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n-Harmonic calculus and applications

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Abstract
We define and study a functional calculus for vector valued n-harmonic functions. As applications, we obtain a generalization of the well-known von-Neumann's inequality to several variables. We also use weighted algebras analogues of the classical theorems of N. Wiener and P. Lévy on absolutely Fourier series in order to obtain multi-dimensional versions of N. Wiener and P. Lévy theorems.

Keywords Hermitian algebra · Functional calculus · n-Harmonic function · von-Neumann inequality · Wiener theorem · Lévy theorem.

Mathematics Subject Classification 46H30 · 46J99

1 Preliminaries and introduction

Let $(A, \|.\|)$ be a complex algebra with unit. If $x \in A$ the symbols $Sp_A(x)$ and $\rho_A(x)$ denote the spectrum of x and its spectral radius, respectively. Let $x \longmapsto x^*$ be an involution on A. An element h of A is called hermitian if $h^* = h$. The set of all hermitian elements of A will be denoted by $H(A)$. The real and imaginary parts of an element x of A are denoted by Rex and Imx, respectively, i.e., $Rex = (x + x^*)/2$, $Imx = (x - x^*)/2i$. We say that a Banach algebra A is hermitian if the spectrum of every element of $H(A)$ is real ([12], Definition 5.1, p. 23). For elements h and k of $H(A)$, we write $h \geq k$ to indicate that $h - k$ is positive, i.e., $Sp_A(h-k) \subset [0, +\infty[$. Let x be an element of A. We denote by $|x|$ the square root of the spectral radius of the element x^*x, i.e., $|x| = \rho_A(x^*x)^{\frac{1}{2}}$. In ([12], Theorem 5.2, (5.4) and (5.8), p. 23-25), V. Pt àk proved the following result: If A is hermitian, then $|.|$ is an algebra seminorm on A such that $\rho_A(x) \leq |x|$, for every $x \in A$. The following result of Shirali- Ford ([16], Theorem 1, p. 275) will be needed throughout the paper:

$$A \text{ is hermitian} \Longrightarrow x^*x \geq 0, \text{ for every } x \in A. \tag{1}$$

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Throughout the paper, e will denote the unit of A, and for scalars r we often write simply r for the element re of A. Also $Sp(A)$ denotes the spectrum of A, that is the set of non-zero continuous characters of A. Let n be a positive integer and let A^n denote the cartesian product of n copies of A. Let $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ be a commutative family of elements of A (a $c.f.e.$ in short). Then the full sub-algebra B generated by $\mathbf{a}$ is a unital commutative algebra. Let $\widehat{\mathbf{a}}$ denote the Gel'fand transformation defined by:

$$\widehat{\mathbf{a}}(\chi) = (\chi(a_1), \ldots, \chi(a_n)) \in \mathbb{C}^n, \text{ for every } \chi \in Sp(B).$$

The image $\widehat{\mathbf{a}}(SpB) \subset \mathbb{C}^n$ is therefore a nonempty compact subset of $\mathbb{C}^n$. It is called the *simultaneous spectrum* or *the joint spectrum* of $\mathbf{a}$ and denoted by $Sp(A, \mathbf{a})$ or just $Sp(\mathbf{a})$ ([2], Definition 7, p. 100). One has $Sp(\mathbf{a}) \subset \prod_{i=1}^{n} Sp_A(a_i)$.

Let A be a complex unital Banach algebra and A' the topological dual of A. Let Ω be an open subset of $\mathbb{C}^n$ and $f : \Omega \longrightarrow A$ be an A-valued function. Then f is said to be holomorphic if $\varphi(f(z))$ is holomorphic on Ω in the classical sense for every $\varphi \in A'$. The set of all holomorphic A-valued functions on Ω is denoted by $\mathcal{H}(\Omega, A)$. Since the dual A' separates the points of A, the most results of complex function theory ([13, 14]), such as Cauchy's integral, Taylor expansion, Cauchy estimates and so on, are applicable to $\mathcal{H}(\Omega, A)$. It is clear that $\mathcal{H}(\Omega, A)$ is a complex unital algebra. Moreover, if f is an element of $\mathcal{H}(\Omega, A)$ and if $f(z)$ is invertible for every $z \in \Omega$, then the function f^{-1} defined by $f^{-1}(z) = f(z)^{-1}$ for each $z \in \Omega$ is an element of $\mathcal{H}(\Omega, A)$.

A continuous A-valued function $f : \Omega \longrightarrow A$ is a said to be *n-harmonic* if f is harmonic in each complex variable that is if $z_j = x_j + iy_j$, f should satisfy the n differential equations:

$$\frac{\partial^2 f}{\partial^2 x_j} + \frac{\partial^2 f}{\partial^2 y_j} = 0, \quad \text{for } j = 1, \ldots, n.$$

The set of all *n-harmonic* A-valued functions on Ω is denoted by $h(\Omega, A)$. If f is holomorphic on Ω, then it is holomorphic in each variable, so we have $\mathcal{H}(\Omega, A) \subset h(\Omega, A)$. Let A be an involutive complex Banach algebra. An A-valued function $f : \Omega \longrightarrow A$ is said to be *hermitian* if $f(z) = f(z)^*$, for every $z \in \Omega$. We denote by Ref (resp. Imf) the real part of f (resp. the imaginary part of f) defined by $Ref(z) = Re(f(z))$ (resp. $Imf(z) = Im(f(z))$), for every $z \in \Omega$.

Let $z_0 \in \mathbb{C}$ and $r > 0$, the open (resp. closed) disc with center z_0 and radius r is denoted by $D(z_0, r)$ (resp. $\overline{D}(z_0, r)$); its boundary, denoted by $T(z_0, r)$, is the circle with center z_0 and radius r. If $z^0 = \left(z_1^0, \ldots, z_n^0\right) \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n)$ is a multi-index $\left(\mathbb{R}_+^*\right)^n$, the open polydisc with center z^0 and radius r is the set

$$D^n\left(z^0, r\right) = \prod_{j=1}^{n} D\left(z_j^0, r_j\right).$$

Its closure is denoted by $\overline{D}^n\left(z^0, r\right)$. The notation $T^n\left(z^0, r\right)$ denotes the torus of $\mathbb{C}^n$ with center z^0 and radius r, that is $T^n\left(z^0, r\right) = \prod_{j=1}^{n} T\left(z_j^0, r_j\right)$. Let $\mathbb{Z}^+ = \{0, 1, 2, ..\}$. A multi-index α is an element of $\left(\mathbb{Z}^+\right)^n$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we write w^α for the monomial (power product) $w_1^{\alpha_1} \ldots w_n^{\alpha_n}$. We also consider the following differential form, on $\mathbb{C}^n$, $dz = dz_1 \ldots dz_n$. In the sequel, all algebras considered here are complex and unital ones.

