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Indirect boundary stabilization of strongly coupled degenerate hyperbolic systems

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Abstract
In this paper, we consider the energy decay of a damped hyperbolic system of two degenerate wave equations coupled by velocities when only one equation is directly damped by a linear boundary feedback. To this aim, we first prove that the proposed system is well-posed using the semigroup theory. Then, under the hypothesis that the coupling coefficient is positive and small, we show that the total energy of the whole system decays exponentially. The explicit energy decay rate is established by using the energy multiplier method.

Keywords Coupled degenerate wave equations · Stabilization · Exponential decay · Multiplier techniques

Mathematics Subject Classification 35B40 · 35L80 · 93D15 · 93D23

1 Introduction

The stabilization problems for scalar wave equations have received considerable attention in the literature, with numerous contributions achieved over the past several years. We refer, for example, to [1–16] and the articles citing them.

Recently, the subject of indirect stabilization of coupled wave equations has received a lot of attention of many authors. This notion introduced by Russell [17] concerns stabilization questions for coupled equations with a reduced number of feedbacks. This means that some equations of the coupled system are not directly damped, but one then hopes that the coupling effects will be sufficient so that the full system is stabilized. This kind of damped systems

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is very important from the applications in control theory point of view, since it may be impossible or too expensive to damp each equation because of engineering or biological constraints.

For this reason, the theory of indirect stabilization for coupled wave equations has been investigated extensively. See, in particular [18–34] and the rich references therein. We also refer to the two monographs [35] and [36] for a comprehensive review.

However, all the previous results concern nondegenerate problems. On the other hand, from [37, Theorem 4.5], we know that the linearly damped scalar degenerate wave equation is exponentially stable.

In this paper, the question we are interested in is to determine if it is still possible to achieve the exponential stability of a system of two degenerate waves by means of only one damping.

More precisely, for given $\beta > 0$, we investigate the stabilization of the following model of coupled degenerate wave equations with only one boundary damping:

$$\begin{cases} v_{tt} - (a(x)v_x)_x + bu_t = 0, & \text{in } (0, +\infty) \times (0, 1), \\ u_{tt} - (a(x)u_x)_x - bv_t = 0, & \text{in } (0, +\infty) \times (0, 1), \\ Bv(0) = Bu(0) = 0, & \text{on } (0, +\infty), \\ v(1) = 0, & \text{on } (0, +\infty), \\ u_t(1) + u_x(1) + \beta u(1) = 0, & \text{on } (0, +\infty), \\ v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), & \text{in } (0, 1), \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & \text{in } (0, 1), \end{cases} \tag{1}$$

where $a \in C([0, 1]) \cap C^1((0, 1])$ is positive on $]0, 1]$ but vanishes at zero, $b > 0$ is the coupling parameter and

$$Bz(t, x) := \begin{cases} z(t, x), & \text{in the case,} \quad 0 < \mu_a < 1, \\ (a(x)z_x)(t, x), & \text{in the case,} \quad 1 \leq \mu_a < 2. \end{cases}$$

In view of the results in [38], since the coupling acts here in a stronger way (through velocities), we are interested in finding conditions on the parameters of the system such that the energy of this linearly damped system decays exponentially. Indeed, different from the case of couplings through displacements [18–20], it is shown in [38] that the damping properties are fully transferred from the damped equation to the undamped one by the coupling in velocities.

Here, in agreement with [38], our main result (see theorem 9) asserts that a single feedback is sufficient to guarantee that the energy of the full system (1) decays exponentially to 0 at infinity. This extends the energy decay result in [37] for the single degenerate wave equation to the system of two degenerate wave equations which are coupled through the velocities.

To the best of our knowledge, this is the first paper where the asymptotic behavior of solutions to strongly coupled degenerate wave equations is studied with the main particularity that only one equation is effectively damped by a boundary feedback acting on one end only.

In order to study the damped system (1), we will assume that the coupling parameter b is sufficiently small and the function a satisfies the following assumptions:

$$\begin{cases} \text{(i)} & a(x) > 0 \quad \forall x \in]0, 1], \; a(0) = 0, \\ \text{(ii)} & \mu_a := \sup\limits_{0<x\leq 1} \dfrac{x\,|a'(x)|}{a(x)} < 2, \; \text{and} \\ \text{(iii)} & a \in \mathcal{C}^{[\mu_a]}([0, 1]), \end{cases} \tag{2}$$

where $[\cdot]$ stands for the integer part.

The rest of the paper is organized as follows. In Sect. 2, we introduce the appropriate functional spaces that are naturally associated with degenerate problems and preliminary results used throughout the paper. Section 3 is devoted to the proof of the well posedness of the considered system. In Sect. 4, we study the boundary stabilization problem proving its exponential stability.

2 Preliminary results

Let $a \in \mathcal{C}([0, 1]) \cap \mathcal{C}^1(]0, 1])$ be a function satisfying assumptions (2). At first, as in [37], we introduce some weighted Sobolev spaces that are naturally associated with degenerate operators. We denote by $H_a^1(0, 1)$ the space of all functions $u \in L^2(0, 1)$ such that

$$\begin{cases} \text{(i)} \quad u \text{ is locally absolutely continuous in }]0,1], \text{ and} \\ \text{(ii)} \ \sqrt{a} u_x \in L^2(0,1). \end{cases}$$

It is easy to see that $H_a^1(0, 1)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H_a^1(0,1)} = \int_0^1 \big(a(x)u'(x)v'(x) + u(x)v(x)\big)\, dx \quad \forall u, v \in H_a^1(0,1)$$

and associated norm

$$\|u\|_{H_a^1(0,1)} = \left\{ \int_0^1 \left(a(x) \left|u'(x)\right|^2 + |u(x)|^2\right) dx \right\}^{\frac{1}{2}} \quad \forall u \in H_a^1(0,1).$$

Next, we define

$$H_a^2(0,1) := \left\{ u \in H_a^1(0,1) \mid au' \in H^1(0,1) \right\}.$$

Note that if $u \in H_a^2(0, 1)$, then au' is continuous on $[0, 1]$.

