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A note on the spectral mapping theorem for point spectrum

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Abstract
The spectral mapping theorem, in conjunction with the concept of a spectral set, is utilized to construct reducing pairs of subspaces for a given operator. By leveraging the theory of functional calculus, we revisit the well-established spectral mapping theorem for various spectra, with a particular focus on the point spectrum. Concluding this discussion, we present an application wherein we collect and establish properties of the point spectrum of a compact exponentially m-isometry. Also, we examine the behavior of solutions of a system of differential equations associated with exponentially m-isometric matrices.

Keywords Spectral mapping theorems · Point spectrum · Functional calculus · m-Isometries · System of differential equations

Mathematics Subject Classification 47A10 · 47A60 · 47B37

1 Introduction

Let $\mathcal{X}$ be a non-trivial complex Banach space. Denote by $\mathcal{L}(\mathcal{X})$ the algebra of bounded linear operators on $\mathcal{X}$. For an operator $T \in \mathcal{L}(\mathcal{X})$, let $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$ and $\sigma_p(T)$ denote its spectrum, approximate point spectrum, surjectivity spectrum and point spectrum respectively. A very basic property of the spectrum of an operator on a complex Banach space is that it is a nonempty compact subset of $\mathbb{C}$. Recall also that the approximate point spectrum $\sigma_{ap}(T)$ is a nonempty closed subset of $\mathbb{C}$ that includes the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ and the following inclusions $\overline{\sigma_p(T)} \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$ hold. It is worth noting that the point spectrum may be empty. It is also well known that $\sigma_{\rm ap}(T) = \sigma_{\rm su}(T^*)$ and $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T^*)$. Additionally, it is essential to recall that a neighborhood of a subset A in $\mathbb{C}$ is defined as any open set $\mathcal{G}$ such that $A \subset \mathcal{G}$.

Within the realm of advanced studies encompassing operator theory and functional analysis, it is consistently demonstrated that if $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial, and $p(T)$ represents the operator obtained by formally substituting an operator $T \in \mathcal{L}(\mathcal{X})$ for z in $p(z)$, then $\sigma(p(T)) = p(\sigma(T))$. This result stands as a highly specialized case within a powerful theory, allowing the association of a broad class $\mathcal{F}(T)$ of analytic functions (which

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includes all polynomials) such that for each $f \in \mathcal{F}(T)$ an operator $f(T)$ can be defined whose spectrum $\sigma(f(T))$ coincides with $f(\sigma(T))$.

The following result proves to be very useful as it encompasses the algebraic properties of operators $f(T)$. This collection of properties is commonly known as "functional calculus."

Theorem 1.1 [1, Theorem 6.29] *The mapping $f \mapsto f(T)$, from $\mathcal{F}(T)$ to $\mathcal{L}(\mathcal{X})$, is an algebraic homomorphism. That is, for each pair f, g in $\mathcal{F}(T)$ and all scalars ξ and β in $\mathbb{C}$ we have*

$$(\xi f + \beta g)(T) = \xi f(T) + \beta g(T) \quad \textit{and} \quad (fg)(T) = f(T)g(T)$$

Moreover:

(1) *If $S \in \mathcal{L}(\mathcal{X})$ commutes with T, then S commutes with $f(T)$. In particular, $Tf(T) = f(T)T$ for each $f \in \mathcal{F}(T)$.*
(2) *If $f \in \mathcal{F}(T)$, then $f \in \mathcal{F}(T^*)$ and $f(T^*) = f(T)^*$.*
(3) *If a function f satisfies $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ for all λ in a neighborhood of $\sigma(T)$, then $f \in \mathcal{F}(T)$ and $f(T) = \sum_{n=0}^{\infty} a_n T^n$.*

Further comprehensive presentations of functional calculus can be explored in the classic monographs authored by Dunford and Schwartz [9] as well as Hille and Phillips [14].

The spectral mapping theorem is employed in tandem with the concept of a spectral set to construct reducing pairs of subspaces for a given operator. Using the theory of functional calculus, we are prepared to revisit the well-established spectral mapping theorem for the spectrum.

