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ORIGINAL ARTICLE

Convection in a Two-Layer System with a Deformable Interface Under Low Gravity Conditions

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Abstract The onset of thermal convection in a *system* of *two horizontal* layers of immiscible liquids of similar densities is studied under low gravity conditions. A constant heat flux is prescribed at both rigid boundaries. A generalized *Boussinesq* approach that allows correct accounting for the interface deformation is used. The long-wave perturbations emerge under low-gravity conditions; either monotonic or oscillatory modes are critical depending on the problem. Moreover, two different modes of the monotonic instability exist. For the first instability mode, the convection dominates, whereas the interface remains nearly undeformable. The second monotonic instability mode is substantially related to interface deformations. The system of non-linear amplitude equations describing the behavior of long-wave regimes at finite-amplitude interface deflection and finite supercriticalities is obtained. The analytical and numerical investigations of these equations show that the stable non-trivial stationary solutions are absent, and after a transient at least one of the layers is split into the areas not connected to each other. The nonlinear regimes of cellular convection are studied numerically by the Level Set method.

Keywords Two-layer system · Thermal convection · Generalized Boussinesq approach · Stability analysis

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Introduction

Multilayer hydrodynamic systems are widely prevalent in technological applications and life-support systems in a microgravity environment. However, the behavior of liquid systems under these conditions has not been comprehensively studied. This work is devoted to the investigation of the onset and nonlinear regimes of convection in a two-layer system of immiscible liquids with a deformable interface under external heat flux boundary conditions in low gravity.

A linear stability analysis of a two-layer system with a undeformable interface heated from below and having perfectly conducting external boundaries (Gershuni and Zhukhovitskii 1982) showed that under certain conditions, the conductive state is unstable with respect to the oscillatory perturbations of a finite wavelength. In order to take the interface deformability into account, the Generalized Boussinesq approximation was developed in Rasenat et al. (1989) with assumption that the layer thicknesses and all parameters of the liquids except the densities are equal. The investigation revealed the existence of monotonic and oscillatory modes of instability with a finite wavelength. Later, the same problem was solved in Lobov et al. (1996) for fluids with different properties. The observed monotonic long-wave instability was found to be related to deformations of the interface (in the particular case considered by Rasenat S. et al., this mode is absent). It was shown that in a wide range of parameters the long-wave perturbations are critical. Thus, the system of horizontal layers of immiscible liquids with perfectly conducting external boundaries can be unstable with respect to the long-wave monotonic perturbations associated with the interface deformability. A single layer with prescribed heat flux at the boundaries provides another example of the system, where the monotonic long wave convection emerges. The stability of two horizontal layers of immiscible liquids was investigated under the conditions of a fixed heat flux at the external boundaries and the non-deformable interface (Gershuni and Zhukhovitskii 1986). It was shown that when the layer thickness ratio is close to zero or unity, i.e., if the system is similar to a single layer with a prescribed heat flux at the external boundaries, the long-wave monotonic instability takes place. It has also been found that the long-wave monotonic perturbations can emerge at the intermediate values of the layer thickness ratio close to 0.5. The present paper describes the nontrivial interaction of the above-mentioned modes.

Formulation of the Problem

Consider a system of two horizontal layers of immiscible liquids with a deformable interface. The layers of equal thickness are bounded by solid surfaces, z = $\pm h$, at which a constant vertical heat flux is imposed. We choose a coordinate system so that the z-axis is directed vertically upwards, and the x- and y-axes in the plane of the unperturbed liquid-liquid interface.

It is known that the Boussinesq approximation cannot be used to calculate the deformation of the liquid-liquid interface (Drasin and Reid 1981). The present paper is concerned with the study of liquids of similar densities. In this case, the deformation of the interface can be correctly taken into account by solving the problem in the framework of the generalized Boussinesq approximation, which states that, in the limit $Ga^* \to \infty\,(\mathrm{Ga}^* = gh^3/\nu^2)$, not only the relative temperature inhomogeneity density $\beta_*\theta$ goes to zero, but also the relative difference of densities of liquids $\delta = (\rho_{02} - \rho_{01})/(\rho_{02} + \rho_{01})$ is a small parameter. Here, β_* is the characteristic value of the thermal expansion coefficient, θ is the characteristic temperature difference, and ρ_{01}, ρ_{02} are the densities of the lower and upper liquids.

