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On graded-(m, n)-prime ideals of commutative graded rings

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Abstract
Let G be an abelian group written addditively and R be a commutative graded ring of type G with identity and m, n be positive integers. The main purpose of this paper is to introduce the class of graded-(m, n)-prime ideals which lies properly between the classes of graded-prime and graded-(m, n)-closed ideals introduced recently by the authors in Ahmed et al. (Moroccan J Algebra Geom Appl 1(2):1-10, 2022). A proper graded ideal I of R is called graded-(m, n)-prime if for some homogeneous elements $a, b \in R$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$. Several characterizations of this new class of graded ideals with several original examples are given. Moreover, we defend the actions of graded-(m, n)-prime ideals in several extensions of graded rings, especially in idealization of graded modules and amalgamation of graded rings and similarly to graded-primary decomposition, we introduce the graded-(m, n)-decomposition of graded ideals and we prove that every graded ideal in a graded-n-Noetherian ring has a graded-(m, n)-decomposition. Finally, the graded-(m, n)-prime avoidance theorem is given.

Keywords Graded-(m, n)-prime ideal · Graded-(m, n)-closed ideal · Graded-n-absorbing ideal · Avoidance theorem

Mathematics Subject Classification 13A02 · 13A15

1 Introduction

In this article, all rings under consideration are assumed to be commutative with nonzero identity and all modules are assumed to be nonzero unital. R will always represent such a ring, M will represent such an R-module. Also, G will represent an abelian group with an identity element denoted by 0. Recently, there have been various generalizations of graded-prime ideals in several papers. Among the many recent generalizations of the notion of graded-prime ideals in the literature, we find the following; in [1], the authors introduced the notion of graded-2-absorbing ideals, and this idea is generalized also by the authors

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in a paper to the concept of graded-n-absorbing ideals, see [2]. According to Hamoda [2, Definition 2.1], a proper graded ideal I of R is called a graded-n-absorbing ideal if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in h(R)$, there are n of the x_i's whose product is in I. Thus a graded-1-absorbing ideal is just a graded-prime ideal.

Let m and n be positive integers. Afterwards, the structure of graded-(m, n)-closed ideals is first introduced by the authors in [3]. A proper graded ideal I of a ring R is called a graded-(m, n)-closed ideal of R if whenever $a^m \in I$ for some $a \in h(R)$, then $a^n \in I$. On the other hand, the concept of graded-1-absorbing primary ideals is introduced. According to Abu-Dawwas [4], a proper graded ideal I of a graded ring R is said to be a graded-1-absorbing primary if for non-unit elements $a, b, c \in h(R)$ such that $abc \in I$, then either $ab \in I$ or $c \in Gr(I)$, where $Gr(I)$ is the graded-radical of R. Following this paper, a subclass of graded-1-absorbing primary ideals is given in [5] and studied also in [6]. A proper ideal I of R is called graded-1-absorbing prime if for non-unit elements $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $c \in I$.

Inspired from the ideal concepts cited above, in this paper, we introduce the notion of graded-(m, n)-prime ideals which is a structure lies between a graded-prime and graded-primary ideals, i.e. graded-prime ideal $\Rightarrow$ graded-(m, n)-prime ideal $\Rightarrow$ graded-primary ideal and a generalization of the concept of (m, n)-prime ideals introduced by the authors in [7] to the context of graded ring theory. According to Khashan [7], a proper ideal of a ring R is said to be an (m, n)-prime ideal where m, n are positive integers if for $a, b \in R$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$.

Our paper is organized as follows. In Sect. 3, we discuss all relationships among the ideal types listed above and the new one by supporting many examples (Example 3.2, Remark 3.3, Examples 3.4 and 3.5). Moreover, many characterizations of graded-(m, n)-prime ideals of graded rings are proved. We determine all graded-(m, n)-prime ideals of some special graded rings such as graded-domains and gr-zero dimensional rings. Let I be a graded ideal of a graded ring R and n a positive integer. We define I to be of graded-maximum length n if any ascending chain $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ of graded ideals of a graded ring R terminates and n is the largest integer such that $I_n = I_{n+1} = \dots$. Moreover, a graded ring R is called graded-n-Noetherian if every graded ideal of R has a maximum graded length at most n. Analogous to graded-primary ideal case, we introduce the graded-(m, n)-decomposition of I which is an expression for I as a finite intersection of graded-(m, n)-prime ideals. It is proved that every graded ideal in a graded-n-Noetherian ring has a graded-(m, n)-decomposition (Theorem 3.30).

In Sect. 4, We defend the actions of graded-(m, n)-prime ideals in graded-localizations, quotient of graded rings, finite direct product of crossed products, idealization of graded modules and amalgamation of graded rings. For a graded ideal I of R, we introduce the set $\Im(I)^{gr} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : I \text{ is graded-}(m, n)\text{-prime}\}$ and study some of its properties (Theorem 4.13). Analogous to the graded-prime avoidance theorem, the last section is devoted to state and prove the graded-(m, n)-prime avoidance theorem (Theorem 5.3).

2 Preliminaries

We devote this section to recall some basic properties and terminology related to graded ring theory. Unless otherwise stated, G will denote an abelian group with an identity element denoted by 0. By a graded ring R of type G (or sometimes a G-graded ring), we mean a ring graded by G, that is, a direct sum of subgroups R_g of R such that $R_g R_h \subseteq R_{g+h}$ for every $g, h \in G$. The set $h(R) = \cup_{g \in G} R_g$ is the set of homogeneous elements of R. A nonzero element $x \in R$ is called homogeneous if it belongs to one of the R_g, homogeneous of degree g if $x \in R_g$. A graded ring R is said to be a crossed product if R_g contains an inversible element for all $g \in G$. An ideal I of R is said to be a graded ideal (or sometimes called homogeneous ideal) if the homogeneous components of every element of I belong to I, equivalently, if I is generated by homogeneous elements.

If I is a graded ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_g := (R_g + I)/I$. Let R be a graded ring and I, J a graded ideals of R and x_g a homogeneous element of R. Then, it is well known that $I + J, IJ, I \cap J$ and $(I : x_g) = \{a \in R : ax_g \in I\}$ are graded ideals of R.

Suppose that R is a graded ring and M is an R-module. By a graded R-module M, we mean an R-module graded by G, that is, a direct sum of subgroups M_g of M such that $R_g M_h \subseteq M_{g+h}$ for every $g, h \in G$. The set $h(M) = \cup_{g \in G} M_g$ is the set of homogeneous elements of M. A submodule N of M is called graded if $N = \oplus_{g \in G}(N \cap M_g)$, equivalently, if N is generated by homogeneous elements. If N is a graded submodule of a graded A-module M, then M/N is a graded A-module, where $(M/N)_g := (M_g + N)/N$.

Let R be a graded ring and let M a graded R-module. If S is a multiplicatively closed set of homogeneous elements of R, then $S^{-1}R$ is a graded ring and $S^{-1}M$ is a graded $S^{-1}R$-module, where

$$\left(S^{-1}R\right)_g = \left\{\frac{a}{s} \mid a \in R_h, s \in R_k \cap S \text{ and } h - k = g\right\}$$

and

$$\left(S^{-1}M\right)_g = \left\{\frac{m}{s} \mid m \in M_h, s \in R_k \cap S \text{ and } h - k = g\right\}.$$

Let R and R' be two graded rings, a ring homomorphism $f : R \to R'$ is called graded if $f(R_g) \subseteq R'_g$ for all $g \in G$. A graded ring isomorphism is a bijective graded ring homomorphism.

Let R_1 and R_2 be two graded rings. Then $R = R_1 \times R_2$ is a graded ring with homogeneous elements $h(R) = \cup_{g \in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. It is well known that an ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ for some ideals I_1 of R_1 and I_2 of R_2. Also it is easily seen that $I_1 \times I_2$ is a graded ideal of $R_1 \times R_2$ if and only if I_1, I_2 are graded ideals of R_1 and R_2, respectively.

A direct system $\left(R_\lambda, \phi_{\mu\lambda}\right)$ of graded rings is a direct system of rings such that each R_λ is a graded ring and each $\phi_{\mu\lambda}$ is a homomorphism of graded rings. If $\left(R_\lambda^g\right)_{g \in G}$ is the graduation of R_λ and if we have that $R = \varinjlim R_\lambda$, $R^g = \varinjlim R_\lambda^g$, then $(R^g)_{g \in G}$ is a graduation of R and R is a graded ring. If $\phi_\lambda : R_\lambda \longrightarrow R$ is the canonical mapping, ϕ_λ is a homomorphism of graded rings.

Let I be a graded ideal, I is said to be a graded-prime ideal if whenever $xy \in I$ for some $x, y \in h(R)$, then $x \in I$ or $y \in I$, equivalently, if R/I is a graded-domain, that is, if every nonzero homogeneous element of R/I is regular. Note that, when G is a torsionfree abelian group then I is graded-prime if and only if I is a prime ideal. A homogeneous-prime element generates a graded-prime ideal of R. A graded ideal I is said to be graded-maximal if $I \neq R$ and if it is maximal among graded ideals, equivalently, if R/I is a graded-field, that is, if every nonzero homogeneous element of R/I is invertible and a graded ring is said to be graded-local if it has a unique graded-maximal ideal.

If R is a graded ring, P is a graded-prime ideal in R, and $S = h(R) \setminus P$ is the saturated multiplicative set consisting of the homogeneous elements of $R \setminus P$, then $S^{-1}R$ is a graded-local ring with unique graded-maximal ideal $S^{-1}P$, $S^{-1}R$ is said to be the graded-localization of R and will be denoted by $R_{[P]}$. A graded R-module M is called graded-Noetherian if it satisfies the ascending chain condition (a.c.c.) on graded submodules; equivalently, if each graded submodule of R is finitely generated. A graded ring R is called graded-Noetherian if it is graded-Noetherian as a graded R-module. It is clear that a Noetherian graded ring is graded-Noetherian but the converse is not true in general, see [8, Example 1.1.22].

