

Comparison of the Effect of Horizontal Vibrations on Interfacial Waves in a Two-Layer System of Inviscid Liquids to Effective Gravity Inversion

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Abstract We study the waves at the interface between two thin horizontal layers of immiscible liquids subject to high-frequency tangential vibrations. Nonlinear governing equations are derived for the cases of two- and three-dimensional flows and arbitrary ratio of layer thicknesses. The derivation is performed within the framework of the long-wavelength approximation, which is relevant as the linear instability of a thin-layers system is long-wavelength. The dynamics of equations is integrable and the equations themselves can be compared to the Boussinesq equation for the gravity waves in shallow water, which allows one to compare the action of the vibrational field to the action of the gravity and its possible effective inversion.

Keywords Interfacial waves · Two-layer liquid system · Longitudinal vibrations · Boussinesq equation

Introduction

The first experimental studies on the usage of vibrations for stabilizing otherwise unstable configurations of multi-phase fluid systems were reported by Wolf (1961, 1970).

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Under the weightlessness conditions, the demand for such a control tool even increases because of the necessity to maintain stratification of fluids or control convection for diverse technological systems, which is well highlighted by the ongoing research on the subject (Thiele et al. 2006; Mialdun et al. 2008; Shklyaev et al. 2009; Nepomnyashchy and Simanovskii 2013; Gaponenko and Shevtsova 2016; Bratsun et al. 2016; Lappa 2016; Smorodin et al. 2017; Lyubimova et al. 2017).

Wolf's experimental observations of wave patterns on the interface between immiscible fluids subject to horizontal vibrations received their first solid theoretical basis with the linear instability analysis of the flat state of the interface (Lyubimov and Cherepanov 1986; Khenner et al. 1998, 1999). In Fig. 1, one can see the sketch of the system for which the instability was theoretically revealed for strong enough vibrations. Later on, the nonlinear dynamics of interfacial waves for the high-viscosity case and one liquid layer was studied analytically by Shklyaev et al. (2009) and Benilov and Chugunova (2009). In Wolf's experiments (Wolf 1961, 1970), the viscous boundary layer in the most viscous liquid was an order of magnitude thinner than the liquid layer, meaning the approximation of inviscid liquid to be relevant. The analytical treatment of the nonlinear dynamics of interfacial waves in inviscid liquids was made possible in Goldobin et al. (2014, 2015), where the governing equations for two-dimensional flows were reported. The equations were derived below the instability threshold within the framework of the long-wavelength approximation for the case of equal thickness of layers. Simultaneously, the understanding of strongly nonlinear regimes of the dynamics of the interface in *low-viscosity* liquids above the instability threshold was significantly advanced in (Lyubimova et al. 2017) by means of numerical simulation accompanied by analytical estimates.

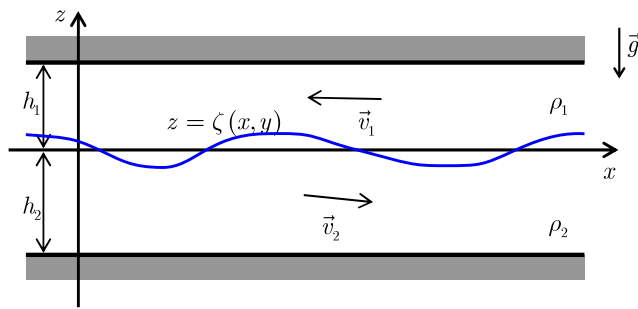


Fig. 1 Sketch of the coordinate frame and two-layer liquid system subject to longitudinal vibrations

In this paper we extend the derivation of governing equations for inviscid liquids to the case of arbitrary ratio of layer thicknesses and admit the variation of flow in the horizontal direction orthogonal to vibrations, i.e., consider a 3-d problem. The equations to be derived for both ‘normal’ and ‘inverted’ (the heavy liquid overlies the light one) configurations of the system. The comparison of these equations with the Boussinesq equation for the gravity waves in shallow water is of interest and suggests noteworthy interpretations of the action of the vibration field on the system.

Problem Statement and Governing Equations

We consider a system of two horizontal layers of immiscible inviscid liquids, confined between two impermeable horizontal boundaries (see Fig. 1). The system is subject to high-frequency longitudinal vibrations of linear polarization; the velocity of vibrational motion of the system is $(b/2)e^{i\omega t} + c.c.$ (here and hereafter, “c.c.” stands for complex conjugate). The density of the upper liquid ρ_1 is smaller than the density of the lower one ρ_2 . The layer thicknesses are h_1 and h_2 (see Fig. 1). We choose the horizontal coordinate x along the direction of vibrations, the z -axis is vertical with origin at the unperturbed interface between layers.

In this system, at the limit of infinitely long layers, the state with flat interface $z = \zeta(x, y) = 0$ is always possible. If the layers are limited in horizontal directions by impermeable lateral boundaries, the interface will be nearly flat at a distance from these boundaries. For inviscid fluids, this state (the ground state) is featured by spatially homogeneous pulsating velocity fields \mathbf{v}_{j0} in both layers;

$$\mathbf{v}_{j0} = a_j(t)\mathbf{e}_x, \quad a_j(t) = A_j e^{i\omega t} + c.c., \quad (1)$$

$$A_1 = \frac{(h_1 + h_2)\rho_2 b}{2(h_2\rho_1 + h_1\rho_2)}, \quad A_2 = \frac{(h_1 + h_2)\rho_1 b}{2(h_2\rho_1 + h_1\rho_2)},$$

where $j = 1, 2$ and \mathbf{e}_x is the unit vector of the x -axis. (The shift of the time offset results in a complex multiplier for b and A_j ; therefore, one can choose the time offset so that b

and A_j will be real.) The result (01) follows from the condition of zero pressure jump across the uninflected interface and the condition of the total fluid flux through the vertical cross-section which is $\int_{-h_2}^{+h_1} v^{(x)} dz = (h_1 + h_2)b \cos \omega t$ (which is due to the system motion with velocity $b \cos \omega t$).

Considering flow of inviscid liquid, it is convenient to introduce potential ϕ_j of the velocity field;

$$\mathbf{v}_j = -\nabla\phi_j. \quad (2)$$

The mass conservation law for incompressible liquid, $\nabla \cdot \mathbf{v}_j = 0$, yields the Laplace equation for potential, $\Delta\phi_j = 0$. The kinematic conditions on the top and bottom boundaries

$$\phi_{1z}(z = h_1) = \phi_{2z}(z = -h_2) = 0 \quad (3)$$

and on the interface $z = \zeta(x, y)$

$$\dot{\zeta} = -\phi_{1z} + \nabla\phi_1 \cdot \nabla\zeta, \quad (4)$$

$$\dot{\zeta} = -\phi_{2z} + \nabla\phi_2 \cdot \nabla\zeta \quad (5)$$

are also to be taken into account. (Here and hereafter, the upper dot stands for the time-derivative and letter in subscript denotes partial derivative with respect to the corresponding coordinate.) Eqs. (04) and (05) can be derived from the condition that the points of zero value of the distance function $F = z - \zeta(x, y)$, which correspond to the position of the interface, move with liquid, i.e., the Lagrangian derivative (material derivative) $dF/dt = \partial F/\partial t + \mathbf{v} \cdot \nabla F$ is zero on the interface: $-\dot{\zeta} + v^{(z)} - \mathbf{v} \cdot \nabla\zeta = 0$, and this holds for both liquids.