Within the framework of the generalized Boussinesq approximation, the system of governing equations and boundary conditions describing natural convection in a two-layer system of immiscible liquids reads as follows:

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \vec{\mathrm{v}}_j}{\partial t} + \vec{\mathrm{v}}_j\cdot\nabla\vec{\mathrm{v}}_j\right) = -\nabla p_j + \nu_j\Delta\vec{\mathrm{v}}_j + \mathrm{Ra}\beta_j T_j\vec{\gamma} \quad (1)$$

$$\frac{\partial T_j}{\partial t} + \vec{\mathrm{v}}_j\cdot\nabla T_j = \chi_j\Delta T_j, \quad \mathrm{div}\,\vec{\mathrm{v}}_j = 0 \quad (2)$$

$$\begin{aligned}
z = -1 :&\quad \vec{\mathrm{v}}_1 = 0, \quad \frac{\partial T_1}{\partial z} = -A_1\\
z = 1 :&\quad \vec{\mathrm{v}}_2 = 0, \quad \frac{\partial T_2}{\partial z} = -A_2\\
z = \zeta :&\quad [\mathrm{v}] = 0, \quad [T] = 0, \quad [\kappa\nabla T]\cdot\vec{n} = 0\\
&\quad (\mathrm{Ga}\zeta - [p])\,\vec{n} + [\bar{\bar{\sigma}}\cdot\vec{n}] = -\vec{n}\,\mathrm{Ca}\,\mathrm{div}\,\vec{n}\\
&\quad \frac{\partial\zeta}{\partial t} + \vec{\mathrm{v}}\cdot\nabla\zeta = \mathrm{v}_z
\end{aligned} \quad (3)$$

where $j = 1{,}2$, index 1 corresponds to the lower liquid, index 2 corresponds to the upper liquid, the square brackets $[f] = f_1 - f_2$ designate a jump of the variable f at the interface, $\vec{\gamma}$ is a unit vector directed along the axis z, $\vec{n}$ is a vector normal to the surface separating the media and directed toward the second medium, and $\bar{\bar{\sigma}}$ is the viscous stress tensor.

The system of equations and boundary conditions (Eqs. 1, 2 and 3) are written in the dimensionless form. We use h^2/χ_*, h, χ_*/h, A_*h, $\rho\nu_*\chi_*/h^2$ as the units of time, length, speed, temperature and pressure. Here, $\nu_*, \kappa_*, \beta_*, \chi_*, A_*$ are the average arithmetic values of viscosity, thermal conductivity, thermal expansion, thermal diffusivity and temperature gradients. These values are taken as units of measurement and are calculated by the formulas:

$$\nu_* = \frac{\nu_{1^*} + \nu_{2^*}}{2}, \quad \kappa_* = \frac{\kappa_{1^*} + \kappa_{2^*}}{2}, \quad \beta_* = \frac{\beta_{1^*} + \beta_{2^*}}{2},$$

$$\chi_* = \frac{\chi_{1^*} + \chi_{2^*}}{2}, \quad A_* = \frac{A_{1^*} + A_{2^*}}{2}$$

This choice of measurement units allows us to write the following relations for the dimensionless coefficients of viscosity, thermal conductivity, thermal expansion and thermal diffusivity:

$$\nu_1 + \nu_2 = 2, \quad \kappa_1 + \kappa_2 = 2, \quad \beta_1 + \beta_2 = 2, \quad \chi_1 + \chi_2 = 2,$$

and the dimensionless conductive state temperature gradients are given by

$$A_1 = \kappa_2, \quad A_2 = \kappa_1.$$

The boundary value problem (Eqs. 1, 2 and 3) contains several dimensionless parameters: the Prandtl number, the Rayleigh number, the capillary number and the Galileo number, which are determined by the formulas

$$\mathrm{Pr} = \frac{\upsilon_*}{\chi_*}, \quad \mathrm{Ra} = \frac{g\beta_* A_* h^4}{\upsilon_*\chi_*}, \quad \mathrm{Ca} = \frac{\alpha h}{\upsilon_*\chi_*\rho_{0^*}},$$

$$\mathrm{Ga} = \frac{(\rho_{02} - \rho_{01})\,gh^3}{\eta_*\chi_*}.$$

where η_* is the average dynamic viscosity, ρ_{0^*} is the average density, and α is the surface tension coefficient.

The positive Galileo numbers correspond to the potentially unstable stratification (the lighter liquid is at the bottom), and the negative ones to the stable stratification. The positive Rayleigh numbers correspond to heating from below, i.e., the potentially unstable temperature stratification, and the negative ones to heating from above.

Note that, according to the generalized Boussinesq approximation, the Galileo number Ga is a finite value, as opposed to the asymptotically large Ga*.

Of the full set of eight independent dimensionless parameters, five characterize the properties of liquids. If the liquid parameters are taken to be fixed, then there remain three independent parameters: Ra, Ga, Ca.