In the following proposition, we collect useful properties of the above functional spaces which will play an important role in order to evaluate several boundary terms, see [37, Proposition 2.5].

Proposition 1 *Assume that a is a function satisfying* (2)*. Then the following assertions hold true:*

1. *For every* $u \in H_a^1(0, 1)$
$$\lim_{x \downarrow 0} x|u(x)|^2 = 0. \tag{3}$$
Moreover, if $\mu_a \in [0, 1[$*, then u is absolutely continuous in* $[0, 1]$.
2. *For every* $u \in H_a^2(0, 1)$
$$\lim_{x \downarrow 0} x a(x)|u'(x)|^2 = 0. \tag{4}$$
3. *For all* $u \in H_a^2(0, 1)$ *and for all* $v \in H_a^1(0, 1)$
$$\lim_{x \downarrow 0} a(x)u'(x)v(x) = 0, \tag{5}$$
assuming, in addition, $v(0) = 0$ *if* $\mu_a \in [0, 1[$.

4. *If $\mu_a \in [1,2[$, for every $u \in H_a^2(0,1)$*

$$\lim_{x\downarrow 0} a(x)u'(x) = 0. \tag{6}$$

In view of Proposition 1, we see that the boundary conditions imposed at $x = 0$ make sense for any classical solution of (1). Such conditions are of Dirichlet type if $\mu_a \in [0,1[$, whereas they are of Neumann/Dirichlet type at $x = 0$ and $x = 1$, respectively, if $\mu_a \in [1,2[$.

In order to express the boundary conditions of the first component of the solution of (1) in the functional setting, we define the space $H^1_{a,0}(0,1)$ depending on the value of μ_a, as follows:

(i) For $0 \leq \mu_a < 1$, we define

$$H^1_{a,0}(0,1) := \left\{u \in H^1_a(0,1) \mid u(0) = u(1) = 0\right\}.$$

(ii) For $1 \leq \mu_a < 2$, we define

$$H^1_{a,0}(0,1) := \left\{u \in H^1_a(0,1) \mid u(1) = 0\right\}.$$

Let us recall the following version of Poincaré's inequality, which is proved in [37, Proposition 2.2].

Lemma 2 *Assume (2) holds. Then*

$$\int_0^1 |u(x)|^2\, dx \leq C_a \int_0^1 a(x)\left|u'(x)\right|^2 dx, \quad \forall u \in H^1_{a,0}(0,1), \tag{7}$$

where

$$C_a = \frac{1}{a(1)} \min\left\{4, \frac{1}{2-\mu_a}\right\}. \tag{8}$$

Then set

$$\|u\|_{H^1_{a,0}(0,1)} := \left\{\int_0^1 a(x)\left|u'(x)\right|^2 dx\right\}^{\frac{1}{2}} \quad \forall u \in H^1_{a,0}(0,1).$$

which, thanks to Lemma 2, defines a norm on $H^1_{a,0}(0,1)$ that is equivalent to $\|\cdot\|_{H^1_a(0,1)}$.

Finally, we define

$$H^2_{a,0}(0,1) := H^2_a(0,1) \cap H^1_{a,0}(0,1).$$

Observe that all functions $u \in H^2_{a,0}(0,1)$ satisfy the above homogeneous boundary conditions at both $x = 0$ and $x = 1$.

3 Well-posedness

In this section, we first provide existence and uniqueness results of solutions for the damped hyperbolic system (1). Let us denote by $W^1_a(0,1)$ the space $H^1_a(0,1)$ itself if $\mu_a \in [1,2)$ and, if $\mu_a \in [0,1)$, the closed subspace of $H^1_a(0,1)$ consisting of all the functions $u \in H^1_a(0,1)$ such that $u(0) = 0$. Moreover, we set

$$W^2_a(0,1) = H^2_a(0,1) \cap W^1_a(0,1).$$

Note that $W^2_a(0,1) = H^2_a(0,1)$ when $\mu_a \in [1,2)$.

In the Hilbert space $W_a^1(0,1)$, we consider the following inner product

$$\langle u, v\rangle_{W_a^1} = \left(\int_0^1 \left(u(x)v(x) + a(x)u_x(x)v_x(x)\right)dx + \beta a(1)u(1)v(1)\right),$$
$$\forall u, v \in W_a^1(0,1),$$

and the associated norm

$$\|u\|_{W_a^1(0,1)} = \left(\int_0^1 \left(|u(x)|^2 + a(x)|u_x(x)|^2\right)\,dx + \beta a(1)|u(1)|^2\right)^{\frac{1}{2}},\ \forall u \in W_a^1(0,1).$$

First of all, recall the following preliminary results (see [37, Proposition 2.5 and Proposition 4.3]).

