherical shells $A_j=\{q\in\mathbb{H}\mid r_j<|q-\alpha_j|<R_j,\ r_j,R_j\in\mathbb{R}^+\}\subseteq U$ with center in real points α_j, and whose boundaries do not intersect $\sigma_S(T)$.

Since an analogue of Lemma 2.23 holds also for Laurent power series expansions, in [10] we could prove that for open sets $U\supset\sigma_S(T)$ satisfying (a) and (b) the integral (10) does not depend on the choice of the imaginary unit $I\in\mathbb{S}$ and on U.

3 The S-Spectral Radius

In this section we give the definition of S-spectral radius which is the analogue of the spectral radius for the Riesz-Dunford case. The main result of this section is Theorem 3.10. This theorem is based on the S-spectral mapping theorem for the powers T^n, $n \in \mathbb{N}$, of a quaternionic bounded linear operator T, which can be proved using some algebraic properties of quaternionic polynomials. In Sect. 4 we will generalize the S-spectral mapping theorem to a wider class of s-regular functions.

Definition 3.1 (The S-spectral radius of T) Let V be a quaternionic Banach space and $T \in \mathcal{B}(V)$. We call S-spectral radius of T the non negative real number

$$r_S(T) := \sup\{|s| \,:\, s \in \sigma_S(T)\}.$$

Before we can state and prove the S-spectral radius theorem, we need two preliminary lemmas on quaternionic polynomials. For the sequel, it is useful to recall that any quaternion $q = \mathrm{Re}[q] + I_q|\mathrm{Im}[q]|$ is associated to the 2-sphere defined by $[q]$ which reduces to q only when q is real.

Lemma 3.2 *Let $n \in \mathbb{N}$ and $q, s \in \mathbb{H}$. Let*

$$P_{2n}(q) := q^{2n} - 2\mathrm{Re}[s^n]q^n + |s^n|^2.$$

Then

$$P_{2n}(q) = Q_{2n-2}(q)(q^2 - 2\mathrm{Re}[s]q + |s|^2) = (q^2 - 2\mathrm{Re}[s]q + |s|^2)Q_{2n-2}(q), \quad (23)$$

where $Q_{2n-2}(q)$ is a polynomial of degree $2n-2$ in q.

Proof First of all we observe that

$$P_{2n}(s) = s^{2n} - 2\mathrm{Re}[s^n]s^n + |s^n|^2 = s^{2n} - (s^n + \bar{s}^n)s^n + s^n\bar{s}^n = 0.$$

Moreover, the substitution of s by any s' on the same 2-sphere leaves the coefficients of the polynomial $P_{2n}(q)$ unchanged, and $P_{2n}(s') = 0$. We conclude, see [18], that the whole 2-sphere defined by s is solution to the equation $P_{2n}(q) = 0$. The statement follows from the factorization theorem, see [18], and the fact that the second degree polynomial $q^2 - 2\mathrm{Re}[s]q + |s|^2$ has real coefficients. □

Lemma 3.3 *Let $n \in \mathbb{N}$ and $q, p \in \mathbb{H}$. Let λ_j, $j = 0, 1, \dots, n-1$ be the solutions of $\lambda^n = p$ in the complex plane L_{I_p}. Then*

$$q^{2n} - 2\mathrm{Re}[p]q^n + |p|^2 = \prod_{j=0}^{n-1}(q^2 - 2\mathrm{Re}[\lambda_j]q + |\lambda_j|^2). \quad (24)$$

Proof The equation $\lambda^n = p$ can be solved in the complex plane $x + I_p y$ containing $p = p_0 + I_p p_1$ where it admits n solutions $\lambda_j = \lambda_{j0} + I_p\lambda_{j1}$, $j = 0, 1, \dots, n-1$. By reason of degree, these are the only solutions to the equation in the complex plane L_{I_p}. Note that if we take any $p' = p_0 + I p_1$, $I \in \mathbb{S}$ in the 2-sphere of p then the solutions to the equation $\lambda^n = p'$ are $\lambda'_j = \lambda_{j0} + I\lambda_{j1}$, $j = 0, 1, \dots, n-1$, $I \in \mathbb{S}$. We consider the polynomial $P_{2n}(q) = q^{2n} - 2\mathrm{Re}[p]q^n + |p|^2$ and we observe that $q = \lambda_j$ is a root of $P_{2n}(q) = 0$, in fact

$$P_{2n}(\lambda_j) = \lambda_j^{2n} - 2\mathrm{Re}[p]\lambda_j^n + |p|^2 = p^2 - 2\mathrm{Re}[p]p + |p|^2 = 0.$$

The substitution of p by p' on the same 2-sphere leaves P_{2n} unchanged and it is immediate that $P_{2n}(\lambda'_j) = 0$ when I varies in $\mathbb{S}$. This proves that the roots of $P_{2n}(q) = 0$ lye on the 2-spheres of λ_j, $j = 0, \dots, n-1$. The statement follows from the factorization theorem, see [18]. □

Let us introduce an important subclass of $\mathcal{R}(U)$, (the set of s-regular functions on U) for the purpose to guarantee that the product of two s-regular functions is still an s-regular function.

Definition 3.4 Let $U \subset \mathbb{H}$ be an open set. We define

$$\mathcal{N}(U) = \{f \in \mathcal{R}(U) \mid f(U \cap L_I) \subseteq L_I, \forall I \in \mathbb{S}\}.$$

Remark 3.5 Observe that if f is a polynomial (resp. a convergent series on U) with real coefficients, then $f \in \mathcal{N}(\mathbb{H})$ (resp. $f \in \mathcal{N}(U)$).