Linear Stability Analysis

The problem stated above admits a stationary solution corresponding to the conductive state $(\vec{v}_j = 0)$ with a flat horizontal interface $\zeta = 0$, and the temperature distribution in the layers has the form:

$$T_1 = -z\,A_1, \quad T_2 = -z\,A_2.$$

Let us formulate the problem of stability of this state with respect to small perturbations. Linearizing the full problem, we obtain the following boundary-value problem for small perturbations of the conductive state:

$$\begin{aligned}
&\frac{1}{\mathrm{Pr}}\frac{\partial \vec{v}_j}{\partial t} = -\nabla p_j + \nu_j \Delta \vec{v}_j + \mathrm{Ra}\beta_j T_j \vec{\gamma} \\
&\frac{\partial T_j}{\partial t} = \chi_j \Delta T_j + A_j\left(\vec{v}_j\vec{\gamma}\right), \quad \mathrm{div}\,\vec{v}_j = 0
\end{aligned} \tag{4}$$

$$\begin{aligned}
z = -1:&\quad \vec{v}_1 = 0, \quad \frac{\partial T_1}{\partial z} = 0 \\
z = 1:&\quad \vec{v}_2 = 0, \quad \frac{\partial T_2}{\partial z} = 0 \\
z = 0:&\quad \vec{v}_1 = \vec{v}_2, \quad -A_1\zeta + T_1 = -A_2\zeta + T_2 \\
&\quad \kappa_1 \frac{\partial T_1}{\partial z} = \kappa_2 \frac{\partial T_2}{\partial z}, \quad \nu_1 \vec{\tau}\cdot\left(\frac{\partial \vec{v}_1}{\partial z} + \nabla \mathrm{v}_{1z}\right) \\
&\quad = \nu_2 \vec{\tau}\cdot\left(\frac{\partial \vec{v}_2}{\partial z} + \nabla \mathrm{v}_{2z}\right) \\
&\quad - (p_1 - p_2) - \mathrm{Ga}\zeta + 2\left(\nu_1 \frac{\partial \mathrm{v}_{1z}}{\partial z} - \nu_2 \frac{\partial \mathrm{v}_{2z}}{\partial z}\right) \\
&\quad = -\mathrm{Ca}\Delta\zeta \quad \frac{\partial \zeta}{\partial t} = \vec{v}\cdot\vec{\gamma}
\end{aligned} \tag{5}$$

where $\vec{\tau}$ is a unit vector in the horizontal plane. The conditions of zero heat flux at the rigid boundaries $z = \mp 1$ correspond to the physical situation when thermal conductivity of walls is much lower than that of the fluid (Gershuni and Zhukhovitskii 1976).

We consider normal perturbations proportional to $\exp\left(\lambda t + i\vec{k}\vec{r}\right)$, where λ is an increment, $\vec{k}$ is the wave vector in the (x, y)-plane. For the amplitudes of normal perturbations, the spectral problem is

$$\begin{aligned}
&\frac{\lambda}{\mathrm{Pr}} u_j = -ikq_j + \nu_j D u_j \\
&\frac{\lambda}{\mathrm{Pr}} w_j = -q_j' + \nu_j D w_j + Ra\beta_j\theta_j \\
&\lambda\theta_j = \chi_j D\theta_j + A_j w_j, \quad w_j' + iku_j = 0
\end{aligned} \tag{6}$$

$$\begin{aligned}
z = -1:&\quad u_1 = w_1 = 0, \quad \theta_1' = 0 \\
z = 1:&\quad u_2 = w_2 = 0, \quad \theta_2' = 0 \\
z = 0:&\quad u_1 = u_2, \quad w_1 = w_2 = \lambda\zeta \\
&\quad \theta_1 - A_1\zeta = \theta_2 - A_2\zeta, \quad \kappa_1\theta_1' = \kappa_2\theta_2' \\
&\quad \nu_1\left(u_1' + ikw_1\right) = \nu_2\left(u_2' + ikw_2\right) \\
&\quad - (q_1 - q_2) - \mathrm{Ga}\,\zeta + 2\left(\nu_1 w_1' - \nu_2 w_2'\right) \\
&\quad\quad = -\mathrm{Ca}\,k^2\zeta
\end{aligned} \tag{7}$$

For the z-dependent amplitudes of normal perturbations, we introduce the following designations: u for the projection of velocity in the direction of the wave vector, w for the vertical velocity component, θ for temperature, and q for pressure. The stroke denotes differentiation with respect to the coordinate z, $D = d^2/dz^2 - k^2$. The eigenvalue of the spectral problem (Eqs. 6 and 7) is the increment λ as a function of the problem parameters and the wave number k.

