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On estimates for the quaternion linear canonical transform in the space $L^2(\mathbb{R}^2, \mathcal{H})$

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Abstract
This paper is an exposition of some estimates which have a number of applications to interpolation theory. In particular some recent problems in image processing and singular integral operators require the computation of suitable estimates. In Abilov et al. (Comput Math Math Phys 48:2146, 2008) , Abilov et al. proved two useful estimates for the Fourier transform in the space of square integral multivariable functions on certain classes of functions characterized by the generalized continuity modulus, and these estimates are proved by Abilov for only two variables, using a translation operator. The purpose of this paper is to study these estimates for measurable sets from complex domain to hyper complex domain by using quaternion algebras, associated with the quaternion linear canonical transform, constructed by the generalized Steklov function.

Keywords Quaternion linear canonical transform · Lipschitz class · Dini–Lipschitz class · Titchmarsh theorem · Estimates

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1 Introduction

The integral Fourier transform, as well as Fourier series, are widely used in various fields of calculus, computational mathematics, mathematical physics, etc. Certain applications of

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this transform are described in a number of fundamental monographs (for example, see [25, 26]). Numerical estimates of the Fourier transform are presented in [30].

The classical linear canonical transform (LCT) is considered as a generalization of the Fourier transform (FT), and was first proposed in the 1970 s by Collins [9] and Moshinsky and Quesne [21]. It is an effective processing tool for chirp signal analysis, such as the parameter estimation, sampling progress for non bandlimited signals with nonlinear Fourier atoms [20], and the LCT filtering [27].

In this paper we will give our results in a more general context, that of quaternion linear canonical transform (QLCT). There are many studies in the literature that are concerned with the QLCT (see, for example, [5, 7, 16–18, 29]). They established some important properties of the QLCT, such as the uncertainty principle, the inversion formula and the study of generalized swept-frequency filters.

Recently several results of estimation have been proved in several different versions and for several different types of transforms (for example for the Fourier transform [3, 13], for the Bessel transform, [10], for the Dunkl transform [11], for the Laguerre Hypergroup transform [22]). In [1] the authors estimated the integral $\int_{|x|\geq R} |\mathcal{F}\{f\}(x)|^2 dx$ in certain classes of functions in $L^2(\mathbb{R}^n)$ where $\mathcal{F}\{f\}$ is the Fourier transform (FT) of f. Since the QLCT is a generalization of the FT, so for this reason we want in this paper, to estimate the integral $\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega$, where $\mathcal{L}_{A_1,A_2}\{f\}$ stands for the QLCT transform of f and $N \geq 1$.

In order to describe our results, we first need to introduce some facts about harmonic analysis related to the QLCT. We cite here, as briefly as possible, some properties. For more details we refer to [2, 6, 8, 12, 14, 16–18, 28, 29].

The quaternion algebra $\mathcal{H}$ was first invented by W. R. Hamilton in 1843 for extending complex numbers to a 4D algebra [24]. A quaternion $q \in \mathcal{H}$ can be written in this form

$$q = q_0 + \underline{q} = q_0 + iq_1 + jq_2 + kq_3$$

where i, j, k satisfy Hamilton's multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k.$$
$$jk = -kj = i, \quad ki = -ik = j.$$

Using Hamilton's multiplication rules, the multiplication of two quaternions $p = p_0 + \underline{p}$ and $q = q_0 + \underline{q}$ can be expressed as

$$pq = p_0q_0 + p_0\underline{q} + q_0\underline{p} + \underline{pq}.$$

We define the conjugation of $q \in \mathcal{H}$ by $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$. Clearly, $q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. So the modulus of a quaternion q is defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

In this paper, we study the quaternion-valued signal $f : \mathbb{R}^2 \to \mathcal{H}$ that can be expressed as

$$f(x) = f_0(x) + if_1(x) + jf_2(x) + kf_3(x)$$

where $x = x_1e_1 + x_2e_2 \in \mathbb{R}^2$ and $f_0,\ f_1,\ f_2$ and f_3 are real-valued functions. For $1 \leq r < \infty$, the quaternion modulus $L^r(\mathbb{R}^2, \mathcal{H})$ is defined as

$$L^r = L^r(\mathbb{R}^2, \mathcal{H}) = \{f/f : \mathbb{R}^2 \to \mathcal{H},\ \|f\|^r_{L^r(\mathbb{R}^2,\mathcal{H})} = \int_{\mathbb{R}^2} |f(x)|^r dx < \infty\}.$$

Let $f \in L^r(\mathbb{R}^2, \mathcal{H})$. The quaternion Fourier transform (QFT) of f is defined by

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix_1\omega_1} f(x) e^{-jx_2\omega_2} dx.$$