Proposition 3 *Assume* (2) *holds. Then*

$$u^2(1) \le \max\left\{2, \frac{1}{a(1)}\right\}\|u\|^2_{H_a^1(0,1)} \quad \forall u \in H_a^1(0,1). \tag{9}$$

We also have

$$\int_0^1 |u(x)|^2\,dx \le 2|u(1)|^2 + C_a' \int_0^1 a(x)\,|u_x(x)|^2\,dx \quad \forall u \in W_a^1(0,1), \tag{10}$$

where

$$C_a' = \frac{1}{a(1)}\min\left\{4, \frac{2}{2-\mu_a}\right\}. \tag{11}$$

In view of (10), observe that

$$|u|_{W_a^1(0,1)} := \left(\int_0^1 \left(a(x)|u_x(x)|^2\right)\,dx + \beta a(1)|u(1)|^2\right)^{\frac{1}{2}}, \quad \forall u \in W_a^1(0,1), \tag{12}$$

defines a norm on $W_a^1(0,1)$ that is equivalent to $\|\cdot\|_{W_a^1(0,1)}$.

In order to prove the well-posedness of the system (1) and establish an exponential decay result, we will need the following (see [37, Proposition 4.3]).

Proposition 4 *Assume* (2) *holds and* $\beta > 0$. *Then, we have*

$$|u|^2_{W_a^1(0,1)} \ge c_{a,\beta}\|u\|^2_{L^2(0,1)} \quad \forall u \in W_a^1(0,1), \tag{13}$$

where

$$c_{a,\beta} = \min\left(\frac{1}{C_a'}, \frac{\beta a(1)}{2}\right).$$

Moreover, we also have

$$\frac{c_{a,\beta}}{c_{a,\beta}+1}\left(\|u\|^2_{H_a^1(0,1)} + \beta a(1)u^2(1)\right) \le |u|^2_{W_a^1(0,1)} \le \gamma_{a,\beta}\|u\|^2_{H_a^1(0,1)} \quad \forall u \in W_a^1(0,1), \tag{14}$$

where

$$\gamma_{a,\beta} = \max\left(2\beta a(1), 1 + \frac{2\beta}{2-\mu_a}\right).$$

Now, let us define the energy space $\mathcal{H}_a^\beta$ by

$$\mathcal{H}_a^\beta = H_{a,0}^1(0,1) \times L^2(0,1) \times W_a^1(0,1) \times L^2(0,1).$$

It is easy to see that $\mathcal{H}_a^\beta$ is a Hilbert space, equipped with the scalar product defined by

$$\begin{aligned}\langle U, \widetilde{U}\rangle_{\mathcal{H}_a^\beta} = \int_0^1 &\big(a(x)u_{1,x}(x)\widetilde{u}_{1,x}(x) + u_2(x)\widetilde{u}_2(x) + a(x)u_{3,x}(x)\widetilde{u}_{3,x}(x)\\ &+ u_4(x)\widetilde{u}_4(x)\big)\,dx + \beta a(1)u_3(1)\widetilde{u}_3(1),\end{aligned}$$

for all $U = (u_1, u_2, u_3, u_4)$, $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) \in \mathcal{H}_a^\beta$. The expression $\|\cdot\|_{\mathcal{H}_a^\beta}$ will denote the corresponding norm.

We are now ready to study the well posedness of system (1) by using semigroup theory. For this, we define the unbounded linear operator $\mathcal{A}_a^\beta : D(\mathcal{A}_a^\beta) \subset \mathcal{H}_a^\beta \to \mathcal{H}_a^\beta$ by

$$D(\mathcal{A}_a^\beta) = \left\{\begin{array}{c}(u_1, u_2, u_3, u_4) \in H_{a,0}^2(0,1) \times H_{a,0}^1(0,1) \times W_a^2(0,1) \times W_a^1(0,1)\\ \mid u_{3,x}(1) + u_4(1) + \beta u_3(1) = 0\end{array}\right\}$$

and

$$\mathcal{A}_a^\beta U = \big(u_2, \big(au_{1,x}\big)_x - bu_4, u_4, \big(au_{3,x}\big)_x + bu_2\big), \quad \forall U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a^\beta).$$

Setting $U(t) = (v(t), v_t(t), u(t), u_t(t))$, then system (1) can be transformed into the first order evolution equation on the Hilbert space $\mathcal{H}_a^\beta$ as follows

$$\begin{cases} U'(t) &= \mathcal{A}_a^\beta U(t) \quad t \geq 0,\\ U(0) &= U_0,\end{cases} \tag{15}$$

where $U(0) = (v_0, v_1, u_0, u_1)$.

In view of Proposition 1, if $U = (v, v_t, u, u_t) \in D\left(\mathcal{A}_a^\beta\right)$, then U satisfies the boundary conditions $Bv(0) = Bu(0) = 0$ at $x = 0$ and the Dirichlet boundary condition $v(1) = 0$ at $x = 1$. Notice also that $u_x(1)$, $u_t(1)$ and $\beta u(1)$ are well defined for all $U = (v, v_t, u, u_t) \in D\left(\mathcal{A}_a^\beta\right)$ because of the classical Sobolev embedding theorem.

The next result holds.