The Conductive State Stability with Respect to Long Wave Perturbations

The problem (Eqs. 6 and 7) does not have an analytical solution for an arbitrary set of parameters. However, in the case long-wave perturbations, a sufficiently complete analytical study can be performed. When $k = 0$, the problem (Eqs. 6 and 7) admits a two-parameter manifold of solutions corresponding to neutral perturbations:

$$\begin{aligned}
&u_{0j} = w_{0j} = 0, \ \zeta_0 = C_1 \\
&\theta_{01} = C_2 + \frac{1}{2}\left(A_1 - A_2\right)C_1, \ \theta_{02} = C_2 - \frac{1}{2}\left(A_1 - A_2\right)C_1
\end{aligned} \tag{8}$$

The perturbations with $C_1 = 0$ correspond to the invariance of the problem concerning the temperature shift. Such a mode of neutral perturbations is typical for the problems with a fixed heat flux at the boundaries.

The perturbations with $C_2 = 0$ corresponds to the shift interface as a whole. When $k = 0$, a doubly degenerate level with $\lambda = 0$ exists. When $k \neq 0$, this level should be split into two real levels, or a couple of complex-conjugate levels should appear. Thus, the real part of the increment may be either positive or negative, i.e., depending on the parameters problem, the long-wave instability will occur or the long-wave perturbations will damp. To clarify the question of what case is actually realized, we seek the solution of the spectral problem (Eqs. 6 and 7) and the increment in the form of a powers series with respect to the wave number. Calculations show that in the power series expansion of the increment $\lambda = \lambda_1 k + \lambda_2 k^2 + \ldots$, the first term λ_1 is identically zero, and for λ_2 we obtain the quadratic equation

$$\lambda_2^2 + B\lambda_2 + C = 0. \tag{9}$$

The coefficients of this equation are the cumbersome functions of problem parameters and their explicit form is not provided here.

The coefficients B and C do not depend on the Prandtl and capillary numbers, but contain Ra, Ga, and viscosity, thermal conductivity and volume expansion ratios. Unlike the case of ideal heat transfer boundary, the long-wave instability in our investigation also depends on the diffusivity ratio. The main distinction from the results obtained in Lobov et al. (1996) is that the equation for λ_2 is a quadratic equation (not a linear), which reflects the existence of two interacting long wave instability modes.

Let us consider the simplest special case when all properties of liquids, except for densities, are identical. Equation 9 in this case takes the simple form

$$\lambda_2^2 - \left(\frac{1}{45}\mathrm{Ra} + \frac{1}{24}\mathrm{Ga} - 1\right)\lambda_2 + \frac{1}{17280}\mathrm{Ga}\ (\mathrm{Ra} - 720) = 0. \tag{10}$$

The map of stability with respect to long-wave perturbations is shown in Fig. 1. The parameter region, in which all perturbations decay, is bounded by three straight lines: line 1 ($\mathrm{Ra} = 720$) and line 3 ($\mathrm{Ga} = 0$) are the boundaries of instability with respect to monotonic long-wave perturbations, and line 2 $(\mathrm{Ra} = -15\ \mathrm{Ga}/8 + 45)$ is the boundary of instability with respect to oscillatory long-wave perturbations, whose frequency is given by $\omega = (1/96)\sqrt{-\mathrm{Ga}\,(\mathrm{Ga} + 360)}$.

For $\mathrm{Ga} = 0$, the critical Rayleigh number is equal to 45, which coincides with the threshold of the conductive state stability of the horizontal layer of a homogeneous liquid. This is likely since we are dealing here with the liquid that fills the whole layer. Recall that the surface tension does not affect the long-wave perturbations. The critical movements are the convective rolls covering the entire system, so that the interface is involved in the movement (Fig. 2).

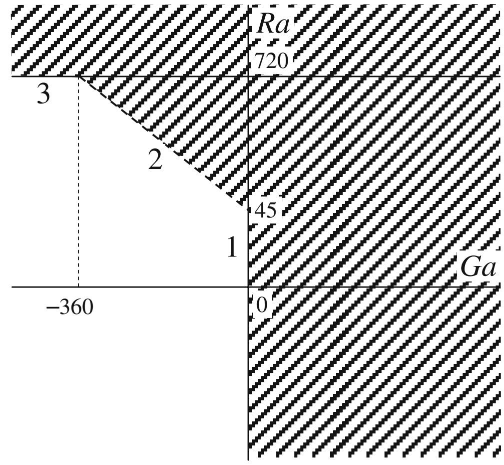

Fig. 1 The map of the long-wave stability (the *crosshatched region* corresponds to the instability)

For Galileo numbers greater than zero, the upper liquid is heavier than the lower one and, as a result, at all Rayleigh's number numbers there takes place Rayleigh-Taylor instability.