The inner product of $f, g \in L^2(\mathbb{R}^2, \mathcal{H})$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(x)\overline{g(x)} dx.$$

Clearly, $\|f\|_2^2 = \langle f, f \rangle$. For all $\theta \in \mathbb{R}$ we have

$$|e^{i\theta}| = |e^{j\theta}| = 1. \tag{1}$$

Now, we define a norm of $\mathcal{F}(f)$ as

$$|\mathcal{F}(f)(\omega)|_Q = \left(|\mathcal{F}(f_0)(\omega)|^2 + |\mathcal{F}(f_1)(\omega)|^2 + |\mathcal{F}(f_2)(\omega)|^2 + |\mathcal{F}(f_3)(\omega)|^2\right)^{1/2}.$$

Furthermore, we obtain the $L^r(\mathbb{R}^2, \mathcal{H})$-norm

$$\|\mathcal{F}(f)\|_{Q,r} = \left(\int_{\mathbb{R}^2} |\mathcal{F}(f)(\omega)|_Q^r d^2\omega\right)^{1/r}.$$

For $f \in L^1(\mathbb{R}^2, \mathcal{H})$, we have

$$\|\mathcal{F}(f)\|_{Q,\infty} \le \|f\|_1. \tag{2}$$

(QFT Plancherel) If $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^2(\mathbb{R}^2, \mathcal{H})$, then

$$\int_{\mathbb{R}^2} |f(x)|^2 dx = \int_{\mathbb{R}^2} |\mathcal{F}(f)(\omega)|^2 d\omega. \tag{3}$$

Moreover,

$$\int_{\mathbb{R}^2} |f(x)|^2 dx = \int_{\mathbb{R}^2} |\mathcal{F}(f)(\omega)|_Q^2 d\omega,$$

then we can rewrite the QFT Plancherel as follows

$$\|\mathcal{F}(f)\|_{Q,2} = \|f\|_2. \tag{4}$$

Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathcal{F}\{f\}(\omega)|_Q^2 \, d\omega &= \int_{\mathbb{R}^2} \left(|\mathcal{F}\{f_0\}(\omega)|^2 + |\mathcal{F}\{f_1\}(\omega)|^2\right. \\ &\quad \left.+ |\mathcal{F}\{f_2\}(\omega)|^2 + |\mathcal{F}\{f_3\}(\omega)|^2\right) d\omega \\ &= \left(\int_{\mathbb{R}^2} \left|\mathcal{F}_{\mathrm{Q}}\{f_0\}(\omega)\right|^2 d\omega + \int_{\mathbb{R}^2} \left|\mathcal{F}_{\mathrm{Q}}\{f_1\}(\omega)\right|^2 d\omega\right. \\ &\quad \left.+ \int_{\mathbb{R}^2} \left|\mathcal{F}_{\mathrm{Q}}\{f_2\}(\omega)\right|^2 d\omega + \int_{\mathbb{R}^2} |\mathcal{F}\{f_3\}(\omega)|^2 d\omega\right)\end{aligned}$$

Applying 3 into the right-hand side of the above identity gives

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathcal{F}\{f\}(\omega)|_Q^2 \, d\omega &= \int_{\mathbb{R}^2} |f_0(x)|^2 \, dx + \int_{\mathbb{R}^2} |f_1(x)|^2 \, dx \\ &\quad + \int_{\mathbb{R}^2} |f_2(x)|^2 \, dx + \int_{\mathbb{R}^2} |f_3(x)|^2 \, dx\end{aligned}$$

Since $f_i(x)$, $i = 0, 1, 2, 3$, is real-valued, the above equation can be written in the form

$$\int_{\mathbb{R}^2} |\mathcal{F}\{f\}(\omega)|_Q^2 \, d\omega = \int_{\mathbb{R}^2} \left(f_0^2(x) + f_1^2(x) + f_2^2(x) + f_3^2(x)\right) dx.$$

Suppose that $\mathcal{F}(f) \in L^1(\mathbb{R}^2, \mathcal{H})$ and $\mathcal{F}(\frac{\partial^n f}{\partial x_1^n}) \in L^1(\mathbb{R}^2, \mathcal{H})$. Then

$$\mathcal{F}(\frac{\partial^n f}{\partial x_1^n} i^{-n})(w) = w_1^n \mathcal{F}(f)(w), \quad \forall n \in \mathbb{N}. \tag{5}$$

Moreover, if $\mathcal{F}(f) \in L^1(\mathbb{R}^2, \mathcal{H})$ and $\mathcal{F}(\frac{\partial^m f}{\partial x_2^m}) \in L^1(\mathbb{R}^2, \mathcal{H})$, then

$$\mathcal{F}(\frac{\partial^m f}{\partial x_2^m})(w) = \mathcal{F}(f)(w)(j w_2)^m, \quad \forall m \in \mathbb{N}. \tag{6}$$