Definition 3.6 Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. We will denote by $\mathcal{N}_{\sigma_S(T)}$ the set of functions for which there exists a domain $U \subset \mathbb{H}$ as in Definition 2.16 and such that $f \in \mathcal{N}(U)$, where $\overline{U}$ is contained in the set of s-regularity of f.

Lemma 3.7 *Let $U \subset \mathbb{H}$ be an open set. Let $f \in \mathcal{N}(U)$, $g \in \mathcal{R}(U)$, then $fg \in \mathcal{R}(U)$. In particular, if $f, g \in \mathcal{N}(U)$, then $fg \in \mathcal{N}(U)$*

Proof Consider $I \in \mathbb{S}$ and set $z = x + Iy$. The restriction $f_I(z)$ of f equals $F(z)$ with $F : U \cap L_I \to L_I$ holomorphic and we have that:

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(fg)(z) = \frac{\partial F}{\partial x}(z)g(z) + F(z)\frac{\partial g}{\partial x}(z) + I\frac{\partial F}{\partial y}(z)g(z) + IF(z)\frac{\partial g}{\partial y}(z)$$

and since I commutes with $F(z)$ we obtain:

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(fg)(z) = \left(\frac{\partial F}{\partial x}(z) + I\frac{\partial F}{\partial y}(z)\right)g(z) + F(z)\left(\frac{\partial g}{\partial x}(z) + I\frac{\partial g}{\partial y}(z)\right) = 0.$$

The second part of the statement follows from the fact that both f and g take L_I to itself for all $I \in \mathbb{S}$. □

The following result is an important property of our functional calculus and will be used to prove our S-spectral mapping theorem.

Theorem 3.8 *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. If $\phi \in \mathcal{N}_{\sigma_S(T)}$ and $g \in \mathcal{R}_{\sigma_S(T)}$, then $(\phi g)(T) = \phi(T) g(T)$.*

Proof Denote by U a domain as in Definition 2.16 on which both ϕ and g are s-regular. Observe that ϕg is s-regular on U thanks to Lemma 3.7. Let G_1 and G_2 be two open sets as in Definition 2.16 such that $G_1 \cup \partial G_1 \subset G_2$ and $G_2 \cup \partial G_2 \subset U$. Take $s \in \partial(G_1 \cap L_I)$ and $t \in \partial(G_2 \cap L_I)$ and observe that, for $I \in \mathbb{S}$, we have

$$g(s) = \frac{1}{2\pi} \int_{\partial(G_2 \cap L_I)} S^{-1}(t,s)\, dt_I\ g(t).$$

Now consider

$$\begin{aligned}(\phi g)(T) &= \frac{1}{2\pi} \int_{\partial(G_1 \cap L_I)} S^{-1}(s,T)\, ds_I \phi(s) g(s) \\ &= \frac{1}{2\pi} \int_{\partial(G_1 \cap L_I)} S^{-1}(s,T)\, ds_I \phi(s) \left[\frac{1}{2\pi} \int_{\partial(G_2 \cap L_I)} S^{-1}(t,s)\, dt_I g(t) \right]\end{aligned}$$

for the vectorial version of the Fubini theorem we have

$$(\phi g)(T) = \frac{1}{2\pi} \int_{\partial(G_2 \cap L_I)} \left[\frac{1}{2\pi} \int_{\partial(G_1 \cap L_I)} S^{-1}(s,T)\, ds_I \phi(s) S^{-1}(t,s) \right] dt_I g(t).$$

Finally, observe that $S^{-1}(t,s)$ is s-regular in the variable s on the S-spectrum of T and $\phi(s) S^{-1}(t,s)$ is s-regular in the variable s thanks to Lemma 3.7, so we have

$$(\phi g)(T) = \frac{1}{2\pi} \int_{\partial(G_2 \cap L_I)} \phi(T) S^{-1}(t,T) dt_I g(t) = \phi(T) g(T),$$

where we have taken $\phi(T)$ out of the integral. □

Theorem 3.9 (A particular case of the S-spectral mapping theorem) *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. Then*

$$\sigma_S(T^n) = (\sigma_S(T))^n = \{s^n \in \mathbb{H} : s \in \sigma_S(T)\}.$$

Proof Recall that

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2\mathrm{Re}[s]T + |s|^2 \mathcal{I} \text{ is not invertible}\}$$

and

$$\sigma_S(T^n) = \{p \in \mathbb{H} : T^{2n} - 2\mathrm{Re}[p]T^n + |p|^2 \mathcal{I} \text{ is not invertible}\}.$$

Since, by Lemma 3.2 and Theorem 3.8, the operator $T^{2n} - 2\mathrm{Re}[s^n]T^n + |s^n|^2 \mathcal{I}$ can be factorized as

$$T^{2n} - 2\mathrm{Re}[s^n]T^n + |s^n|^2 \mathcal{I} = \mathcal{Q}_{2n-2}(T)(T^2 - 2\mathrm{Re}[s]T + |s|^2 \mathcal{I}),$$

we deduce that if $T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$ is not injective also $T^{2n} - 2\mathrm{Re}[s^n]T^n + |s^n|^2\mathcal{I}$ is not injective. This proves that $(\sigma_S(T))^n \subseteq \sigma_S(T^n)$. Let us now consider $p \in \sigma_S(T^n)$. By Lemma 3.3 and Theorem 3.8 we can write

$$T^{2n} - 2\mathrm{Re}[p]T^n + |p|^2\mathcal{I} = \prod_{j=0}^{n-1}(T^2 - 2\mathrm{Re}[\lambda_j]T + |\lambda_j|^2\mathcal{I}),$$

and since $T^{2n} - 2\mathrm{Re}[p]T^n + |p|^2\mathcal{I}$ is not invertible at least one of the operators $T^2 - 2\mathrm{Re}[\lambda_j]T + |\lambda_j|^2\mathcal{I}$ for some j is not invertible, proving that $\sigma_S(T^n) \subseteq (\sigma_S(T))^n$. □

We can now conclude this section with the S-spectral radius theorem.