When $\mathrm{Ga} < 0$, the upper liquid is lighter than the lower liquid, which stabilizes the layers. The interaction of two perturbation modes (thermal and deformation) causes the appearance of waves at the interface and, consequently, the oscillatory nature of the instability.

At large in modulus, negative values of the Galileo number ($\mathrm{Ga} < -360$), one more monotonic mode becomes responsible for the instability. It is characterized by the two-level structure of critical motions, and the interface in this case is not deformed. Both the velocity components and the vertical derivatives of temperature perturbations become zero at the interface, so that the

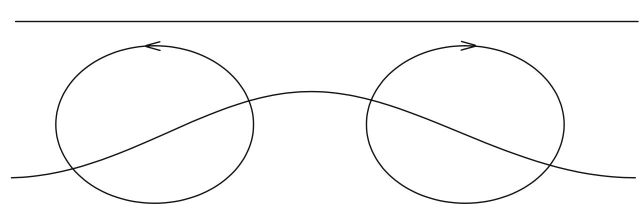

Fig. 2 The structure of critical flows at $\mathrm{Ga} = 0$

long-wavelength convection develops independently in each half of the layer. The critical Rayleigh number in every layer is the same as in the layer of a homogeneous liquid with solid boundaries under fixed heat flux conditions (Fig. 3). Taking into account changes in the scales, this means that the critical Rayleigh number must be 16 times greater than that at Ga = 0, which is truly observed. Note that in this situation the pressure is a continuous function, and the vorticities of the upper and lower layers have the same sign, i.e., the horizontal velocity components in different layers near the interface have opposite directions.

Thus, there are two types of the monotonous instability. The first type is characterized by convection in each of the layers, so that the interface remains undeformed. The second type of the instability is sufficiently related to the deformation of the interface, and the vertical velocity of the interface is comparable with the horizontal velocity of the liquid. The oscillatory mode is of a mixed-mode nature: at large Rayleigh number, the interface is not almost deformed, and as the Rayleigh number decreases, the deformation increases.

To find out whether the long-wave perturbations are most dangerous, it is necessary to get the next term of the increment expansion, viz. λ_4. Calculations show that for the monotonic instability (curves 1 and 3), the long-wave perturbations with $k \neq 0$ are less dangerous than the perturbations with $k = 0$. For the oscillatory instability, the situation is more complicated: the instability type depends on Pr, Ca, Ga, which means that the perturbations with $k = 0$ are more dangerous than the long-wave perturbations with $k \neq 0$ if $\mathrm{Ra}_2 > 0$, where Ra_2 is given by the expression

$$\mathrm{Ra}_2 = \left(\frac{5}{22528} + \frac{13}{21504\,\mathrm{Pr}}\right)\mathrm{Ga}^2 + \left(\frac{1847}{19712} + \frac{155}{896\,\mathrm{Pr}}\right)\mathrm{Ga} + \frac{15}{8}\mathrm{Ca} + \frac{510}{77} \tag{11}$$

The value of Ra_2 is the coefficient at k^2 in the powers series of the critical Rayleigh number with respect to k. The analysis of formula (Eq. 11) shows that for sufficiently large capillarities

$$\mathrm{Ca} > \mathrm{Ca}_* = \frac{102419\,\mathrm{Pr}^2 + 579380\,\mathrm{Pr} + 1057100}{560\,\mathrm{Pr}\,(105\,\mathrm{Pr} + 286)}$$

the value of Ra_2 is positive throughout the parameter range, where the long-wavelength oscillatory instability exists. When $\mathrm{Ca} < \mathrm{Ca}_*$, the value Ra_2 is positive near the ends of the interval of long-wave oscillatory instability and negative in the central part of this interval.

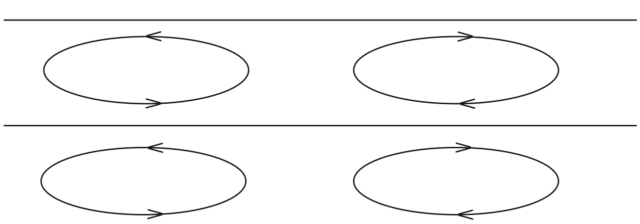

Fig. 3 The structure of critical flows at Ga < 0

The Stability of the Conductive State with Respect to Finite Wavelength Perturbations

The instability of the conductive state with respect to perturbations of finite wave length was studied numerically by the differential sweep method.

