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The Levy-Gromov Isoperimetric Inequality in Convex Manifolds with Boundary

Frank Morgan

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Abstract We observe after Bayle and Rosales that the Levy-Gromov isoperimetric inequality generalizes to convex manifolds with boundary and certain singularities.

Keywords Isoperimetric inequality · Levy-Gromov

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The Levy-Gromov isoperimetric inequality [4, Sect. 2.2], [2, 34.3.2] provides a sharp lower bound on the perimeter required to enclose given volume in a closed manifold M in terms of a positive lower bound on the Ricci curvature, by comparison with the sphere. Our Theorem 1 shows that the result and its proof generalize to compact, convex manifolds with boundary. The heart of the proof is the observation that the region of prescribed volume is covered by rays normal to the enclosing, area-minimizing surface, yielding an inequality for the volume in terms of the surface and its mean curvature after Heintze and Karcher [6]. In a convex manifold with boundary, the surface should meet the boundary of M orthogonally, the normal rays should still cover the region, and the same result should hold. Remark 4 shows that cone-type singularities with sufficiently small link are negligible.

The case of smooth boundary was proved by Bayle and Rosales [1, Theorem 4.8], using the second variation formula. One could use smoothing to deduce from their result our Theorem 1, except for our characterization of when equality holds.

Colin Adams suggested to me another line of proof for the smooth boundary case: apply the classical Levy-Gromov inequality to the double, after delicate smoothing which preserves the lower bound on curvature. Kris Tapp pointed out to me that such

F. Morgan (✉)
Department of Mathematics and Statistics, Williams College, Williamstown, 01267 MA, USA
e-mail: Frank.Morgan@williams.edu

smoothing is provided by Kronwith [7] and Guijarro [5]. This approach loses the characterization of equality even for the smooth case.

So let M be a compact convex body with boundary B in a smooth n-dimensional Riemannian manifold ($n \geq 2$). The (partially free boundary) isoperimetric problem seeks a region of prescribed volume fraction $0 < V < 1$ of least perimeter, not counting perimeter inside B. We review some results from geometric measure theory (see [9, 11]). The existence of a perimeter-minimizing or *isoperimetric* region follows from standard compactness arguments. Let S be the closure of the part of the boundary of the region in the interior of M. At an interior point p, S has a tangent cone, area-minimizing without volume constraint; if such a cone is a plane, then S is a smooth, constant-mean-curvature hypersurface at p. (Actually it is known that the interior singular set has Hausdorff dimension at most $n - 8$.) At boundary points of S, tangent "cones" still exist, and one expects S to be normal to B at most regular points of B, although we do not need such delicate regularity.

Theorem 1 (Levy-Gromov for Convex Manifolds with Boundary) *Let M be a compact convex body in an n-dimensional smooth Riemannian manifold* ($n \geq 2$), *with Ricci curvature bounded below by $n - 1$. Let λ denote the ratio of the volume of M to the volume of the unit sphere. Then the isoperimetric profile $P(V)$, the least perimeter to enclose given volume fraction* $0 < V < 1$, *satisfies*

$$P \geq \lambda P_0, \tag{1}$$

where P_0 is the isoperimetric profile of the unit sphere. Equality holds for some V only if M is a suspension of a space form with boundary and M is locally isometric to a round sphere.

For example, equality holds if M is the smaller region of S^2 between two great semicircles of longitude, the suspension of an arc of the equator.

We provide a sketch of the whole proof. The main new ingredient for the generalization appears in paragraph three, the observation that shortest paths from an interior point to an isoperimetric surface meet the surface on the interior, so that normal rays from the interior of the surface cover the interior of the region.

Proof sketch For given volume fraction $0 < V < 1$, let P be the perimeter of a minimizing hypersurface S in M and let P_0 be the perimeter of the hypersphere S_0 in the unit sphere. By replacing V by $1 - V$ if necessary, which changes the sign of the (constant) mean curvatures, we may assume that the mean curvature of S is greater than or equal to that of S_0.

The idea is to estimate V by the volume of the union of rays normal to S at regular points of S. Consider a shortest path γ from a point p to the surface S, meeting S at a point q. We claim that if q is an interior point, then q must be a regular point of S.

Let C be an (area-minimizing) tangent cone to S at q. Consider a sphere Σ about the vertex of C, with the tangent to γ entering at the south pole. Since γ is a shortest path, C lies in the far halfspace bounded by the hyperplane normal to γ and intersects the sphere Σ only in the northern hemisphere. Unless C meets Σ only in the equator, moving the vertex of C towards the north pole (in the direction of the continuation of γ) would decrease area to first order, a contradiction. Therefore C meets Σ only in the equator, C is contained in a hyperplane, and C must be a hyperplane. Consequently q is a regular point of S. Moreover, as a shortest path, γ must meet S normally.

We claim that q cannot be a boundary point; suppose it were. As in the previous paragraph, C must be contained in the hyperplane normal to γ at q. Actually, here the situation is even more favorable because when you move the vertex of C in the direction of the continuation of γ, you can throw away everything outside the convex tangent cone C' to M at q. Without this effect, since C lies in the hyperplane normal to γ at q, the area of C would not change to first order. With this effect, area would decrease to first order, unless the direction of motion lies in $\partial C'$, which implies that γ is tangent to the boundary of M at q, a contradiction of convexity.