The graded height of a graded-prime ideal P denoted by $gr\text{-}ht(P)$, is defined as the length of the longest chain of graded-prime ideals contained in P. The graded Krull dimension of a graded ring R is denoted by $gr\text{-}dim(R)$ and defined as follows:

$$\text{gr-dim(R)} = \max\{\text{gr-ht(P)}\,|\,P \in \text{gr-Spec(R)}\}.$$

Let I be a proper graded ideal of R. Then the graded-radical of I is denoted by $\mathrm{Gr}(I)$ and it is defined as follows:

$$\mathrm{Gr}(I) = \left\{ a = \sum_{g\in G} a_g \in R : \forall g \in G,\ \exists n_g > 0 \text{ such that } a_g^{n_g} \in I \right\}.$$

Note that $\mathrm{Gr}(I)$ is a graded ideal of R, it is the intersection of all the graded-prime ideals of R containing I, and we have gr-Nil(R) $= \mathrm{Gr}(0)$, see [9, Proposition 2.5]. We refer the reader to [9, Proposition 2.4] for the basic properties of the graded-radical. According to [10, Definition 1.5], a proper graded ideal I is said to be graded-primary if whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in \mathrm{Gr}(I)$.

For more informations and other terminology on graded rings and modules, we refer [11] and [12] to the reader.

3 Graded-(m, n)-prime ideals

We begin this section by giving some elementary properties of graded-(m, n)-prime ideals and by investigating graded-(m, n)-prime ideals in several classes of graded rings. In particular, we determine the graded-(m, n)-prime ideals of graded rings in which every power of a graded-prime ideal is graded-primary.

Definition 3.1 Let I be a proper graded ideal of a graded ring R and m, n be positive integers. Then I is called a graded-(m, n)-prime in R if for some $a, b \in h(R)$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$.

After we begin our study and discussion about the relationships existing between this new concept and all the graded classical ideals existing already in the literature, we give some examples of graded ideals which are graded-(m, n)-prime but not (m, n)-prime emphasizing the non-trivial nature of our generalization to the graded ring theory context. It is clear that any graded-(m, n)-prime ideal I in a graded ring R is both graded-primary and graded-(m, n)-closed. Hence, $P = Gr(I)$ is the smallest graded-prime ideal of R containing I. In this case, we call I a graded-P-(m, n)-prime ideal of R.

Example 3.2 Let $R = \mathbb{Z}[i]$ be the Gaussian integer ring with its natural graduation of type $\mathbb{Z}_2$; $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $I =< 2 >$. Then I is not (2, 1)-closed ideal of R because $(1+i)^2 = 2i \in I$ and $(1+i) \notin I$, so it is not (2, 1)-prime. Likewise, $J =< 4 >$ is not (4, 3)-closed ideal since $(1+i)^4 = -4 \in J$, but $(1+i)^3 = 2i - 2 \notin J$ and so it is not (4, 3)-prime. Still, it's simple to see that I is graded-(2, 1)-closed (since it is a graded-prime) and J is graded-(4, 3)-closed ideals of R.

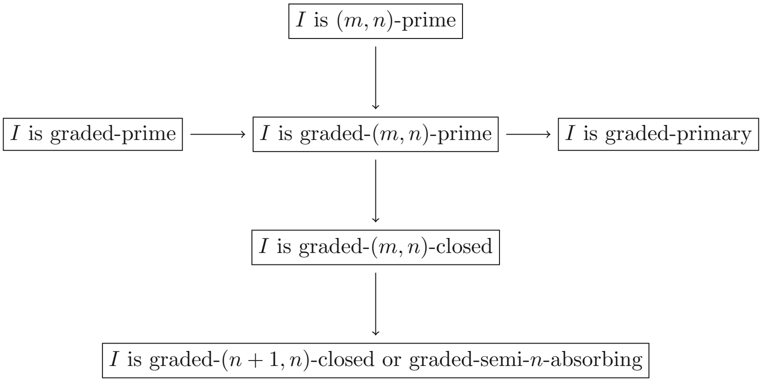

Fig. 1 The relations between the graded-prime like notions for a graded ideal I

Next, we justify the relationship between graded-(m, n)-prime ideals and some other kinds of graded ideals.

Remark 3.3 Let I be a proper graded ideal of R and m, n be positive integers.

1. I is a graded-prime ideal of R if and only if I is a graded-(1, 1)-prime ideal.
2. If I is graded-(m, n)-prime in R, then it is graded-$\left(m', n'\right)$-prime where $n \leq n'$ and $m' \leq m$.
3. If I is a graded-(m, n)-prime in R, then $(I : x)$ a is graded-(m, n)-prime ideal in R for all $x \in h(R)\backslash I$.
4. If I is a graded-1-absorbing prime (resp. if I is a graded-prime) ideal of R, then I is a graded-(m, n)-prime ideal for $n \geq 2$ (resp. for all n). Indeed, let $a, b \in h(R)$ with $a^m b \in I$ and $b \notin I$. Then a is nonunit. If b is unit, then $a^m = a \cdot a^{m-2} \cdot a \in I$ and since I is graded-1-absorbing prime, we have $a^{m-1} = a \cdot a^{m-2} \in I$ or $a \in I$. Repeat this approach to obtain $a^2 \in I$ and so $a^n \in I$ for all $n \geq 2$, (if I is graded-prime, then $a \in I$) as required. The converse is also true if I is a graded-radical ideal, i.e $I = Gr(I)$.
5. In general, we may find a graded-n-absorbing ideal that is not graded-(m, n)-prime for all integers m and n. For example, the graded ideal $18\mathbb{Z}[i]$ is graded-3-absorbing in $\mathbb{Z}[i]$ which is not graded-(m, n)-prime for all integers m and n since it is not graded-primary.
6. If I is graded-(m, n)-prime in R, then I is a graded-$(n + 1, n)$-closed ideal of R (which can be also called graded-semi-n-absorbing). Indeed, let $a \in h(R)$ such that $a^{n+1} \in I$. Suppose $n \lneqq m$ so that $a^m \in I$. Then $a^n \in I$ as I is graded-(m, n)-closed in R. On the other hand, suppose $m \leq n$ and note that $a^m a^{n+1-m} \in I$. Then by assumption, either $a^n \in I$ or $a^{n+1-m} \in I$ and the result follows since $n + 1 - m \leq n$.

Next, we describe the location of the concept of graded-(m, n)-prime ideals for all positive integers m and n by the following diagram: Fig. 1 in which the arrows are irreversible as we can see in the following Example 3.4.

Example 3.4 1. The graded ideal $I = 8\mathbb{Z}[i]$ is a graded-(5, 3)-prime that is not graded-prime in $\mathbb{Z}[i]$. Indeed, let $a, b \in \mathbb{Z}$ such that $a^5 b \in I$. Then $ab \in 2\mathbb{Z}$ and so $2 \mid a$ or $2 \mid b$. if $2 \mid a$, then $a^3 \in I$. If $2 \nmid a$, then clearly we have $b \in 8\mathbb{Z}[i] = I$.
2. The graded ideal $I = 16\mathbb{Z}[i]$ is graded-primary and clearly graded-(3, 2)-closed in $\mathbb{Z}[i]$. However, I is not graded-(3, 2)-prime since for example, $2^3 \cdot 2 \in I$ but $2^2, 2 \notin I$.

3. In contrast to the case of graded-(m,n)-closed ideals, if $n \geq m$, then a proper graded ideal need not be a graded-(m,n)-prime. For example, the graded ideal $I = 32\mathbb{Z}[i]$ is not graded-(3, 4)-prime in $\mathbb{Z}[i]$ as $2^3.2^2 \in I$ but $2^4, 2^2 \notin I$.
4. In general, if $(I : x_g) \neq I$ and $(I : x_g)$ is a graded-(m,n)-prime ideal in R for all $x_g \in h(R)\backslash I$, then I need not be graded-(m,n)-prime. Consider the graded ideal $I = \overline{8}\mathbb{Z}_{16}[i]$ in the $\mathbb{Z}_2$-graded ring $R = \mathbb{Z}_{16}[i]$. Then, in particular for all $\overline{x_0} \in R_0 = \mathbb{Z}_{16}$ such that $(I : x_0) \neq I$, we have $(I : \overline{x_0}) = \overline{4}\mathbb{Z}_{16}[i]$ or $\overline{2}\mathbb{Z}_{16}[i]$ which are clearly a graded-(3, 2)-prime ideals of $\mathbb{Z}_{16}[i]$. But, I is not graded-(3, 2)-prime as $\overline{2}^3.\overline{2} \in I$ where $\overline{2}^2, \overline{2} \notin I$.

Let R be a ring and let $\{X_1, X_2, \ldots\}$ be (commuting) algebraically (respectively, analytically) independent indeterminates over R. For $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, let $X^s = X_1^{s_1} \ldots X_n^{s_n}$. Then the polynomial ring $P = R[X_1, \ldots, X_n]$ is graded by $\mathbb{Z}$ via $P_m = \left\{\sum_{s\in\mathbb{N}^n} r_s X^s | r_s \in R \text{ and } \sum_{i=1}^n s_i = m\right\}$ and $P_m = 0$ for $m < 0$.

Example 3.5 The graded ideal $M = \langle X_1, X_2\rangle$ is a graded-maximal ideal of the graded ring $R = K[X_1, X_2]$ considered with its natural total graduation of type $\mathbb{Z}$ where K is a field and so $M^2 = \left\langle X_1^2, X_1X_2, X_2^2\right\rangle$ is graded-M-primary. On the other hand, M^2 is not graded-(2, 1)-prime in $K[X_1, X_2]$ since for example, $(X_1 - X_2)^2 \in M^2$ but $(X_1 - X_2) \notin M^2$.