Theorem 3.10 (The S-spectral radius theorem) *Let V be a quaternionic Banach space, let $T \in \mathcal{B}(V)$ and let $r_S(T)$ be its S-spectral radius. Then*

$$r_S(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Proof For every $s \in \mathbb{H}$ such that $|s| > r_S(T)$ the series $\sum_{n\geq 0} T^n s^{-1-n}$ converges in $\mathcal{B}(V)$ to the S-resolvent operator $S^{-1}(s,T)$. So the sequence $T^n s^{-1-n}$ is bounded in the norm of $\mathcal{B}(V)$ and

$$\limsup_{n\to\infty} \|T^n\|^{1/n} \leq r_S(T). \tag{25}$$

Theorem 3.9 implies $\sigma_S(T^n) = (\sigma_S(T))^n$, so we have

$$(r_S(T))^n = r_S(T^n) \leq \|T^n\|,$$

from which we get

$$r_S(T) \leq \liminf \|T^n\|^{1/n}. \tag{26}$$

From (25), (26) we obtain

$$r_S(T) \leq \liminf_{n\to\infty} \|T^n\|^{1/n} \leq \limsup_{n\to\infty} \|T^n\|^{1/n} \leq r_S(T). \tag{27}$$

The chain of inequalities (27) also proves the existence of the limit. □

4 The S-Spectral Mapping and the Composition Theorems

We collect in the following Lemma some useful properties of s-regular functions that will be used to prove the main results of this section.

Lemma 4.1 *Let $U \subset \mathbb{H}$ be an open set.*

a) *Suppose that $P(q)$, $Q(q)$ are polynomials in the quaternionic variable q with real coefficients and assume that $Q(q)$ has no zeros in U. Define $F(q) = (Q(q))^{-1}P(q)$ (or $F(q) = P(q)(Q(q))^{-1}$) then $F \in \mathcal{N}(U)$.*

b) *If* $f \in \mathcal{N}(U)$ *then* $f^2 \in \mathcal{N}(U)$.
c) *Let* U, U' *be two open sets in* $\mathbb{H}$ *and* $f \in \mathcal{N}(U')$, $g \in \mathcal{N}(U)$ *with* $g(U) \subseteq U'$. *Then* $f(g(q))$ *is s-regular in* U.

Proof Part a) trivially follows by replacing q by $z = x + Iy$ and observing that

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) F(x + Iy) = 0$$

for all $I \in \mathbb{S}$. To prove b), consider L_I for any $I \in \mathbb{S}$ and the restriction $f_I(z) = F(z)$, where $F : U \cap L_I \to L_I$ is a holomorphic function. This implies that also the function f^2 belongs to $\mathcal{N}(U)$. Finally, to prove c) set $q = x + Iy$. By hypothesis, $g(x + Iy) = \alpha(x, y) + I\beta(x, y)$, where α and β are real valued functions and

$$f(g(x + Iy)) = f(\alpha(x, y) + I\beta(x, y)) \subseteq L_I.$$

The function $f(g(x + Iy))$ is holomorphic on each plane L_I thus it satisfies the condition

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(g(x + Iy)) = 0$$

for all $I \in \mathbb{S}$ and so $f(g(q))$ is s-regular. □

Theorem 4.2 (The general version of the S-spectral mapping theorem) *Let* V *be a quaternionic Banach space*, $T \in \mathcal{B}(V)$ *and* $f \in \mathcal{N}_{\sigma_S(T)}$. *Then*

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : s \in \sigma_S(T)\}.$$

Proof Since $f \in \mathcal{N}_{\sigma_S(T)}$, there exists a domain $U \subset \mathbb{H}$ containing $\sigma_S(T)$, satisfying the requirements in Definition 2.16 and such that $f \in \mathcal{N}(U)$. Let us fix $\lambda \in \sigma_S(T)$. For $q \notin [\lambda]$, let us define the function $\tilde{g}(q)$ by

$$\tilde{g}(q) = (q^2 - 2\mathrm{Re}[\lambda]q + |\lambda|^2)^{-1}(f^2(q) - 2\mathrm{Re}[f(\lambda)]f(q) + |f(\lambda)|^2).$$

Observe that the assumption $f \in \mathcal{N}(U)$ implies that $f^2 \in \mathcal{N}(U)$ by Lemma 4.1(b), so also $f^2(q) - 2\mathrm{Re}[f(\lambda)]f(q) + |f(\lambda)|^2 \in \mathcal{N}(U)$. The function $(q^2 - 2\mathrm{Re}[\lambda]q + |\lambda|^2)^{-1} \in \mathcal{N}(U \setminus \{[\lambda]\})$, by Lemma 4.1(a), thus $\tilde{g}(q) \in \mathcal{N}(U \setminus \{[\lambda]\})$ by Lemma 3.7. We can extend $\tilde{g}(q)$ to an s-regular function whose domain is U: if the 2-sphere $[\lambda]$ is not reduced to a real point, then we define

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \notin [\lambda], \\ \frac{\partial}{\partial x} f(\mu) \frac{f(\mu) - \overline{f(\mu)}}{\mu - \overline{\mu}} & \text{if } q = \mu = \lambda_0 + I\lambda_1 \in [\lambda],\ I \in \mathbb{S}. \end{cases}$$

If the 2-sphere $[\lambda]$ is reduced to the real point λ, we define

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \neq \lambda, \\ (\frac{\partial}{\partial x} f(\lambda))^2 & \text{if } q = \lambda \in \mathbb{R}. \end{cases}$$