Therefore the union of rays normal to S at regular points of S cover the region of volume fraction V. By the calculus estimate of Heintze-Karcher (see [6, Theorem 2.2] or [9, Theorem 18.4]), the volume λV enclosed by S satisfies

$$\frac{\lambda V}{P} \leq \frac{V}{P_0},$$

as desired. If equality holds for some V, then a pencil of geodesics normal to S must be isometric to a pencil of normals to a round subsphere of a round sphere (local version of [6, Theorem 4.6]), S and S_0 have the same mean curvature, the argument works in both directions, M is the round suspension of a convex space form with boundary, and equality holds for all V.

Technical Note While the existence of a tangent cone depends on so-called monotonicity, which is not standard at nonsmooth free boundary, we just need a non-zero tangent object (weak limit of homothetic expansions), which follows from lower density bounds (which can be proved for example as in [10, Lemma 3.6]) and trivial upper density bounds. □

Corollary 2 *Let M be a convex subregion of the nD unit hemisphere with volume λ times the volume of the hemisphere. Then the isoperimetric profile $P(V)$ (least perimeter to enclose given volume fraction $0 < V < 1$) satisfies*

$$P \geq \lambda P_1, \tag{2}$$

where P_1 is the isoperimetric profile of the unit hemisphere. If equality holds for some V, then M is the suspension of a convex subset of an equatorial hypersphere (and hence equality holds for all V).

Proof Corollary 2 follows immediately from Theorem 1 because $P_1 = 0.5 P_0$ and the λ of the corollary is twice the λ of the theorem. □

As a further corollary we have an isoperimetric result stated in Morgan [10, Remark 3.11], a paper which dealt primarily with the *surface* of a polytope. Corollary 3 was proved earlier without uniqueness by Lions and Pacella [8, Theorem 1.1]. See also [12, Remark after Theorem 10.6] and for the smooth case [14, Theorem 4.11].

Corollary 3 *Let P^n be a solid (convex) polytope in R^n. For small prescribed volume, isoperimetric regions are balls about a vertex.*

Proof The proof for solid polytopes follows the proof for surfaces of polytopes [10, Theorem 3.2]. The proof applies an isoperimetric comparison theorem for warped products [15, Theorems 3.7, 3.9] to the cone C over a vertex p of P, viewed as a warped product $C = (0, \infty) \times_f M$ (with $f(t) = t$). The theorem infers an isoperimetric inequality on the product from an isoperimetric inequality on slices in one factor. Specifically, given (2), the theorem concludes that the isoperimetric profiles of C and the cone over the unit hemisphere, namely a Euclidean halfspace, also satisfy (2). In particular, since balls about the vertex (half-balls in the halfspace) are isoperimetric in the latter, balls about the vertex are isoperimetric in the former. If uniqueness fails, some slice is nontrivial and equality must hold in (2) for some $0 < V < 1$, so that by Corollary 2 M is the suspension of an equatorial hypersphere and p would not be a vertex of P.

By variational and limit arguments [10, 3.5–3.7], there is an $a > 0$ such that an isoperimetric region R of small volume in P, scaled up to be an isoperimetric region of unit volume in a homothetic expansion of P, has area at least a inside a unit ball about each of its points. By translation of components, we may assume that R is contained in the union of disjoint balls about the vertices of P. By the first paragraph of this proof, we may assume that R consists of balls about vertices v. Since for such a ball, $A = c_v V^t$ with $t = (n-2)/(n-1) < 1$, by concavity of V^t a single ball is best. □

Remark on Singularities 4 (J. Petean) Theorem 1 generalizes to allow M to have interior and boundary singularities with sufficiently sharp tangent cones. More technically, although further generalizations are possible, we will make the following assumptions:

(1) M is a connected compact Lipschitz neighborhood retract in some Euclidean space, the classical singular domain of geometric measure theory [3, Sect. 4.1.29];
(2) M is a smooth n-dimensional submanifold with boundary except for a (compact) singular set X of $(n-1)$-dimensional Hausdorff measure 0, intrinsically convex at all regular boundary points;
(3) at every point of X, M has an oriented tangent cone, the cone over a compact, path-connected rectifiable subset Q of the unit sphere;
(4) the complement of every intrinsic metric ball in Q of radius $\pi/2$ is contained in a ball of radius less than $\pi/2$.

Then the conclusions of Theorem 1 continue to hold.

Proof To extend the proof, it suffices to show that a shortest path γ from an interior regular point p of M to an isoperimetric surface S never hits the singular set X.

Suppose that an interior point of γ lies in X. Since γ is a shortest path, the distance between the backward and forward unit tangent vectors to γ in Q must be at least π, in contradiction of (4). Suppose that the endpoint q of γ lies in the singular set X. By hypothesis (4), the complement of the metric ball in Q of radius $\pi/2$ about the unit tangent to γ is contained in a ball D of radius less than $\pi/2$ about some point w in Q. Since γ is a shortest path, an (area-minimizing) oriented tangent cone C to S at q must lie in the cone over D. Moving the vertex of C in the direction of w would decrease area to first order, a contradiction. □

Remark 4 provides an alternate proof of some cases of Petean [13, Theorem 1.1] (his X is our M) and was suggested by him. One of his cases not covered occurs when his M (our Q) is CP^2, normalized to have Ricci curvature between 3 and 4.5; the three axis complex lines are pairwise at distance $\pi/\sqrt{2}$ and our hypothesis (4) fails.

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