Note that if I is a graded ideal of a graded ring R, then $P = \{a = \sum_{g\in G} a_g \in R : \forall g \in G,\ a_g^n \in I\}$ need not be an ideal of R and if a is a homogeneous element, then $a \in P$ if and only if $a^n \in I$. For example, consider the graded ideal $I = \left\langle X_1^2, X_2^2\right\rangle$ in the graded ring $K[X_1, X_2]$, where K is a ring, with his natural total graduation. Then $X_1, X_2 \in \left\{f \in K[X_1, X_2] : f^2 \in I\right\}$ but $X_1 - X_2 \notin \left\{P \in K[X_1, X_2] : P^2 \in I\right\}$ as $(X_1 - X_2)^2 \notin I$. However, for certain types of graded ideals I such as graded-(m,n)-prime, in particular, graded-radical (i.e. $I = Gr(I)$) ideals, the set P is a graded ideal of R.

Lemma 3.6 *Let m and n be positive integers and I be a graded-P-(m,n)-prime ideal of a graded ring R. Then $Gr(I) = P = \{a = \sum_{g\in G} a_g \in R : \forall g \in G,\ a_g^n \in I\}$.*

Proof Let $a = \sum_{g\in G} a_g \in P$ and let k be the smallest positive integer such that $a_g^k \in I$ for every $g \in G$. Now, $a_g \cdot a_g^{k-1} \in I$ implies $a_g^m \cdot a_g^{k-1} \in I$. Since I is graded-(m,n)-prime and $a_g^{k-1} \notin I$, then $a_g^n \in I$ for every $g \in G$ and so $a \in Gr(I)$, then $Gr(I) \subseteq \{a = \sum_{g\in G} a_g \in R : \forall g \in G,\ a_g^n \in I\}$. The other containment is clear. □

Proposition 3.7 *Let m and n be positive integers and I be a graded ideal of a graded ring R. If $M = \{a = \sum_{g\in G} a_g \in R : \forall g \in G,\ a_g^n \in I\}$ is a graded-maximal ideal of R, then I is a graded-M-(m,n)-prime ideal of R.*

Proof Clearly, I is proper graded ideal of R. Let $a^m b \in I$ for $a, b \in h(R)$ such that $a^n \notin I$. Then $a \notin M$ and so $a^m \notin M$. Since a is a homogeneous element and M is a graded-maximal ideal of R, then $M + Ra^m = R$ and so $1 = t + ra^m$ for some $t \in M_0$ and $r \in R_{-m}$. Thus, $1 = 1^n = (t + ra^m)^n = t^n + sa^m$ for some $s \in R_{-m}$. Hence, $b = b \cdot 1 = bt^n + bsa^m \in I$ and I is a graded-(m,n)-prime in R. Moreover, $Gr(I) = M$ by Lemma 3.6. □

Corollary 3.8 *Let m, n, k be positive integers. If $I = M^k$ for a graded-maximal ideal M of R and $k \leq n$, then I is graded-M-(m,n)-prime in R.*

Proof It is obvious that for $k \leq n$ we have $\{a = \sum_{g\in G} a_g \in R :$ for all $g \in G$, $a_g^n \in I = M^k\} = M$. Thus, I is graded-M-(m,n)-prime in R by Proposition 3.7. □

Nevertheless, if $\{a = \sum_{g\in G} a_g \in R : \text{for all } g \in G,\ a_g^n \in I\}$ is a graded-prime ideal which is not graded-maximal of R, then I need not be a graded-(m, n)-prime. Indeed, for a field K and the graded ideal $I = P^3$ of $R = K[X_1, X_2]/\langle X_1^2 X_2\rangle$ where $P = \langle \bar{X}_1\rangle$, we have $\{a = \sum_{g\in G} a_g \in R : \forall g \in G,\ a_g^3 \in I\} = P$ is a graded-prime ideal of R which is not graded-maximal. But, I is not graded-$(m, 3)$-prime in the graded ring $R = K[X_1, X_2]/\langle X_1^2 X_2\rangle$, see Example 3.16. Also, if $n \lneq k$, then Corollary 3.8 may not be true, see Example 3.4.

Following [13], a proper graded ideal Q of a graded ring R is called graded-uniformly-primary, if there exists a positive integer k such that whenever $a, b \in h(R)$ such that $ab \in Q$ and $b \notin Q$, then $a^k \in Q$. Moreover, a graded-uniformly-primary ideal Q has order n and write $h\text{-}o(Q) = n$ if n is the smallest positive integer for which the aforementioned property holds. While clearly every graded-uniformly-primary ideal is graded-primary, the converse is not true, as shown by the following example.

Example 3.9 Consider the graded ring $K[X_1, X_2, \ldots]$ of type $\mathbb{Z}$, where K is a field, it is the inductive limit of the direct system of graded rings $K[X_1, X_2, \ldots X_n]$, $n \in \mathbb{N}$ of type $\mathbb{Z}$, the graded ideal $\left(\{X_i^2\}_{i=1}^{\infty}, \{X_1 X_i\}_{i=1}^{\infty}\right) K[X_1, X_2, \ldots]$ is a graded-primary ideal that is not graded-uniformly-primary.

For positive integers m and n, if I is graded-(m, n)-prime in R, then clearly I is graded-uniformly-primary. Moreover, the two concepts coincide if $h\text{-}o(I) \leq n$.

Proposition 3.10 *Let $\{m_i, n_i\}_{i=1}^k$ be positive integers and let $\{I_i\}_{i=1}^k$ be graded-P-(m_i, n_i)-prime ideals of a graded ring R. Then $\bigcap_{i=1}^k I_i$ is a graded-P-(m, n)-prime ideal of R for all $m \leq \min\{m_1, m_2, \ldots, m_k\}$ and $n \geq \max\{n_1, n_2, \ldots, n_k\}$.*

Proof Suppose that I_i is graded-P-(m_i, n_i)-prime in R for all $i \in \{1, 2, \ldots, k\}$. Let $a^m b \in \bigcap_{i=1}^k I_i$ and $b \notin \bigcap_{i=1}^k I_i$ for some $a, b \in h(R)$. Then $b \notin I_j$ for some $j \in \{1, 2, \ldots, k\}$. Since $a^{m_j} b \in I_j$, then by assumption $a^{n_j} \in I_j$ and so $a \in P$. By Lemma 3.6, we have for all $i \in \{1, 2, \ldots, k\}$, $P = \{a = \sum_{g\in G} a_g \in R : \text{for all } g \in G,\ a^{n_i} \in I_i\}$. Thus, $a^n \in \bigcap_{i=1}^k I_i$ as a is a homogeneous element and since $n \geq \max\{n_1, n_2, \ldots, n_k\}$. Since also $Gr(\bigcap_{i=1}^k I_i) = \bigcap_{i=1}^k Gr(I_i) = P$, then $\bigcap_{i=1}^k I_i$ is a graded-P-(m, n)-prime ideal of R. □

Remark 3.11
1. In general, if I and J are two graded-(m, n)-prime ideals with $Gr(I) \neq Gr(J)$, then $I \cap J$ need not be graded-(m, n)-prime. For example, the ideals $2\mathbb{Z}[i]$ and $3\mathbb{Z}[i]$ are graded-(m, n)-prime ideal for all positive integers n and m (since they are graded-prime), but $2\mathbb{Z}[i] \cap 3\mathbb{Z}[i] = 6\mathbb{Z}[i]$ is not graded-(m, n)-prime as it is not graded-primary.
2. If I and J are two graded-P-(m, n)-prime ideals, then IJ or $I^k (k \leq m)$ need not be graded-P-(m, n)-prime. For instance, consider the graded ring $R = \mathbb{Z} + pX\mathbb{Z}[X]$, with its natural graduation of type $\mathbb{N}$, where p is a prime integer and the graded ideal $P = pX\mathbb{Z}[X]$ of R. Since P is graded-prime, it is graded-P-(m, n)-prime for all m, n. However, $P^k (k \leq m)$ is not graded-P-(m, n)-prime as $p^m X^m \in P^k$, which is a homogeneous element of degree m but neither $p^n \in P^k$ nor $X^m \in P^k$.

Next, we provide further characterizations of graded-(m, n)-prime ideals.

Theorem 3.12 *Let I be a proper graded ideal of a graded ring R and let m and n be positive integers. Then the following statements are equivalent.*

1. *I is a graded-(m, n)-prime ideal of R.*

2. *$I = (I : a^m)$ for all $a \in h(R)$ such that $a^n \notin I$.*
3. *If $a \in h(R)$ and K is a graded ideal of R with $a^m K \subseteq I$, then $a^n \in I$ or $K \subseteq I$.*

Proof (1) $\Rightarrow$ (2) Let $a \in h(R)$ such that $a^n \notin I$ and let $b \in (I : a^m)$, since $(I : a^m)$ is a graded ideal as a^m is a homogeneous element of degree m, we may suppose that $b \in h(R)$. Then $a^m b \in I$ implies $b \in I$ as I is graded-(m, n)-prime in R. Thus, $(I : a^m) \subseteq I$ and so $I = (I : a^m)$.
(2) $\Rightarrow$ (3) Let $a \in h(R)$ and K be a graded ideal of R with $a^m K \subseteq I$ and suppose $a^n \notin I$. Then by (2) $K \subseteq (I : a^m) = I$ as needed.
(3) $\Rightarrow$ (1) It is straightforward. □

Recall that a graded integral domain is said to be a graded-principal ideal domain (gr-PID, for short) if every graded ideal is principal. Considering the aforementioned theorem, several equivalent characterizations of graded-(m, n)-prime ideals in a gr-PID are given in the following Corollary.