Let us consider the first case. Given the 2-sphere $[\lambda]$, on each plane L_I, $I \in \mathbb{S}$ the function $\tilde{g}$ has the two singularities $\lambda_0 \pm I\lambda_1 \in [\lambda]$. If we set $z = x + Iy$, we can compute the limit of $\tilde{g}$ on the plane L_I for $z \to \mu = \lambda_0 + I\lambda_1$ and for $z \to \overline{\mu} = \lambda_0 - I\lambda_1$. Observe that f restricted to the plane L_I is a holomorphic function from $U \cap L_I$ with values in the complex plane L_I, and, by Remark 2.12, $f(\lambda_0 + I\lambda_1) = \eta_K(\lambda_0, \lambda_1) + I\theta_K(\lambda_0, \lambda_1)$. However, by their definition, $\eta_K, \theta_K : U \cap L_K \to L_K$ for all $K \in \mathbb{S}$, so η_K, θ_K are real valued functions η, θ depending only on λ_0, λ_1 and we can write $f(\lambda_0 + I\lambda_1) = \eta(\lambda_0, \lambda_1) + I\theta(\lambda_0, \lambda_1)$. We deduce that $\mathrm{Re}[f(\lambda)] = \mathrm{Re}[f(\mu)]$ and $|f(\lambda)|^2 = |f(\mu)|^2$ for any choice of μ and λ on the same 2-sphere. So we have:

$$\lim_{z\to\mu} g_I(z) = \lim_{z\to\mu} (z^2 - 2\mathrm{Re}[\mu]z + |\mu|^2)^{-1}(f^2(z) - 2\mathrm{Re}[f(\mu)]f(z) + |f(\mu)|^2)$$
$$= \lim_{z\to\mu} \frac{(f(z) - f(\mu))(f(z) - \overline{f(\mu)})}{(z-\mu)(z-\overline{\mu})} = f'(\mu)\frac{f(\mu) - \overline{f(\mu)}}{\mu - \overline{\mu}},$$

and similarly we can calculate the limit when $z \to \overline{\mu}$. Note that the derivative $f'(\mu)$ coincides with $\frac{\partial}{\partial x} f(\mu)$ since f is an s-regular function (see [7]). In the second case, assume that $\lambda \in \mathbb{R}$. Consider any $J \in \mathbb{S}$ and the restriction of f to the plane L_J. Then $f : U \cap L_J \to L_J$ is a holomorphic function and $f(\lambda) \in \mathbb{R}$, indeed $f(\lambda) \in L_J$ for all $J \in \mathbb{S}$. Let us set $z = x + Jy$. We have:

$$\lim_{z\to\lambda} g_J(z) = \lim_{z\to\lambda} (z^2 - 2\mathrm{Re}[\lambda]z + |\lambda|^2)^{-1}(f^2(z) - 2\mathrm{Re}[f(\lambda)]f(z) + |f(\lambda)|^2)$$
$$= \lim_{z\to\lambda} \frac{(f(z) - f(\lambda))^2}{(z-\lambda)^2} = f'(\lambda)^2,$$

so the value of the limit is independent of the plane L_J. The function $g_I : U \cap L_I \to L_I$ is extended by continuity to $U \cap L_I$, so it is holomorphic on $U \cap L_I$ for all $I \in \mathbb{S}$. We conclude that the function $g : U \to \mathbb{H}$ is an s-regular function.

Now, using the auxiliary function g, defined on U and s-regular, thanks to Theorem 3.8 we can write

$$f^2(T) - 2\mathrm{Re}[f(\lambda)]f(T) + |f(\lambda)|^2\mathcal{I} = (T^2 - 2\mathrm{Re}[\lambda]T + |\lambda|^2\mathcal{I})g(T).$$

If $f^2(T) - 2\mathrm{Re}[f(\lambda)]f(T) + |f(\lambda)|^2\mathcal{I}$ admits a bounded inverse

$$B := (f^2(T) - 2\mathrm{Re}[f(\lambda)]f(T) + |f(\lambda)|^2\mathcal{I})^{-1} \in \mathcal{B}(V)$$

then we have

$$(T^2 - 2\mathrm{Re}[\lambda]T + |\lambda|^2\mathcal{I})g(T)B = \mathcal{I},$$

i.e. $g(T)B$ is the inverse of $T^2 - 2\mathrm{Re}[\lambda]T + |\lambda|^2\mathcal{I}$. Thus $f(\sigma_S(T)) \subseteq \sigma_S(f(T))$. Now we take $p \in \sigma_S(f(T))$ such that $p \notin f(\sigma_S(T))$. We define the function

$$h(q) := (f^2(q) - 2\mathrm{Re}[p]f(q) + |p|^2)^{-1}$$

which is s-regular on $\sigma_S(T)$. By Theorem 3.8 we obtain

$$h(T)(f^2(T) - 2\mathrm{Re}[p]f(T) + |p|^2\mathcal{I}) = \mathcal{I}$$

this means that $p \notin \sigma_S(f(T))$, but this contradicts the assumption. So $p \in f(\sigma_S(T))$. □

Theorem 4.3 *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. Assume that $f_n \in \mathcal{R}_{\sigma_S(T)}$, for all $n \in \mathbb{N}$ and let $U \supset \sigma_S(T)$ be as in Definition 2.16. If f_n converges uniformly to f on $U \cap L_I$, $I \in \mathbb{S}$, then $f_n(T)$ converges to $f(T)$ in $\mathcal{B}(V)$.*