Proposition 5 *Assume (2) holds and consider $\beta > 0$. Then $\mathcal{A}_a^\beta$ is a maximal dissipative operator on $\mathcal{H}_a^\beta$.*

Proof For all $U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a^\beta)$, we have

$$\begin{aligned}\langle \mathcal{A}_a^\beta U, U\rangle_{\mathcal{H}_a^\beta} = &\int_0^1 a(x)u_{1,x}(x)u_{2,x}(x)\,dx + \int_0^1 \Big((a(x)u_{1,x}(x))_x - bu_4(x)\Big)u_2(x)\,dx\\ &+ \int_0^1 a(x)u_{3,x}(x)u_{4,x}(x)\,dx + \int_0^1 \Big((a(x)u_{3,x}(x))_x + bu_2(x)\Big)u_4(x)\,dx\\ &+ \beta a(1)u_3(1)u_4(1).\end{aligned}$$

Integrating by parts and using (5), we get

$$\begin{aligned}\langle \mathcal{A}_a^\beta U, U\rangle_{\mathcal{H}_a^\beta} &= [a(x)u_{1,x}(x)u_2(x)]_0^1 + [a(x)u_{3,x}(x)u_4(x)]_0^1 + \beta a(1)u_3(1)u_4(1)\\ &= a(1)u_4(1)\big(u_{3,x}(1) + \beta u_3(1)\big)\\ &= -a(1)u_4^2(1) \leq 0,\end{aligned}$$

which implies that $\mathcal{A}_a^\beta$ is dissipative. In order to show that $\mathcal{A}_a^\beta$ is maximal dissipative, it remains to prove that $R(I - \mathcal{A}_a^\beta) = \mathcal{H}_a^\beta$. Let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}_a^\beta$. We look for an element $U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a^\beta)$ such that

$$U - \mathcal{A}_a^\beta U = F \Leftrightarrow \begin{cases} u_1 - u_2 & = f_1, \\ u_2 - \left(au_{1,x}\right)_x + bu_4 & = f_2, \\ u_3 - u_4 & = f_3, \\ u_4 - \left(au_{3,x}\right)_x - bu_2 & = f_4. \end{cases} \tag{16}$$

Suppose that we have found u_1 and u_3 with the appropriate regularity. Therefore, the first and the third equations in (16) give

$$\begin{cases} u_2 & = u_1 - f_1, \\ u_4 & = u_3 - f_3. \end{cases} \tag{17}$$

Then, it is clear that $u_2 \in H^1_{a,0}(0,1)$ and $u_4 \in W^1_a(0,1)$. By using (16) and (17) the functions u_1 and u_3 satisfy the following system:

$$\begin{cases} u_1 - \left(a(x)u_{1,x}\right)_x + bu_3 & = f_1 + f_2 + bf_3, \\ u_3 - \left(a(x)u_{3,x}\right)_x - bu_1 & = -bf_1 + f_3 + f_4. \end{cases} \tag{18}$$

Solving system (18) is equivalent to finding $(u_1, u_3) \in H^2_{a,0}(0,1) \times W^2_a(0,1)$ such that

$$\begin{cases} \int_0^1 \left(u_1\phi_1 + a(x)u_{1,x}\phi_{1,x} + bu_3\phi_1\right) dx = \int_0^1 \left(f_1 + f_2 + bf_3\right)\phi_1\, dx, \\ \int_0^1 \left(u_3\phi_2 + a(x)u_{3,x}\phi_{2,x} - bu_1\phi_2\right) dx = \int_0^1 \left(-bf_1 + f_3 + f_4\right)\phi_2\, dx, \end{cases} \tag{19}$$

for all $(\phi_1, \phi_2) \in C_c^\infty(0,1) \times C_c^\infty(0,1)$.

To this aim, introduce the bilinear form $\Lambda : \left(H^1_{a,0}(0,1) \times W^1_a(0,1)\right)^2 \to \mathbb{R}$ given by

$$\begin{aligned} \Lambda\left((u_1, u_3), (\phi_1, \phi_2)\right) = & \int_0^1 \left(u_1\phi_1 + a(x)u_{1,x}\phi_{1,x} + u_3\phi_2 + a(x)u_{3,x}\phi_{2,x}\right) dx \\ & + \int_0^1 b\left(u_3\phi_1 - u_1\phi_2\right) dx + (\beta + 1)a(1)u_3(1)\phi_2(1) \end{aligned}$$

and the linear form $L : H^1_{a,0}(0,1) \times W^1_a(0,1) \to \mathbb{R}$ given by

$$L\left(\phi_1, \phi_2\right) = \int_0^1 \Big(\left(f_1 + f_2 + bf_3\right)\phi_1 + \left(-bf_1 + f_3 + f_4\right)\phi_2\Big)\, dx + a(1)\phi_2(1)f_3(1).$$

From (7), (13) and the definition of $|\cdot|_{W^1_a(0,1)}$, one can show that Λ is a continuous bilinear form on $H^1_{a,0}(0,1) \times W^1_a(0,1)$ and L is a continuous linear functional on $H^1_{a,0}(0,1) \times W^1_a(0,1)$. Furthermore, it is easy to see that Λ is also coercive on $H^1_{a,0}(0,1) \times W^1_a(0,1)$. As a consequence, by the Lax-Milgram Theorem, there exists a unique $(u_1, u_3) \in H^1_{a,0}(0,1) \times W^1_a(0,1)$ such that

$$\Lambda\left((u_1, u_3), (\phi_1, \phi_2)\right) = L\left(\phi_1, \phi_2\right), \quad \forall\, (\phi_1, \phi_2) \in H^1_{a,0}(0,1) \times W^1_a(0,1). \tag{20}$$

Now, we will prove that $(u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a^\beta)$ and solves (16). Since $C_c^\infty(0,1) \times C_c^\infty(0,1) \subset H^1_{a,0}(0,1) \times W^1_a(0,1)$, (20) holds for every $(\phi_1, \phi_2) \in C_c^\infty(0,1) \times C_c^\infty(0,1)$.