Corollary 3.13 *Let R be a gr-PID and let m, n be positive integers. Then the following are equivalent.*

1. *I is a graded-(m, n)-prime ideal of R.*
2. *$I = (I : a^m)$ for all $a \in h(R)$ such that $a^n \notin I$.*
3. *If $a \in h(R)$ and K is a graded ideal of R with $a^m K \subseteq I$, then $a^n \in I$ or $K \subseteq I$.*
4. *If J and K are graded ideals of R with $J^m K \subseteq I$, then $J^n \subseteq I$ or $K \subseteq I$.*
5. *$I = (I : J^m)$ for every graded ideal J of R such that $J^n \nsubseteq I$.*
6. *If J is a graded ideal of R and $b \in h(R)$ with $J^m b \subseteq I$, then $J^n \subseteq I$ or $b \in I$.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious using Theorem 3.12.
(3) $\Rightarrow$ (4) Since R is a gr-PID and J is graded, $J = \langle a \rangle$ for some $a \in h(R)$. Hence, the claim is clear.
(4) $\Rightarrow$ (5) is clear.
(5) $\Rightarrow$ (6) Assume that $J^m b \subseteq I$ and $J^n \nsubseteq I$. Then $b \in (I : J^m) = I$ by (5), as needed.
(6) $\Rightarrow$ (1) Let $a^m b \in I$ and $a^n \notin I$. Put $J = \langle a \rangle$. Hence $J^m b$ and $J^n \nsubseteq I$ which imply using (6) that $b \in I$. Hence I is a graded-(m, n)-prime ideal of R. □

In the following theorem, we figure out when the powers of a principal graded-prime ideal are graded-(m, n)-prime in graded rings in which every power of a graded-prime ideal is graded-primary

Theorem 3.14 *Let R be a graded ring such that every power of a graded-prime ideal is graded-primary. Let m, n and k be positive integers and $I = \langle p^k \rangle$ where p is a homogeneous-prime element of R. Then I is a graded-(m, n)-prime ideal of R if and only if $k \leq n$.*

Proof Assume that $I = \langle p^k \rangle$ is a graded-(m, n)-prime ideal of R. By way of contradiction, let's say that $k \gneqq n$. If $k \leq m$, then $p^m \in I$ but $p^n \notin I$, a contradiction. If $m \leq k$, then $p^m p^{k-m} \in I$ but $p^n \notin I$ and $p^{k-m} \notin I$ which is also a contradiction. Hence, $k \leq n$. On the other hand, suppose $k \leq n$ and let $a, b \in h(R)$ such that $a^m b \in I$ and $b \notin I$. Since by assumption I is graded-primary, then $a^m \in Gr(I) = \langle p \rangle$. It follows that $a \in \langle p \rangle$ and so $a^n \in \langle p^n \rangle \subseteq \langle p^k \rangle = I$. So, I is a graded-(m, n)-prime ideal of R. □

Corollary 3.15 *Let R be either a graded-domain or a gr-dim(R)$= 0$ and m, n, k be positive integers and $I = \langle p^k \rangle$, where p is a homogeneous-prime element of R. Then I is a graded-(m, n)-prime ideal of R if and only if $k \leq n$.*

If some power of a graded-prime ideal of R is not graded-primary, then Theorem 3.30 need not be true in general.

Example 3.16 Consider the non graded-domain $R = K[X_1, X_2]/\left\langle X_1^2 X_2\right\rangle$ (since $X_1^2 X_2$ is a homogeneous polynomial) where K is any field. Then the graded ideal $P = \langle \bar{X_1}\rangle$ is graded-prime in R as $\langle X_1\rangle$ is graded-prime in $K[X_1, X_2]$ containing $\left\langle X_1^2 X_2\right\rangle$. Now, we show that $I = P^3$ is not graded-primary in R. Indeed, we have $\bar{X_1^2}\bar{X_2} = \bar{0} \in I$ but $\bar{X_2} \notin Gr(I)$ as $X_2 \notin \langle X_1\rangle$ in $K[X_1, X_2]$. If $\bar{X_1^2} \in I$ and $\varphi : K[X_1, X_2] \to R$ is the projection mapping, then $X_1^2 = \varphi^{-1}\left(\overline{X_1^2}\right) \in \varphi^{-1}(I) = \left\langle X_1^3, X_1^2 X_2\right\rangle$ which is impossible. Thus, also $\overline{X_1^2} \notin I$ and $I = P^3$ is not graded-primary in R. Hence, I is not graded-(m, n)-prime in R for all positive integers m and n and so in particular for all $k = 3 \leq n$.

Now, in the purpose to contruct examples of graded-(m, n)-closed ideal of R that are not a graded-(m, n)-prime, we direct our attention to determine when the powers of a principal graded-prime ideal in a graded-domain are graded-(m, n)-closed.

The following result present the graded version of [14, Theorem 3.1].

Theorem 3.17 *Let R be a graded-domain, m and n integers with $1 \leq n < m$, and $I = \left\langle p^k\right\rangle$, where p is a homogeneous-prime element of R and k is a positive integer. Then the following assertions are equivalent.*

1. *I is a graded-(m, n)-closed ideal of R.*
2. *$k = mq + r$, where q and r are integers such that $q \geq 0, 1 \leq r \leq n, q(m \bmod n)+ r \leq n$, and if $q \neq 0$, then $m = n + c$ for an integer c with $1 \leq c \leq n - 1$.*

Proof The proof is omitted since it is just like the ungraded case. □

Corollary 3.18 *Let R be a graded-domain, n a positive integer, and $I = \left\langle p^k\right\rangle$, where p is a homogeneous-prime element of R and k is a positive integer. Then the following statements are equivalent.*

1. *I is a graded-$(n + 1, n)$-closed ideal of R.*
2. *$k = (n+1)q + r$, where q and r are integers such that $q \geq 0, 1 \leq r \leq n$, and $q + r \leq n$.*

Corollary 3.19 *Let R be a graded-domain and $I = \left\langle p^k\right\rangle$, where p is a homogenous-prime element of R and k is a positive integer. Then I is a graded-$(3, 2)$-closed ideal of R if and only if $k \in \{1, 2, 4\}$.*

Next, we expand these results to products of homogeneous-prime powers. Note that if $p_1, \ldots, p_n$ are nonassociate homogeneous-prime elements of a graded-domain R, then $\left\langle p_1^{k_1}\right\rangle \cap \ldots \cap \left\langle p_n^{k_n}\right\rangle = \left\langle p_1^{k_1} \ldots p_n^{k_n}\right\rangle$ for all positive integers $k_1, \ldots, k_n$.

Theorem 3.20 *Let R be a graded-domain, m and n integers with $1 \leq n < m$, and $I = \left\langle p_1^{k_1} \ldots p_i^{k_i}\right\rangle$, where $p_1, \ldots, p_i$ are nonassociate homogeneous-prime elements of R and $k_1, \ldots, k_i$ are positive integers. Then the following assertions are equivalent.*

1. *I is a graded-(m, n)-closed ideal of R.*
2. *$\left\langle p_j^{k_j}\right\rangle$ is a graded-(m, n)-closed ideal of R for every $1 \leq j \leq i$.*

Before proving Theorem 3.20, we first establish the following two Lemmas.

Lemma 3.21 *Let R be a graded ring, $m_1, \ldots, m_k, n_1, \ldots, n_k$ positive integers, and $I_1, \ldots, I_k$ be ideals of R such that I_i is graded-(m_i, n_i)-closed for $1 \leq i \leq k$.*

1. *$I_1 \cap \ldots \cap I_k$ is graded-(m, n)-closed for all positive integers $m \leq \min\{m_1, \ldots, m_k\}$ and $n \geq \min\{m, \max\{n_1, \ldots, n_k\}\}$.*
2. *$I_1 \ldots I_k$ is graded-(m, n)-closed for all positive integers $m \leq \min\{m_1, \ldots, m_k\}$ and $n \geq \min\{m, n_1 + \ldots + n_k\}$.*

Proof (1) Let $x^m \in I_1 \cap \ldots \cap I_k$ for $x \in h(R)$, $m \leq \min\{m_1, \ldots, m_k\}$, and $1 \leq i \leq k$. Then $x^m \in I_i$, and thus $x^{m_i} \in I_i$, so $x^{n_i} \in I_i$ since I_i is graded-(m_i, n_i)-closed. Hence $x^n \in I_1 \cap \ldots \cap I_k$ for $n \geq \max\{n_1, \ldots, n_k\}$. Thus $x^n \in I_1 \cap \ldots \cap I_k$ for $n \geq \min\{m, \max\{n_1, \ldots, n_k\}\}$.
(2) Let $x^m \in I_1 \ldots I_k$ for $x \in h(R)$, $m \leq \min\{m_1, \ldots, m_k\}$, and $1 \leq i \leq k$. Then $x^m \in I_i$, and thus $x^{m_i} \in I_i$, so $x^{n_i} \in I_i$ since I_i is graded-(m_i, n_i)-closed. Hence $x^{n_1+\ldots+n_k} \in I_1 \ldots I_k$, so $x^n \in I_1 \ldots I_k$ for $n \geq n_1 + \ldots + n_k$. Thus $x^n \in I_1 \ldots I_k$ for $n \geq \min\{m, n_1 + \ldots + n_k\}$. □

Lemma 3.22 *Let R be a graded ring, m and n positive integers, and $I_1, \ldots, I_k$ be graded-(m, n)-closed ideals of R.*

1. *$I_1 \cap \ldots \cap I_k$ is a graded-(m, n)-closed ideal of R.*
2. *If $I_1, \ldots, I_k$ are pairwise comaximal, then $I_1 \ldots I_k$ is a graded-(m, n)-closed ideal of R.*

Proof It is an immediate consequence of the Lemma 3.21. □

Proof of Theorem 3.20 (1) $\Rightarrow$ (2) Let $I_j = \left\langle p_j^{k_j} \right\rangle$. Suppose that $x^m \in I_j$ for $x \in h(R)$. Let $y = x\left(p_1^{k_1} \ldots p_i^{k_i}\right)/p_j^{k_j} \in h(R)$. Then $y^m \in I$, and hence $y^n \in I$ since I is graded-(m, n)-closed. By construction, $y^n \in I$ if and only if $x^n \in I_j$. Thus I_j is a graded-(m, n)-closed ideal of R for every $1 \leq j \leq i$.

(2) $\Rightarrow$ (1) This is clear by Lemma 3.22 since $\left\langle p_1^{k_1} \right\rangle \cap \ldots \cap \left\langle p_i^{k_i} \right\rangle = \left\langle p_1^{k_1} \ldots p_i^{k_i} \right\rangle$. □

In view of Theorems 3.14 and 3.17, we have the following corollary.