After substitution of the potential flow, the Euler equation takes the following form:

$$\nabla \left(-\dot{\phi}_j + \frac{1}{2} (\nabla\phi_j)^2 \right) = \nabla \left(-\frac{1}{\rho_j} p_j - gz \right),$$

where g is the gravity acceleration. The latter equation provides the expression for the pressure field in the volume of two liquids for a given flow field;

$$p_j = p_{j0} + \rho_j \left(\dot{\phi}_j - \frac{1}{2} (\nabla\phi_j)^2 - gz \right). \quad (6)$$

Now the stress on the interface is remaining to be brought into account to make the equation system self-contained by providing required boundary conditions for ϕ_j on the interface between two liquids. The pressure jump across the interface is caused by the surface tension;

$$z = \zeta(x, y) : p_1 - p_2 = -\alpha \nabla \cdot \mathbf{n} \quad \left(\text{where } \mathbf{n} := \frac{\nabla F}{|\nabla F|} \right), \quad (7)$$

where α is the surface tension coefficient and \mathbf{n} is the unit vector normal to the interface.

The linear stability analysis revealed the marginal vibration-induced instability of the flat-interface state to be

long-wavelength (Lyubimov and Cherepanov 1986; Goldobin et al. 2015). Hence, we restrict our consideration to the case of the long-wavelength approximation, $|\partial_x v| \ll |\partial_z v|$.

Governing Equations for Long-Wavelength Patterns

Derivation of Equations for 2-Dimensional Flow

In this section we derive the governing equation for long-wavelength patterns. We employ the standard method of multiple scales with small parameters ω^{-1} and l^{-1} , where l is the reference horizontal length of patterns, $\partial_x \sim l^{-1}$. The hierarchy of small parameters and the orders of magnitude of fields will be established in the course of derivation.

Within the long-wavelength approximation, the solutions to the Laplace equation for $\phi_j(x, t)$ satisfying boundary conditions (03) in the most general form read

$$\phi_1 = -a_1(t)x + \Phi_1(x, t) - \frac{1}{2}(h_1 - z)^2 \Phi_{1xx}(x, t) + \frac{1}{4!}(h_1 - z)^4 \Phi_{1xxxx}(x, t) - \dots, \tag{8}$$

$$\phi_2 = -a_2(t)x + \Phi_2(x, t) - \frac{1}{2}(h_2 + z)^2 \Phi_{2xx}(x, t) + \frac{1}{4!}(h_2 + z)^4 \Phi_{2xxxx}(x, t) - \dots. \tag{9}$$

Here the ground state (the flat-interface state) is represented by the terms $-a_j(t)x$; $\Phi_j(x, t)$ describe perturbation flow, they are yet arbitrary functions of x and t . After substitution of p_j from expression (06) and ϕ_j from expressions (8)–(9), the condition of stress balance on the interface (07) reads

$$\begin{aligned} p_{1\infty} - p_{2\infty} + \rho_1 \left[-\dot{a}_1 x + \dot{\Phi}_1 - \frac{(h_1 - \zeta)^2}{2} \dot{\Phi}_{1xx} - \frac{1}{2} \left(-a_1 + \Phi_{1x} - \frac{(h_1 - \zeta)^2}{2} \Phi_{1xxx} \right)^2 - \frac{((h_1 - \zeta)\Phi_{1xx})^2}{2} + \dots \right] \\ - \rho_2 \left[-\dot{a}_2 x + \dot{\Phi}_2 - \frac{(h_2 + \zeta)^2}{2} \dot{\Phi}_{2xx} - \frac{1}{2} \left(-a_2 + \Phi_{2x} - \frac{(h_2 + \zeta)^2}{2} \Phi_{2xxx} \right)^2 - \frac{((h_2 + \zeta)\Phi_{2xx})^2}{2} + \dots \right] \\ + (\rho_2 - \rho_1)g\zeta = \alpha \frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}}. \end{aligned}$$

Here “...” stand for terms $\mathcal{O}_1(\dot{\Phi}_j h_j^4 / l^4) + \mathcal{O}_2(a_j \Phi_j h_j^4 / l^5) + \mathcal{O}_3(\Phi_j^2 h_j^4 / l^6)$. The difference of constants $p_{1\infty} - p_{2\infty}$ is to be determined from the condition that in the area of vanishing perturbations of the pulsation flow, i.e. $\Phi_j(x, t) = const$, the interface remains flat, i.e. $\zeta(x, t) = 0$. This condition yields $p_{1\infty} - p_{2\infty} - (\rho_1 a_1^2(t) - \rho_2 a_2^2(t)) / 2 = 0$. Choosing measure units for length: $L = \sqrt{\alpha / [(\rho_2 - \rho_1)g]}$, for time: $T = L/b$, and for the fluid densities: ρ_* —which means replacement

$$(x, z) \rightarrow (Lx, Lz), \quad t \rightarrow Tt, \quad \zeta \rightarrow L\zeta, \tag{10}$$

$$\Phi_j \rightarrow (L^2/T)\Phi_j, \quad \rho_i \rightarrow \rho_*\rho_i$$

in equations—one can rewrite the last equation in the dimensionless form

$$\begin{aligned} B \left[\frac{\rho_1 a_1^2 - \rho_2 a_2^2}{2} + \rho_1 \dot{\Phi}_1 - \frac{\rho_1 (h_1 - \zeta)^2}{2} \dot{\Phi}_{1xx} - \frac{\rho_1}{2} \left(a_1 - \Phi_{1x} + \frac{1}{2}(h_1 - \zeta)^2 \Phi_{1xxx} \right)^2 - \frac{\rho_1}{2} ((h_1 - \zeta)\Phi_{1xx})^2 - \rho_2 \dot{\Phi}_2 \right. \\ \left. + \frac{\rho_2 (h_2 + \zeta)^2}{2} \dot{\Phi}_{2xx} + \frac{\rho_2}{2} \left(a_2 - \Phi_{2x} + \frac{1}{2}(h_2 + \zeta)^2 \Phi_{2xxx} \right)^2 + \frac{\rho_2}{2} ((h_2 + \zeta)\Phi_{2xx})^2 + \dots \right] + \zeta = \frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}}. \tag{11} \end{aligned}$$