The A-valued harmonic functional calculus for an element of an involutive Banach algebra is defined and studied in [5]. Here we construct an A -valued n-harmonic calculus for an arbitrary n-tuple of elements of an involutive Banach algebra elements. This calculus consists in giving a sense to $f(\mathbf{a})$ whenever $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ and f is an A-valued n-harmonic function on a neighbourhood U of the simultaneous spectrum $Sp(\mathbf{a})$ of $\mathbf{a}$. To that aim, we need to introduce a functional calculus for holomorphic A-functions. The paper is organized as follows. In Sect. 2, we introduce a vector-valued Cauchy transform. This allows us to introduce a functional calculus for holomorphic A-functions. We show that this calculus is continuous (namely the mapping $f \longmapsto C[f](\mathbf{a})$ is continuous) and satisfies the spectral mapping theorem. Sect. 3 relies highly on Sect. 2 where we define and study a vector-valued n-harmonic calculus. The most important properties of this calculus are studied. The last section is devoted to applications. The first one is a generalization of the well-known von-Neumann's inequality to several variables. The second concerns the classical and famous theorems of N. Wiener [19] and P. Lévy [11].

For more details on holomorphic functional calculus, we refer the reader to [1, 18].

2 A vector-valued Cauchy transform

We first introduce a vector-valued Cauchy transform by means of appropriate vector-valued kernel. In particular, we obtain a functional calculus for holomorphic A-functions which will be useful to us later.

Definition 2.1 Let A be a complex unital Banach algebra, Ω an open subset of $\mathbb{C}^n$, $z^0 = \left(z_1^0, \ldots, z_n^0\right) \in \Omega$, $r = (r_1, \ldots, r_n) \in \left(\mathbb{R}_+^*\right)^n$ such that $\overline{D}^n\left(z^0, r\right) \subset \Omega$. If $f \in \mathcal{H}(\Omega, A)$ and $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ be a $c.f.e.$ with $Sp(\mathbf{a}) \subset D^n\left(z^0, r\right)$, then

$$f(\mathbf{a}) = \frac{1}{(2\pi i)^n} \int_{T^n(z^0, r)} f(z) C(\mathbf{a}, z) dz,$$

where

$$C(\mathbf{a}, z) = \prod_{j=1}^{n} (z_j - a_j)^{-1} = \prod_{j=1}^{n} C(a_j, z_j).$$

If we denote by $\Phi_{\mathbf{a}}(f)$ the element $f(\mathbf{a})$, one has a linear mapping of $\mathcal{H}(\Omega, A)$ into A, denoted by:

$$\Phi_{\mathbf{a}} : \mathcal{H}(\Omega, A) \longrightarrow A : f \longmapsto f(\mathbf{a}) \tag{2}$$

Suppose f is a function on Ω into A and $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$. The function $\mathbf{a}f$ is defined by:

$$\mathbf{a}f(z) = (a_1 f(z), \ldots, a_n f(z)), \text{ for every } z \in \Omega.$$

We say that $\mathbf{a}$ and f are commuting if $\mathbf{a}f(z) = f(z)\mathbf{a}$ for all z in Ω that is:

$$(a_1 f(z), \ldots, a_n f(z)) = (f(z)a_1, \ldots, f(z)a_n), \text{ for every } z \in \Omega.$$

If $\mathbf{a}$ commutes with every element of $\mathcal{H}(\Omega, A)$, then we say that $\mathbf{a}$ and $\mathcal{H}(\Omega, A)$ commute. Now $f, g \in \mathcal{H}(\Omega, A)$ and $\mathbf{a} = (a_1, \ldots, a_n)$ be a $c.f.e.$ as described in Definition 2.1. Then as in the classical case, one has $(fg)(\mathbf{a}) = f(\mathbf{a})g(\mathbf{a})$ if $\mathbf{a}$ and g are commuting. Unless otherwise stated we assume that $\mathbf{a}$ and $\mathcal{H}(\Omega, A)$ are commuting.

Let $\mathcal{C}(T^n, A)$, where $T^n = T^n(0, 1)$, be the algebra of all A-valued continuous functions on T^n. For every $f \in \mathcal{C}(T^n, A)$, we define

$$C[f](\mathbf{a}) = \frac{1}{(2\pi i)^n} \int_{T^n(z^0, r)} f(z) C(\mathbf{a}, z) dz.$$

The mapping $\mathcal{C}(T^n, A) \longrightarrow A : f \longmapsto C[f](\mathbf{a})$ is obviously a linear map. Now, since the mapping $z \longmapsto \|C(\mathbf{a}, z)\|$ is continuous, and therefore bounded, on $T^n\left(z^0, r\right)$, there exists a positive constant M such that:

$$\|C[f](\mathbf{a})\| \leq M\,|f|_{T^n(z^0,r)}\,, \text{ for every } f \in \mathcal{C}\left(T^n(z^0, r), A\right),$$

where

$$|f|_{T^n(z^0,r)} = \sup\left\{\|f(z)\| : z \in T^n(z^0, r)\right\}.$$

It follows that the mapping

$$f \longmapsto C[f](\mathbf{a})$$

is continuous from $\left(\mathcal{C}(T^n, A),\ |.|_{T^n(z^0,r)}\right)$ into $(A, \|.\|)$. It is called the Cauchy transform of f at $\mathbf{a}$ ([17], p. 49). If $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ be a $c.f.e.$ with $Sp(\mathbf{a}) \subset D^n(0, 1)$, then for every $f \in \mathcal{H}(\Omega, A)$, we have $C[f](\mathbf{a}) = \Phi_{\mathbf{a}}(f)$. Moreover if $P(z) = z_1^{\gamma_1} \ldots z_n^{\gamma_n}$ is a monomial, then $C[P](\mathbf{a})$ is actually $a_1^{\gamma_1} \ldots a_n^{\gamma_n}$, where $\gamma_1, .., \gamma_n \in \mathbb{Z}_+$. Whence for every analytic polynomial P, one has $C[P](\mathbf{a}) = P(\mathbf{a})$. So we have the following:

Theorem 2.2 *Let A be a complex unital Banach algebra, Ω an open subset of $\mathbb{C}^n$, $z^0 = \left(z_1^0, \ldots, z_n^0\right) \in \Omega$, $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ such that $\overline{D}^n\left(z^0, r\right) \subset \Omega$ and $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ be a $c.f.e.$ with*

$$Sp(\mathbf{a}) \subset D^n\left(z^0, r\right).$$

Then there exists a continuous linear map $\Theta_{\mathbf{a}}$ from $\mathcal{C}\left(T^n(z^0, r), A\right)$ into A with the following properties:

(1) *For every analytic polynomial P, one has*

$$\Theta_{\mathbf{a}}(P) = P(\mathbf{a})$$

(2) $\Theta_{\mathbf{a}}/_{\mathcal{H}(\Omega,A)}$ *is multiplicative and $\Theta_{\mathbf{a}}\left(z_j\right) = a_j$, $(j = 1, \ldots, n)$, here z_j denotes the j-th coordinate projection $\mathbb{C}^n \longrightarrow \mathbb{C}$.*

Remark 2.3 (1) Let $\Omega = D^n(0, R)$, where $R = (R_1, \ldots, R_n) \in \mathbb{R}_+^n$, and $f \in \mathcal{H}(\Omega, A)$ with the the Taylor expansion

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha, \text{ for every } z \in \Omega,$$

where (a_α) is a sequence in A. If $x = (x_1, \ldots, x_n) \in A^n$ be $c.f.e.$ with $\rho\left(x_j\right) < R_j$ $(j = 1, \ldots, n)$, then

$$f(\mathbf{a}) = \sum_{\alpha} a_\alpha x^\alpha.$$

(2) Let $f \in \mathcal{H}(\Omega, A)$ be such that $f(z)^{-1}$ exists, for every $z \in \Omega$. Then $f^{-1}(x) = f(x)^{-1}$, i.e.,

$$f(x)^{-1} = \frac{1}{(2\pi i)^n} \int_{T^n(z^0,r)} f(z)^{-1} C(x, z) dz.$$

Let Ω_1 and Ω_2 be two open subsets of $\mathbb{C}$. Suppose $f \in \mathcal{H}(\Omega_1, A)$ and $g \in \mathcal{H}(\Omega_2, A)$ satisfy the condition that for every compact set K_2 in Ω_2, there exists a compact K_1 in Ω_1 such that $Sp_A(g(z))$ is contained in K_1, for every z in K_2. Let $z_0 \in \Omega_2$ and $r \in \mathbb{Z}_+$ such that $\overline{D}(z_0, r) \subset \Omega_2$. Let K_1 be a compact set in Ω_1, which contains each $Sp_A(g(z))$ for every $z \in \overline{D}(z_0, r)$. As in ([2], Proposition 3, p. 29), we choose a suitable positively oriented simple closed rectifiable contour Γ_1 such that the interior domain $int(\Gamma_1)$ of Γ_1 contains K_1 and $int(\Gamma_1) \cup \Gamma_1 \subset \Omega_1$. Then for $z \in \overline{D}(z_0, r)$, we have

$$Sp_A(g(z)) \subset K_1 \subset int(\Gamma_1) \cup \Gamma_1 \subset \Omega_1.$$

It follows, from ([2], Proposition 4, p. 29), that:

$$f(g(z)) = \frac{1}{2\pi i} \int_{\Gamma_1} f(w)(we - g(z))^{-1} dw \tag{3}$$

Since for any fixed w on Γ_1,

$$z \longmapsto (we - g(z))^{-1}$$

is analytic on $D(z_0, r)$, it follows from 2) of Remark 2.3 that, for $w \in \Gamma_1$ and $z \in D(z_0, r)$,

$$(we - g(z))^{-1} = \frac{1}{2\pi i} \int_{T(z_0,r)} (we - g(u))^{-1} (u - z)^{-1} du$$

which shows that, for every $z \in D(z_0, r)$, one has

$$f(g(z)) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{T(z_0,r)} f(w)(we - g(u))^{-1} (u - z)^{-1} du dw.$$

Let φ be any bounded functional on A. As in ([20], Lemma 2.4, p. 297), we can prove that $\varphi(f(g(z)))$ is analytic on $D(z_0, r)$. It follows, from ([15], Definition 3. 30, p. 78), that the "composite function" $h = f \circ g$ defined by $h(z) = f(g(z))$ for every $z \in \Omega_2$ is an element of $\mathcal{H}(\Omega_2, A)$. More generally, we obtain a theorem of composition of functions for holomorphic functional calculus given by:

Proposition 2.4 *Let Ω_1 and Ω_2 be two open subsets of $\mathbb{C}$. Let $f \in \mathcal{H}(\Omega_1, A)$ and $g \in \mathcal{H}(\Omega_2, A)$ satisfy the condition that for every compact set K_2 in Ω_2, there exists a compact K_1 in Ω_1 such that $Sp_A(g(z))$ is contained in K_1, for every z in K_2. Let $a \in A$ with $Sp_A(a) \subset \Omega_2$ and $Sp_A(g(a)) \subset \Omega_1$. Then $f \circ g(a) = f(g(a))$.*

Proof Let Γ_2 be a positively oriented simple closed rectifiable contour such that $Sp_A(a) \subset int(\Gamma_2)$ and $int(\Gamma_2) \cup \Gamma_2 \subset \Omega_2$. By our assumption, there exists a compact set K_1 in Ω_1 such that $Sp_A(g(z)) \subset K_1$ for all $z \in int(\Gamma_2) \cup \Gamma_2$. Now, since $int(\Gamma_2) \cup \Gamma_2$ is a compact subset of Ω_2 ([3], Definition 3.1, p. 45), one can choose a suitable positively oriented simple closed rectifiable contour Γ_1 so that both $Sp_A(g(a))$ and K_1 are contained in $int(\Gamma_1)$ and $int(\Gamma_1) \cup \Gamma_1 \subset \Omega_1$. Then in the same manner as before, one has:

$$f(g(a)) = \frac{1}{2\pi i} \int_{\Gamma_1} f(w)(we - g(a))^{-1} dw.$$

By 2) of Remark 2.3, one has:

$$(we - g(a))^{-1} = \frac{1}{2\pi i} \int_{\Gamma_2} (we - g(z))^{-1} (ze - a)^{-1} dz.$$

It follows that:

$$f(g(a)) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} f(w)(we - g(z))^{-1} (ze - a)^{-1} dz dw.$$