Proof Let W be a domain as in Definition 2.16 such that $\sigma_S(T) \subset \overline{W} \subset U$. Then $f_n \to f$ converges uniformly on $\partial(W \cap L_I)$ and consequently

$$f_n(T) = \frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(s,T)\,ds_I\,f_n(s)$$

converges in the uniform topology of operators to

$$f(T) = \frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(s,T)\,ds_I\,f(s).$$

□

Theorem 4.4 *Let V be a quaternionic Banach space and let $T \in \mathcal{B}(V)$. Suppose that $f \in \mathcal{N}_{\sigma_S(T)}$, $\phi \in \mathcal{N}_{f(\sigma_S(T))}$ and define $F(s) = \phi(f(s))$. Then $F \in \mathcal{R}_{\sigma_S(T)}$ and $F(T) = \phi(f(T))$.*

Proof The statement $F \in \mathcal{R}_{\sigma_S(T)}$ follows from Lemma 4.1(c). Let $U \supset \sigma_S(f(T))$ be a domain as in Definition 2.16 whose boundary is denoted by ∂U. Suppose that $U \cup \partial U$ is contained in the domain in which ϕ is s-regular. Let W be a neighborhood of $\sigma_S(T)$ as in Definition 2.16 whose boundary is denoted by ∂W and suppose that $W \cup \partial W$ is contained in the domain where f is s-regular. Finally suppose that $f(W \cup \partial W) \subset U$. Let $I \in \mathbb{S}$ and define the operator

$$S^{-1}(\lambda, f(T)) = \frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\,ds_I\,S^{-1}(\lambda, f(s)),$$

where

$$S^{-1}(\lambda, f(s)) = -(f(s)^2 - 2\mathrm{Re}[\lambda]f(s) + |\lambda|^2)^{-1}(f(s) - \overline{\lambda}). \tag{28}$$

By applying Lemmas 4.1 and 3.7 and with some easy calculation it follows that $S^{-1}(\lambda, f(s))$ is s-regular in the variable s and it is right s-regular in the variable λ.

Take $\lambda \in \mathbb{R}$, so that also $S(\lambda, f(s))$ is an s-regular function and observe that

$$S^{-1}(\lambda, f(s))S(\lambda, f(s)) = S(\lambda, f(s))S^{-1}(\lambda, f(s)) = 1$$

so by Theorem 3.8 the operator $S^{-1}(\lambda, f(T))$ satisfy the equation:

$$S(\lambda, f(T))S^{-1}(\lambda, f(T)) = S^{-1}(\lambda, f(T))S(\lambda, f(T)) = \mathcal{I}. \tag{29}$$

Observe also that when λ is not necessarily a real number, identity (29) remains valid as it can be easily shown by replacing $S^{-1}(\lambda, f(T))$ and $S(\lambda, f(T))$ by their explicit expressions

$$S^{-1}(\lambda, f(T)) = -(f(T)^2 - 2\mathrm{Re}[\lambda] f(T) + |\lambda|^2)^{-1}(f(T) - \overline{\lambda})$$

and

$$S(\lambda, f(T)) = -(f(T) - \overline{\lambda})^{-1}(f(T)^2 - 2\mathrm{Re}[\lambda] f(T) + |\lambda|^2)$$

in (29) and verifying that we get an identity. Consequently we obtain

$$\begin{aligned}
\phi(f(T)) &= \frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(\lambda, f(T))\, d\lambda_I \phi(\lambda)\\
&= \frac{1}{2\pi}\int_{\partial(W\cap L_I)} \left(\frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\, ds_I S^{-1}(\lambda, f(s))\right) d\lambda_I \phi(\lambda)\\
&= \frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\, ds_I \left(\frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(\lambda, f(s))\, d\lambda_I \phi(\lambda)\right)\\
&= \frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\, ds_I \phi(f(s))\\
&= \frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,T)\, ds_I F(s) = F(T),
\end{aligned}$$

so this concludes the proof. □

5 Functional Calculus for Unbounded Operators

Let V be a quaternionic Banach space and $T = T_0 + \sum_{j=1}^{3} e_j T_j$ where $T_j : \mathcal{D}(T_j) \to V$ are linear operators for $j = 0, 1, 1, 3$ where at least one of the T_j's is an unbounded operator, $\mathcal{D}(T_j)$ denotes the domain of T_j. In this case we have to define the extended S-spectrum of T as

$$\overline{\sigma}_S(T) := \sigma_S(T) \cup \{\infty\}.$$

Let us consider $\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ endowed with the natural topology. Precisely, a set is open if and only if it is union of open discs $D(q,r)$ with center at points in $q \in \mathbb{H}$ and radius r, for some r, and/or union of sets the form $D'(\infty, r) \cup \{\infty\}$, for some r, where $D'(\infty, r) = \{q \in \mathbb{H} \mid |q| > r\}$.

We recall the following definitions:

Definition 5.1 We say that f is an s-regular function at ∞ if $f(q)$ is an s-regular function in a set $D'(\infty, r)$ and $\lim_{q\to\infty} f(q)$ exists and it is finite. We define $f(\infty)$ to be the value of this limit.

Remark 5.2 We now that if T is a linear and bounded quaternionic operator then $\sigma_S(T)$ is a compact nonempty set, but for unbounded operators, as in the classical case, the S-spectrum can be $\sigma_S(T)=\emptyset$, $\sigma_S(T)=\mathbb{H}$; moreover $\sigma_S(T)$ can be bounded or unbounded. In the sequel we will assume that $\rho_S(T)\neq\emptyset$.

Definition 5.3 Let V be a quaternionic Banach space. We consider the linear closed densely defined operator $T:\mathcal{D}(T)\subset V\to V$ where $\mathcal{D}(T)$ denotes the domain of T. Let us assume that

1) $\mathcal{D}(T)$ is dense in V,
2) $T-\overline{s}\mathcal{I}$ is densely defined in V,
3) $\mathcal{D}(T^2)\subset\mathcal{D}(T)$ is dense in V,
4) $T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I}$ is one-to-one with range V.