Corollary 3.23 *Let R be a graded-domain, m and n positive integers and $I = \langle p^k \rangle$ where p is a homogeneous-prime element of R and k is a positive integer. Then I is a graded-(m, n)-closed ideal of R that is not a graded-(m, n)-prime ideal of R if and only if the following hold.*

1. *$k \lneqq n$.*
2. *$k = mq + r$, where $q, r \in \mathbb{N}$ such that $q \geq 0$ and $1 \leq r \leq n$, $q(m \bmod n) + r \leq n$, and if $a \neq 0$, then $m = n + c$ for an integer c with $1 \leq c \leq n - 1$.*

Remark 3.24 Let R be a graded ring such that every power of a graded-prime ideal is graded-primary, for instance, a graded-domain or a gr-dim(R)= 0 and m and n are positive integers. If $I = \left\langle p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \right\rangle$ where $p_1, p_2, \ldots, p_r$ are non-associate homogeneous-prime elements of R and $k_1, k_2, \ldots, k_r$ are positive integers, then it is clear that I is not graded-primary in R. Thus, I is not graded-(m, n)-prime in R. Explicitly, we can take the graded ring $R = \mathbb{Z}[i]$ with its natural graduation of type $\mathbb{Z}_2$.

Note that Theorem 3.20 and Remark 3.24 give plenty examples of graded-(m, n)-closed ideals that are not graded-(m, n)-prime.

Corollary 3.25 *Let R be a gr-PID, I a proper graded ideal of R and m and n positive integers. Then I is graded-(m, n)-prime in R if and only if I is generated by a power less than or equal n of a homogeneous-prime element in R.*

Afterward, we introduce a new subclass of graded-Noetherian rings.

Definition 3.26 Let I be a graded ideal of a graded ring R. Then I is said to be of graded-maximum length n if any ascending chain $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ of graded ideals of R terminates and n is the largest integer such that $I_n = I_{n+1} = \ldots$. In addition, R is called graded-n-Noetherian if every graded ideal of R has a graded-maximum length at most n.

Clearly, any graded-n-Noetherian ring is graded-Noetherian. But the converse is not true in general as shown by the following example; consider the graded-Noetherian ring $\mathbb{Z}[i]$ which is not graded-n-Noetherian for any positive integer n. Moreover, a graded-1-Noetherian ring is a graded-field clearly as every graded ideal is graded-prime. If we consider the ideal $24\mathbb{Z}[i]$ of the the $\mathbb{Z}_2$-graded ring $\mathbb{Z}[i]$, then $24\mathbb{Z}[i] \subseteq 12\mathbb{Z}[i] \subseteq 6\mathbb{Z}[i] \subseteq 2\mathbb{Z}[i] \subseteq \mathbb{Z}[i]$ is the chain of graded-maximum length $n = 4$. In general, we have:

Example 3.27 Let R be a gr-PID and $I = \left\langle p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \right\rangle$ where $p_1, p_2, \ldots, p_r$ are non-associate homogeneous-prime elements R. Then I is of graded-maximal length $k_1 + k_2 + \ldots + k_r$.

Proof By induction on r. If $r = 1$, then $I = \left\langle p_1^{k_1} \right\rangle \subseteq \left\langle p_1^{k_1 - 1} \right\rangle \subseteq \ldots \subseteq \langle p_1 \rangle \subseteq R$ is the chain of graded-maximum length $n = k_1$. Suppose the result is true for $r - 1$. Then

$$I = \left\langle p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \right\rangle \subseteq \left\langle p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r - 1} \right\rangle \subseteq \left\langle p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r - 2} \right\rangle \subseteq \ldots$$
$$\subseteq \left\langle p_1^{k_1} p_2^{k_2} \ldots p_{r-1}^{k_{r-1}} \right\rangle \subseteq_1 \ldots \subseteq_{k_1 + k_2 + \ldots + k_{r-1}} R$$

is the chain of graded-maximum length $n = k_1 + k_2 + \ldots + k_r$ as desired. □

Thus, if $k = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ for distinct homogeneous-prime $p_1, p_2, \ldots, p_r$ elements, then the graded ring $\mathbb{Z}_k[i]$ is graded-n-Noetherian where $n = k_1 + k_2 + \ldots + k_r$.

Recall that a graded ideal I of a graded ring R is called graded-irreducible if whenever $I = K \cap L$ for some graded ideals K and L of R, then either $I = K$ or $I = L$, see [10, 15]. Next, we show that for $m, n \in \mathbb{N}$, if I is a graded-irreducible ideal of length n in a graded ring R, then I is graded-(m, n)-prime in R.

Proposition 3.28 *Let m, n be positive integers and I be a proper graded ideal of R of graded-maximum length n. If I is graded-irreducible in R, then it is graded-(m, n)-prime.*

Proof Let $a, b \in h(R)$ such that $a^m b \in I$. For each i consider the graded ideal $I_i = \{x \in R : a^i x \in I\}$. Then $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ and so $I_n = I_{n+1} = \ldots$ as I is of graded-maximum length n. So, if $n \leq k$ and $a^k x \in I$, then $a^n x \in I$ for any $x \in h(R)$. Now, let $Q = I + bR$ and $L = I + a^n R$ which are two graded ideals of R. Hence it is clear that $I \subseteq Q \cap L$. Let $y \in Q \cap L$, say, $y = x_1 + r_1 b = x_2 + r_2 a^n$ where $x_1, x_2 \in I$. So $r_2 a^n - r_1 b \in I$ and then $r_2 a^{n+m} - r_1 b a^m \in I$. Since $a^m b \in I$, so $r_2 a^{n+m} \in I$ and so $r_2 a^n \in I$. Hence, $y = x_2 + r_2 a^n \in I$ and so $Q \cap L \subseteq I$. Hence, $I = Q \cap L$ and by assumption, either $I = Q$ or $I = L$. If $I = Q$, then $b \in I$ and if $I = L$, then $a^n \in Q$ and therefore I is graded-(m, n)-prime. □

Definition 3.29 Let R be a graded ring, I a proper graded ideal of R and m, n positive integers. A graded-(m, n)-decomposition of I is an expression for I as a finite intersection of graded-(m, n)-prime ideals, say $I = \bigcap_{i=1}^{k} Q_i$ where Q_i is graded-P_i-(m, n)-prime for all i. In addition, such a graded-(m, n)-decomposition of I is called minimal if

1. $P_1, P_2, \ldots, P_k$ are different graded-prime ideals of R, and
2. For all $j = 1, 2, \ldots, n$, we have $I \neq \bigcap_{\substack{i=1 \\ i \neq j}}^{k} Q_i$.

We say that I is graded-(m, n)-decomposable in R precisely when it has a graded-(m, n)-decomposition. By Proposition 3.10, the intersection of graded-P-(m, n)-prime ideals is graded-P-(m, n)-prime. Thus, similar to the case of graded-primary decomposition of graded ideals, any graded-(m, n)-decomposition of a graded ideal can be reduced to a minimal one.

Since any graded-(m, n)-prime ideal is graded-primary, then any graded-(m, n)-decomposable ideal is graded-decomposable. However, the converse is not true as for example, the ideal $72\mathbb{Z}[i] = 2^3\mathbb{Z}[i] \cap 3^2\mathbb{Z}[i]$ is decomposable in $\mathbb{Z}[i]$ but not graded-$(3, 2)$-decomposable. Indeed, $2^3\mathbb{Z}[i]$ is not graded-$(3, 2)$-prime by Theorem 3.14 and any graded-$(3, 2)$-prime ideal in $\mathbb{Z}[i]$ is a power of a graded-prime ideal.

Let $I = \bigcap_{i=1}^{k} Q_i$ be a minimal graded-primary decomposition of a graded ideal I of a graded ring R where $Gr(Q_i) = P_i$ for each $i = 1, 2, \ldots, k$. Recall that $\{P_1, P_2, \ldots, P_k\}$ is called the set of associated homogeneous-prime ideals of I (denoted by gr-ass(I)) which is independent of the choice of minimal graded-primary decomposition of I. Moreover, it is well-known that a graded-prime ideal P of R is a minimal graded-prime ideal of I if and only if P is a minimal member of gr-ass(I).

Now, obviously any minimal graded-(m, n)-decomposition of I is a minimal graded-primary decomposition. So, if $I = \bigcap_{i=1}^{k} Q_i$ is any minimal graded-(m, n)-decomposition of I where $Gr(Q_i) = P_i$ for each $i = 1, 2, \ldots, k$, then gr-ass$(I) = \{P_1, P_2, \ldots, P_k\}$.

Theorem 3.30 *Let m, n be positive integers. If a graded ring R is graded-n-Noetherian, then any graded-ideal of R is graded-(m, n)-decomposable.*

Proof Assume that R is a graded-n-Noetherian and let I be a proper graded ideal of R. Then I is of graded-maximal length n. Since R is graded-Noetherian, it is well-known that I is a finite intersection of graded-irreducible ideals. Now, it remains to use Proposition 3.28, since every graded-irreducible ideal is graded-(m, n)-prime. □

4 Graded-(m, n)-prime ideals in extensions of graded rings, idealization of graded modules and amalgamation of graded rings

The purpose of this section is to defend the actions of graded-(m, n)-prime ideals in the graded localizations, quotient of graded rings, direct product of crossed products, idealization of graded modules and amalgamation of graded rings. Furthermore, for a graded ideal I of a graded ring R, we study some properties of the set $\Im(I)^{gr} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : I$ is graded-(m, n)-prime$\}$.