The continuity of in (w, z) on $\Gamma_1 \times \Gamma_2$ ([2], Proposition 6, p. 11) of the function:

$$(w, z) \longmapsto f(w)(we - g(z))^{-1} (ze - a)^{-1}$$

allows us to change the order of integration. Thus

$$f\left(g(a)\right)=\left(\frac{1}{2\pi i}\right)^{2}\int_{\Gamma_2}\left[\int_{\Gamma_1}f(w)\left(we-g(z)\right)^{-1}dw\right](ze-a)^{-1}dz.$$

Now, by (3),

$$f\left(g(z)\right)=\frac{1}{2\pi i}\int_{\Gamma_1}f(w)\left(we-g(z)\right)^{-1}dw.$$

Whence

$$\begin{aligned}f\left(g(a)\right)&=\frac{1}{2\pi i}\int_{\Gamma_2}f(g(z))(ze-a)^{-1}dz\\&=f\circ g(a).\end{aligned}$$

This completes the proof. □

Now, we examine one of the most powerful properties of holomorphic functional calculus. That is the spectral mapping theorem:

Proposition 2.5 *Let A be a complex unital Banach algebra and $\mathbf{a}$ be a c.f.e and Ω as described in Definition 2.1. Then the mapping $\Phi_{\mathbf{a}}$ defined by (2) is an algebra homomorphism of $\mathcal{H}(\Omega, A)$ into A such that*

$$\widehat{\Phi_{\mathbf{a}}(f)}(\chi)=(\chi\circ f)(\chi(\mathbf{a})), \text{ for every } \chi\in Sp(A).$$

Proof Let $\chi\in Sp(A)$. Then $\widehat{\Phi_{\mathbf{a}}(f)}(\chi)=\chi\left(\Phi_{\mathbf{a}}(f)\right)$ and

$$\begin{aligned}\chi\left(\Phi_{\mathbf{a}}(f)\right)&=\frac{1}{(2\pi i)^n}\int_{T^n(z^0,r)}\chi\left(f(z)C(\mathbf{a},z)\right)dz\\&=\frac{1}{(2\pi i)^n}\int_{T^n(z^0,r)}\chi\left(f(z)\right)\chi\left(C(\mathbf{a},z)\right)dz.\end{aligned}$$

Therefore, taking into account the fact that $\chi\left(C(\mathbf{a},z)\right)=\left(C(\chi(\mathbf{a}),z)\right)$, we have

$$\begin{aligned}\chi\left(\Phi_{\mathbf{a}}(f)\right)&=\frac{1}{(2\pi i)^n}\int_{T^n(z^0,r)}(\chi\circ f)(z)\left(C(\chi(\mathbf{a}),z)\right)dz\\&=(\chi\circ f)(\chi(\mathbf{a})).\end{aligned}$$

□

Proposition 2.6 *Let A be a complex unital Banach algebra and Ω be an open subset of $\mathbb{C}^n$ as described in Definition 2.1. If $f\in\mathcal{H}(\Omega, A)$ and $\mathbf{a}=(a_1,\ldots,a_n)\in A^n$ be a c.f.e. with $Sp(\mathbf{a})\subset D^n\left(z^0,r\right)$, then*

(1)

$$Sp_A\left(f(\mathbf{a})\right)\subset\bigcup_{\lambda\in Sp(\mathbf{a})}Sp_A\left(f(\lambda)\right).$$

(2) *If $f=\widetilde{f}e$, where $\widetilde{f}$ is a holomorphic scalar function on Ω, then*

$$Sp_A\left(f(\mathbf{a})\right)=f\left(Sp(\mathbf{a})\right).$$

Proof (1) Observe first that if f has no inverse on $Sp(\mathbf{a})$, then $g = f^{-1}$is holomorphic in an open set Ω_1 such that $Sp(\mathbf{a}) \subset \Omega_1 \subset \Omega$. Since $fg = e$ in Ω_1, it follows that $f(\mathbf{a})\, g(\mathbf{a}) = e$ and $f(\mathbf{a})$ is invertible. Now fix $\beta \in \mathbb{C}$. Then $\beta \in Sp_A(f(\mathbf{a}))$ if and only if $f(\mathbf{a}) - \beta e$ is not invertible in A. From the above, there exists $\lambda \in Sp(\mathbf{a})$ such that $f(\lambda) - \beta e$ is not invertible in A, that is $\beta \in Sp_A(f(\lambda))$. (2) If $f = \widetilde{f}e$, where $\widetilde{f}$ is a holomorphic scalar function on Ω, then $\bigcup_{\lambda \in Sp(\mathbf{a})} Sp(f(\lambda)) = f(Sp(\mathbf{a}))$. Furthermore, for $\chi \in Sp(A)$, one has

$$\widehat{f(\mathbf{a})}(\chi) = (\chi \circ f)(\chi(\mathbf{a})) = f(\chi(\mathbf{a})).$$

Therefore $f(Sp(\mathbf{a})) \subset Sp_A(f(\mathbf{a}))$. □

Remark 2.7 The inclusion 1) of Proposition 2.6. can be strict as the simple example shows. Let $x \in A$ be an invertible element such that $Sp_A(x) = \{1, 2\}$ and put $f(z) = x^{-1}z$. Since $f(x) = e$, we have $Sp_A(f(x)) = \{1\}$ but $Sp_A(f(1)) = \left\{\frac{1}{2}, 1\right\}$.

3 A vector-valued *n*-hamonic functional calculus

In this section we define a functional calculus for A-valued n-harmonic functions of several variables and describe some of its properties. Let $z \in D^n(z^0, r)$ and $w \in T^n(z^0, r)$. Then the Poisson kernel $P^n(z, w)$ is the product

$$P^n(z, w) = P(z_1, w_1) \ldots P(z_n, w_n),$$

where $P(z_i, w_i)$ is the classical Poisson kernel for the disk $D(z_i^0, r_i)$. Note that

$$\begin{aligned} P(z_i, w_i) &= Re\left[\left(w_i + z_i - 2z_i^0\right)(w_i - z_i)^{-1}\right] \\ &= (\overline{w_i} - \overline{z_i})^{-1}\left[r_i^2 - \left(\overline{z_i} - \overline{z_i^0}\right)\left(z_i - z_i^0\right)\right](w_i - z_i)^{-1} \\ &= \overline{C(z_i, w_i)}\left[r_i^2 - \left(\overline{z_i} - \overline{z_i^0}\right)\left(z_i - z_i^0\right)\right]C(z_i, w_i). \end{aligned}$$