Figures 4, 5 and 6 show the neutral curves and the stability map for Ca = 0, Pr = 1. As follows from the above results, the long-wave perturbations are most dangerous at relatively small in magnitude, negative values of the Galileo number, so that the instability boundary is defined by $\mathrm{Ra} = -15\mathrm{Ga}/8 + 45$. As soon as the Galileo number reaches the value approximately equal to −27.1 (this value is calculated from (Eq. 11) for Ca = 0, Pr = 1, $\mathrm{Ra}_2 = 0$), the minimum appears on the neutral curve in the region of small, but finite wave numbers, i.e., the long-wave perturbations cease to be most dangerous. With a further increase of the Galileo modulus, this minimum shifts toward the larger values of the wave number and becomes deeper compared to the threshold of the long-wave instability (the neutral curves for Ga = −240 are shown in Fig. 4). When the

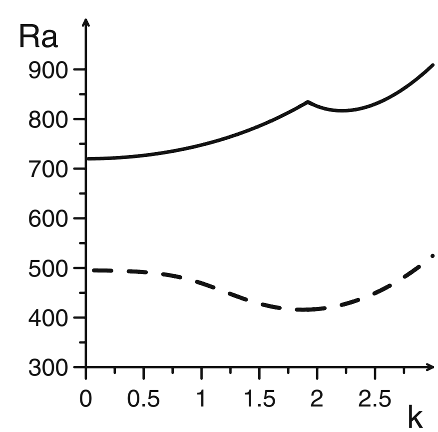

Fig. 4 Neutral curves of monotonic (*solid line*) and oscillatory (*dashed line*) instability for Ga = −240

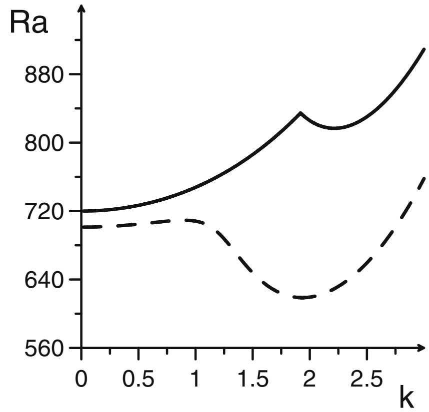

Fig. 5 Neutral curves of monotonic (*solid line*) and oscillatory (*dashed line*) instability for Ga = −350

value of the Galileo number is approximately −295.6, Ra_2 again changes the sign and the local minimum of the neutral curve is observed at $k = 0$. However, in this case the cellular perturbations appear to be most dangerous, and the neutral curve becomes bimodal (see, Fig. 5, which corresponds to Ga = −350).

The neutral curves of the monotonic cellular instability are shown in Figs. 4 and 5 by solid lines. In the examined range of the Galileo numbers, the monotonic perturbations are less dangerous than the oscillatory perturbations. However, with a further increase of the Galileo modulus, the long-wave monotonic perturbations become most dangerous as it has been shown in the previous section.

The stability map illustrating the above results is shown in Fig. 6. The boundary of the monotonic cellular mode for all values of the Galileo number lies much higher than the others modes and is not shown in the figure.

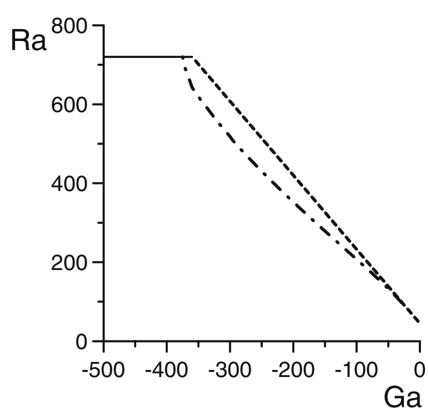

Fig. 6 Stability map (*dashed line* is the boundary of the long-wavelength oscillatory instability, *dot–dash line* is the boundary of the cellular oscillatory instability, *solid line* corresponds to long-wavelength monotonic instability)

Supercritical Regimes of Convection

The long-wave nature of the supercritical flow regimes allows us to construct an analytical theory for the finite amplitudes of the interface deviation from the equilibrium state under the constraint of smallness of the interface slope.

We introduce a slow coordinate on the assumption that all the fields in the horizontal direction vary slowly. This can be conveniently accomplished by representing the operator of differentiation on the horizontal coordinates as a power series expansion in a small parameter ε, which characterizes the ratio of the vertical to the horizontal scale of the problem:

$$\nabla = \vec{\gamma}\frac{\partial}{\partial z} + \varepsilon\nabla_1 + \ldots \tag{12}$$

It can be easily shown that the long-wave modes evolve slowly over time, so for the temporal derivative we can also introduce an expansion in terms of the same small parameter:

$$\frac{\partial}{\partial t} = \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + \ldots \tag{13}$$