Here the dimensionless vibration parameter

$$B \equiv \frac{\rho_* b^2}{\sqrt{\alpha(\rho_2 - \rho_1)g}} = B_0 + B_1 \tag{12}$$

(ρ_j is dimensional here), where B_0 is the critical value of the vibration parameter above which the flat-interface state becomes linearly unstable, B_1 is a small deviation of the vibration parameter from the critical value. Further, kinematic conditions (04) and (05) turn into

$$\dot{\zeta} = \left(-(h_1 - \zeta)\Phi_{1x} + \frac{1}{3!}h_1^3 \Phi_{1xxx} - a_1\zeta + \dots \right)_x, \tag{13}$$

$$\dot{\zeta} = \left((h_2 + \zeta)\Phi_{2x} - \frac{1}{3!}h_2^3 \Phi_{2xxx} - a_2\zeta + \dots \right)_x. \tag{14}$$

Here “...” stand for $\mathcal{O}_1(\Phi_j h_j^2 \zeta / l^3) + \mathcal{O}_2(\Phi_j h_j^4 / l^5)$. Equations 11, 13, and 14 form a self-contained equation system.

It is convenient to distinguish two main time-modes in fields: the average over vibration period part and the pulsation part;

$$\zeta = \eta(\tau, x) + \xi(\tau, x)e^{i\omega t} + c.c. + \dots,$$

$$\Phi_j = \varphi_j(\tau, x) + \psi_j(\tau, x)e^{i\omega t} + c.c. + \dots,$$

where τ is a “slow” time related to the average over vibration period evolution and “...” stand for higher powers of $e^{i\omega t}$.

In order to develop an expansion in small parameter ω^{-1} , we have to adopt certain hierarchy of smallness of parameters, fields, etc. We adopt small deviation from the instability threshold $B_1 \sim \omega^{-1}$. Then $\eta \sim \omega^{-1}$ and $\partial_x \sim \omega^{-1/2}$ (Lyubimov and Cherepanov 1986; Goldobin et al. 2015). It is as well established (Lyubimov and Cherepanov 1986) that for finite wavelength perturbations (finite $k \neq 0$) $B_0(k) = B_0(0) + Ck^2 + \mathcal{O}(k^4)$. Generally, the expansion of exponential growth rate of perturbations in series of B_1 near the instability threshold possesses a non-zero linear part, and $B_0(k) - B_0(0) \sim k^2$; therefore, $\partial_\tau \sim \mathcal{O}_1(B_1) + \mathcal{O}_2(k^2) \sim \omega^{-1}$. The order of magnitude of ξ , φ_j and ψ_j is more convenient to be determined in the course of development of the expansion.

Collecting in Eqs. 13–14 terms with $e^{i\omega t}$, one finds

$$i\omega\xi + \xi_\tau = \left(- (h_1 - \eta)\psi_{1x} + \frac{1}{3!}h_1^3\psi_{1xxx} + \xi\varphi_{1x} - A_1\eta + \dots \right)_x, \quad (15)$$

$$i\omega\xi + \xi_\tau = \left((h_2 + \eta)\psi_{2x} - \frac{1}{3!}h_2^3\psi_{2xxx} + \xi\varphi_{2x} - A_2\eta + \dots \right)_x, \quad (16)$$

where “...” stand for $\mathcal{O}_1((\xi\varphi + \eta\psi)h_j^2/l^4) + \mathcal{O}_2(\psi h_j^4/l^6)$. Constant with respect to t terms sum-up to

$$\eta_\tau = \left(-(h_1 - \eta)\varphi_{1x} + \xi\psi_{1x}^* + c.c. - A_1\xi^* + c.c. + \dots \right)_x, \quad (17)$$

$$\eta_\tau = \left((h_2 + \eta)\varphi_{2x} + \xi\psi_{2x}^* + c.c. - A_2\xi^* + c.c. + \dots \right)_x, \quad (18)$$

where the superscript “*” stands for complex conjugate and “...” stand for $\mathcal{O}_1((\eta\varphi + \xi\psi)h_j^2/l^4) + \mathcal{O}_2(\varphi h_j^4/l^6)$. The difference of Eqs. 15 and 16 yields $\psi_j \sim \omega^{-1/2}$, the difference of Eqs. 17 and 18 yields $\varphi_j \sim \omega^{-1}$. For dealing with non-linear terms in the consideration that follows, it is convenient to extract the first correction to ψ_j explicitly, i.e. write $\psi_j = \psi_j^{(0)} + \psi_j^{(1)} + \dots$, where $\psi_j^{(1)} \sim \omega^{-1}\psi_j^{(0)} \sim \omega^{-3/2}$. Equation 15 (or (16)) yields in the leading order ($\sim \omega^{-3/2}$)

$$\xi = \frac{i}{\omega}(h_1\psi_{1x} + A_1\eta)_x = -\frac{i}{\omega}(h_2\psi_{2x} + A_2\eta)_x \sim \omega^{-5/2}. \quad (19)$$

Considering the difference of Eqs. 16 and 15, one has to keep in mind, that we are interested in localized patterns for which $\Phi_{jx}(x = \pm\infty) = 0, \zeta(x = \pm\infty) = 0$. Hence, this

difference can be integrated with respect to x , taking the form

$$h_1\psi_{1x} + h_2\psi_{2x} - \eta(\psi_1 - \psi_2)_x - \frac{1}{6}(h_1^3\psi_1 + h_2^3\psi_2)_{xxx} - \xi(\varphi_1 - \varphi_2)_x + (A_1 - A_2)\eta + \dots = 0,$$

which yields in the first two orders of smallness

$$(h_1\psi_1^{(0)} + h_2\psi_2^{(0)})_x = -(A_1 - A_2)\eta = -\frac{(\rho_2 - \rho_1)(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)}\eta, \quad (20)$$

$$(h_1\psi_1^{(1)} + h_2\psi_2^{(1)})_x = (\psi_1^{(0)} - \psi_2^{(0)})_x\eta + \frac{1}{6}(h_1^3\psi_1^{(0)} + h_2^3\psi_2^{(0)})_{xxx}. \quad (21)$$

The difference and the sum of Eqs. 17 and 18 yield in the leading order, respectively,

$$h_1\varphi_1 = -h_2\varphi_2, \quad (22)$$

$$\eta_\tau = -h_1\varphi_{1xx} = h_2\varphi_{2xx}. \quad (23)$$

Let us now consider Eq. 11. We will collect groups of terms with respect to power of $e^{i\omega t}$ and the order of smallness in ω^{-1} . $\sim \omega^{+\frac{1}{2}}e^{i\omega t}$:

$$i\omega B_0(\rho_1\psi_1^{(0)} - \rho_2\psi_2^{(0)}) = 0.$$

We introduce

$$\psi^{(0)} \equiv \rho_j\psi_j^{(0)}. \quad (24)$$

The last equation and Eq. 20 yield

$$\psi_x^{(0)} = -\frac{\rho_1\rho_2(\rho_2 - \rho_1)(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^2}\eta. \quad (25)$$

The next group of terms to be formally collected should be formed by the contributions of the order ω^0 and proportional to $e^{i\omega t}$, i.e., $\sim \omega^0 e^{i\omega t}$:

No contributions.