Let $A_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a real matrix parameter such that $\det(A_s) = 1$, for $s = 1, 2$. The two-sided (sandwich) QLCT of $f \in L^1(\mathbb{R}^2, \mathcal{H})$ is defined by

$$\mathcal{L}_{A_1,A_2}\{f\}(\omega) = \int_{\mathbb{R}^2} K_{A_1}^i(x_1, \omega_1) f(x) K_{A_2}^j(x_2, \omega_2) dx,$$

where the kernel functions of the QLCT above are given by

$$K_{A_1}^i(x_1, \omega_1) = \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{(i/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega_1 + (d_1/b_1)\omega_1^2 - (\pi/2))} & \text{for } b_1 \neq 0, \\ \sqrt{d_1} e^{i(c_1 d_1/2)\omega_1^2} & \text{for } b_1 = 0. \end{cases} \tag{7}$$

$$K_{A_2}^j(x_2, \omega_2) = \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{(j/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega_2^2 - (\pi/2))} & \text{for } b_2 \neq 0, \\ \sqrt{d_2} e^{j(c_2 d_2/2)\omega_2^2} & \text{for } b_2 = 0. \end{cases} \tag{8}$$

Let the kernel function K_A be defined by (7) or (8). Then

- $K_A(-x, \omega) = K_A(x, -\omega)$.
- $K_A(-x, -\omega) = K_A(x, \omega)$.
- $\overline{K_A(x, \omega)} = K_A^{-1}(\omega, x)$.

From the definition of the QLCT, we can easily see that when $b_1 b_2 = 0$ and $b_1 = b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. Therefore, in this work, we always assume $b_1 b_2 \neq 0$.

(**Inversion formula**) The (Two-sided) inverse quaternion linear canonical transform of $g \in L^1(\mathbb{R}^2, \mathcal{H})$

$$\mathcal{L}_{A_1,A_2}^{-1}\{g\}(\omega) = \int_{\mathbb{R}^2} K_{A_1^{-1}}^i(x_1, \omega_1) g(x) K_{A_2^{-1}}^j(x_2, \omega_2) dx. \tag{9}$$

(**Hausdorff–Young inequality**) If $1 \le r < 2$ and letting r' be such that $1/r + 1/r' = 1$ then for all $f \in L^r(\mathbb{R}^2, \mathcal{H})$ it holds that

$$\|\mathcal{L}_{A_1,A_2}\{f\}\|_{Q,r'} \le \frac{|b_1 b_2|^{-1/2+1/r'}}{2\pi} \|f\|_r. \tag{10}$$

(**Plancherel theorem of QLCTs**) Let $f \in L^2(\mathbb{R}^2, \mathcal{H})$. Then

$$\|\mathcal{L}_{A_1,A_2}\{f\}\|_{Q,2} = \|f\|_2.$$

(**Shift property**) For a quaternion function $f \in L^1(\mathbb{R}^2, \mathcal{H})$, we denote by $\tau_k f(x)$ the shifted (translated) function defined by $\tau_k f(x) = f(x-k)$, where $k = k_1 e_1 + k_1 e_1 \in \mathbb{R}^2$. Then we obtain

$$\mathcal{L}_{A_1,A_2}\{\tau_k f\}(\omega) = e^{-ia_1c_1k_1^2/2+ic_1k_1\omega_1} \times \mathcal{L}_{A_1,A_2}\{f\}(\omega_1 - a_1k_1, \omega_2 - a_2k_2)e^{-ja_2c_2k_2^2/2+jc_2k_2\omega_2}. \quad (11)$$

(**Modulation property**). We define a modulation operator $\mathbb{M}_{\omega_0} f$ by

$$\mathbb{M}_{\omega_0} f(x) = e^{ix_1u_0} f(x) e^{jx_2v_0}$$

with $\omega_0 = u_0 e_1 + v_0 e_1$. So

$$\mathcal{L}_{A_1,A_2}\{\mathbb{M}_{\omega_0} f\}(\omega) = \mathcal{L}_{A_1,A_2}\{e^{ix_1u_0} f(x) e^{jx_2v_0}\}(\omega) = e^{-ib_1d_1u_0^2/2+id_1u_0\omega_1}\mathcal{L}_{A_1,A_2}\{f\}(\omega_1 - u_0b_1, \omega_2 - v_0b_2)e^{-jb_2d_2v_0^2/2+jd_2v_0\omega_2}. \quad (12)$$