If we put

$$\Delta\left(z_i, z_i^0\right) = r_i^2 - \left(\overline{z_i} - \overline{z_i^0}\right)\left(z_i - z_i^0\right),$$

then

$$P(z_i, w_i) = \overline{C(z_i, w_i)}\, \Delta\left(z_i, z_i^0\right) C(z_i, w_i)$$

and

$$P(z, w) = \overline{C(z, w)} \prod_{i=1}^{n} \Delta\left(z_i, z_i^0\right) C(z, w).$$

We also put

$$\Delta^n\left(z, z^0\right) = \prod_{i=1}^{n} \Delta\left(z_i, z_i^0\right)$$

Let A be a complex unital Banach algebra with continuous involution $x \longmapsto x^*$, $z^0 = \left(z_1^0, \ldots, z_n^0\right) \in \mathbb{C}^n$, $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. If $\mathbf{x} = (x_1, \ldots, x_n) \in A^n$ be a $c.f.e.$ with $Sp_A(\mathbf{x}) \subset D^n\left(z^0, r\right)$, then the A-valued Poisson kernel is defined by the equality:

$$P(\mathbf{x}, w) = C(\mathbf{x}, w)^* \Delta^n\left(\mathbf{x}, z^0\right) C(\mathbf{x}, w), \ w \in T^n\left(z^0, r\right),$$

where

$$\Delta^n\left(\mathbf{x}, z^0\right)=\prod_{i=1}^n \Delta\left(x_i, z_i^0\right)=\prod_{i=1}^n\left[r_i^2-\left(x_i^*-\overline{z_i^0}\right)\left(x_i-z_i^0\right)\right].$$

If $n=1$, then

$$\begin{aligned} P\left(x_1, w\right) & =C\left(x_1, w\right)^* \Delta^1\left(x_1, z^0\right) C\left(x_1, w\right) \\ & =\left(w-x_1^*\right)^{-1}\left[r^2-\left(x_1^*-\overline{z_1^0}\right)\left(x_1-z^0\right)\right]\left(w-x_1\right)^{-1} \\ & =Re\left[\left(w+x_1-2 z_1^0\right)\left(w-x_1\right)^{-1}\right)\right] \geq 0 . \end{aligned}$$

Definition 3.1 Let A be a complex unital Banach algebra with continuous involution $x \longmapsto x^*$, Ω an open subset of $\mathbb{C}^n$, $z^0=\left(z_1^0, \ldots, z_n^0\right) \in \Omega$, $r=\left(r_1, \ldots, r_n\right) \in \mathbb{R}_{+}^n$ such that $\overline{D}^n\left(z^0, r\right) \subset \Omega$, $\mathbf{x}=\left(x_1, \ldots, x_n\right) \in A^n$ be a $c.f.e.$ with $Sp(\mathbf{x}) \subset D^n\left(z^0, r\right)$ and $f \in h(\Omega, A)$. Then the element of A given by the Poisson integral formula:

$$\frac{1}{(2 \pi)^n} \int_{T^n\left(z^0, r\right)} f(w) P(\mathbf{x}, w) \frac{\left|d w_1\right|}{r_1} \cdots \frac{\left|d w_n\right|}{r_n}$$

is denoted by $P[f](\mathbf{x})$.

If we denote by $\Psi_{\mathbf{x}}(f)$ or just $f(\mathbf{x})$ the element $P[f](\mathbf{x})$, one has a mapping of $h(\Omega, A)$ into A, noted $\Psi_{\mathbf{x}}$, given by:

$$\Psi_{\mathbf{x}}: h(\Omega, A) \longrightarrow A: f \longmapsto \Psi_{\mathbf{x}}(f)=P[f](\mathbf{x})=f(\mathbf{x}).$$

Then $\Psi_{\mathbf{x}}$ is an involutive homomorphism from $h(\Omega, A)$ into A that extends the algebra homomorphism $\Theta_{\mathbf{a}}$ given by the Cauchy transform ([18], Proposition 9, p. 103). Furthermore, if K is a compact neighbourhood contained in Ω and containing $Sp(\mathbf{x})$, then the mapping $\Psi_{\mathbf{x}}$ is continuous with respect to the uniform convergence on K.

Proposition 3.2 *Let A be a hermitian Banach algebra with continuous involution $x \longmapsto x^*$, Ω and $\mathbf{x}$ as described in Definition 3.1. If $\mathbf{x}$ is normal, then, for every $f \in h(\Omega, A)$, one has*

$$\widehat{\Psi_{\mathbf{x}}(f)}(\chi)=(\chi \circ f)(\chi(\mathbf{x})), \textit{ for every } \chi \in Sp(A).$$

Corollary 3.3 *Let A be a hermitian Banach algebra with continuous involution $x \longmapsto x^*$, Ω and $\mathbf{x}$ as described in Definition 3.1. If $\mathbf{x}$ is normal, then*

(1) $Sp_A(f(\mathbf{x})) \subset \bigcup_{\lambda \in Sp(\mathbf{x})} Sp_A(f(\lambda))$, *for every* $f \in h(\Omega, A)$.

(2) *If $f=\widetilde{f}e$, where $\widetilde{f}$ is a harmonic scalar function on Ω, then*

$$Sp_A(f(\mathbf{x}))=f(Sp(\mathbf{x})).$$

Remark 3.4 The proofs of Proposition 3.2 and Corollary 3.3 are similar to those of Proposition 2.5 and Proposition 2.6. Here the hypothesis that A is hermitian is used to get

$$\chi(P(\mathbf{x}, w))=P(\chi(\mathbf{x}), w), \text{ for every } \chi \in Sp(A)$$

which results from the fact that every character χ of a hermitian algebra A is hermitian ([4], (i), Theorem 1.4.1, p. 11), i.e.,

$$\chi\left(x^*\right)=\overline{\chi(x)}, \text{ for every } x \in A.$$

4 Some applications

In this section, we give some applications of functional calculi as explored in the preceding sections. Its applications concern a generalization of von Neumann's theorem ([5], Théorème 6, p. 506), N. Wiener and P. Lé vy theorems ([9], Theorem 4.2, p. 337 and Theorem 5.1, p. 339) and ([6], Theorem 3.1 and Theoerem 3.2). We obtain an analog of Neumann's theorem for A-valued holomorphic functions of several variables. Afterward we use weighted algebras analogues of the classical theorems of N. Wiener and P. L évy on absolutely Fourier series and we get multi-dimensional versions of N. Wiener and P. Lévy theorems given in ([8], Theorem l, 347 and Theorem 2, p. 349).