Hence, we have (19) which is equivalent to (18). This yields $(u_1, u_3) \in H_a^2(0,1) \times H_a^2(0,1)$ and thus $(u_1, u_3) \in H_{a,0}^2(0,1) \times W_a^2(0,1)$.

Coming back to (18), we deduce after an integration by parts, together with (5), that

$$\begin{aligned}
&\int_0^1 (u_1 + bu_3)\,\phi_1\,dx + \int_0^1 a(x)u_{1,x}\phi_{1,x}\,dx + \int_0^1 (u_3 - bu_1)\,\phi_2\,dx \\
&\quad + \int_0^1 a(x)u_{3,x}\phi_{2,x}\,dx - a(1)u_{3,x}(1)\phi_2(1) \\
&\quad = \int_0^1 (f_1 + f_2 + bf_3)\,\phi_1\,dx + \int_0^1 (-bf_1 + f_3 + f_4)\,\phi_2\,dx, \\
&\quad \forall(\phi_1, \phi_2) \in H_{a,0}^1(0,1) \times W_a^1(0,1).
\end{aligned}$$

This combined with (20) leads to:

$$a(1)\phi_2(1)\Big(u_{3,x}(1) + (\beta+1)u_3(1) - f_3(1)\Big) = 0, \quad \forall \phi_2 \in W_a^1(0,1).$$

Using the fact that $a(1) > 0$ and the function ϕ_2 defined by $\phi_2(x) = x$ for all $x \in (0,1)$ is in $W_a^1(0,1)$, we infer that

$$u_{3,x}(1) + (\beta+1)u_3(1) - f_3(1) = 0.$$

Finally, recalling (17), we deduce that $(u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a^\beta)$ and solves (16). The proof is thus complete. □

By using the Hille-Yosida theorem (see [39, Theorem 4.5.1] or [40, Theorem A.7]), we deduce that the operator $\mathcal{A}_a^\beta$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}_a^\beta}\right)_{t\ge 0}$. The solution of the Cauchy problem (15) admits the following representation

$$U(t) = e^{t\mathcal{A}_a^\beta}U_0, \quad t \ge 0,$$

which leads to the well-posedness of (15). Hence, we have the following result.

Corollary 6 *Assume (2) holds and consider $\beta > 0$. For any $U_0 \in \mathcal{H}_a^\beta$, there exists a unique solution $U \in C^0([0,+\infty); \mathcal{H}_a^\beta)$ of problem (15). Moreover, if $U_0 \in D(\mathcal{A}_a^\beta)$, then*

$$U \in C^0([0,+\infty); D(\mathcal{A}_a^\beta)) \cap C^1([0,+\infty); \mathcal{H}_a^\beta).$$

4 Exponential stability

In this section, we study the exponential stability of system (1). To this aim we first define its energy as

$$\begin{aligned}
\mathcal{E}_{v,u}(t) = \frac{1}{2}\bigg[&\int_0^1 \left\{v_t^2(t,x) + a(x)v_x^2(t,x) + u_t^2(t,x) + a(x)u_x^2(t,x)\right\} dx \\
&+ \beta a(1)u^2(t,1)\bigg], \quad \forall t \ge 0.
\end{aligned} \tag{21}$$

In particular, it is possible to prove that the energy is a non increasing function.

Lemma 7 *Assume (2) holds. Let $U = (v, v_t, u, u_t)$ be a regular solution of system (1). Then, the energy $\mathcal{E}_{v,u}$ associated to (v, u) satisfies*

$$\frac{d\mathcal{E}_{v,u}}{dt}(t) = -a(1)u_t^2(t, 1), \quad \forall t \geq 0. \tag{22}$$

Proof Multiplying the first and the second equation of (1) by v_t and u_t respectively, integrating by parts over $(0, 1)$, we get

$$\begin{aligned} 0 &= \int_0^1 v_t(t, x) \left\{ v_{tt}(t, x) - (a(x)v_x)_x (t, x) + bu_t(t, x) \right\} dx \\ &= \int_0^1 \{v_t(t, x)v_{tt}(t, x) + a(x)v_x(t, x)v_{tx}(t, x) + bv_t(t, x)u_t(t, x)\} dx \\ &\quad - [a(x)v_t(t, x)v_x(t, x)]_{x=0}^{x=1} \end{aligned} \tag{23}$$

and

$$\begin{aligned} 0 &= \int_0^1 u_t(t, x) \left\{ u_{tt}(t, x) - (a(x)u_x)_x (t, x) - bv_t(t, x) \right\} dx \\ &= \int_0^1 \{u_t(t, x)u_{tt}(t, x) + a(x)u_x(t, x)u_{tx}(t, x) - bu_t(t, x)v_t(t, x)\} dx \\ &\quad - [a(x)u_t(t, x)u_x(t, x)]_{x=0}^{x=1}. \end{aligned} \tag{24}$$

Adding (23) and (24), by using the boundary conditions, we obtain

$$\begin{aligned} 0 = \int_0^1 \Big[& v_t(t, x)v_{tt}(t, x) + a(x)v_x(t, x)v_{tx}(t, x) \\ & + u_t(t, x)u_{tt}(t, x) + a(x)u_x(t, x)u_{tx}(t, x) \Big] dx - a(1)u_t(t, 1)u_x(t, 1). \end{aligned}$$

Using the fact that $u_x(t, 1) = -u_t(t, 1) - \beta u(t, 1)$, we get the desired equation (22). □

From (22), it follows that system (1) is dissipative. Now we address the question how fast this energy decays. Precisely, we give an exponential stabilization estimate based on a direct application of the multiplier method.