The S-resolvent operator is defined by

$$S^{-1}(s,T)=-(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}(T-\overline{s}\mathcal{I}). \tag{30}$$

Observe that the operator $S^{-1}(s,T)$ is the restriction to the dense subspace $\mathcal{D}(T)$ of V of a bounded linear operator defined on V. This fact follows by the commutation relation $(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}Tv=T(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}v$ which holds for all $v\in\mathcal{D}(T)$ since the polynomial operator $T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I}:\mathcal{D}(T^2)\to V$ has real coefficients. The operator $T(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}:V\to\mathcal{D}(T)$ is continuous for those $s\in\mathbb{H}$ such that $(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}\in\mathcal{B}(V)$. We will intend the operator in (30) extended to all V as $S^{-1}(s,T):=(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}\overline{s}-T(T^2-2T\mathrm{Re}[s]+|s|^2\mathcal{I})^{-1}$. So the S-resolvent set $\rho_S(T)$ of T consists of those $s\in\mathbb{H}$ such that $S^{-1}(s,T)\in\mathcal{B}(V)$ and the S-spectrum $\sigma_S(T)$ of T is defined by $\sigma_S(T)=\mathbb{H}\setminus\rho_S(T)$.

Definition 5.4 Let $T:\mathcal{D}(T)\to V$ be a linear closed operator as in Definition 5.3. Let $U\subset\mathbb{H}$ be an open set that contains the S-spectrum of T. Suppose that U also satisfies the condition i)–iv) in Definition 2.16. Assume that $\overline{U}$ and ∞ are contained in an open set in which f is s-regular. A function f is said to be locally s-regular on $\overline{\sigma}_S(T)$ if there exists an open set U as above such that f is s-regular on U and at infinity.

We will denote by $\mathcal{R}_{\overline{\sigma}_S(T)}$ the set of locally s-regular functions on $\overline{\sigma}_S(T)$.

Remark 5.5 As we have pointed out in Remark 2.17, the open set U related to $f\in\mathcal{R}_{\overline{\sigma}_S(T)}$ need not to be connected. Moreover, as in the classical functional calculus, U can depend on f and can be unbounded.

Definition 5.6 Consider $k\in\mathbb{H}$ and the homeomorphism

$$\Phi:\overline{\mathbb{H}}\to\overline{\mathbb{H}}$$

defined by

$$p=\Phi(s)=(s-k)^{-1},\qquad \Phi(\infty)=0,\qquad \Phi(k)=\infty.$$

Definition 5.7 Let $T : \mathcal{D}(T) \to V$ be a linear closed operator as in Definition 5.3 with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{R}_{\overline{\sigma}_S(T)}$. Let us consider

$$\phi(p) := f(\Phi^{-1}(p))$$

and the operator defined by

$$A := (T - k\mathcal{I})^{-1}, \quad \text{for some } k \in \rho_S(T) \cap \mathbb{R}.$$

We define the operator $f(T)$ as

$$f(T) = \phi(A). \tag{31}$$

Remark 5.8 Observe that, if $k \in \mathbb{R}$, we have that:

i) the function $\phi = f(\Phi^{-1}(p))$ is s-regular because it is the composition of the function f which is s-regular and $\Phi^{-1}(p) = p^{-1} + k$ which is s-regular with real coefficients;
ii) in the case $k \in \rho_S(T) \cap \mathbb{R}$ we have that $(T - k\mathcal{I})^{-1} = -S^{-1}(k, T)$.

To prove the fundamental Theorem 5.10 we need the following result proved in [5].

Theorem 5.9 *If $k \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and Φ, ϕ are as above, then $\Phi(\overline{\sigma}_S(T)) = \sigma_S(A)$ and $\phi(p) = f(\Phi^{-1}(p))$ determines a one-to-one correspondence between $f \in \mathcal{R}_{\overline{\sigma}_S(T)}$ and $\phi \in \mathcal{R}_{\overline{\sigma}_S(A)}$. The relation between the S-resolvent operators of T and of A is given by*

$$S^{-1}(s, T) = p\mathcal{I} - S^{-1}(p, A)p^2. \tag{32}$$

The proof of the next important result is analogous to the one of Theorem 4.12 in [8]; but thanks to the Cauchy formula with s-regular kernel (2.13) we can replace the functional calculus in Definition 3.15 in [8] with the one in Definition 2.21 that holds for more general domains.

Theorem 5.10 *Let V be a quaternionic Banach space and let $T : \mathcal{D}(T) \to V$ be a linear closed operator as in Definition 5.3 with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{R}_{\overline{\sigma}_S(T)}$. We have the following:*

i) *The operator $f(T)$ defined in (31) is independent of $k \in \rho_S(T) \cap \mathbb{R}$.*
ii) *Let W be an open set as in Definition 5.4 such that $\overline{\sigma}_S(T) \subset W$ and let f be an s-regular function on $W \cup \partial W$. Let $I \in \mathbb{S}$ and $W \cap L_I$ be such that its boundary $\partial(W \cap L_I)$ is positively oriented and consists of a finite number of rectifiable Jordan curves. Then*

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(W \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s). \tag{33}$$

Proof Part i) of the statement follows from the validity of formula (33) since the integral is independent of k.

To prove part ii), consider $k \in \rho_S(T) \cap \mathbb{R}$ and assume that the set W is such that $k \notin \overline{(W \cap L_I)}$, $\forall I \in \mathbb{S}$. Otherwise, by the Cauchy theorem, we can replace W by W', on which f is s-regular, such that $k \notin \overline{(W' \cap L_I)}$, without altering the value of the integral (33). Moreover, the integral (33) is independent of the choice of $I \in \mathbb{S}$.