Theorem 1.2 (The spectral mapping theorem for spectrum) *If $T \in \mathcal{L}(\mathcal{X})$, then for every function $f \in \mathcal{F}(T)$ we have $\sigma(f(T)) = f(\sigma(T))$.*

The spectral mapping theorem introduced earlier pertains to the entire spectrum. Of course, numerous other results exist concerning various classical spectra, which have been established by several authors. These include findings related to the approximate point spectrum and the surjectivity spectrum, as affirmed in the following Theorem [2].

Theorem 1.3 *Let $T \in \mathcal{L}(\mathcal{X})$, and let $f : \Omega \to \mathbb{C}$ be an analytic function on a neighborhood Ω of $\sigma(T)$. Then*

$$\sigma_{\text{su}}(f(T)) = f(\sigma_{\text{su}}(T)) \quad \textit{and} \quad \sigma_{\text{ap}}(f(T)) = f\left(\sigma_{\text{ap}}(T)\right).$$

Let us denote by

$$\sigma_{\text{es}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular }\}.$$

The set $\sigma_{\text{es}}(T)$ is called the essentially semi-regular spectrum of T and has been investigated by Rakočević [19] and Müller [16]. The following theorem has been established for this spectra.

Theorem 1.4 *Given an operator $T \in \mathcal{L}(\mathcal{X})$, and an analytic function f defined on a neighborhood Ω of $\sigma(T)$, then*

$$f(\sigma_{\text{es}}(T)) = \sigma_{\text{es}}(f(T)).$$

Recall the definition of upper semi-Fredholm and lower semi-Fredholm operators for a bounded operator $T \in \mathcal{L}(\mathcal{X})$: an operator T is considered upper semi-Fredholm if its range $\mathsf{R}(T)$ is closed and the dimension of its kernel, denoted by $\alpha(T)$, is finite. On the other hand, T is termed lower semi-Fredholm if the codimension of its range, denoted by $\beta(T)$, is finite. The sets of all upper semi-Fredholm and lower semi-Fredholm operators are respectively denoted by $\Phi_+(\mathcal{X})$ and $\Phi_-(\mathcal{X})$. The class of all Fredholm operators, denoted by $\Phi(\mathcal{X})$, comprises operators that belong to both $\Phi_+(\mathcal{X})$ and $\Phi_-(\mathcal{X})$.

Thanks to [24], we recall that for $T \in \mathcal{L}(\mathcal{X})$, the ascent, $\boldsymbol{a}(T)$, and the descent, $\boldsymbol{d}(T)$, are defined by $\boldsymbol{a}(T) = \inf\{n \geq 0 : \mathsf{N}(T^n) = \mathsf{N}(T^{n+1})\}$ and $\boldsymbol{d}(T) = \inf\{n \geq 0 : \mathsf{R}(T^n) = \mathsf{R}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be ∞.

Two significant classes of operators within Fredholm theory are the upper semi-Browder operators, denoted as $\mathcal{B}_+(\mathcal{X})$, and the lower semi-Browder operators, denoted as $\mathcal{B}_-(\mathcal{X})$. The former comprises operators $T \in \Phi_+(\mathcal{X})$ with finite ascent, while the latter consists of operators $T \in \Phi_-(\mathcal{X})$ with finite descent. These classes, $\mathcal{B}_+(\mathcal{X})$ and $\mathcal{B}_-(\mathcal{X})$, have been subjects of investigation by various authors, as documented in [13, 15, 18]. The class of Browder operators, denoted as $\mathcal{B}(\mathcal{X})$, is defined as the intersection of $\mathcal{B}_+(\mathcal{X})$ and $\mathcal{B}_-(\mathcal{X})$.

These classes of operators lay the groundwork for defining several spectra. For instance, the upper semi-Browder spectrum of $T \in \mathcal{L}(\mathcal{X})$ is defined as

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(\mathcal{X})\},$$

while the lower semi-Browder spectrum of $T \in \mathcal{B}(\mathcal{X})$ is expressed as

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

and finally, the Browder spectrum of $T \in \mathcal{L}(\mathcal{X})$ is defined as

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(\mathcal{X})\}.$$

For these spectra, the following theorem was established in [11, 17, 20, 23].