Prior to the precise statement of our main result, we first recall the following result (See [37, Proposition 4.4]).

Proposition 8 *Assume (2) holds and consider $\beta > 0$. Then, for every $\lambda \in \mathbb{R}$, the variational problem*

$$\int_0^1 a(x)z_x\varphi_x \, dx + \beta a(1)z(1)\varphi(1) = \lambda a(1)\varphi(1) \quad \forall \varphi \in W_a^1(0, 1) \tag{25}$$

admits a unique solution $z \in W_a^1(0, 1)$ which satisfies the elliptic estimates

$$|z|^2_{W_a^1(0,1)} \leq \frac{a(1)}{\beta}\lambda^2 \text{ and } \|z\|^2_{L^2(0,1)} \leq \frac{a(1)}{\beta c_{a,\beta}}\lambda^2. \tag{26}$$

Moreover, $z \in W_a^2(0, 1)$ and solves

$$\begin{cases} -(a(x)z_x)_x = 0, \\ z_x(1) + \beta z(1) = \lambda. \end{cases} \tag{27}$$

Let us also introduce the following notations:

$$M_{a,b} = \min\left\{1 - \frac{bC_a}{2}, \frac{1}{2}\left(1 - \frac{6-\mu_a}{2-\mu_a}b\right), 2\left(1 - \frac{4}{(2-\mu_a)a(1)}b\right)\right\} \tag{28}$$

and

$$b_a = \min\left\{\frac{2}{C_a}, \frac{2-\mu_a}{6-\mu_a}, \frac{(2-\mu_a)a(1)}{4}\right\}.$$

Observe that, by condition (2) (ii) and assuming that $0 < b < b_a$, we have that $M_{a,b} > 0$.

Now, we are in position to state our main stability result.

Theorem 9 *Assume (2) holds and that $\beta > 0$ is given. Suppose $0 < b < b_a$. Then for any $U_0 = (v_0, v_1, u_0, u_1) \in \mathcal{H}_a^\beta$, the solution $U = (v, v_t, u, u_t)$ of (1) satisfies the uniform exponential decay*

$$\mathcal{E}_{v,u}(t) \le \mathcal{E}_{v,u}(0)e^{1-\frac{t}{M_{a,b,\beta}}} \quad \forall t \in \left[M_{a,b,\beta}, +\infty\right), \tag{29}$$

where $M_{a,b,\beta} > 0$ is given in (72) and is independent of U_0.

Remark 1 As far as we know, this result on exponential stabilization seems to be new even for strongly coupled hyperbolic systems with nondegenerate variable coefficients.

The next lemmas are technical results to be used in the proof of Theorem 9 given below.

Lemma 10 *Assume (2) holds and that $\beta > 0$ is given. Let $U_0 = (v_0, v_1, u_0, u_1) \in D(\mathcal{A}_a^\beta)$ and $U = (v, v_t, u, u_t)$ be the solution of (1). Then for every $0 \le S \le T$ and $\varepsilon > 0$ the following inequality holds:*

$$b\int_S^T\int_0^1 v_t^2\,dx\,dt \le b\int_S^T\int_0^1 u_t^2\,dx\,dt + \frac{\varepsilon a(1)}{2}\int_S^T v_x^2(t,1)\,dt + \left(2 + \frac{1}{2\varepsilon}\right)\mathcal{E}_{v,u}(S). \tag{30}$$

Proof Multiplying the second equation of (1) by v_t, integrating by parts over $(S,T)\times(0,1)$, we get

$$\begin{aligned}
0 &= \int_S^T\int_0^1 v_t(t,x)\Big(u_{tt}(t,x) - (a(x)u_x(t,x))_x - bv_t(t,x)\Big)\,dx\,dt \\
&= \int_S^T\int_0^1 v_t(t,x)u_{tt}(t,x)\,dx\,dt - \int_S^T [a(x)u_x(t,x)v_t(t,x)]_{x=0}^{x=1}\,dt \\
&\quad + \int_S^T\int_0^1 a(x)u_x(t,x)v_{tx}(t,x)\,dx\,dt - b\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt \\
&= \int_S^T\int_0^1 v_t(t,x)u_{tt}(t,x)\,dx\,dt + \int_S^T\int_0^1 a(x)u_x(t,x)v_{tx}(t,x)\,dx\,dt \\
&\quad - b\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt,
\end{aligned}$$

because $a(x)u_x(t,x)v_t(t,x)$ vanishes at $x = 1$ and, owing to (5), also at $x = 0$.

After integrating by parts on time, this gives

$$
\begin{aligned}
0 = &\left[\int_0^1 v_t(t,x)u_t(t,x)\,dx\right]_S^T - \int_S^T\int_0^1 v_{tt}(t,x)u_t(t,x)dx\,dt \\
&+\left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_S^T \\
&-\int_S^T\int_0^1 a(x)v_x(t,x)u_{tx}(t,x)\,dx\,dt - b\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt. \qquad (31)
\end{aligned}
$$

Next, we multiply the first equation of (1) by u_t and integrate the resulting equation over $(S,T)\times(0,1)$. This gives, after a suitable integration by parts,

$$
\begin{aligned}
0 = &\int_S^T\int_0^1 u_t(t,x)v_{tt}(t,x)dx\,dt - \left[\int_S^T a(x)v_x\,u_t\,dt\right]_0^1 \\
&+\int_S^T\int_0^1 a(x)v_x(t,x)u_{tx}(t,x)\,dx\,dt \\
&+b\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt \\
= &\int_S^T\int_0^1 u_t(t,x)v_{tt}(t,x)dx\,dt - a(1)\int_S^T v_x(t,1)u_t(t,1)\,dt \\
&+\int_S^T\int_0^1 a(x)v_x(t,x)u_{tx}(t,x)\,dx\,dt + b\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt, \qquad (32)
\end{aligned}
$$

because $a(x)v_x(t,x)u_t(t,x)$ vanishes at $x=0$ owing to (5).