The temperature, pressure and velocity fields are also sought as a power series of the same small parameter

$$T_j = T_j^{(0)} + \varepsilon T_j^{(1)} + \varepsilon^2 T_j^{(2)} + \ldots,$$

$$p_j = p_j^{(0)} + \varepsilon p_j^{(1)} + \varepsilon^2 p_j^{(2)} + \ldots \tag{14}$$

$$\vec{\mathrm{v}}_j = \varepsilon\vec{u}_j + \varepsilon^2\vec{u}_j^{(2)} + \ldots + \vec{\gamma}\left(\varepsilon^2 w_j + \ldots\right) \tag{15}$$

In the leading order of the power-series expansion in ε the temperature field corresponds to a heat-conducting regime and is defined accurate up to an additive term θ, which is an arbitrary function of slow coordinates and slow time. The deviation of the interface from the horizontal position ζ is also an arbitrary function of coordinates. The equations for θ and ζ are derived from the solubility conditions for higher order expansions and have the form:

$$\frac{\partial\zeta}{\partial t} = -\frac{\mathrm{Ra}}{24}\frac{\partial}{\partial x}\left(\left(\zeta^2-1\right)^2\frac{\partial\theta}{\partial x}\right) + \frac{1}{24}\frac{\partial}{\partial x}\left(\left(\zeta^2-1\right)^3\frac{\partial}{\partial x}\left(\mathrm{Ga}\zeta - \mathrm{Ca}K\right)\right) \tag{16}$$

$$\frac{\partial\theta}{\partial t} = \left(1-\frac{\mathrm{Ra}}{45}\right)\frac{\partial^2\theta}{\partial x^2} - \frac{1}{48}\frac{\partial}{\partial x} \times\left(\left(\zeta^2-1\right)^2\frac{\partial}{\partial x}\left(\mathrm{Ga}\zeta - \mathrm{Ca}K\right)\right) \tag{17}$$

where $K = -\frac{\partial^2\zeta}{\partial x^2}$ is the interface curvature in the long-wave approximation. The equations are written for the