We have that $\mathcal{V} \cap L_I := \Phi^{-1}(W \cap L_I)$ is an open set that contains $\sigma_S(T)$ and its boundary $\partial(\mathcal{V} \cap L_I) = \Phi^{-1}(\partial(W \cap L_I))$ is positively oriented and consists of a finite number of rectifiable Jordan curves. Using the relation (32) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(s,T)\,ds_I\, f(s)\\
&= -\frac{1}{2\pi}\int_{\partial(\mathcal{V}\cap L_I)} \left(p\mathcal{I} - S^{-1}(p,A)p^2\right)p^{-2}dp_I\phi(p)\\
&= -\frac{1}{2\pi}\int_{\partial(\mathcal{V}\cap L_I)} p^{-1}dp_I\phi(p) + \frac{1}{2\pi}\int_{\partial(\mathcal{V}\cap L_I)} S^{-1}(p,A)dp_I\phi(p)\\
&= -\mathcal{I}\phi(0) + \phi(A)
\end{aligned}$$

now by definition $\phi(A) = f(T)$ and $\phi(0) = f(\infty)$ we obtain

$$\frac{1}{2\pi}\int_{\partial(W\cap L_I)} S^{-1}(s,T)\,ds_I\, f(s) = -\mathcal{I}f(\infty) + f(T).$$

□

In the following theorem we show some algebraic properties that can be deduced easily.

Theorem 5.11 *Let V be a quaternionic Banach space and let $T : \mathcal{D}(T) \to V$ be a linear closed operator as in Definition 5.3 with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. If f and $g \in \mathcal{R}_{\sigma_S(T)}$, then*

$$(f+g)(T) = f(T) + g(T).$$

If $g \in \mathcal{R}_{\overline{\sigma}_S(T)}$ and $f \in \mathcal{N}_{\overline{\sigma}_S(T)}$, then

$$(fg)(T) = f(T)g(T).$$

Proof Observe that $fg \in \mathcal{R}_{\sigma_S(T)}$ thanks to Lemma 3.7. Let $\phi(\mu) = f(\Phi^{-1}(\mu))$ and $\psi(\mu) = g(\Phi^{-1}(\mu))$. Thanks to Lemma 3.7 and Lemma 4.1 the product $\phi\psi$ is s-regular. By definition we have

$$f(T) = \phi(A), \qquad g(T) = \psi(A).$$

By Theorem 3.8 we get

$$(\phi+\psi)(A) = \phi(A) + \psi(A), \qquad (\phi\psi)(A) = \phi(A)\psi(A)$$

so we get the statement. □

Theorem 5.12 *Let V be a quaternionic Banach space and let $T : \mathcal{D}(T) \to V$ be a linear closed operator as in Definition 5.3 with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. If $f \in \mathcal{N}_{\sigma_S(T)}$, then*

$$\sigma_S(f(T)) = f(\overline{\sigma}_S(T)).$$

Proof Let $\phi(\mu) = f(\Phi^{-1}(\mu))$. For the S-spectral mapping theorem we have $\phi(\sigma_S(A)) = \sigma_S(\phi(A))$ and for Theorem 5.9 we also have $\Phi(\overline{\sigma}_S(T)) = \sigma_S(A)$. So we obtain

$$\phi(\Phi(\overline{\sigma}_S(T)) = \phi(\sigma_S(A)) = \sigma_S(\phi(A)) = \sigma_S(f(T)).$$

On the other hand

$$\phi(\Phi(\overline{\sigma}_S(T)) = f(\Phi^{-1}(\Phi(\overline{\sigma}_S(T))) = f(\overline{\sigma}_S(T)).$$ □

6 Examples

In this subsection we collect some examples which give an idea of the applications of our functional calculus especially to the theory of quaternionic evolution operators. The material of this subsection is still under investigation and is the subject of a forthcoming paper.

We begin with an application of Theorems 5.10 and 5.12.

Example 6.1 Let V be a quaternionic Banach space. Let $T : \mathcal{D}(T) \to V$ be a linear closed operator as in Definition 5.3 such that T^{-1} is a bounded operator. From the definition of S-resolvent operator we get $S^{-1}(0, T) = -T^{-1}$ so 0 belongs to $\rho_S(T)$. Moreover the function $f(q) = q^{-1}$ is s-regular in neighborhood W of $\sigma_S(T)$ such that $0 \notin \overline{W}$. Since $f(q) = q^{-1} \to 0$ as $q \to \infty$, for Theorem 5.10, we have

$$T^{-1} = \frac{1}{2\pi} \int_{\partial(W \cap L_I)} S^{-1}(s, T)\, ds_I\, s^{-1}$$

and thanks to Theorem 5.12 we obtain:

$$\sigma_S(T^{-1}) = \{\lambda^{-1} : \lambda \in \overline{\sigma}_S(T)\}.$$

Definition 6.2 A family $\{\mathcal{U}(t)\}_{t\geq 0}$ of quaternionic bounded linear operators on a quaternionic Banach space V will be called a strongly continuous semigroup if

- $\mathcal{U}(t+\tau) = \mathcal{U}(t)\mathcal{U}(\tau),\ t, \tau \geq 0,$
- $\mathcal{U}(0) = \mathcal{I}$,
- for every $v \in V$, $\mathcal{U}(t)v$ is continuous in $t \in [0, \infty]$.

If, in addition, the map $t \to \mathcal{U}(t)$ is continuous in the uniform operator topology, the family $\{\mathcal{U}(t)\}_{t\geq 0}$ is called a uniformly continuous semigroup in $\mathcal{B}(V)$.