(**Time-frequency shift**). Let a quaternion function $f \in L^1(\mathbb{R}^2, \mathcal{H})$. Then we obtain that

$$\begin{aligned}&\mathcal{L}_{A_1,A_2}\{\mathbb{M}_{\omega_0}\tau_k f\}(\omega)\\ &= \mathcal{L}_{A_1,A_2}\{e^{ix_1u_0} f(x-k) e^{jx_2v_0}\}(\omega)\\ &= e^{-i(a_1c_1k_1^2+b_1d_1u_0^2)/2+i(c_1k_1+d_1u_0)\omega_1 - ib_1c_1k_1u_0}\\ &\quad \mathcal{L}_{A_1,A_2}\{f\}(\omega_1 - a_1k_1 - u_0b_1, \omega_2 - a_2k_2 - v_0b_2)\\ &\quad e^{-j(a_2c_2k_2^2+b_2d_2v_0^2)/2+j(c_2k_2+d_2v_0)\omega_2 - jb_2c_2k_2v_0}. \end{aligned} \quad (13)$$

For a function f on $L^1(\mathbb{R}^2, \mathcal{H})$ and for any $h_1, h_2 \in \mathbb{R}$, we define the operator Δ_{h_1,h_2} by

$$\Delta_{h_1,h_2} f(x) = e^{i\frac{a_1h_1}{b_1}x_1} f(x_1+h_1, x_2+h_2)e^{j\frac{a_2h_2}{b_2}x_2} - e^{i\frac{a_1h_1}{b_1}x_1} f(x_1+h_1, x_2) \quad (14)$$
$$- f(x_1, x_2+h_2)e^{j\frac{a_2h_2}{b_2}x_2} + f(x_1, x_2).$$

Definition 1.1 Let $f(x) = f(x_1, x_2)$ belongs to $L^2(\mathbb{R}^2, \mathcal{H})$. We say that f is in the Lipschitz space $\mathrm{Lip}_{A_1,A_2}(\alpha_1, \alpha_2)$ if

$$\|\Delta_{h_1,h_2} f(x)\|_2 = O(h_1^{\alpha_1} h_2^{\alpha_2}), \quad (15)$$

as h_1, h_2 tend to zero, $0 < \alpha_1, \alpha_2 \leq 1$.

In $L^2(\mathbb{R}^2, \mathcal{H})$, consider the operator

$$F_h^{A_1,A_2} f(x_1, x_2) = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} e^{i\frac{a_1\xi}{b_1}x_1 + i\frac{a_1\xi^2}{2b_1}} f(x_1+\xi, x_2+\eta) \times e^{j\frac{a_2\eta}{b_2}x_2 + j\frac{a_2\eta^2}{2b_2}} d\xi d\eta, \quad h > 0. \quad (16)$$

Observe that if $a_1 = a_2 = 0$, then

$$F_h^{A_1,A_2} f(x_1, x_2) = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} f(x_1+\xi, x_2+\eta) d\xi d\eta.$$

This is analogous to the Steklov operator.

Let the function $f \in L^2(\mathbb{R}^2, \mathcal{H})$. The finite differences of the order m ($m \in 1, 2, 3, \ldots$) are defined as follows:

$$\Delta_h^m f(x_1, x_2) = (I - F_h^{A_1,A_2})^m f(x_1, x_2),$$

here I is the unit operator, and the mth order generalized continuity modulus of the function f is defined by the formula

$$w_m(f, \delta)_2 = \sup_{0<h\leq\delta} \|\Delta_h^m f\|_2,$$

where $\delta > 0$.
For a function f on $L^2(\mathbb{R}^2, \mathcal{H})$, we define the function g_f by

$$g_f(x) = e^{-i(a_1/2b_1)x_1^2} f(x) e^{-j(a_2/2b_2)x_2^2}.$$

From the definition of g_f, we easily obtain

$$\mathcal{F}\{g_f\}^2(w) = \sqrt{b_1 b_2} e^{-i(d_1/2b_1)w_1^2 + i\pi/4} \mathcal{L}_{A_1,A_2}\{f\}(b_1 w_1, b_2 w_2) e^{-j(d_2/2b_2)w_2^2 + j\pi/4}. \tag{17}$$

Consider in $L^2(\mathbb{R}^2, \mathcal{H})$ the operator

$$D_{A_1,A_2} f(x) = e^{-i(a_1/2b_1)x_1^2} D g_f(x) e^{-j(a_2/2b_2)x_2^2}, \tag{18}$$

where $D = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, $D^0_{A_1,A_2} f = f$, $D^r_{A_1,A_2} f = D_{A_1,A_2}(D^{r-1}_{A_1,A_2} f)$, $r = 1, 2, \ldots$.
In view of formulas (5), (6) and (18), we have

$$\mathcal{L}_{A_1,A_2}\{D_{A_1,A_2} f\}^2(w) = -\left(\left(\frac{w_1}{b_1}\right)^2 + \left(\frac{w_2}{b_2}\right)^2\right) \mathcal{L}_{A_1,A_2}\{f\}^2(w),$$

and hence

$$\mathcal{L}_{A_1,A_2}\{D^r_{A_1,A_2} f\}^2(w) = (-1)^r \left(\left(\frac{w_1}{b_1}\right)^2 + \left(\frac{w_2}{b_2}\right)^2\right)^r \mathcal{L}_{A_1,A_2}\{f\}^2(w). \tag{19}$$