4.1 Analog of von Neumann's theorem.

The spectral inequality of von Neumann (cf. [10], Theorem 1, p. 276) is well-known. Its asserts that, given a contraction T on a Hilbert space $\mathcal{H}$, i.e., $\|T\| \leq 1$ and a complex function f analytic on the open unit disk D. If $f(D) \subset D$, then $f(T)$ is also a contraction on $\mathcal{H}$. In ([7], Theorem 3.1, p. 933), the third author showed that hermitian algebras are the natural framework of the last inequality. He also obtained an extension to analytic A-valued functions ([5], Théorème 1, p. 498). Here, we obtain a generalization of the von-Neumann's inequality to several variables.

In the sequel, A will denote a hermitian Banach algebra with continuous involution $x \longmapsto x^*$ and $D^n = D^n(0, 1)$. We consider:

$$\mathcal{H}N_A(D^n) = \left\{f \in \mathcal{H}(D^n, A) : f(z) \text{ is normal, for every } z \in D^n\right\}$$

$$\mathcal{H}_A(D^n) = \left\{\begin{array}{c} f \in \mathcal{H}N_A(D^n) : f(z)f(w) = f(w)f(z), \\ \text{for every } z, w \in D^n \end{array}\right\}$$

$$B_A\left(D^n\right) = \left\{f \in \mathcal{H}N_A(D^n) : |f(z)| < 1, \text{ for every } z \in D^n\right\}$$

$$P_A\left(D^n\right) = \left\{g \in \mathcal{H}N_A(D^n) : Reg(z) > 0, \text{ for every } z \in D^n\right\},$$

where $Reg(z)$ designates the real part of $g(z)$.

As a first application of the n-harmonic functional calculus, we have the following result:

Theorem 4.1 *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ *be a* $c.f.e.$ *such that* $|a_i| < 1$, *for every* $i = 1, \ldots, n$. *If* $P(\mathbf{a}, w) > 0$ *for every* w *in the torus* $T^n(0, 1)$, *then* $Reg(\mathbf{a}) > 0$, *with* g *in* $P_A\left(D^n\right)$.

Proof Since $g \in P_A\left(D^n\right)$, one has $g \in \mathcal{H}N_A(D^n)$ and its real part Reg is an A-valued harmonic function on D^n. Let $\mathbf{a} \in A^n$ be a $c.f.e.$ such that $|a_i| < 1$, for every $i = 1, \ldots n$. Choose positive numbers r_i and r'_i with $|a_i| < r_i < r'_i < 1$. It is easy to verify that $Sp_A(\mathbf{a}) \subset D^n(0, r)$, where $r = (r_1, \ldots, r_n)$. By hypothesis, g in $P_A\left(D^n\right)$. It follows that $Reg(z) > 0$, for every $z \in \overline{D}^n\left(0, r'\right)$, i.e., $Reg(z)$ is a positive and invertible element of A. Consider the function

$$\psi(z) = \rho\left([Reg(z)]^{-1}\right), \text{ for every } z \in \overline{D}^n\left(0, r'\right).$$

As the spectral radius $x \longmapsto \rho(x)$ is upper-semicontinuous on A, the function ψ is therefore upper semicontinuous on $\overline{D}^n\left(0, r'\right)$. So ψ has a maximum on $\overline{D}^n\left(0, r'\right)$. Therefore, there exists $\delta > 0$ such that:

$$\rho\left([Reg(z)]^{-1}\right) \leq \frac{1}{\delta}, \text{ for every } z \in \overline{D}^n\left(0, r'\right).$$

Whence $\rho\left(Reg(z)\right) > \delta$, for every $z \in \overline{D}^n\left(0, r'\right)$. It follows that $Reg(z) > \delta$, for every $z \in D^n\left(0, r'\right)$. Consider h defined by:

$$h(z) = Reg(z) - \delta e, \text{ for every } z \in D^n\left(0, r'\right).$$

Thus, by Definition 3.1, one has:

$$h(\mathbf{a}) = \frac{1}{(2\pi)^n}\int_{T^n(0,r)} h(w)P(\mathbf{a}, w)\frac{|dw_1|}{r_1}\dots\frac{|dw_n|}{r_n}.$$

By our assumption, $P(\mathbf{a}, w) \geq 0$, for every $w \in T^n(0, 1)$. Then, since $\mathbf{a}$ and $\mathcal{H}(D^n, A)$ are commuting, we have

$$h(w)P(\mathbf{a}, w) \geq 0, \text{ for every } w \in T^n(0, r).$$

Indeed, for a fixed $w \in T^n(0, r)$, one has $h(w) > 0$ and $P(\mathbf{a}, w) \geq 0$. Thus there exists $u, v \in H(A)$ such that

$$h(w) = u^2 \text{ and } P(\mathbf{a}, w) = v^2.$$

Moreover u and v commutes since h and $\mathbf{a}$ are commuting. It follows that $h(w)P(\mathbf{a}, w) \in H(A)$ and

$$h(w)P(\mathbf{a}, w) = u^2v^2 = uv\,(uv)^* \geq 0 \text{ by (1)}.$$

So $h(\mathbf{a}) \geq 0$. Finally, since $h(\mathbf{a}) = Reg(\mathbf{a}) - \delta$, we have $Reg(\mathbf{a}) - \delta \geq 0$, i.e., $Sp_A(Reg(\mathbf{a}) - \delta) \subset [0, +\infty[$. Whence $Sp_A\left(Reg(\mathbf{a})\right) \subset [\delta, +\infty[$. Thus $Reg(\mathbf{a})$ is a hermitian element of A and $Sp_A\left(Reg(\mathbf{a})\right) \subset]0, \infty[$ for $\delta > 0$. So $Reg(\mathbf{a}) > 0$. This completes the proof. □

As in the complex case, the reader can prove that the relations

$$g(z) = (e + f(z))\,(e - f(z))^{-1} \text{ and } f(z) = (g(z) - e)(g(z) + e)^{-1}$$

establish a bijection between the functions f in $B_A\left(D^n\right)$ and the functions g in $P_A\left(D^n\right)$. Using this fact, we obtain an equivalent version of Theorem 4.1 given by:

Theorem 4.2 *Let* $f \in B_A\left(D^n\right)$ *and* $\mathbf{a} = (a_1, \dots, a_n) \in A^n$ *be a c.f.e. such that* $|a_i| < 1$, *for every* $i = 1, \dots, n$. *If* $P(\mathbf{a}, w) > 0$, *for every* $w \in T^n(0, 1)$, *then* $|f(\mathbf{a})| < 1$.