Theorem 1.5 *Let $T \in \mathcal{L}(\mathcal{X})$, and suppose that the function f is defined on a neighborhood Ω of $\sigma(T)$. If Σ is one of the spectra $\sigma_{\mathrm{ub}}(T)$, $\sigma_{\mathrm{lb}}(T)$, and $\sigma_{\mathrm{b}}(T)$, then*

$$f(\Sigma(T)) = \Sigma(f(T)).$$

Shifting our focus, let's now delve into the spectral mapping theorem concerning point spectrum. It's noteworthy that Halmos [12], Brezis [7], and Garcia et al. [10] laid the groundwork for the spectral mapping theorem for point spectrum, primarily focusing on its polynomial version. Recent inquiries from esteemed colleagues have prompted a closer examination of the nuanced validity of this theorem in a broader context. Insights from esteemed experts, Professor Dan TIMOTIN and Professor Pietro AIENA, provide valuable perspectives on this matter.

Professor Pietro AIENA reflected on the theorem's general applicability, stating, "But I think that in general, it does not hold for the point spectrum. But I am not completely sure; I have never seen that." Meanwhile, Professor Dan TIMOTIN pinpointed a key challenge: "Just a quick remark. It's the possible absence of point spectrum that usually prevents a nice spectral mapping theorem for it. For instance, if you take any operator T without point spectrum, while f is the identically zero function, then the spectrum of $f(T)$ is $\{0\}$, but f applied to the spectrum of T is the empty set."

Therefore, the purpose of this note is to present and revisit a new, more general version of the spectral mapping theorem, focusing on the point spectrum, as affirmed by the following theorem.

Theorem 1.6 (The spectral mapping theorem for point spectrum) *Let $T \in \mathcal{L}(\mathcal{X})$ be an operator, Ω a neighborhood of $\sigma(T)$, and f a complex analytic function within Ω that is non-constant in any component of Ω. Then*

$$f\left(\sigma_p(T)\right) = \sigma_p(f(T)).$$

We employ functional calculus to elucidate this general version. Consequently, the proof provided herein warrants dissemination within the mathematical community. This proof represents a slight variation on Halmos' proof for the polynomial case [12] and, according to an expert, it can also be found in [21]. Furthermore, we give in Theorem 2.1 an application of the general version, which delves into the properties of the set of eigenvalues of compact exponentially m-isometries. We will also discuss the diagonalizability of an exponentially m-isometric matrix based on the behavior of the solutions an associated system of differential equations.

2 Proof of Theorem 1.6 and an application

Proof of Theorem 1.6

The direct inclusion always holds true for any analytic function f on Ω, even if it does not satisfy the non-constant condition in any component of Ω. Indeed, let $\xi \in \sigma_p(T)$. Then, there exists $x \neq 0$ such that $Tx = \xi x$. Consequently, $\xi \in \sigma(T)$. Define the function $g : \Omega \to \mathbb{C}$ by

$$g(\lambda) = \begin{cases} \frac{f(\lambda)-f(\xi)}{\lambda-\xi} & \text{if } \lambda \neq \xi \\ f'(\xi) & \text{if } \lambda = \xi \end{cases}$$

Clearly, g is analytic on Ω and satisfies

$$f(\lambda) - f(\xi) = g(\lambda)(\lambda - \xi).$$

According to Theorem 1.1, this implies

$$f(T) - f(\xi)I = g(T)(T - \xi I).$$

As $(T - \xi I)x = 0$, $f(\xi) \in \sigma_p(f(T))$.