Denote by $W^{2,k}_{2,\phi}(\mathbb{R})$ the class of functions $f \in L^2(\mathbb{R}^2, \mathcal{H})$ having the generalized derivatives $\dfrac{\partial f}{\partial x_1}$, $\dfrac{\partial^2 f}{\partial x_1 \partial x_2}$,…in the sense of Levi (see [19, 23]) in $L^2(\mathbb{R}^2, \mathcal{H})$ are estimated by

$$\omega_k(D^r f; \delta) = O(\phi(\delta^k)).$$

where $\phi(t)$ is a continuous steadily increasing function on $[0, +\infty)$ and $\phi(0) = 0$.

2 Some new estimates for quaternion linear canonical transform

In order to prove the main result, we shall need some preliminary results.

Lemma 2.1 *If f belongs to $L^2(\mathbb{R}^2, \mathcal{H})$, then*

$$\|\Delta_h^m f\|_2 \leq 2^m \|f\|_2. \tag{20}$$

Proof Using the inequality (16), we have $\|F_h^{A_1,A_2}(f)\|_2 \leq \|f\|_2$. Then $\|\Delta_h^1 f\|_2 \leq 2\|f\|_2$. Thus the result follows easily by using the recurrence for m. □

Lemma 2.2 *If quaternion function $f \in L^2(\mathbb{R}^2, \mathcal{H})$, then*

$$\mathcal{L}_{A_1,A_2}\{F_h^{A_1,A_2} f\}^2(w) = \frac{\sin(w_1 h/b_1)}{w_1 h/b_1} \frac{\sin(w_2 h/b_2)}{w_2 h/b_2} \mathcal{L}_{A_1,A_2}\{f\}^2(w). \tag{21}$$

Proof Let $f \in L^2(\mathbb{R}^2, \mathcal{H})$. Taking into account the formula (1), we have

$$\begin{aligned}
J &= \mathcal{L}_{A_1,A_2}\{F_h^{A_1,A_2} f\}^2(w) \\
&= \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} e^{i\frac{a_1\xi^2}{2b_1}} \mathcal{L}_{A_1,A_2}\{e^{i\frac{a_1\xi}{b_1}x_1} f(x_1+\xi, x_2+\eta) e^{j\frac{a_2\eta}{b_2}x_2}\}^2(w) e^{j\frac{a_2\eta^2}{2b_2}} d\xi d\eta \\
&= \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} e^{i\frac{a_1\xi^2}{2b_1}} e^{i\frac{\omega_1 h_1}{b_1} - i\frac{a_1\xi^2}{2b_1}} \mathcal{L}_{A_1,A_2}\{f\}(\omega) e^{j\frac{\omega_2 h_2}{b_2} - j\frac{a_2\eta^2}{2b_2}} e^{j\frac{a_2\eta^2}{2b_2}} d\xi d\eta \\
&= \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} e^{i\frac{\omega_1\xi}{b_1}} \mathcal{L}_{A_1,A_2}\{f\}(\omega) e^{j\frac{\omega_2\eta}{b_2}} d\xi d\eta \\
&= \left(\frac{1}{2h} \int_{-h}^{h} e^{i\frac{\omega_1\xi}{b_1}} d\xi\right) \mathcal{L}_{A_1,A_2}\{f\}(\omega) \left(\frac{1}{2h} \int_{-h}^{h} e^{j\frac{\omega_2\eta}{b_2}} d\eta\right).
\end{aligned}$$

It is easily seen that

$$\frac{1}{2h} \int_{-h}^{h} e^{i\frac{\omega_1\xi}{b_1}} d\xi = \frac{\sin(w_1 h/b_1)}{w_1 h/b_1}$$

and

$$\frac{1}{2h} \int_{-h}^{h} e^{j\frac{\omega_2\eta}{b_2}} d\eta = \frac{\sin(w_2 h/b_2)}{w_2 h/b_2}.$$

So that the transform of $F_h^{A_1,A_2} f(x)$ is given as

$$\frac{\sin(w_1 h/b_1)}{w_1 h/b_1} \frac{\sin(w_2 h/b_2)}{w_2 h/b_2} \mathcal{L}_{A_1,A_2}\{f\}^2(w).$$