Proposition 4.1 *Let $f : R_1 \rightarrow R_2$ be a graded ring homomorphism and m, n be positive integers.*

1. *If J is a graded-(m, n)-prime ideal of R_2, then $f^{-1}(J)$ is a graded-(m, n)-prime ideal of R_1.*

2. *If f is a graded epimorphism and I is a graded-(m, n)-prime ideal containing* $\operatorname{Ker} f$*, then $f(I)$ is a graded-(m, n)-prime ideal of R_2.*

Proof (1) Let $a, b \in h(R_1)$ such that $a^m b \in f^{-1}(J)$ and $b \notin f^{-1}(J)$. Since f is a graded homomorphism so the elements $f(a)$ and $f(b)$ are two homogeneous elements of R_2. Then $f(a^m b) = f(a)^m f(b) \in J$ and $f(b) \notin J$ imply $f(a)^n = f(a^n) \in J$. Therefore $a^n \in f^{-1}(J)$, as desired.
(2) Let $a := f(x), b := f(y) \in h(R_2)$ for some two homogeneous elements x and y of R_1 such that $a^m b \in f(I)$ and $b \notin f(I)$. Then it is clear that $f(x^m y) \in f(I)$ and so $x^m y \in I$ as $\operatorname{Ker}(f) \subseteq I$. Since I is graded-(m, n)-prime, we have that $x^n \in I$ or $y \in I$. Hence, $a^n = f(x^n) \in f(I)$ or $b = f(y) \in f(I)$. □

Based on Proposition 4.1, we have the following Corollary:

Corollary 4.2 *Let R be a graded ring and m, n positive integers. Then the following statements hold.*

1. *If I is a graded-(m, n)-prime ideal of R and R' is a graded subring of R, then $I \cap R'$ is a graded-(m, n)-prime ideal of R'.*
2. *If $I \subseteq J$ are two graded proper ideals of R, then J/I is a graded-(m, n)-prime ideal of R/I if and only if J is a graded-(m, n)-prime ideal of R.*

Corollary 4.3 *Let I be a proper ideal of a graded ring R, X be an indeterminate and m, n be positive integers. Then the following statements hold.*

1. *$< I, X >$ is a graded-(m, n)-prime ideal of $R[X]$ if and only if I is a graded-(m, n)-prime ideal of R if and only if I is an (m, n)-prime ideal of R.*
2. *If $I[X]$ is a graded-(m, n)-prime ideal of $R[X]$, then I is a (m, n)-prime ideal of R.*

Proof
1. The isomorphisms $R[X]/ < X > \cong R$ and $\langle I, X > / < X > \cong I$ are graded, we conclude by Corollary 4.2(2) that $\langle I, X\rangle$ is a graded-(m, n)-prime ideal of $R[X]$ if and only if I is a graded-(m, n)-prime ideal of R if and only if I is a (m, n)-prime ideal of R since $I \subset (R[X])_0 = R$.
2. The same proof as (1) using (1) and Corollary 4.2(1).

□

Throughout the subsequent, $h\text{-}Z_I(R)$ denotes the set $\{x \in h(R) : xy \in I$ for some $y \in R \backslash I\}$. Next, we study the relationship between graded-(m, n)-prime ideals and their graded-localizations.

Proposition 4.4 *Let I be a proper graded ideal of a graded ring R, S a multiplicatively closed subset of homogeneous elements of R such that $I \cap S = \varnothing$ and m, n be positive integers.*

1. *If I is a graded-P-(m, n)-prime ideal of R, then $S^{-1}I$ is a graded-$S^{-1}P$-(m, n)-prime ideal of $S^{-1}R$.*
2. *If $S^{-1}I$ is a graded-$\bar{P}$-(m, n)-prime ideal of $S^{-1}R$ and $S \cap h\text{-}Z_I(R) = \varnothing$, then I is a graded-$(\bar{P} \cap R)$-(m, n)-prime ideal of R.*

Proof
1. Let $\left(\frac{a}{s_1}\right)^m \left(\frac{b}{s_2}\right) \in S^{-1}I$ for $\frac{a}{s_1}, \frac{b}{s_2} \in h(S^{-1}R)$. So $(ua)^m b \in I$ for some $u \in S$ which gives either $(ua)^n \in I$ or $b \in I$. Thus, either $\left(\frac{a}{s_1}\right)^n = \frac{u^n a^n}{u^n s_1^n} \in S^{-1}I$ or $\frac{b}{s_2} \in S^{-1}I$. Now, since $Gr(I) = P$, then $Gr(S^{-1}I) = S^{-1}Gr(I) = S^{-1}P$.

2. Let $a, b \in h(R)$ such that $a^m b \in I$. Then $\frac{a^m b}{1} = \left(\frac{a}{1}\right)^m \left(\frac{b}{1}\right) \in S^{-1}I$. Since $S^{-1}I$ is graded-(m,n)-prime, then either $\left(\frac{a}{1}\right)^n \in S^{-1}I$ or $\left(\frac{b}{1}\right) \in S^{-1}I$. Thus, there are some homogeneous elements $u, v \in S$ such that $ua^n \in I$ or $vb \in I$. Then our supposition results in $a^n \in I$ or $b \in I$. In addition, as $Gr(I)$ is a graded-prime ideal of R, we have $S^{-1}Gr(I) = Gr(S^{-1}I) = \bar{P}$ which implies that $Gr(I) = S^{-1}Gr(I) \cap R = Gr(S^{-1}I) \cap R = \bar{P} \cap R$, as desired.

□

Corollary 4.5 *Let I be a proper graded ideal of a graded ring R, P a graded-prime ideal of R with $I \subseteq P$ and m, n positive integers. Then I is a graded-Q-(m,n)-prime ideal of R if and only if $I_{[P]}$ is a graded-$Q_{[P]}$-(m,n)-prime ideal of $R_{[P]}$.*

Proof $\Rightarrow$) Immediately using Proposition 4.4(1).
$\Leftarrow$) Let $a, b \in h(R)$ such that $a^m b \in I$. Consider the graded ideals $J_1 = \{r \in R : ra^n \in I\}$, $J_2 = \{r \in R : rb \in I\}$. Now, since I is graded-(m,n)-prime $\left(\frac{a}{1}\right)^m \left(\frac{b}{1}\right) \in I_{[P]}$ implies that $\left(\frac{a}{1}\right)^n \in I_{[P]}$ or $\left(\frac{b}{1}\right) \in I_{[P]}$. Thus, there are $u, v \in R\backslash P$ such that $ua^n \in I$ or $vb \in I$. If $ua^n \in I$, then $J_1 \nsubseteq P$. Furthermore, $J_1 \nsubseteq L$ for every graded-prime ideal L such that $I \nsubseteq L$ as $I \subseteq J_1$. Hence, $J = R$ and $a^n \in I$. If $vb \in I$, then likewise, $J_2 = R$ and $b \in I$. Since also obviously $Gr(I_P) = Q_{[P]}$, then I is a graded-$Q_{[P]}$-(m,n)-prime ideal of R. □

Let R be a graded ring and P a graded-prime ideal of R. For a positive integer n, the graded k^{th} symbolic power of P is the graded ideal $P^{(k)} = P^k R_{[P]} \cap R = \varphi^{-1}\left(P^k R_{[P]}\right)$ where $\varphi : R \to R_{[P]}$ is the natural canonical map. Hence, $P^{(k)} = \Big\{a = \sum_{g \in G} a_g \in R : sa_g \in P^k, \forall g \in G$ for some $s \in h(R)\backslash P\}$. Note that if P is a graded-prime, then $P^{(k)}$ is the smallest graded-P-primary ideal containing P^k.

Corollary 4.6 *Let m, k be a positive integers and P be a graded-prime ideal of a graded ring R. Then for all $k \leq n$, $P^{(k)}$ is the smallest graded-P-(m,n)-prime ideal containing P^k.*

Proof Since $PR_{[P]}$ is graded-maximal in $R_{[P]}$ and $k \leq n$, then $P^k R_{[P]} = \left(PR_{[P]}\right)^k$ is a graded-(m,n)-prime ideal of $R_{[P]}$ for any positive integer m by Corollary 3.8. Hence, $P^{(k)} = P^k R_{[P]} \cap R$ is a graded-(m,n)-prime ideal of R by Proposition 4.1(1). Now, it is clear that $P^k \subseteq P^{(k)}$ since $1 \in h(R)\backslash P$. Let J be another graded-P-(m,n)-prime ideal with $P^k \subseteq J$ and let $r = \sum_{g \in G} r_g \in P^{(k)}$. Then $sr_g \in P^k$, $\forall g \in G$ for some $s \in h(R)\backslash P$. Since $P^k \subseteq J$, then $sr_g \in J$, $\forall g \in G$ and so $s^m r_g \in J$, $\forall g \in G$. Thus, either $s \in P = \{x \in R : x^n \in J\}$ or $r_g \in J$, $\forall g \in G$ since J is graded-P-(m,n)-prime. Now, as we take $s \in h(R)\backslash P$, then $r_g \in J$, $\forall g \in G$. So $r \in G$. Therefore, $P^{(k)} \subseteq J$ and $P^{(k)}$ is the smallest graded-P-(m,n)-prime ideal containing P^k. □

Theorem 4.7 *Let $R_1, R_2, \ldots, R_k$ be crossed products, $R = R_1 \times R_2 \times \ldots \times R_k$ and $I_1, I_2, \ldots, I_k$ be graded ideals of $R_1, R_2, \ldots, R_k$, respectively. For any positive integers m and n, we have $I_1 \times I_2 \times \ldots \times I_k$ is a graded-(m,n)-prime ideal of R if and only if there exists $i \in \{1, 2, \ldots, k\}$ such that I_i is a graded-(m,n)-prime ideal of R_i and $I_j = R_j$ for all $j \neq i$.*

Proof Assume $I_1 \times I_2 \times \ldots \times I_k$ is a graded-(m,n)-prime in R. Suppose that I_1 and I_2 are proper and choose two homogeneous elements $a_1 \in (I_1)_g$ and $a_2 \in (I_2)_g$. Since the R_i are crossed products, in particular R_1 and R_2, there exist two inversibles elements a_3 and a_4 of degree g in $(R_1)_g$ and $(R_2)_g$, respectively. Then $(a_1, a_4, 0, \ldots, 0)$ and