Combining (31) and (32), we obtain

$$
\begin{aligned}
&b\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt - b\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt \\
&\quad=\left[\int_0^1 v_t(t,x)u_t(t,x)\,dx\right]_S^T + \left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_S^T \\
&\quad -a(1)\int_S^T v_x(t,1)u_t(t,1)\,dt. \qquad (33)
\end{aligned}
$$

By the Young inequality, we have

$$
\begin{aligned}
&\left|\int_0^1 v_t(t,x)u_t(t,x)\,dx\right| + \left|\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right| \\
&\le \frac{1}{2}\int_0^1 \left\{v_t^2(t,x) + a(x)v_x^2(t,x) + u_t^2(t,x) + a(x)u_x^2(t,x)\right\}\,dx \\
&\le \mathcal{E}_{v,u}(t), \quad \forall t\in(S,T).
\end{aligned}
$$

From the fact that the energy is non-increasing, it follows that

$$
\left[\int_0^1 u_t(t,x)v_t(t,x)\,dx\right]_S^T + \left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_S^T \le 2\mathcal{E}_{v,u}(S). \qquad (34)
$$

On the other hand, for $\varepsilon > 0$, using Young's inequality, we have

$$\left|\int_S^T v_x(t,1)u_t(t,1)\,dt\right| \leq \frac{\varepsilon}{2}\int_S^T v_x^2(t,1)\,dt + \frac{1}{2\varepsilon}\int_S^T u_t^2(t,1)\,dt.$$

By the dissipation relation (22), we obtain

$$\left|\int_S^T v_x(t,1)u_t(t,1)\,dt\right| \leq \frac{\varepsilon}{2}\int_S^T v_x^2(t,1)\,dt + \frac{1}{2a(1)\varepsilon}\mathcal{E}_{v,u}(S). \tag{35}$$

Thus, inserting (34) and (35) into (33), we get the required inequality (30). □

Lemma 11 *Assume (2) holds and that $\beta > 0$ is given. Let $U_0 = (v_0, v_1, u_0, u_1) \in D(\mathcal{A}_a^\beta)$ and $U = (v, v_t, u, u_t)$ be the solution of (1). Then for all $0 \leq S \leq T$ and all $\varepsilon > 0$ the following inequality holds:*

$$\begin{aligned}
&\left(\frac{2-\mu_a}{2} - \frac{b}{2}\right)\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt + \left(\frac{2-\mu_a}{2} - \frac{2b}{a(1)}\right)\int_S^T\int_0^1 a(x)u_x^2(t,x)\,dx\,dt \\
&\quad + \frac{2-\mu_a}{2}\beta a(1)\int_S^T u^2(t,1)\,dt \\
&\leq C_\varepsilon^1 \mathcal{E}_{v,u}(S) + \frac{\varepsilon a(1)}{4}\int_S^T v_x^2(t,1)\,dt + \int_S^T h(t)\,dt,
\end{aligned} \tag{36}$$

where

$$C_\varepsilon^1 = 2d_{a,\beta}^1 + 1 + \frac{1}{4\varepsilon} \text{ with } d_{a,\beta}^1 = \frac{2\max(2, \frac{\mu_a}{2\beta})}{\min\{1, a(1)\}} \tag{37}$$

and

$$\begin{aligned}
h(t) &= (1 + a(1))u_t^2(t,1) + \left[a(1)\beta(1+\beta-\mu_a) + \frac{\mu_a b}{2}\right]u^2(t,1) \\
&\quad + (2\beta - \frac{\mu_a}{2})a(1)u_t(t,1)u(t,1), \quad t \in (S,T).
\end{aligned} \tag{38}$$

Proof We multiply the second equation of (1) by $2xu_x$ and we integrate by parts over $(S,T)\times(0,1)$ as follows:

$$\begin{aligned}
0 &= \int_S^T\int_0^1 2xu_x(t,x)\left(u_{tt}(t,x) - (a(x)u_x(t,x))_x - bv_t(t,x)\right)\,dx\,dt \\
&= 2\left[\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} - 2\int_S^T\int_0^1 xu_{tx}(t,x)u_t(t,x)\,dx\,dt \\
&\quad - b\int_S^T\int_0^1 2xu_x(t,x)v_t(t,x)\,dx\,dt \\
&\quad - 2\int_S^T\int_0^1 \left(xa'(x)u_x^2(t,x) + xa(x)u_x(t,x)u_{xx}(t,x)\right)\,dx\,dt \\
&= 2\left[\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} - b\int_S^T\int_0^1 2xu_x(t,x)v_t(t,x)\,dx\,dt \\
&\quad - 2\int_S^T\int_0^1 xa'(x)u_x^2(t,x)\,dx\,dt - \int_S^T\int_0^1 \left(x\left(u_t^2(t,x)\right)_x + xa(x)\left(u_x^2(t,x)\right)_x\right)\,dx\,dt.
\end{aligned} \tag{39}$$