This completes the proof. □

Corollary 2.3 *For any function f in $L^2(\mathbb{R}^2, \mathcal{H})$, we have*

$$\mathcal{L}_{A_1,A_2}\{\Delta_h^m f\}^2(w) = \left(1 - \frac{\sin(w_1 h/b_1)}{w_1 h/b_1} \frac{\sin(w_2 h/b_2)}{w_2 h/b_2}\right)^m \mathcal{L}_{A_1,A_2}\{f\}^2(w). \tag{22}$$

In the next, in order to describe our results we will use the following notation:

- $G = \{(w_1, w_2) : (\frac{w_1}{b_1})^2 + (\frac{w_2}{b_2})^2 \geq N^2\}$.
- $|w| = (\frac{w_1}{b_1})^2 + (\frac{w_2}{b_2})^2$.
- $\varphi_h(\frac{w_1}{b_1}, \frac{w_2}{b_2}) = \frac{\sin(w_1 h/b_1)}{w_1 h/b_1} \frac{\sin(w_2 h/b_2)}{w_2 h/b_2}$.
- $I_G = \left(\int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)$.

In the following result, we estimate the integral

$$\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega$$

in certain classes of functions in $L^2(\mathbb{R}^2, \mathcal{H})$.

Theorem 2.4 *For functions $f \in L^2(\mathbb{R}^2, \mathcal{H})$ in the class $W^{2,k}_{2,\phi}$,*

$$\sup_{f\in W^{2,k}_{2,\phi}} \left(\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{\frac{1}{2}} = O\left(N^{-2r}\phi^2((\frac{\pi}{4N})^k)\right)$$

where $r = 1, \ldots$; $k = 1, 2, \ldots$; and $\phi(t)$ is any nonnegative function defined on the interval $[0, \infty)$.

Proof Let $f \in W^{2,k}_{2,\phi}$. Thanks to Hölder's inequality, we obtain

$$\begin{aligned}
J^{'} &= \int_G \left(1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \\
&= \int_G \left(1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^{\frac{1}{k}} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^{2-\frac{1}{k}} d^2\omega \\
&\leq \left(\int_G \left(\left|1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{\frac{1}{2k}} \left(\int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{1-\frac{1}{2k}} \\
&= \left(\int_G |w|^{2r} \frac{\left(\left|1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k}}{|w|^{2r}} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{\frac{1}{2k}} \left(\int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{1-\frac{1}{2k}} \\
&\leq N^{-\frac{2r}{k}} \left(\int_G |w|^{2r} \left(\left|1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{\frac{1}{2k}} I_G^{1-\frac{1}{2k}} \qquad (23)
\end{aligned}$$

where $I_G = \left(\int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)$.

In view of Holder inequality, we have that

$$\int_G |w|^{2r} \left(\left|1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \leq \|\Delta_h^k D^r f(x)\|_2^2. \qquad (24)$$

Using the inequalities (23) and (24),

$$\int_G \left(1-\varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \leq N^{-\frac{2r}{k}} \|\Delta_h^k D^r f(x)\|_2^{\frac{1}{k}} I_G^{1-\frac{1}{2k}}$$

Now, let us estimate the integral

$$\int_G \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega))|^2 d^2\omega.$$

It is easy to see that, for this purpose, it is sufficient to consider the domain of integration

$$E = \left\{(w_1, w_2) : \frac{w_1}{b_1} \geq 0, \frac{w_2}{b_2} \geq 0, |w| \geq N^2\right\}$$

Divide the domain E into the two subdomains

$$E_1 = \left\{(w_1, w_2) : \frac{w_1}{b_1} \geq \frac{w_2}{b_2},\ |w| \geq N^2\right\} \text{ and } E_2 = \left\{(w_1, w_2) : \frac{w_1}{b_1} < \frac{w_2}{b_2},\ |w| \geq N^2\right\}.$$

Then,

$$\begin{aligned} I &= \int_E \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \\ &= \int_{E_1} \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega + \int_{E_2} \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \end{aligned}$$

Since $|\sin x| \leq |x|$, $\frac{w_1}{b_1} \geq \frac{N}{\sqrt{2}}((w_1, w_2) \in E_1)$ and $\frac{w_2}{b_2} \geq \frac{N}{\sqrt{2}}((w_1, w_2) \in E_2)$ it is clear that

$$\begin{aligned} |I| &= \int_{E_1} \left|\frac{\sin(w_1 h/b_1)}{w_1 h/b_1}\right| |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega + \int_{E_2} \left|\frac{\sin(w_2 h/b_2)}{w_2 h/b_2}\right| |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \\ &\leq \frac{\sqrt{2}}{Nh}\left(\int_{E_1} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega + \int_{E_2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\ &\leq \frac{\sqrt{2}}{Nh} \int_E |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \\ &\leq \frac{\sqrt{2}}{Nh} \int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \end{aligned}$$