Further, $\sim \omega^{-\frac{1}{2}}e^{i\omega t}$:

$$i\omega B_1(\underbrace{\rho_1\psi_1^{(0)} - \rho_2\psi_2^{(0)}}_{=0}) + i\omega B_0(\rho_1\psi_1^{(1)} - \rho_2\psi_2^{(1)}) + B_0(\underbrace{\rho_1\psi_1^{(0)} - \rho_2\psi_2^{(0)}}_{=0})_\tau + i\omega B_0\frac{1}{2}(h_2^2\rho_2\psi_{2xx}^{(0)} - h_1^2\rho_1\psi_{1xx}^{(0)}) = 0.$$

(We marked the combinations which are known to be zero from the leading order of expansion.) Hence,

$$\rho_1\psi_1^{(1)} - \rho_2\psi_2^{(1)} = \frac{1}{2}(h_1^2 - h_2^2)\psi_{xx}^{(0)}. \quad (26)$$

The last equation and Eq. 21 yield

$$\begin{aligned} \psi_{1x}^{(1)} &= \frac{\rho_2 - \rho_1}{\rho_1(h_1\rho_2 + h_2\rho_1)} \psi_x^{(0)} \eta \\ &\quad + \frac{3h_2h_1^2\rho_1 - 2h_2^3\rho_1 + h_1^3\rho_2}{6\rho_1(h_1\rho_2 + h_2\rho_1)} \psi_{xxx}^{(0)} \\ &= -\frac{\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^3} \eta^2 \\ &\quad - \frac{\rho_2(3h_2h_1^2\rho_1 - 2h_2^3\rho_1 + h_1^3\rho_2)(\rho_2 - \rho_1)(h_1 + h_2)}{12(h_1\rho_2 + h_2\rho_1)^3} \eta_{xx}, \end{aligned} \quad (27)$$

$$\begin{aligned} \psi_{2x}^{(1)} &= -\frac{\rho_1(\rho_2 - \rho_1)^2(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^3} \eta^2 \\ &\quad - \frac{\rho_1(3h_1h_2^2\rho_2 - 2h_1^3\rho_2 + h_2^3\rho_1)(\rho_2 - \rho_1)(h_1 + h_2)}{12(h_1\rho_2 + h_2\rho_1)^3} \eta_{xx}. \end{aligned} \quad (28)$$

$$\sim \omega^{-1}(e^{i\omega t})^0;$$

$$B_0[-\rho_2(A_2\psi_{2x}^{(0)*} + c.c.) + \rho_1(A_1\psi_{1x}^{(0)*} + c.c.)] + \eta = 0. \quad (29)$$

Substituting (24) and (25) into the last equation, one finds

$$\left[-\frac{B_0\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2}{2(h_1\rho_2 + h_2\rho_1)^3} + 1 \right] \eta = 0.$$

Thus we obtain the solvability condition, which poses a restriction on B_0 ; this restriction determines the linear instability threshold

$$B_0 = \frac{2(h_1\rho_2 + h_2\rho_1)^3}{\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2}. \quad (30)$$

Since the threshold B_0 is an important and experimentally measurable characteristic of the system, we provide it here also in original *dimensional* terms (cf. Eqs. 10 and 12):

$$B_0 = \frac{2\rho_*(h_1\rho_2 + h_2\rho_1)^3}{\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2} \sqrt{\frac{(\rho_2 - \rho_1)g}{\alpha}},$$

$$b_0^2 = \frac{2(h_1\rho_2 + h_2\rho_1)^3g}{\rho_1\rho_2(\rho_2 - \rho_1)(h_1 + h_2)^2}.$$

$$\sim \omega^{-2}(e^{i\omega t})^0;$$

$$\begin{aligned} &B_1 \underbrace{[-\rho_2(A_2\psi_{2x}^{(0)*} + c.c.) + \rho_1(A_1\psi_{1x}^{(0)*} + c.c.)]}_{=-\eta/B_0} \\ &+ B_0 \left[\rho_1\varphi_{1\tau} - \rho_2\varphi_{2\tau} - \rho_1|\psi_{1x}^{(0)}|^2 + \rho_2|\psi_{2x}^{(0)}|^2 \right. \\ &\quad \left. + \rho_1 \left(A_1\psi_{1x}^{(1)*} + c.c. - A_1\frac{h_1^2}{2}\psi_{1xxx}^{(0)*} + c.c. \right) \right. \\ &\quad \left. - \rho_2 \left(A_2\psi_{2x}^{(1)*} + c.c. - A_2\frac{h_2^2}{2}\psi_{2xxx}^{(0)*} + c.c. \right) \right] = \eta_{xx}. \end{aligned} \quad (31)$$

Substituting $\psi_j^{(n)}$ from Eqs. 24–27 and using Eq. 22, one can rewrite the last equation as

$$\begin{aligned} &-\frac{B_1}{B_0}\eta + B_0 \left[\frac{h_1\rho_2 + h_2\rho_1}{h_2}\varphi_{1\tau} \right. \\ &\quad \left. - \frac{3\rho_1\rho_2(\rho_2 - \rho_1)^3(h_1 + h_2)^2}{4(h_1\rho_2 + h_2\rho_1)^4}\eta^2 \right. \\ &\quad \left. + \frac{\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2(h_1^3\rho_2 + h_2^3\rho_1)}{6(h_1\rho_2 + h_2\rho_1)^4}\eta_{xx} \right] = \eta_{xx}. \end{aligned} \quad (32)$$

Together with Eq. 23 the latter equation form the final system of governing equations for long-wavelength perturbations of the flat-interface state:

$$\begin{cases} B_0 \frac{h_1\rho_2 + h_2\rho_1}{h_1h_2} (h_1\varphi_1)_\tau = \left[1 - \frac{1}{3} \frac{h_1^3\rho_2 + h_2^3\rho_1}{h_1\rho_2 + h_2\rho_1} \right] \eta_{xx} \\ \quad + \frac{3}{2} \frac{\rho_2 - \rho_1}{h_1\rho_2 + h_2\rho_1} \eta^2 + \frac{B_1}{B_0} \eta, \\ \eta_\tau = -(h_1\varphi_1)_{xx}. \end{cases} \quad (33)$$

Notice, equation system (33) is valid for B_1 small compared to B_0 , otherwise one cannot stay within the long-wavelength approximation. It is only rarely possible to use long-wavelength approximation for finite deviations from the linear instability threshold and derive certain information on the system dynamics (Goldobin and Lyubimov 2007) or its properties (Pimenova et al. 2015; Goldobin et al. 2015); typically the long-wavelength analysis is inconclusive for such conditions.