Now, let us establish the other inclusion. Let $\xi \in \sigma_p(f(T))$. Given the assumption made on f, we observe that $f - \xi$ does not vanish identically in any component of Ω. Moreover, we have

$$\xi \in \sigma_p(f(T)) \subset f(\sigma(T)).$$

Put

$$A := f^{-1}(\{\xi\}) \cap \sigma(T).$$

Since $\sigma(T)$ is a compact subset of Ω and $f - \xi$ does not vanish identically in any component of Ω, the non-empty set A is finite. Consider $\alpha_1, \ldots, \alpha_n$ as the zeros of $f - \xi$ in $\sigma(T)$, each counted with respect to its multiplicity. Write

$$f(\lambda) - \xi = g(\lambda)\,(\lambda - \alpha_1)\cdots(\lambda - \alpha_n)\,,$$

where g is analytic on Ω and has no zeros on $\sigma(T)$. The classical spectral mapping theorem ensures the invertibility of $g(T)$. Moreover, we have

$$f(T) - \xi I = g(T)\,(T - \alpha_1 I)\cdots(T - \alpha_n I)\,.$$

As ξ stands as an eigenvalue of $f(T)$ and $f(T) - \xi I$ lacks injectivity, it follows from the last equation that at least one of the operators $T - \alpha_i I$ fails to be injective. The respective α_i resides in $\sigma_p(T)$, and due to $f(\alpha_i) = \xi$, we conclude the proof of the second inclusion, thereby achieving the intended equality.

An Application

Let $\mathcal{H}$ denote an infinite-dimensional complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ represent the set of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, we define the $\Delta_{T,m}$ operator as:

$$\Delta_{T,m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k}.$$

An operator T is termed an m-isometry if $\Delta_{T,m} = 0$.

J. Alger and M. Stankus have published excellent papers on m-isometric operators [3–5]. They showed that m-isometries have interesting spectral properties. For instance, they demonstrated that if T is an invertible m-isometry, its spectrum lies on the unit circle.

In recent studies, several authors have delved deeper into the spectral properties of m-isometries and their associated classes. For example, Aouichaoui has expanded upon the concept of m-isometries, providing a comprehensive analysis of their spectral properties, as detailed in [6]. Furthermore, within the framework of functional calculus for m-isometries, Salehi and Hedayatian introduced the notion of exponentially m-isometric operators in [22], defining them as operators T for which $\Delta_{e^T,m} = 0$.

We now present a significant application, wherein we collect and establish properties of the point spectrum of a compact exponentially m-isometry.

Theorem 2.1 *Let $T \in \mathcal{L}(\mathcal{H})$ be a compact exponentially m-isometry. Then the following properties hold:*

(1) *$\sigma_p(T)$ is a non-empty subset of the imaginary axis.*
(2) *If $\lambda \in \sigma_p(T)$, then $\bar{\lambda} = -\lambda \in \sigma_p(T^*) + 2i\pi\mathbb{Z}$.*
(3) *If $\sigma(T)$ consists of a finite number of points, then*

$$\sigma_p(T^*) = \{-\lambda : \lambda \in \sigma_p(T)\}.$$

In particular, $\sigma_p(T^*)$ is also a non-empty subset of the imaginary axis.

Proof (1) From [22, Corollary 3.11] and [8, Corollary 6.13], it follows that the point spectrum of T is non-empty. Additionally, as mentioned earlier, the point spectrum of any invertible m-isometric operator is a subset of the unit circle. Consequently, Theorem 1.6 implies that the point spectrum of every exponentially m-isometric operator lies on the imaginary axis.
(2) Let $\lambda \in \sigma_p(T)$. Then by Theorem 1.6, $e^\lambda \in \sigma_p(e^T)$. Let x be a non-zero vector such that $e^T x = e^\lambda x$. By induction, we obtain $e^{kT} x = e^{k\lambda} x$, for all $k \geq 0$. We have

$$\begin{aligned} 0 &= \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} e^{kT^*} e^{kT} x \\ &= \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} e^{kT^*} e^{k\lambda} x \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} e^{k(T^*+\lambda)} x. \end{aligned}$$

It follows that

$$(e^{T^*+\lambda} - I)^m x = 0.$$

Hence, $e^{-\lambda} = e^{\bar{\lambda}} \in \sigma_p\left(e^{T^*}\right)$. Again, applying Theorem 1.6 yields

$\bar{\lambda} = -\lambda \in \sigma_p(T^*) + 2i\pi\mathbb{Z}$.