Consequently,

$$\int_G \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right) |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \leq N^{-\frac{2r}{k}} \|\Delta_h^k D^r f(x)\|_2^{\frac{1}{k}} I_G^{1-\frac{1}{2k}} + \frac{4\sqrt{2}}{Nh} I_G.$$

Setting $h = \frac{\pi^2}{N}$.

$$\left(1 - \frac{4\sqrt{2}}{Nh}\right) \int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \leq N^{-\frac{2r}{k}} \|\Delta_h^k D^r f(x)\|_2^{\frac{1}{k}} I_G^{1-\frac{1}{2k}}.$$

Hence,

$$\int_G |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega = O\left(N^{-4r} \|\Delta_h^k D^r f(x)\|_2^2\right)$$

and we have

$$\|\Delta_h^k D^r f(x)\|_2^2 = O\left(N^{-2r} \phi\left[\left(\frac{\pi^2}{N}\right)^k\right]\right),$$

which yields the desired result. □

Theorem 2.5 *Let $\phi(t) = t^\alpha$ ($\alpha > 0$). Then the next conditions are equivalent:*

$$f \in W_{2,t^\alpha}^{2,k} \tag{25}$$

and

$$\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega = O(N^{-2r-2k\alpha}) \tag{26}$$

where $m = 1, 2 \ldots .$; $r = 1, \ldots .$; $k = 1, 2, \ldots .$; *and* $0 < \alpha < m$.

Proof It follows from Theorem 2.4 that (25) entails (26).
Suppose now that

$$\left(\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)^{\frac{1}{2}} = O(N^{-2r-k\alpha}).$$

Thus, by Parseval's identity, we have

$$\|\Delta_h^k D^r f(x)\|_2^2 = \int_{\mathbb{R}^2} |\omega|^{2r} \left(\left|1 - \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega.$$

Divide this integral into two, $\underbrace{\int_{\mathbb{R}^2}} = \underbrace{\int_{|\omega|<N^2}}_{I_1} + \underbrace{\int_{|\omega|>N^2}}_{I_2}$, where $N = [h^{-1}]$, and estimate each of them. Firstly, we estimate I_2, since

$$\left|1 - \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right| \leq 2,$$

it follows that

$$\begin{aligned}
I_2 &= \int_{|\omega|>N^2} |\omega|^{2r} \left(\left|1 - \varphi_h\left(\frac{w_1}{b_1}, \frac{w_2}{b_2}\right)\right|\right)^{2k} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega \\
&\leq \int_{|\omega|>N^2} |\omega|^{2r}\, (2)^{2k}\, |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega. \\
&= O\left(\int_{|\omega|>N^2} |\omega|^{2r} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\
&= O\left(\sum_{l=0}^{\infty} \int_{(N+l)^2\leq|\omega|<(N+l+1)^2} |\omega|^{2r} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\
&= O\left(\sum_{l=0}^{\infty} (N+l+1)^{4r} \int_{N^2\leq|\omega|} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\
&\quad - \left(\sum_{l=0}^{\infty} (N+l+1)^{4r} \int_{|\omega|\geq(N+l+1)^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\
&= O\left(N^{4r} \int_{N^2\leq|\omega|} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)| d^2\omega\right) \\
&\quad + \left(\sum_{l=0}^{\infty} [(N+l+1)^{4r} - (N+l)^{4r}] \int_{|\omega|\geq(N+l+1)^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right) \\
&= O\left(N^{4r} \int_{N^2\leq|w|} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)| d^2\omega\right) \\
&\quad + \left(\sum_{l=0}^{\infty} (N+l)^{4r-1} \int_{(N+l)^2\leq|\omega|} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega\right)
\end{aligned}$$

$$
\begin{aligned}
&= O\left(N^{4r}N^{-4r-2k\alpha}+\sum_{l=0}^{\infty}(N+l)^{4r-1}(N+l)^{-4r-2k\alpha}\right)\\
&= O(N^{-2k\alpha})+O(N^{-2k\alpha})\\
&= O(N^{-2k\alpha})\\
&= O(h^{2k\alpha}),
\end{aligned}
$$

i.e.,

$$I_2 = O(h^{2k\alpha}).$$

Secondly, we estimate I_1, since

$$0 \le 1-\frac{\sin(y)}{y} \le \frac{y^2}{6}, \quad |\sin(y)| \le |y|, \quad y\in\mathbb{R},$$

and

$$1-\varphi_h\left(\frac{w_1}{b_1},\frac{w_2}{b_2}\right)=\left(1-\frac{\sin(w_1h)}{w_1h}\right)\frac{\sin(w_2h)}{w_2h}+\left(1-\frac{\sin(w_2h)}{w_2h}\right),$$