$(a_3, a_2, 0, \ldots, 0)$ are two homogeneous elements of degree g of $R = R_1 \times R_2 \times \ldots \times R_k$ and $(a_1, a_4, 0, \ldots, 0)^m (a_3, a_2, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_k$ but neither $(a_1, a_4, 0, \ldots, 0)^n \in I_1 \times I_2 \times \ldots \times I_k$ nor $(a_3, a_2, 0, \ldots, 0) \in I_1 \times I_2 \times \ldots \times I_k$. So, there is $i \in \{1, 2, \ldots, k\}$ such that $I_j = R_j$ for all $j \neq i$. Without loss of generality, suppose that $I_j = R_j$ for all $j \neq 1$. We prove that I_1 is a graded-(m, n)-prime ideal of R_1. Let $a, b \in h(R_1)$ and $a^m b \in I_1$. Then $(a, 0, \ldots, 0)^m (b, 0, \ldots, 0) \in I_1 \times R_2 \times \ldots \times R_k$ which implies that $(a, 0, \ldots, 0)^n \in I_1 \times R_2 \times \ldots \times R_k$ or $(b, 0, \ldots, 0) \in I_1 \times R_2 \times \ldots \times R_k$. Hence $a^n \in I_1$ or $b \in I_1$ and I_1 is a graded-(m, n)-prime ideal of R_1. Conversely, assume that I_1 is a graded-(m, n)-prime ideal of R_1 and $I_j = R_j$ for all $j \neq 1$. Assume that $(a_1, a_2, \ldots, a_k)^m (b_1, b_2, \ldots, b_k) \in I_1 \times R_2 \times \ldots \times R_k$ but $(b_1, b_2, \ldots, b_k) \notin I_1 \times R_2 \times \ldots \times R_k$ for some $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in h(R)$. Then $a_1^m b_1 \in I_1$ and $b_1 \notin I_1$ imply that $a_1^n \in I$. Thus $(a_1, a_2, \ldots, a_k)^n \in I_1 \times R_2 \times \ldots \times R_k$, as desired. □

As a particular consequence, we have the following corollary. Note that if I and J are graded-(m, n)-prime ideals of R_1 and R_2, respectively, then I and J are proper and so $I \times J$ is never graded-(m, n)-prime ideal in $R_1 \times R_2$.

Corollary 4.8 *Let R_1 and R_2 be two crossed products, $R = R_1 \times R_2$ and I, J be graded ideals of R_1, R_2, respectively. For any positive integers m and n, we have $I \times J$ is a graded-(m, n)-prime ideal of R if and only if one of the following statements is satisfied:*

1. *I is a graded-(m, n)-prime ideal of R_1 and $J = R_2$.*
2. *J is a graded-(m, n)-prime ideal of R_2 and $I = R_1$.*

Let R be a ring and M an R-module. The following ring construction called the trivial ring extension of R by M (also called the idealization of M) was introduced by Nagata [16, page 2]. It is the set of pairs (r, m) with pairwise addition and multiplication given by $(r, e)(q, f) = (rq, rf + qe)$, denoted by $R \propto M$ whose underlying abelian group is $A \times M$ This construction has been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. For more informations, the reader is referred to [17–20]. Now, by taking R a graded ring, and M a graded R-module, Then $R \propto M$ is a graded ring by $(R \propto M)_g = R_g \oplus M_g$ for all $g \in G$, see [21]. Recently, many papers have studied the transfer of several graded properties in the idealization of graded modules, see for instance [22–24].

Proposition 4.9 *Let I be a proper graded ideal of a graded ring R, N be a proper graded submodule of a graded R-module M and m, n be positive integers. Then*

1. *I is a graded-(m, n)-prime ideal of R if and only if $I \propto M$ is a graded-(m, n)-prime ideal of $R \propto M$.*
2. *If $I \propto N$ is a graded-(m, n)-prime ideal of $R \propto M$, then I is a graded-(m, n)-prime ideal of R.*

Proof (1) Let I be a graded-(m, n)-prime ideal of R and $(a_g, x_g)^m (b_g, y_g) \in I \propto M$ for some $(a_g, x_g), (b_g, y_g) \in h(R \propto M)$. Then $a_g^m b_g \in I$ which implies either $a_g^n \in I$ or $b_g \in I$. Hence, either $(a_g, x_g)^n \in I \propto M$ or $(b_g, y_g) \in I \propto M$. Conversely, if $a_g^m b_g \in I$ for some $a_g, b_g \in h(R)$, then $(a_g, 0)^m (b_g, 0) \in I \propto M$ which implies $(a_g, 0)^n \in I \propto M$ or $(b_g, 0) \in I \propto M$, and so $a_g^n \in I$ or $b_g \in I$, as desired.
(2) Omitted since it is similar to the proof of the converse part of (1). □

We note that the converse of (2) of Proposition 4.4 is not true in general as shown by the following example.

Example 4.10 Let $R = \mathbb{Z}[i] \propto \mathbb{Z}[i]$ be the graded idealization of the Gaussian integer ring considered with his natural $\mathbb{Z}_2$-graduation (a new graduation of the same type of its natural graduation), so R is a $\mathbb{Z}_2$-graded ring. While $2\mathbb{Z}[i]$ is a graded-(2, 1)-prime ideal in $\mathbb{Z}[i]$, the graded ideal $< (2, 2) >$ is not so in R. In fact, $(2, 1)^2 = (4, 4) \in 2\mathbb{Z}[i] \propto 2\mathbb{Z}[i]$ but $(2, 1) \notin 2\mathbb{Z}[i] \propto 2\mathbb{Z}[i]$.

Let R and S be two rings, let J be an ideal of S and let $f : R \to S$ be a ring homomorphism. The following ring construction called the amalgamation of R with S along J with respect to f is a subring of $R \times S$ defined by:

$$R \bowtie^f J := \{(r, f(r) + j) | r \in R, j \in J\}$$

This construction generalize Nagata's idealization and the so called amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [25]), denoted by $R \bowtie I$, which is the subring of $R \times R$ given by: $R \bowtie I := \{(r, r + i) | r \in R, i \in I\}$. Now, by taking R and S two graded rings, J a graded ideal of S and $f : R \to S$ a graded ring homomorphism, $R \bowtie^f J$ is a graded ring by $(R \bowtie^f J)_g = \{(r_g, f(r_g) + j_g) | r_g \in R_g, j \in J_g\}$ for all $g \in G$. If I is a graded ideal of R and K is a graded ideal of $f(R) + J$, then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ are graded ideals of $R \bowtie^f J$, see [26, 27].

Next, we give a characterization about when the graded ideals $I \bowtie^f J$ and $\bar{K}^f$ are graded-(m, n)-prime ideals of $R \bowtie^f J$, for any positive integers m and n.

Theorem 4.11 *Let R, S be two graded rings, f a graded homomorphism, J, I and K be the graded ideals cited above. For positive integers m and n, we have:*

1. *$I \bowtie^f J$ is a graded-(m, n)-prime ideal of $R \bowtie^f J$ if and only if I is a graded-(m, n)-prime ideal of R.*
2. *$\bar{K}^f$ is a graded-(m, n)-prime ideal of $R \bowtie^f J$ if and only if K is a graded-(m, n)-prime ideal of $f(R) + J$.*

Proof (1) Assume that $I \bowtie^f J$ is graded-(m, n)-prime in $R \bowtie^f J$ and let $a_g, b_g \in h(R)$ such that $a_g^m b_g \in I$. So $(a_g, f(a_g)_g)^m (b_g, f(b_g)_g) \in I \bowtie^f J$ and then either $(a_g, f(a_g))_g^n \in I \bowtie^f J$ or $(b_g, f(b_g)_g) \in I \bowtie^f J$. Hence, either $a_g^n \in I$ or $b_g \in I$ and I is a graded-(m, n)-prime of R. Conversely, suppose I is graded-(m, n)-prime in R. Let $(a_g, f(a_g) + (j_1)_g)_g, \left(b_g, f(b_g)_g + (j_2)_g\right)_g \in h(R \bowtie^f J)$ such that $\left(a_g, f(a_g)_g + (j_1)_g\right)^m \left(b_g, f(b_g)_g + (j_2)_g\right) \in I \bowtie^f J$. So $a_g^m b_g \in I$ and then either $a_g^n \in I$ or $b_g \in I$. It follows that $\left(a_g, f(a_g)_g + (j_1)_g\right)^n \in I \bowtie^f J$ or $\left(b_g, f(b_g)_g + (j_2)_g\right) \in I \bowtie^f J$ as desired.
(2) Assume that $\bar{K}^f$ is a graded-(m, n)-prime ideal in $R \bowtie^f J$. Let $f(a_g)_g + (j_1)_g, f(b_g)_g + (j_2)_g \in h(f(R) + J)$ such that $\left(f(a_g)_g + (j_1)_g\right)^m \left(f(b_g)_g + (j_2)_g\right) \in K$. Then $\left(a_g, f(a_g)_g + (j_1)_g\right)^m \left(b_g, f(b_g)_g + (j_2)_g\right) \in \bar{K}^f$ and hence by assumption, $\left(a_g, f(a_g)_g + (j_1)_g\right)^n \in \bar{K}^f$ or $\left(b_g, f(b_g)_g + (j_2)_g\right) \in \bar{K}^f$. It follows that $\left(f(a_g) + (j_1)_g\right)^n \in K$ or $\left(f(b_g)_g + (j_2)_g\right) \in K$. Conversely, suppose K is graded-(m, n)-prime in $f(R) + J$. Suppose $\left(a_g, f(a_g) + (j_1)_g\right)^m \left(b_g, f(b_g) + (j_2)_g\right) \in h(\bar{K}^f)$ for $\left(a_g, f(a_g)_g + (j_1)_g\right), \left(b_g, f(b_g) + (j_2)_g\right) \in R \bowtie^f J$. Then $\left(f(a_g)_g + (j_1)_g\right)^m \left(f(b_g)_g + (j_2)_g\right) \in K$ and so $\left(f(a_g)_g + (j_1)_g\right)^n \in K$ or $\left(f(b_g) + (j_2)_g\right) \in K$. Therefore, $\left(a_g, f(a_g) + (j_1)_g\right)^n \in \bar{K}^f$ or $\left(b_g, f(b_g)_g + (j_2)_g\right) \in \bar{K}^f$ and the result follows. □

As a consequence, we have:

Corollary 4.12 *Let I and J be two graded ideals of a graded ring R. Then $I \times J$ is a graded-(m,n)-prime ideal of $R \bowtie J$ if and only if I is a graded-(m,n)-prime ideal of R.*

For a graded ideal I of a graded ring R, we identify:

$$\Re(I)^{gr} = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is graded-}(m,n)\text{-closed}\}$$

Likewise, we let

$$\Im(I)^{gr} = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is graded-}(m,n)\text{-prime}\}$$

and assume $\Im(R)^{gr} = \mathbb{N} \times \mathbb{N}$. It is clear that $\Im(I)^{gr} \subseteq \Re(I)^{gr}$ and this containment in general is proper as we have seen in Example 3.4. Moreover, we have $(1,1) \in \Im(I)^{gr}$ if and only if I is graded-prime.