(3) follows from (1) and [22, Corollary 3.11]. □

In what follows, for $n \geq 1$, we write $\mathcal{M}_n$ for the set of $n \times n$ complex matrices. The matrix exponential, the range, and the kernel of $M \in \mathcal{M}_n$ are denoted by e^M, $\mathsf{R}(M)$, and $\mathsf{N}(M)$, respectively.

In the context of systems of differential equations, finding solutions becomes straightforward when the associated matrix is diagonal. Therefore, it is advantageous to initially explore the possibility of transforming the system into a diagonalizable form. Of course, by the uniqueness of the Dunford-Jordan decomposition, an exponentially m-isometric matrix is diagonalizable if and only if its nilpotent part is null. In the following discussion, as we conclude this paper, we will provide another criterion for the diagonalizability of an exponentially m-isometric matrix based on the behavior of the solutions of an associated system of differential equations.

A continuous dynamical system is a curve $\mathbf{x}(t)$ in a set E that evolves according to some rule as t runs over an interval $I \subseteq \mathbb{R}$. For $M \in \mathcal{M}_n$, consider the homogeneous system

$$\mathbf{x}'(t) - M\mathbf{x}(t) = \mathbf{0}, \quad t \in \mathbb{R}. \tag{1}$$

Theorem 2.2 *Let $M \in \mathcal{M}_n$ be an exponentially m-isometry. Then M is diagonalizable if and only if all solutions of* (1) *are bounded over* $\mathbb{R}$.

Proof "$\Longrightarrow$" If we denote $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ as a basis of $\mathbb{C}^n$ formed by eigenvectors of M and denote $\lambda_1, \ldots, \lambda_n$ as the associated eigenvalues respectively, one can see that the set of vector-valued functions of t $\left\{e^{tM}\mathbf{v}_1, \ldots, e^{tM}\mathbf{v}_n\right\} = \left\{e^{t\lambda_1}\mathbf{v}_1, \ldots, e^{t\lambda_n}\mathbf{v}_n\right\}$ forms a basis for the set of solutions of the homogeneous system (1), i.e., of the null space of the linear transformation that maps $\mathbf{x}(t)$ into $\mathbf{x}'(t) - M\mathbf{x}(t)$. This implies that any solution of (1) is a linear combination of the elements of this basis. Since M is an exponentially m-isometry, it follows from Theorem 2.1 that all the λ_k lie on the imaginary axis. Therefore, all solutions of (1) are bounded over $\mathbb{R}$.

"$\Longleftarrow$" Let $\lambda = i\alpha \in \sigma_p(M)$ and $X \neq 0$ be an eigenvector of M associated with λ. Let $Y \in \mathsf{N}(M - \lambda I_n)^2$ and put $Z = (M - \lambda I_n)Y$.

For any $t \in \mathbb{R}$, we have

$$\begin{aligned} e^{tM}Y &= e^{t\lambda}e^{t(M-\lambda I_n)}Y \\ &= e^{t\lambda}\left(Y + t\,(M - \lambda I_n)\,Y + \sum_{k=2}^{+\infty}\frac{t^k}{k!}\,(M - \lambda I_n)^k\,Y\right) \\ &= e^{t\lambda}(Y + tZ). \end{aligned}$$

By assumption, the solution of (1), $\varphi(t) = e^{tM}Y$ is bounded over $\mathbb{R}$. If $Z \neq 0$, then

$$\|\varphi(t)\| = e^{t\,\mathrm{Re}(\lambda)}\|Y + tZ\| = \|Y + tZ\| \longrightarrow \infty, \quad as \quad t \to \infty$$

which is absurd. Therefore, $(M - \lambda I_n)\,Y = 0$. Thus, we have shown that $\mathsf{N}\,(M - \lambda I_n)^2 \subset \mathsf{N}\,(M - \lambda I_n)$. This implies that for any $k \geq 1$, $\mathsf{N}\,(M - \lambda I_n)^k = \mathsf{N}\,(M - \lambda I_n)$, which in turn implies that M is diagonalizable. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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