Derivation of Equations for 3-Dimensional Flow

In this section we consider interfacial waves and flows which are non-uniform along the y -axis, the horizontal direction perpendicular to the vibration direction.

The Case of Similar Scales of Patterns Along the x - and y -directions ($\partial_y \sim \partial_x$)

Firstly, let us consider the case of $\zeta = \zeta(x, y)$ and $\Phi_j = \Phi_j(x, y)$ for which x - and y - derivatives are of the same order of magnitude. After appropriate changes to equations, Eq. 25 turns into

$$\Delta_2\psi^{(0)} = -\frac{\rho_1\rho_2(\rho_2 - \rho_1)(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^2} \eta_x, \quad (34)$$

where Δ_2 stands for the Laplace operator with respect to the horizontal coordinates, $\Delta_2 \equiv \partial_x^2 + \partial_y^2$. Equation 29 remains unchanged. However, with Eq. 34, after differentiation with respect to x , it yields a new solvability condition;

$$-\frac{B_0\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2}{2(h_1\rho_2 + h_2\rho_1)^3} \psi_{xx}^{(0)} + \Delta_2\psi^{(0)} = 0.$$

The new solvability condition depends on the wave pattern. Specifically, for plane waves $\psi^{(0)}(x, y) \propto e^{i(k_x x + k_y y)}$ an analogue of Squire’s theorem appears to be valid:

$$B_0(\beta) = \frac{B_0(0)}{\cos^2 \beta}, \tag{35}$$

where β is the angle between the wavevector \mathbf{k} and the vibration direction, $B_0(0)$ is the linear instability threshold for 2-dimensional waves, which is determined by Eq. 30. From Eq. 35, the threshold of instability to 3-dimensional waves is increased for a finite value as compared to that for 2-d waves. Hence, 3-dimensional waves with similar scales along x - and y -directions can be considered, only when the system is already unstable with respect to 2-d waves. Since the latter instability leads to an explosive growth of practically any waves Goldobin et al. (2014, 2015), the study for 3-d waves under these circumstances is of marginal interest.

The Case of $\partial_y^2 \sim \omega^{-1} \partial_x^2$

According to Eq. 35, the increase of the linear stability threshold for small β is small and can be comparable to the values B_1 admissible within the framework of our theoretical analysis. The case of plane waves with small β corresponds to wave patterns with $|\partial_y \eta| \ll |\partial_x \eta|$. For this case, one can consider the system dynamics beyond (x, z) -geometry and this dynamics will be still of physical significance (in contrast to the case of $\partial_y \sim \partial_x$), because it is observed below the threshold of the system instability. In what follows we consider the case of $\partial_y^2 \sim \omega^{-1} \partial_x^2$.

Basic principles of the derivations for the 3-d case are similar to that for the case of (x, z) -geometry. Therefore, we do not provide a complete derivation for the 3-d case, but only discuss the equations which change compared to the 2-d case.

In Eq. 11, additional non-negligible terms appear:

$$\begin{aligned}
 B \left[\frac{\rho_1 a_1^2 - \rho_2 a_2^2}{2} + \rho_1 \dot{\Phi}_1 - \frac{\rho_1 (h_1 - \zeta)^2}{2} \dot{\Phi}_{1xx} \right. \\
 - \frac{\rho_1}{2} \left(a_1 - \Phi_{1x} + \frac{1}{2} (h_1 - \zeta)^2 \Phi_{1xxx} \right)^2 - \frac{\rho_1}{2} (\Phi_{1y})^2 \\
 - \frac{\rho_1}{2} ((h_1 - \zeta) \Phi_{1xx})^2 - \rho_2 \dot{\Phi}_2 + \frac{\rho_2 (h_2 + \zeta)^2}{2} \dot{\Phi}_{2xx} \\
 + \frac{\rho_2}{2} \left(a_2 - \Phi_{2x} + \frac{1}{2} (h_2 + \zeta)^2 \Phi_{2xxx} \right)^2 + \frac{\rho_2}{2} (\Phi_{2y})^2 \\
 \left. + \frac{\rho_2}{2} ((h_2 + \zeta) \Phi_{2xx})^2 + \dots \right] + \zeta = \frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}}. \tag{36}
 \end{aligned}$$

The kinematic conditions (13) and (14) turn into

$$\begin{aligned}
 \dot{\zeta} = \left(-(h_1 - \zeta) \Phi_{1x} + \frac{1}{3!} h_1^3 \Phi_{1xxx} - a_1 \zeta + \dots \right)_x \\
 - h_1 \Phi_{1yy} + \dots, \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\zeta} = \left((h_2 + \zeta) \Phi_{2x} - \frac{1}{3!} h_2^3 \Phi_{2xxx} - a_2 \zeta + \dots \right)_x \\
 + h_2 \Phi_{2yy} + \dots. \tag{38}
 \end{aligned}$$

Equations 36–38 form a self-contained equation system.

Collecting in Eqs. 37 and 38 terms with $e^{i\omega t}$, one finds (in place of Eqs. 15 and 16)

$$\begin{aligned}
 i\omega \xi + \xi_\tau = \left(-(h_1 - \eta) \psi_{1x} + \frac{1}{3!} h_1^3 \psi_{1xxx} \right. \\
 \left. + \xi \varphi_{1x} - A_1 \eta + \dots \right)_x - h_1 \psi_{1yy} + \dots, \tag{39}
 \end{aligned}$$

$$\begin{aligned}
 i\omega \xi + \xi_\tau = \left((h_2 + \eta) \psi_{2x} - \frac{1}{3!} h_2^3 \psi_{2xxx} \right. \\
 \left. + \xi \varphi_{2x} - A_2 \eta + \dots \right)_x + h_2 \psi_{2yy} + \dots. \tag{40}
 \end{aligned}$$

Constant with respect to t terms sum-up to

$$\begin{aligned}
 \eta_\tau = \left(-(h_1 - \eta) \varphi_{1x} + \xi \psi_{1x}^* + c.c. - A_1 \xi^* + c.c. + \dots \right)_x \\
 - h_1 \varphi_{1yy} + \dots, \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 \eta_\tau = \left((h_2 + \eta) \varphi_{2x} + \xi \psi_{2x}^* + c.c. - A_2 \xi^* + c.c. + \dots \right)_x \\
 + h_2 \varphi_{2yy} + \dots. \tag{42}
 \end{aligned}$$