For a graded ideal I of a graded ring R, the following are some properties concerning $\Im(I)^{gr}$.

Theorem 4.13 *Let I and J be graded ideals of a graded ring R, and m, n, k and t be positive integers.*

1. *If $(m,n) \in \Im(I)^{gr}$, then $(m', n') \in \Im(I)^{gr}$ for all positive integers m' and n' with $m' \leq m$ and $n' \geq n$.*
2. *If $(m,n) \in \Im(I)^{gr}$, then $(km, tn) \in \Im(I)^{gr}$ for all $t \geq k$.*
3. *If $(m,n) \in \Im(I)^{gr}$ and $(n,k) \in \Re(I)^{gr}$, then $(m,k) \in \Im(I)^{gr}$.*
4. *$(m,n) \in \Im(I)^{gr}$ if and only if $(m+1,n) \in \Im(I)^{gr}$. Hence, $(m,n) \in \Im(I)^{gr}$ if and only if $(t,n) \in \Im(I)^{gr}$ for all $t \geq m$.*
5. *If I and J are proper, then $\Im(I \times J)^{gr} = \phi$. If only one of I and J is proper, then $\Im(I \times J) = \Im(I)^{gr} \cap \Im(J)^{gr}$.*

Proof (1), (2) and (3): Obvious.
(4) Assume that $(m,n) \in \Im(I)^{gr}$ and let $a, b \in h(R)$ such that $a^{m+1}b \in I$ and $b \notin I$. Then $(a^2)^m b \in I$ as $2m \geq m+1$. Since I is graded-(m,n)-prime, then $a^{2n} \in I$. Thus, $a \in Gr(I)$ and so $a^n \in I$ by Lemma 3.6. The converse is clear by (1).
(5) If I and J are proper, then $\Im(I \times J)^{gr} = \varnothing$ by Corollary 4.8. Assume, $I \neq R$ and $J = R$. Then $\Im(I \times J)^{gr} = \Im(I)^{gr} \cap \Im(J)^{gr}$ since $\Im(R)^{gr} = \mathbb{N} \times \mathbb{N}$ and by using Proposition 4.9. □

The converse of (2) of Theorem 4.13 is not true in general as shown by the following Example.

Example 4.14 Consider the ideal $I = \langle p^k \rangle$ where p is a homogeneous-prime element of any graded-domain R, I is graded-(k,k)-prime by Theorem 3.14. But, I is not graded-$(1,1)$-prime as it is not graded-prime in R.

5 Graded-(m, n)-prime avoidance theorem

The purpose of this section is to give the graded-(m,n)-prime avoidance theorem analogous to graded-prime avoidance theorem. A covering $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ is said to be efficient if no I_k is superfluous. Also, $I = I_1 \cup I_2 \cup \ldots \cup I_n$ is an efficient union if none of the I_k may be excluded. It is clear that a covering $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ naturally implies a union $I = (I \cap I_1) \cup (I \cap I_2) \cup \ldots \cup (I \cap I_n)$. We begin by recalling the following lemma which the follow-up will find useful.

Lemma 5.1 *(McCoy) Let* $I = I_1 \cup I_2 \cup \ldots \cup I_n$ *be an efficient union of ideals where* $n \in \mathbb{N} \setminus \{0, 1\}$. *Then* $\bigcap_{i \neq k} I_i = \bigcap_{i=1}^{n} I_i$ *for all k.*

Theorem 5.2 *Let* $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ *be an efficient covering of graded ideals* $I_1, I_2, \ldots, I_n$ *of R where* $2 \leq n$. *Assume that* $Gr(I_i) \nsubseteq Gr((I_j : x))$ *for all* $x \in h(R) \backslash Gr(I_j)$ *whenever* $i \neq j$. *Then no* $I_i (1 \leq i \leq n)$ *is a graded-*(m, n)*-prime ideal of R for all* $n \leq m$.

Proof Suppose on the contrary that I_k is a graded-(m, n)-prime ideal of R for some $1 \leq k \leq n$. First, note that as $I \subseteq \bigcup_{i=1}^{n} I_i$ is an efficient covering, then $I \subseteq \bigcup_{i=1}^{n} (I_i \cap I)$ is also an efficient covering. It follows that $(*)$ $\left(\bigcap_{i \neq k} I_i\right) \cap I = \left(\bigcap_{i=1}^{n} I_i\right) \cap I \subseteq I_k \cap I$, by Lemma 5.1. For all $x \in h(R) \backslash Gr(I_k)$ and $i \neq k$, we have $Gr(I_i) \nsubseteq Gr((I_k : x))$ and so we can choose a homogeneous element $a_i \in Gr(I_i) \backslash Gr((I_k : x))$. Then, there exists the least positive integer m_i such that $a_i^{m_i} \in I_i$ for each $i \neq k$. Write $a = a_1 a_2 \ldots a_{k-1}, b = a_{k+1} a_{k+2} \ldots a_n$ and $m = \max\{m_1, m_2, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n\}$. Then $a^m b^m x \in \left(\bigcap_{i \neq k} I_i\right) \cap I$. In the remaining, we prove that $a^m b^m x \in \left(\left(\bigcap_{i \neq k} I_i\right) \cap I\right) \setminus (I_k \cap I)$. For that, suppose on the way of contradiction that $a^m b^m x \in I_k \cap I$. Then $a^m b^m \in (I_k : x) \subseteq Gr((I_k : x))$. Since $Gr((I_k : x))$ is a graded-prime ideal by Theorem 3.12 (1) and (2), we get either $a = a_1 a_2 \ldots a_{k-1} \in Gr((I_k : x))$ or $b = a_{k+1} a_{k+2} \ldots a_n \in Gr((I_k : x))$. Also, since $Gr((I_k : x))$ is graded-prime, $a_i \in Gr((I_k : x))$ for some $i \neq k$, a contradiction. Thus, $a^m b^m x \notin (I_k \cap I)$, and so $a^m b^m x \in \left(\left(\bigcap_{i \neq k} I_i\right) \cap I\right) \setminus (I_k \cap I)$ which contradicts $(*)$. Therefore, no I_i is a graded-(m, n)-prime ideal for every $1 \leq i \leq n$, as desired. □

Theorem 5.3 (Graded-(m, n)-prime Avoidance Theorem) *Let* $I, I_1, I_2, \ldots, I_n (n \geq 2)$ *be graded-ideals of R such that at most two of* $I_1, I_2, \ldots, I_n$ *are not graded-*(m, n)*-prime and* $Gr(I_i) \nsubseteq Gr((I_j : x))$ *for all* $x \in h(R) \backslash Gr(I_j)$ *whenever* $i \neq j$. *If* $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$, *then* $I \subseteq I_k$ *for some* $1 \leq k \leq n$.

Proof Suppose that $I \nsubseteq I_k$ for all $1 \leq k \leq n$. Without loss of generality, we may assume that $I \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ is an efficient covering of graded ideals of R as any covering can be reduced to an efficient one by omitting any unnecessary terms. It is well-known that a covering of an ideal by two ideals is never efficient. If $n \geq 3$, then no I_k is a graded-(m, n)-prime ideal of R by Theorem 5.2. But our assumption implies that at most two of $I_1, I_2, \ldots, I_n$ are not graded-(m, n)-prime. Thus, $I \subseteq I_k$ for some $1 \leq k \leq n$. □

Corollary 5.4 *Let I be a proper graded ideal of a graded ring R. If the graded-*(m, n)*-prime avoidance theorem holds for R, then the graded-*(m, n)*-prime avoidance theorem holds for* R/I.

Proof Let $J/I, I_1/I, I_2/I, \ldots, I_n/I (n \geq 2)$ be graded ideals of R/I such that at most two of $I_1/I, I_2/I, \ldots, I_n/I$ are not graded-(m, n)-prime and $J/I \subseteq (I_1/I) \cup (I_2/I) \cup \ldots \cup (I_n/I)$. Then, Corollary 4.2 implies that $J \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$ and at most two of $I_1, I_2, \ldots, I_n$ are not graded-(m, n)-prime. Suppose that $Gr(I_i/I) \nsubseteq Gr((I_j/I : x + I))$ for all $x + I \in h(R/I) \backslash Gr((I_j/I))$ whenever $i \neq j$. It is simple to check that if $Gr(I_i) \subseteq Gr((I_j : x))$ for some $x \in h(R)$, then $Gr((I_i/I)) \subseteq Gr((I_j/I : x + I))$ for some $x + I \in h(R/I)$. In addition, note that if $x + I \in h(R/I) \backslash Gr((I_j/I)) = h(R/I) \backslash (Gr(I_j)/I)$, then $x \in h(R) \backslash Gr(I_j)$. Thus, by our assumption $Gr((I_i/I)) \nsubseteq Gr((I_j/I : x + I))$ for all $x + I \in h(R/I)) \backslash Gr((I_j/I))$ whenever $i \neq j$. Hence, we deduce that $Gr(I_i) \nsubseteq Gr((I_j : x))$ for all $x \in h(R) \backslash Gr(I_j)$ whenever $i \neq j$. Hence, Theorem 5.3 implies $J \subseteq I_k$ for some $1 \leq k \leq n$. Therefore, $J/I \subseteq I_k/I$ for some $1 \leq k \leq n$, as desired. □

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