Remark 4.3 (1) In the case where $n = 1$, we have

$$\begin{aligned} P\left(\mathbf{a}, w\right) &= Re\left[(w + a_1)\left((w - a_1)^{-1}\right)\right] \\ &= \left(\overline{w} - a_1^*\right)^{-1}\left(1 - a_1^*a_1\right)(w - a_1)^{-1}. \end{aligned}$$

As $|a_1| < 1$, we have $e - a_1^*a_1 > 0$, so that

$$e - a_1^*a_1 = u^2 \text{ for some } u \in H(A).$$

Hence

$$P\left(a_1, w\right) = \left(\overline{w} - a_1^*\right)^{-1} u\,u\,\left(w - a_1\right)^{-1}.$$

Then, by (1), we have $P\left(a_1, w\right) \geq 0$.

(2) In the case where $n = 2$ and $\mathbf{a} = (a_1, a_2)$, we have:

$$P^2\left(\mathbf{a}, w\right) = P\left(a_1, w_1\right) P\left(a_2, w_2\right)$$

Put $P\left(a_1, w_1\right) = h^2$ and $P\left(a_2, w_2\right) = k^2$, then $P^2\left(\mathbf{a}, w\right) = h^2k^2$ and as

$$Sp_A\left(h^2k^2\right) = Sp_A\left(khhk\right),$$

we obtain $Sp_A\left[P^2\left(\mathbf{a}, w\right)\right] \subset \mathbb{R}^+$.

Remark 4.4 Using Theorem 4.1, we obtain as in [5], the analog of Schawrz's lemma ([5], Théorème 4, p. 502) as well as the analog of Pick's theorem ([5], Théorème 5, p. 504).

4.2 Analogues of Lévy and Wiener's theorems

For $p \in]1, +\infty[$, let $\omega : \mathbb{Z}^k \longrightarrow [1, +\infty[$, $k \in \mathbb{N}^*$ fixed, be a weight on $\mathbb{Z}^k$, i.e., ω satisfies

$$\sum_{m \in \mathbb{Z}^k} \omega^{\frac{1}{1-p}}(m) < +\infty. \tag{4}$$

For $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ and $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$, we will use the notation $(n, t) = n_1 t_1 + \cdots + n_k t_k$. Now, we consider the following weighted space:

$$\mathcal{A}_k^p(\omega) = \left\{ f : \mathbb{R}^k \longrightarrow \mathbb{C} : f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)} : (a_n)_{n \in \mathbb{Z}^k} \in l_\omega^p\left(\mathbb{Z}^k\right) \right\},$$

where $l_\omega^p\left(\mathbb{Z}^k\right)$ stands for the space of all sequences $(a_n)_{n \in \mathbb{Z}}$ with $a_n \in \mathbb{C}$ and

$$\sum_{n \in \mathbb{Z}^k} |a_n|^p \omega(n) < +\infty.$$

In $l_\omega^p\left(\mathbb{Z}^k\right)$, we introduce convolution multiplication given by:

$$a * b = \left\{ \sum_{i \in \mathbb{Z}} a_i b_{n-i} \right\}_n$$

and we suppose that there exists a constant $\gamma = \gamma(\omega) > 0$ such that:

$$\omega^{\frac{1}{1-p}} * \omega^{\frac{1}{1-p}} \leq \gamma \omega^{\frac{1}{1-p}}. \tag{5}$$

Then $l_\omega^p\left(\mathbb{Z}^k\right)$ becomes a Banach algebra ([6], Theorem 3.3). The space $\mathcal{A}_k^p(\omega)$ endowed with the norm $\|.\|_{k,p,\omega}$ defined by:

$$\|f\|_{k,p,\omega} = \left(\sum_{n \in \mathbb{Z}^k} |a_n|^p \omega(n) \right)^{\frac{1}{p}}, \text{ for every } f \in \mathcal{A}_k^p(\omega),$$

and with the classical pointwise multiplication, becomes a Banach algebra. In the sequel, we suppose:

$$\lim_{|n| \longrightarrow +\infty} (\omega(n))^{\frac{1}{n_j}} = 1, \quad \text{for every } j = 1, \ldots k. \tag{6}$$

where $|n| = n_1 + \cdots + n_k$, denotes the length of $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ and

$$\omega(n + m) \leq \omega(n)\omega(m), \text{ for every } n, m \in \mathbb{Z}^k \tag{7}$$

Recall that every character of the algebra $\mathcal{A}_k^p(\omega)$ is an evaluation at some $t^0 \in \mathbb{R}^k$ ([6], Theorem 3.3), where $t^0 = \left(t_1^0, \ldots, t_k^0\right)$ with $0 \leq t_j^0 < 2\pi$, for every $j = 1, \ldots, k$, and so,

$$Sp\left(\mathcal{A}_k^p(\omega)\right) = \left\{ \chi_t : t \in [0, 2\pi[^k \right\},$$

where $\chi_t(f) = f(t)$, for every $f \in \mathcal{A}_k^p(\omega)$, and

$$Sp(f) = \left\{ f(t) : t \in [0, 2\pi[^k \right\}.$$

Also the Jacobson radical of $\mathcal{A}_k^p(\omega)$, denoted by $Rad\left(\mathcal{A}_k^p(\omega)\right)$, is:

$$Rad\left(\mathcal{A}_k^p(\omega)\right) = \bigcap_{\chi \in Sp\left(\mathcal{A}_k^p(\omega)\right)} \ker \chi.$$

Whence $\mathcal{A}_k^p(\omega)$ is semi-simple, i.e., $Rad\left(\mathcal{A}_k^p(\omega)\right) = \{0\}$.

Using the fact that the spectrum of an element f of the algebra $\mathcal{A}_1^p(\omega)$ is nothing other than the set of values of f, we obtain the following generalization of P. Lévy theorem for holomorphic functions of several variables.