The difference of Eqs. 40 and 39 reads

$$\begin{aligned}
 \left(h_1 \psi_{1x} + h_2 \psi_{2x} - \eta (\psi_1 - \psi_2)_x - \frac{1}{6} (h_1^3 \psi_1 + h_2^3 \psi_2)_{xxx} \right. \\
 \left. - \xi (\varphi_1 - \varphi_2)_x + (A_1 - A_2) \eta + \dots \right)_x \\
 + (h_1 \psi_{1y} + h_2 \psi_{2y} + \dots)_y = 0,
 \end{aligned}$$

which yields in the second orders of smallness, in place of Eq. 21,

$$\begin{aligned}
 \left(h_1 \psi_1^{(1)} + h_2 \psi_2^{(1)} \right)_{xx} = \left((\psi_1^{(0)} - \psi_2^{(0)})_x \eta \right. \\
 \left. + \frac{1}{6} (h_1^3 \psi_1^{(0)} + h_2^3 \psi_2^{(0)})_{xxx} \right)_x - \left(h_1 \psi_1^{(0)} + h_2 \psi_2^{(0)} \right)_{yy}. \tag{43}
 \end{aligned}$$

Further, Eqs. 26 and 43 yield

$$\begin{aligned} \psi_{1xx}^{(1)} &= \left(\frac{\rho_2 - \rho_1}{\rho_1(h_1\rho_2 + h_2\rho_1)} \psi_x^{(0)} \eta \right. \\ &\quad \left. + \frac{3h_2h_1^2\rho_1 - 2h_2^3\rho_1 + h_1^3\rho_2}{6\rho_1(h_1\rho_2 + h_2\rho_1)} \psi_{xxx}^{(0)} \right) - \frac{1}{\rho_1} \psi_{yy}^{(0)} \\ &= \left(-\frac{\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^3} \eta^2 \right. \\ &\quad \left. - \frac{\rho_2(3h_2h_1^2\rho_1 - 2h_2^3\rho_1 + h_1^3\rho_2)(\rho_2 - \rho_1)(h_1 + h_2)}{12(h_1\rho_2 + h_2\rho_1)^3} \eta_{xx} \right) \\ &\quad - \frac{1}{\rho_1} \psi_{yy}^{(0)}, \end{aligned} \tag{44}$$

$$\begin{aligned} \psi_{2xx}^{(1)} &= \left(-\frac{\rho_1(\rho_2 - \rho_1)^2(h_1 + h_2)}{2(h_1\rho_2 + h_2\rho_1)^3} \eta^2 \right. \\ &\quad \left. - \frac{\rho_1(3h_1h_2^2\rho_2 - 2h_1^3\rho_2 + h_2^3\rho_1)(\rho_2 - \rho_1)(h_1 + h_2)}{12(h_1\rho_2 + h_2\rho_1)^3} \eta_{xx} \right) \\ &\quad - \frac{1}{\rho_2} \psi_{yy}^{(0)}. \end{aligned} \tag{45}$$

Equation 31 remains unchanged; however, one should substitute fields $\psi_j^{(1)}$ determined by Eqs. 44 and 45 into it. Then, in place of Eq. 32, one obtains

$$\begin{aligned} -\frac{B_1}{B_0} \eta + B_0 \left[\frac{h_1\rho_2 + h_2\rho_1}{h_2} \varphi_{1\tau} \right. \\ \left. - \frac{3\rho_1\rho_2(\rho_2 - \rho_1)^3(h_1 + h_2)^2}{4(h_1\rho_2 + h_2\rho_1)^4} \eta^2 \right. \\ \left. + \frac{\rho_1\rho_2(\rho_2 - \rho_1)^2(h_1 + h_2)^2(h_1^3\rho_2 + h_2^3\rho_1)}{6(h_1\rho_2 + h_2\rho_1)^4} \eta_{xx} \right. \\ \left. + \frac{1}{B_0} \partial_x^{-2} \eta_{yy} \right] = \eta_{xx}. \end{aligned} \tag{46}$$

In an infinite space with the condition that $\eta(x, y, \tau)$ remains finite, the action of operator ∂_x^{-2} is well-defined. Together with Eq. 23, which remains unchanged, the latter equation form the final system of governing equations for 3-d long-wavelength perturbations of the flat-interface state:

$$\begin{cases} B_0 \frac{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1}{\tilde{h}_1\tilde{h}_2} \tilde{h}_1\tilde{\varphi}_{1\tilde{\tau}} = \left[1 - \frac{1}{3} \frac{\tilde{h}_1^3\tilde{\rho}_2 + \tilde{h}_2^3\tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1} \right] \tilde{\eta}_{\tilde{x}\tilde{x}} \\ \quad + \frac{3}{2} \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1} \tilde{\eta}^2 + \frac{B_1}{B_0} \tilde{\eta} - \partial_{\tilde{x}}^{-2} \tilde{\eta}_{\tilde{y}\tilde{y}}, \\ \tilde{\eta}_{\tilde{\tau}} = -\tilde{h}_1\tilde{\varphi}_{1\tilde{x}\tilde{x}}. \end{cases} \tag{47}$$

Here the dimensionless variables and parameters are marked with the tilde sign to distinguish them from original dimensional variables and parameters. (Above in this section, the tilde sign was omitted.) For convenience we explicitly specify how to read rescaling (10) with the tilde-notation: $x = L\tilde{x}$, $t = (L/b)\tilde{t}$, $\rho_i = \rho_*\tilde{\rho}_i$, etc.

Conditions for Applicability of the Long-Wavelength Approximation

On the basis of earlier works of Lyubimov and Cherepanov (1986) and the results for the case $h_1 = h_2$ (Goldobin et al. 2015), we expected the linear instability of the flat-interface state to be long-wavelength for thin enough layers and relayed on this expectation. Now we can see an explicit quantitative condition for the layer to be “thin enough”. The exponential growth rate $\tilde{\lambda}$ of linear normal perturbations ($\tilde{\eta}, \tilde{\varphi}_1 \propto \exp[\tilde{\lambda}\tilde{t} + i(\tilde{k}_x\tilde{x} + \tilde{k}_y\tilde{y})]$) of the trivial state obeys

$$\tilde{\lambda}^2 = \frac{\tilde{h}_1\tilde{h}_2 \left(-\left[1 - \frac{1}{3} \frac{\tilde{h}_1^3\tilde{\rho}_2 + \tilde{h}_2^3\tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1} \right] \tilde{k}_x^4 + \frac{B_1}{B_0} \tilde{k}_x^2 - \tilde{k}_y^2 \right)}{B_0(\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1)}. \tag{48}$$

From Eq. 48 one can see that the long-wavelength perturbations are the most dangerous one and grow for $B_1 > 0$, if the expression in the square brackets is nonnegative,

$$\frac{\tilde{h}_1^3\tilde{\rho}_2 + \tilde{h}_2^3\tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1} \leq 3. \tag{49}$$

Otherwise, the long-wavelength perturbations are not the most dangerous one. Equation 49 imposes limitation from above on h_1 and h_2 . This analysis of equation system (47) only highlights the long-wavelength character of the linear instability, since it deals with the limit of small \tilde{k} and does not provide information on the linear stability for finite \tilde{k} . A comprehensive proof of the long-wavelength character of the instability comes from the research by Lyubimov and Cherepanov (1986).