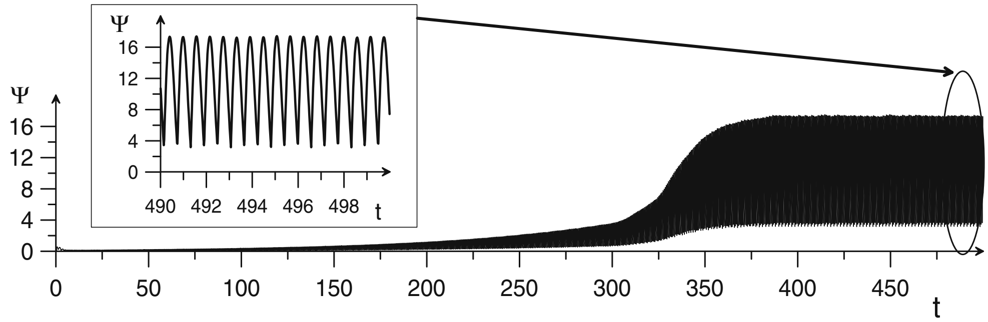

Fig. 7 Time dependence of stream function maximum for Ga = −240, Ra = 419

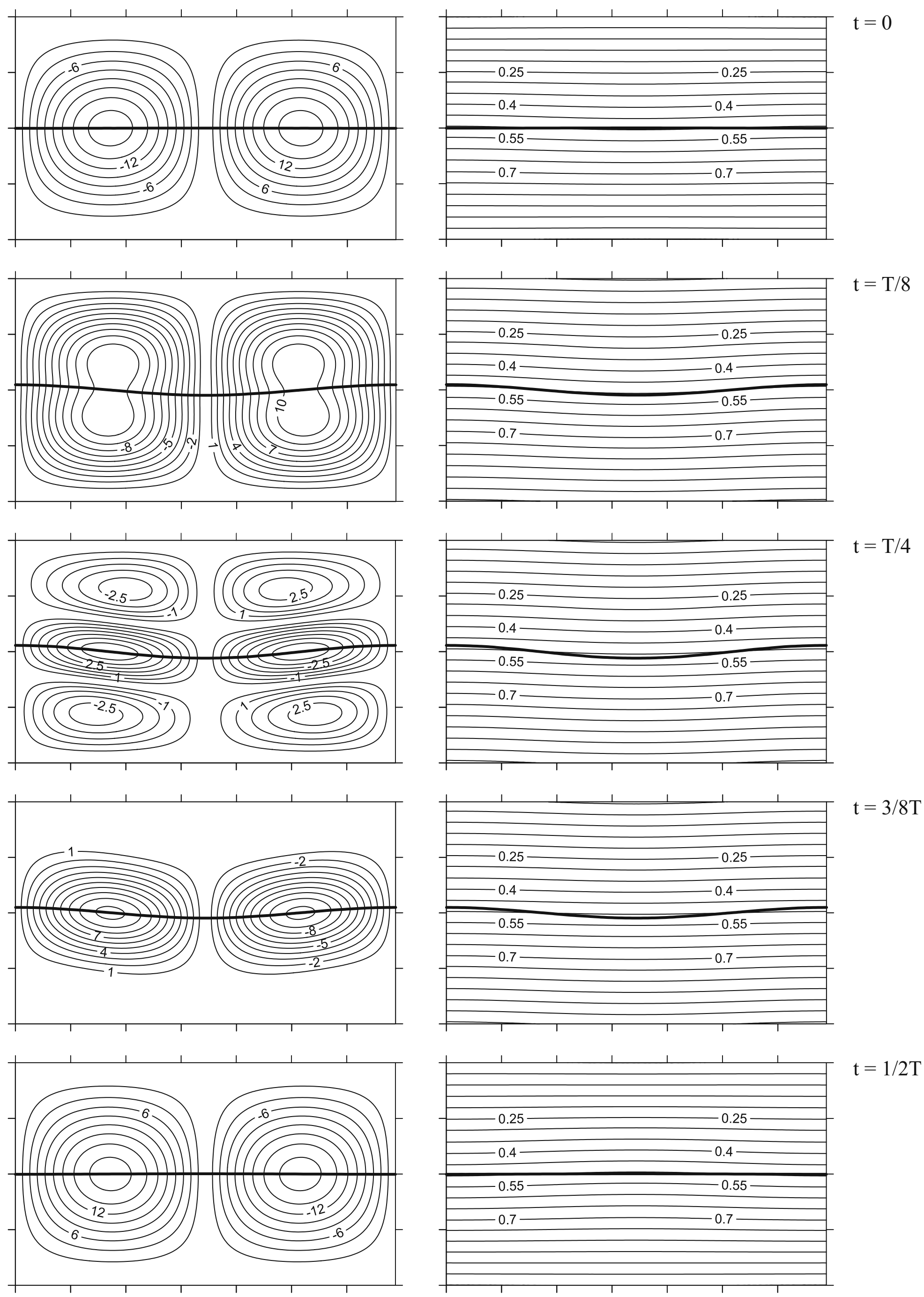

Fig. 8 Temporal evolution of the stream function and temperature fields during the first half of the oscillation period for Ga = −240, Ra = 419 (the stream function in the second half of the oscillation period differs in sign)

case of plane convection, where the solution depends only on a slow horizontal coordinate x. Indices numbering the order of expansion are omitted. In many real situations, the capillary number is large, so that even in the long-wave case when the curvature of the surface section is small, the capillary effects can play a significant role. Therefore, when deriving formulas (Eqs. 16 and 17), we renormalize the capillary number by ε^2 to take account of this effect.

We emphasize that the system of equations (Eqs. 16 and 17) is valid for arbitrary large perturbations of the interface and arbitrary supercriticality. Linearization of the resulting system of equations reproduces the results of the linear theory.

The analysis of equations (Eqs. 16 and 17) has shown that the stable nontrivial stationary solutions are absent. Numerical modeling of the obtained equations demonstrates that under any initial conditions directly after the transition period, at least one of the liquid layers is divided into disconnected areas.

The nonlinear regimes corresponding to the finite values of the wave number were studied numerically using the level set method. Figure 7 shows the dependence of the maximum stream function on time for $\mathrm{Ga} = -240,\ \mathrm{Ra} = 419\ \left(\mathrm{Ra}/\mathrm{Ra}_{cr} = 1.0072,\ \mathrm{Ra}_{cr} = 416\right)$. It can be seen that the regime of the stationary oscillations occurs after a long-time transition. The evolution of the stream function and temperature fields and the form of the interface are shown in Fig. 8. Note that at these values of the parameters, the convective flows lead to a strong interaction between the layers.

Conclusions

We have studied the influence of the interface deformation on the onset and nonlinear regimes of thermal convection in the two-layer fluid system with a vertical temperature gradient. In studying the effect of interface deformation the most interesting case is the system of liquids of similar densities, for which one can use the conventional equations in the Boussinesq approximation taking into account the difference in density only under the condition of the normal stress balance at the interface. The obtained results show that in this case a convective through-flow may occur. It spreads over the entire system, involving in the motion the interface as well. Under the conditions of a fixed heat flux applied at the external boundaries, the instability, as in the case of a single-layer system, can be of long-wave nature. The instability with respect to perturbations with a two-level structure is more dangerous at sufficiently large absolute values of the Galileo number. In the region of the intermediate values of the Galileo number, an intensive exchange of energy between these two types of disturbances can lead to the onset of the oscillatory instability. This energy exchange is also specific to the oscillatory modes at finite supercriticallity.

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