$$
\begin{aligned}
I_1 &= \int_{|\omega|<N^2}|\omega|^{2r}\left(\left|1-\varphi_h\left(\frac{w_1}{b_1},\frac{w_2}{b_2}\right)\right|\right)^{2k}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\\
&\le \int_{|\omega|<N^2}|\omega|^{2r}\,(|\omega|)^{2k}\,|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\\
&= O(h^{4k})\int_{|\omega|<N^2}|\omega|^{2r+2k}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\\
&= O(h^{4k})\sum_{n=0}^{N}\int_{n^2\le|\omega|\le(n+1)^2}|\omega|^{2r+2k}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\\
&= O(h^{4k})\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{n^2\le|\omega|\le(n+1)^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\\
&= O(h^{4k})\sum_{n=0}^{N}(n+1)^{4r+4k}\left[\int_{|\omega|\ge n^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]\\
&\quad -\left[\int_{|\omega|\ge(n+1)^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]\\
&= O(h^{4k})\left[\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{|\omega|\ge n^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]\\
&\quad -\left[\sum_{n=0}^{N}(n+1)^{4r+4k}\int_{|\omega|\ge(n+1)^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]\\
&= O(h^{4k})\left[1+\sum_{n=0}^{N}[(n+1)^{4r+4k}-n^{4r+4k}]\int_{|\omega|\ge n^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]\\
&= O(h^{4k})\left[1+\sum_{n=0}^{N}(n+1)^{4r+4k-1}\int_{|\omega|\ge n^2}|\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2d^2\omega\right]
\end{aligned}
$$

$$= O(h^{4k})\left[1+\sum_{n=1}^{N} n^{4r+4k-1}n^{-4r-2k\alpha}\right] = O(h^{4k})\left[1+\sum_{n=1}^{N} n^{4k-2k\alpha-1}\right]$$
$$= O(h^{4k})O(N^{4k-2k\alpha}) = O(h^{2k\alpha});$$

i.e.,

$$I_1 = O(h^{2k\alpha}).$$

Finally, combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k D^r f(x)\|_2 = O(h^{k\alpha}),$$

which means that $f \in W^{2,k}_{2,t^\alpha}$. Hence the conditions (25) and (26) are equivalent. This proves the Theorem 2.5. □

Remark As in the article [15], the previous definition can be generalized as follows: For any two pure quaternions α and β such that $\alpha^2 = \beta^2 = -1$ used for re placing i and j in (7) and (8), and f in $L^1(\mathbb{R}^2, \mathcal{H})$

$$\mathcal{L}^{\alpha,\beta}_{A_1,A_2}\{f\}(\omega) = \int_{\mathbb{R}^2} K^{\alpha}_{A_1}(x_1,\omega_1) f(x) K^{\beta}_{A_2}(x_2,\omega_2)dx.$$

From linearity of $\mathcal{L}^{\alpha,\beta}_{A_1,A_2}$ we obtain the QLCT for the OPS split $f = f_+ + f_-$ where $f_\pm = \frac{1}{2}(f \pm \alpha f \beta)$

$$\begin{aligned}\mathcal{L}^{\alpha,\beta}_{A_1,A_2}\{f\}(\omega) &= \int_{\mathbb{R}^2} K^{\alpha}_{A_1}(x_1,\omega_1) f(x) K^{\beta}_{A_2}(x_2,\omega_2)dx \\ &= \int_{\mathbb{R}^2} K^{\alpha}_{A_1}(x_1,\omega_1) f_+(x) K^{\beta}_{A_2}(x_2,\omega_2)dx \\ &\quad + \int_{\mathbb{R}^2} K^{\alpha}_{A_1}(x_1,\omega_1) f_-(x) K^{\beta}_{A_2}(x_2,\omega_2)dx\end{aligned}$$

so what was done above for the integral $\int_{|\omega|\geq N^2} |\mathcal{L}_{A_1,A_2}\{f\}(\omega)|^2 d^2\omega$ can be done again for the two integrals $\int_{|\omega|\geq N^2} |\mathcal{L}^{\alpha,\beta}_{A_1,A_2}\{f_+\}(\omega)|^2 d^2\omega$ and $\int_{|\omega|\geq N^2} |\mathcal{L}^{\alpha,\beta}_{A_1,A_2}\{f_-\}(\omega)|^2 d^2\omega$ so that we can find a generalization of our results.

3 Conclusion

In this paper after having given two estimates for the quaternion linear canonical transform which generalize those of Abilov [1] we note so far the difficulty lies only in the fact that for quaternion fields we have no commutativity, whereas for the Fourier transform this does not pose a problem, but even for quaternion fields there is a question which arises, if we can have the same approximation for the right-sided Quaternion linear canonical transform [4], the answer is positive although there will be a slight difference concerning the calculations of certain steps.

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