Final Rescaling and Equations

For consideration of the system dynamics below the linear instability threshold, i.e. for negative B_1 , it is convenient to make further rescaling of coordinates and variables:

$$\begin{aligned} \tilde{x} &\rightarrow x \sqrt{\frac{B_0}{(-B_1)} \left[1 - \frac{1}{3} \frac{\tilde{h}_1^3\tilde{\rho}_2 + \tilde{h}_2^3\tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1} \right]}, \\ \tilde{y} &\rightarrow y \frac{B_0}{(-B_1)} \sqrt{1 - \frac{1}{3} \frac{\tilde{h}_1^3\tilde{\rho}_2 + \tilde{h}_2^3\tilde{\rho}_1}{\tilde{h}_1\tilde{\rho}_2 + \tilde{h}_2\tilde{\rho}_1}}, \end{aligned}$$

$$\begin{aligned} \tilde{t} &\rightarrow t \sqrt{\frac{\tilde{h}_1 \tilde{\rho}_2 + \tilde{h}_2 \tilde{\rho}_1}{\tilde{h}_1 \tilde{h}_2} \frac{B_0^3}{B_1^2} \left[1 - \frac{1}{3} \frac{\tilde{h}_1^3 \tilde{\rho}_2 + \tilde{h}_2^3 \tilde{\rho}_1}{\tilde{h}_1 \tilde{\rho}_2 + \tilde{h}_2 \tilde{\rho}_1} \right]}, \\ \tilde{\eta} &\rightarrow \eta \frac{\tilde{h}_1 \tilde{\rho}_2 + \tilde{h}_2 \tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1} \frac{(-B_1)}{B_0}, \end{aligned} \tag{50}$$

$$\tilde{\varphi}_1 \rightarrow \frac{\varphi}{\tilde{h}_1} \frac{\tilde{h}_1 \tilde{\rho}_2 + \tilde{h}_2 \tilde{\rho}_1}{\tilde{\rho}_2 - \tilde{\rho}_1} \left[1 - \frac{1}{3} \frac{\tilde{h}_1^3 \tilde{\rho}_2 + \tilde{h}_2^3 \tilde{\rho}_1}{\tilde{h}_1 \tilde{\rho}_2 + \tilde{h}_2 \tilde{\rho}_1} \right].$$

Notice, that this means the following transformation of *initial dimensional* coordinates and variables:

$$\begin{aligned} x &\rightarrow x L \sqrt{\frac{B_0}{(-B_1)} \left[1 - \frac{1}{3L^2} \frac{h_1^3 \rho_2 + h_2^3 \rho_1}{h_1 \rho_2 + h_2 \rho_1} \right]}, \\ y &\rightarrow y L \frac{B_0}{(-B_1)} \sqrt{1 - \frac{1}{3L^2} \frac{h_1^3 \rho_2 + h_2^3 \rho_1}{h_1 \rho_2 + h_2 \rho_1}}, \\ t &\rightarrow t \sqrt{\frac{h_1 \rho_2 + h_2 \rho_1}{\rho_* h_1 h_2} \frac{L^3 B_0^3}{b^2 B_1^2} \left[1 - \frac{1}{3L^2} \frac{h_1^3 \rho_2 + h_2^3 \rho_1}{h_1 \rho_2 + h_2 \rho_1} \right]}, \\ \eta &\rightarrow \eta \frac{h_1 \rho_2 + h_2 \rho_1}{\rho_2 - \rho_1} \frac{(-B_1)}{B_0}, \\ \varphi_1 &\rightarrow \varphi \frac{bL}{h_1} \frac{h_1 \rho_2 + h_2 \rho_1}{\rho_2 - \rho_1} \left[1 - \frac{1}{3L^2} \frac{h_1^3 \rho_2 + h_2^3 \rho_1}{h_1 \rho_2 + h_2 \rho_1} \right]. \end{aligned} \tag{51}$$

After the rescaling, equation system (47) (*below the linear instability threshold*) takes a parameterless form;

$$\ddot{\eta} - \eta_{xx} - \eta_{yy} + \left(\frac{3}{2} \eta^2 + \eta_{xx} \right)_{xx} = 0. \tag{52}$$

The derivation of Eq. 52 itself is one of the principle results we report with this paper, as it allows considering the 3-d evolution of quasi-steady patterns in the two-layer liquid system under the action of the vibration field for arbitrary ratio h_1/h_2 (in (Goldobin et al. 2015) the governing equation was derived for 2-d flows and $h_1 = h_2$).

Dynamics Above the Threshold and for Inverted State

Equation system (47) derived in the previous section is relevant in the vicinity of the linear instability threshold, both below and above the threshold. Above the threshold, B_1 is positive and one has to replace $(-B_1)$ with B_1 in rescalings (50)–(51), which leads to the change of the sign of the second term in Eq. 52. Thus, *above the linear instability threshold*, equation system (47) yields

$$\ddot{\eta} + \eta_{xx} - \eta_{yy} + \left(\frac{3}{2} \eta^2 + \eta_{xx} \right)_{xx} = 0. \tag{53}$$

Furthermore, the derivation procedure can be formally repeated for the case where the heavy liquid overlies the light one (we refer to this system state as ‘inverted state’). The correction of the results for this case is straightforward; one can substitute $g \rightarrow -g$ in all the equations of the previous section, but make proper account for rescaling (10), where the real-valued length scale $L = \sqrt{\alpha/[(\rho_2 - \rho_1)g]}$ is to be kept unchanged. After this correction, one finds the same value of B_0 for the critical point of the linear stability of the flat interface state (generally, this does not necessarily mean that the state is stable on one of the sides of the point: it can be unstable on the both sides, but with different instability mechanisms). Within the vicinity of this point, one obtains dimensionless governing equations:

• *below* the critical point B_0 for the *inverted state*,

$$\ddot{\eta} + \eta_{xx} + \eta_{yy} - \left(\frac{3}{2} \eta^2 + \eta_{xx} \right)_{xx} = 0; \tag{54}$$

• *above* the critical point B_0 for the *inverted state*,

$$\ddot{\eta} - \eta_{xx} + \eta_{yy} - \left(\frac{3}{2} \eta^2 + \eta_{xx} \right)_{xx} = 0. \tag{55}$$

For understanding of the general properties of the system dynamics, it is interesting to compare Eqs. 52–55 with the Boussinesq equation for gravity waves in shallow water.