Theorem 4.5 (Multi-dimensional holomorphic version of P. Lévy theorem) *Let $p \in]1, +\infty[$ and ω be a weight on $\mathbb{Z}$ satisfying (5), (6) and (7). Let $f = (f_1, \ldots, f_k)$, where $f_j(t) = \sum\limits_{n \in \mathbb{Z}} a_{n,j} e^{int}$, where $\left(a_{n,j}\right)_{n \in \mathbb{Z}} \subset \mathbb{C}$, for $j = 1, \ldots, k$, is a periodic function such that:*

$$\|f_j\|_{p,\omega} = \left(\sum_{n \in \mathbb{Z}} \left|a_{n,j}\right|^p \omega(n) \right)^{\frac{1}{p}} < +\infty.$$

Let Ω be an open subset of $\mathbb{C}^k$ containing the image of the function f. Let $F \in \mathcal{H}(\Omega, \mathcal{A}_1^p(\omega))$. Then $F(f)$ also can be developed in a trigonometric series $F(f)(t) = \sum\limits_{n \in \mathbb{Z}} b_n e^{int}$, where $(b_n)_{n \in \mathbb{Z}} \subset \mathbb{C}$, such that:

$$\|F(f)\|_{p,\omega} = \left(\sum_{n \in \mathbb{Z}} |b_n|^p \omega(n) \right)^{\frac{1}{p}} < +\infty$$

and, for every $t \in \mathbb{R}$,

$$F(f)(t) = F\left(f_1(t), \ldots, f_k(t)\right)(t) = \sum_{n \in \mathbb{Z}} b_n e^{int}.$$

If moreover F is a holomorphic scalar function on Ω, then, for every $t \in \mathbb{R}$,

$$F\left(f_1(t), \ldots, f_k(t)\right) = \sum_{n \in \mathbb{Z}} b_n e^{int}.$$

Now we consider, in the algebra $\mathcal{A}_k^p(\omega)$, the algebra involution $f \longmapsto f^*$ defined by:

$$f^*(t) = \sum_{n \in \mathbb{Z}^k} \overline{a_{-n}} e^{i(n,t)}, \text{ for every } t \in \mathbb{R}^k.$$

Since the algebra $\mathcal{A}_k^p(\omega)$ is semi-simple, the involution is continuous ([2], Theorem 2, p.191). Moreover $\left(\mathcal{A}_k^p(\omega), \|.\|_{p,\omega}\right)$ is a hermitian algebra.

4.3 Another generalization of Wiener and Lévy theorems

We will now consider complex functions of several variables and analytic functional calculus for a single variable to give generalization of N. Wiener and P. Lévy theorems.

As an immediate consequence, we obtain the following multi-dimen-sional generalization of the N. Wiener theorem.

Theorem 4.6 (Multi-dimensional generalization of N. Wiener theorem) *Let* $p \in]1, +\infty[$ *and* ω *be a weight on* $\mathbb{Z}^k$ *satisfying* (5), (6) *and* (7). *Let* $f(t) = f(t_1, \ldots, t_k)$ *be a* 2π*-periodic function with respect to each variable, represented by a series*

$$f(t) = \sum_{n \in \mathbb{Z}^k} a_n e^{i(n,t)}$$

such that

$$\|f\|_{k,p,\omega} = \left(\sum_{n \in \mathbb{Z}^k} |a_n|^p \, \omega(n) \right)^{\frac{1}{p}} < +\infty.$$

If $f(t)$ *is invertible, for every* $t \in \mathbb{R}^k$, *then the function* f^{-1} *can be developed in a trigonometric series* $f^{-1}(t) = \sum\limits_{n \in \mathbb{Z}^k} b_n e^{i(n,t)}$, *where* $(b_n)_n$ *is a sequence in* $\mathcal{A}_k^p(\omega)$, *such that:*

$$\left\| f^{-1} \right\|_{k,p,\omega} = \left(\sum_{n \in \mathbb{Z}^k} |b_n|^p \, \omega(n) \right)^{\frac{1}{p}} < +\infty.$$

Using holomorphic functional calculus and Theorem 2.2, we also obtain as a consequence, the following multi-dimensional generalization of the Lévy theorem.

Theorem 4.7 (Multi-dimensional generalization of P. Lévy theorem) *Let* $p \in]1, +\infty[$ *and* ω *be a weight on* $\mathbb{Z}^k$ *satisfying* (5), (6) *and* (7). *Let* $f(t) = f(t_1, \ldots, t_k)$ *be a* 2π*-periodic* $\mathcal{A}_k^p(\omega)$*-valued function with respect to each variable, represented by a series* $f(t) = \sum\limits_{n \in \mathbb{Z}^k} a_n e^{i(n,t)}$, *such that*

$$\|f\|_{k,p,\omega} = \left(\sum_{n \in \mathbb{Z}^k} |a_n|^p \, \omega(n) \right)^{\frac{1}{p}} < +\infty.$$

Let Ω *be an open subset of* $\mathbb{C}^n$, $z^0 = \left(z_1^0, \ldots, z_n^0\right) \in \Omega$, $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ *such that* $\overline{D}^n\left(z^0, r\right) \subset \Omega$, *and* $f\left(\mathbb{R}^k\right) \subset D^n\left(z^0, r\right)$. *If* $F \in h(\Omega, \mathcal{A}_k^p(\omega))$, *then*

$$P[F](f) = \frac{1}{(2\pi)^n} \int_{T^n(z^0,r)} F(w) P(f, w) \frac{|dw_1|}{r_1} \cdots \frac{|dw_n|}{r_n}.$$

can be developed in a trigonometric series

$$P[F](f)(t) = F(f(t))(t) = \sum_{n \in \mathbb{Z}^k} b_n e^{i(n,t)},$$

such that:

$$\|P[F](f)\|_{k,p,\omega} = \left(\sum_{n \in \mathbb{Z}^k} |b_n|^p \, \omega(n) \right)^{\frac{1}{p}} < +\infty.$$

If moreover $F \in h(\Omega, \mathbb{C})$ is an n-harmonic scalar function on Ω, then, for every $t \in \mathbb{R}^k$,

$$P\,[F]\,(f)(t) = F(f(t)) = \sum_{n \in \mathbb{Z}^k} b_n e^{i(n,t)}.$$

Remark 4.8 Under the assumptions of the Theorem 4.7, if $F \in h\left(\Omega, \mathcal{A}_k^p(\omega)\right)$, then, by Proposition 3.2, we have:

$$\widehat{P\,[F]\,(f)}\,(\chi) = (\chi \circ F)\,(\chi(f))), \text{ for every } \chi \in Sp(A). \tag{8}$$

This implies that:

$$P\,[F]\,(f)(t) = F(f(t))(t), \text{ for every } t \in \mathbb{R}^k.$$

While if $F \in h(\Omega, \mathbb{C})$, then (8) becomes as follows:

$$\widehat{P\,[F]\,(f)}\,(\chi) = F\,(\chi(f))), \text{ for every } \chi \in Sp(A).$$

So, one has

$$P\,[F]\,(f)(t) = F(f(t)), \text{ for every } t \in \mathbb{R}^k.$$

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