Example 6.3 (Evolution operator) Let V be a quaternionic Banach space, $A \in \mathcal{B}(V)$ and let $U \supset \sigma_S(A)$ be a domain as in the Definition 2.16. Define the operator (for $t \geq 0$)

$$e^{tA} = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, A)\, ds_I e^{ts}. \tag{34}$$

We want to present here just one last result that shows the strong analogy of our quaternionic functional calculus with the Riesz-Dunford functional calculus. An important tool to study the theory of semigroups is its Laplace transform. In the complex case, the Laplace transform of the semigroup e^{tB} is the resolvent operator $(\lambda e - B)^{-1}$, where B is a complex operator. As it is shown by the following result, in the quaternionic case the Laplace transform of the semigroup e^{tA} is the S-resolvent operator $S^{-1}(s, A)$. We anticipate the differentiability property of the semigroup e^{tA}.

Lemma 6.4 *Let $A \in \mathcal{B}(V)$ and let $U \supset \sigma_S(A)$ be a domain as in the Definition* 2.16. *Let e^{tA} be the operator defined in* (34). *Then*

$$\frac{d}{dt} e^{tA} = A e^{tA}.$$

Proof Take $h \in \mathbb{R}$. From definition (34) we get

$$\frac{e^{(t+h)A} - e^{tA}}{h} = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, A)\, ds_I \frac{(e^{(t+h)s} - e^{ts})s}{h} s^{-1}$$

now consider the fact that for any $t, h \in \mathbb{R}$ and for any quaternion $s \in \mathbb{H}$ we have that e^{ts} and e^{hs} commute between them and with s, moreover $e^{(t+h)s} = e^{ts} e^{hs}$ holds. Thus we have

$$\frac{e^{(t+h)A} - e^{tA}}{h} = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, A) ds_I s e^{ts} \frac{(e^{hs} - 1)}{h} s^{-1}.$$

Taking the limit for $h \to 0$ we get

$$\lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h} = A e^{tA}.$$

□

We can now prove the following result.

Theorem 6.5 *Let $A \in \mathcal{B}(V)$ and set $\mathcal{U}(t) = e^{tA}$. Then for $s_0 > \|A\|$ the S-resolvent operator is given by*

$$S^{-1}(s, A) = \int_0^{+\infty} \mathcal{U}(t) e^{-ts}\, dt.$$

Proof We have to prove that

$$S(s, A) \int_0^{\infty} \mathcal{U}(t) e^{-ts}\, dt = \mathcal{I},$$

where

$$S(s,A) = -(A-\overline{s}\mathcal{I})^{-1}(A^2 - 2s_0A + |s|^2\mathcal{I}).$$

Take $\theta > 0$ and consider

$$S(s,A)\int_0^\theta \mathcal{U}(t)e^{-ts}\,dt = -(A-\overline{s}\mathcal{I})^{-1}(A^2 - 2s_0A + |s|^2\mathcal{I})\int_0^\theta e^{tA}e^{-ts}\,dt.$$

Since every bounded linear operator commutes with the integral, we get

$$S(s,A)\int_0^\theta e^{tA}e^{-ts}\,dt = -\int_0^\theta (A-\overline{s}\mathcal{I})^{-1}(A^2 - 2s_0A + |s|^2\mathcal{I})e^{tA}e^{-ts}\,dt. \quad (35)$$

Thanks to Lemma 6.4 we obtain the identities

$$\begin{aligned}
&(A-\overline{s}\mathcal{I})^{-1}(A^2 - 2s_0A + |s|^2\mathcal{I})e^{tA}e^{-ts} \\
&\quad = (A-\overline{s}\mathcal{I})^{-1}e^{tA}(A^2 - A\overline{s} - As + \overline{s}s\mathcal{I})e^{-ts} \\
&\quad = (A-\overline{s}\mathcal{I})^{-1}\{e^{tA}A(A-\overline{s}\mathcal{I})e^{-ts} - e^{tA}(A-\overline{s}\mathcal{I})se^{-ts}\} \\
&\quad = (A-\overline{s}\mathcal{I})^{-1}\left\{\frac{d}{dt}e^{tA}(A-\overline{s}\mathcal{I})e^{-ts} + e^{tA}(A-\overline{s}\mathcal{I})\frac{d}{dt}e^{-ts}\right\} \\
&\quad = \frac{d}{dt}[(A-\overline{s}\mathcal{I})^{-1}e^{tA}(A-\overline{s}\mathcal{I})e^{-ts}]. \quad (36)
\end{aligned}$$

So by identity (36) we can write (35) as

$$\begin{aligned}
S(s,A)\int_0^\theta e^{tA}e^{-ts}\,dt &= -\int_0^\theta \frac{d}{dt}[(A-\overline{s}\mathcal{I})^{-1}e^{tA}(A-\overline{s}\mathcal{I})e^{-ts}]\,dt \\
&= \mathcal{I} - (A-\overline{s}\mathcal{I})^{-1}e^{\theta A}(A-\overline{s}\mathcal{I})e^{-s\theta}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|(A-\overline{s}\mathcal{I})^{-1}e^{\theta A}(A-\overline{s}\mathcal{I})e^{-s\theta}\| \\
&\quad \le \|(A-\overline{s}\mathcal{I})^{-1}\|\|e^{\theta A}\|\|(A-\overline{s}\mathcal{I})\|\|e^{-s\theta}\| \\
&\quad \le \|(A-\overline{s}\mathcal{I})^{-1}\|\|(A-\overline{s}\mathcal{I})\|\,e^{\theta\|A\|}e^{-s_0\theta} \to 0
\end{aligned}$$

for $\theta \to +\infty$ because we have assumed $s_0 > \|A\|$. So we get the statement. □

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