Two-Dimensional Flow ($\partial_y = 0$)

It was demonstrated above that the physical-space scale of variation of fields in the y -direction is large compared to that in the x -direction; therefore, a large lateral horizontal size of container is required to have room for variation of fields with y . Hence, when the lateral size of the container is non-large, the consideration can be restricted to the case of a 2-d flow. For $\partial_y = 0$, Eqs. 52–55 read

$$\ddot{\eta} - s_b \eta_{xx} + s_g \left(\frac{3}{2} \eta^2 + \eta_{xx} \right)_{xx} = 0, \tag{56}$$

where $s_b = +1$ and -1 above and below B_0 , and $s_g = +1$ and -1 for the ‘normal’ and inverted states, respectively.

Let us compare Eq. 56 with the original Boussinesq equation (OBE) for gravity waves in shallow water in the dimensional form (without vibrations and surface tension) (Boussinesq 1872);

$$\ddot{\eta} - gh \eta_{xx} - gh \left(\frac{3}{2} \frac{\eta^2}{h} + \frac{h^2}{3} \eta_{xx} \right)_{xx} = 0, \tag{57}$$

where h is the layer thickness. In the dimensionless form, the same equation is relevant for a two-layer system without vibrations (Choi and Camassa 1999).

The interface dynamics for the normal state below B_0 ($s_b = -1$ and $s_g = +1$) corresponds to the ‘plus’ Boussinesq equation (Manoranjan et al. 1988; Bogdanov and

Zakharov 2002), which differs from Eq. 57, and was thoroughly discussed by Goldobin et al. (2014, 2015). Noticeably, the dynamics above B_0 ($s_b = +1$ and $s_g = +1$) corresponds to Eq. 57 with negative g , which describes the falling of a liquid layer covering a ceiling; the flat-interface state is obviously unstable without any saturation for perturbation growth.

One can notice that the interface dynamics for the inverted state above B_0 is governed by OBE. However, this does not mean that the flat-interface state becomes stable as it is for a shallow water layer subject solely to the gravity field. Indeed, Eq. 56 is rigorously derived for the vibrational system under consideration and the short-wavelength instability inherent to OBE (it can be seen also from Eq. 48) represents the properties of the real physical system. Meanwhile, the validity of OBE is restricted to the description of the gravity waves which are of long wavelength not only for the original fluid dynamics equations, but also for Boussinesq Eq. 56.

Let us elaborate on the latter statement. Equation system (25) in (Boussinesq 1872) for gravity waves in shallow water reads in our terms as

$$\begin{cases} \dot{\eta} + \varphi_{xx} = -(\eta \varphi_x)_x + \frac{1}{6} \varphi_{xxxx}, \\ \dot{\varphi} + \eta = -\frac{1}{2}(\varphi_x)^2 + \frac{1}{2} \dot{\varphi}_{xx}, \end{cases} \quad (58)$$

where the terms in the r.h.s. of equations are small, i.e., both nonlinearity and dispersion are small. Dynamic system (16) does not possess a short-wavelength instability. To the leading corrections owned by nonlinearity and dispersion, for waves traveling in one direction, the latter equation system can be recast as OBE. Thus, the short-wavelength instability of OBE is not an inherent property of the long-wavelength gravity waves in physical systems, but the result of a rough approximation, while for the inverted state of the vibrational system above B_0 this instability actually exists.

To summarize, the normal flat-interface state below the threshold B_0 is free of instabilities to infinitesimal perturbations. For other three cases, the flat-interface state becomes linearly unstable. The similarity between equations for the inverted state above B_0 and the original Boussinesq equation for gravity waves does not mean the former system is as stable as the latter one. Nonetheless, this similarity is remarkable and indicates that the long-wavelength dynamics of the inverted system subject to strong vibrations becomes as stable as the shallow liquid layer subject solely to gravity.

Three-Dimensional Flow

For the system of a large lateral extent in the physical space, flows can be inhomogeneous in the y -direction. For the

case of vibrationless system, the x - and y -directions are equivalent, while for the vibrational system we consider the y -direction can be discriminated. Waves propagate in the y -direction without dispersion and nonlinearity (see Eqs. 52–55; this is valid for the leading order of our expansion, where the propagation of waves in the x -direction is already essentially determined by both dispersion and nonlinearity). Moreover, the system dynamics for the normal state above the threshold B_0 does not match anymore the dynamics of the liquid layer falling from a ceiling; even roughly, the term η_{yy} appears with negative sign in Eq. 53 and with positive sign in the 3-d version of Eq. 57 with negative g . Furthermore, Eq. 55 governing the inverted system above B_0 possesses terms $-\eta_{xx}$ and $+\eta_{yy}$, while for the gravity waves in shallow water these terms will be both with sign “–”. Thus, all the similarities discussed for 2-d flows are broken in the 3-d case.

Conclusion

We have generalized the derivation of the governing equations of conservative dynamics of interfacial waves in a two-layer system of inviscid liquids subject to horizontal vibrations for the case of $h_1 \neq h_2$. Even though the governing equations in the dimensionless form are the same as for $h_1 = h_2$ (Goldobin et al. 2015), the procedure of the construction of expansion has revealed the appearance of a new group of contributions, which vanish for $h_1 = h_2$ (see Eq. 26). The equations are relevant for thin enough layers, $(h_1^3 \rho_2 + h_2^3 \rho_1)/(h_1 \rho_2 + h_2 \rho_1) < 3\alpha/[(\rho_2 - \rho_1)g]$, where the linear instability is long-wavelength.

Further, the consideration has been extended to the case of three-dimensional flows varying in the second horizontal direction. An analogue of Squire’s theorem can be formulated here (35). According to this theorem, the instability threshold for plane perturbations increases with the angle between the wavevector and the vibration direction. Hence, close to the instability threshold of the most dangerous perturbations, only patterns with a “slow” dependence on y are of interest. The governing equations for these patterns have been derived as well; Eqs. 52–55 describe the system dynamics below and above the threshold B_0 for both the normal and inverted states.

For a system confined in the y -direction, where the flows are two-dimensional, equations of the system dynamics can be compared to the Boussinesq equations for the gravity waves in shallow water. Noticeably, above the threshold, the system dynamics for the normal state is similar to the dynamics of the falling of a liquid layer from a ceiling, while the dynamics for the inverted state is reminiscent of the usual gravity waves. However, the latter similarity does not mean the vibrational system to become as stable as a

horizontal liquid layer subject to the gravity field. The vibrational system possesses a short-wavelength instability, which is actually present in the system, while for the gravity waves in shallow water this instability is due to the expansion truncation and is observed beyond the domain of the applicability of the Boussinesq equation to the specific physical system. Nonetheless, the long-wavelength modes of instability of the inverted state become stabilized by vibrations.

For three-dimensional flows, in contrast to the case of the gravity waves in a vibration-free shallow water, two horizontal directions are not equivalent. The sign of η_{yy} -term in equations breaks the similarities in dynamics of some configurations of the vibrational systems and gravity waves in a vibration-free shallow water layer, which were observed for the 2-d case.

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