On the other hand, by integrating by parts and owing to (3)–(4), we have

$$\int_S^T\int_0^1 x\left(u_t^2(t,x)\right)_x\,dx\,dt = -\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt + \int_S^T u_t^2(t,1)\,dt$$

and

$$\int_S^T\int_0^1 xa(x)\left(u_x^2(t,x)\right)_x\,dx\,dt = a(1)\int_S^T u_x^2(t,1)\,dt - \int_S^T\int_0^1 (xa(x))' u_x^2(t,x)\,dx\,dt. \tag{40}$$

Inserting (40) into (39), we get

$$\int_S^T\int_0^1 \Big(u_t^2(t,x) + (a(x) - xa'(x))u_x^2(t,x)\Big)\,dx\,dt = -2\left[\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T}$$
$$+b\int_S^T\int_0^1 2xu_x(t,x)v_t(t,x)\,dx\,dt + \int_S^T \left(u_t^2(t,1) + a(1)u_x^2(t,1)\right)\,dt. \tag{41}$$

Now, we proceed by multiplying the second equation of (1) by u and integrating by parts over $(S,T)\times(0,1)$, to get

$$\begin{aligned}
0 &= \int_S^T\int_0^1 u(t,x)\Big(u_{tt}(t,x) - (a(x)u_x(t,x))_x - bv_t(t,x)\Big)\,dx\,dt\\
&= \left[\int_0^1 u(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} - \int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt\\
&\quad - \int_S^T [a(x)u_x(t,x)u(t,x)]_{x=0}^{x=1}\,dt + \int_S^T\int_0^1 a(x)u_x^2(t,x)\,dx\,dt\\
&\quad - b\int_S^T\int_0^1 u(t,x)v_t(t,x)\,dx\,dt.
\end{aligned}$$

Using the boundary conditions together with (5), this gives

$$-\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt + \int_S^T\int_0^1 a(x)u_x^2(t,x)\,dx\,dt = -\left[\int_0^1 u(t,x)u_t(t,x)\,dx\right]_S^T$$
$$+a(1)\int_S^T u_x(t,1)u(t,1)\,dt + b\int_S^T\int_0^1 u(t,x)v_t(t,x)\,dx\,dt. \tag{42}$$

By adding to (41) the identity (42) multiplied by $\frac{\mu_a}{2}$, we obtain

$$\begin{aligned}
&\frac{2-\mu_a}{2}\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt + \int_S^T\int_0^1 \left(\frac{2+\mu_a}{2}a(x) - xa'(x)\right)u_x^2(t,x)\,dx\,dt\\
&\quad + \frac{2-\mu_a}{2}\beta a(1)\int_S^T u^2(t,1)\,dt\\
&= -\left[\int_0^1 u_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right)dx\right]_S^T + \int_S^T h_0(t)\,dt\\
&\quad + b\int_0^T\int_0^1 v_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right)\,dx\,dt,
\end{aligned} \tag{43}$$

where the function h_0 is given by

$$h_0(t) = (1+a(1))u_t^2(t,1) + a(1)\beta(1+\beta-\mu_a)u^2(t,1) + \left(2\beta - \frac{\mu_a}{2}\right) a(1)u_t(t,1)u(t,1),$$
$$t \in (S,T).$$

Observe that, by the definition of μ_a, we have

$$\frac{2-\mu_a}{2}a(x) \le \frac{2+\mu_a}{2}a(x) - xa'(x).$$

This, combined with (43), gives

$$\begin{aligned}
&\frac{2-\mu_a}{2}\left[\int_S^T\int_0^1 \left(u_t^2(t,x)+a(x)u_x^2(t,x)\right)dx\,dt + \beta a(1)\int_S^T u^2(t,1)\,dt\right]\\
&\le -\left[\int_0^1 u_t(t,x)\left(2xu_x(t,x)+\frac{\mu_a}{2}u(t,x)\right)dx\right]_S^T + \int_S^T h_0(t)\,dt\\
&\quad + b\int_0^T\int_0^1 v_t(t,x)\left(2xu_x(t,x)+\frac{\mu_a}{2}u(t,x)\right)dx\,dt. \qquad (44)
\end{aligned}$$

On the other hand, using Young's inequality, we have

$$\left|\int_0^1 u_t\left(2xu_x+\frac{\mu_a}{2}u\right)dx\right| \le \frac{1}{2}\|u_t\|_{L^2(0,1)}^2 + 2\|xu_x+\frac{\mu_a}{4}u\|_{L^2(0,1)}^2. \qquad (45)$$

Next, we compute:

$$\begin{aligned}
\|xu_x+\frac{\mu_a}{4}u\|_{L^2(0,1)}^2 &= \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\int_0^1 x(u^2)_x\,dx + \frac{\mu_a^2}{16}\int_0^1 u^2\,dx\\
&= \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\left[xu^2\right]_{x=0}^{x=1} + \frac{\mu_a}{4}\left(\frac{\mu_a}{4}-1\right)\int_0^1 u^2\,dx\\
&\le \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\left[xu^2\right]_{x=0}^{x=1}.
\end{aligned}$$

By using the boundary conditions, (3) and (57), we obtain that

$$\|xu_x+\frac{\mu_a}{4}u\|_{L^2(0,1)}^2 \le \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}u^2(1) \le \frac{1}{a(1)}\int_0^1 a(x)u_x^2\,dx + \frac{\mu_a}{4}u^2(1). \qquad (46)$$

Combining (45) and (46), we get

$$\begin{aligned}
\left|\int_0^1 u_t\left(2xu_x+\frac{\mu_a}{2}u\right)dx\right| &\le \frac{1}{2}\|u_t\|_{L^2(0,1)}^2 + \frac{\max(2,\frac{\mu_a}{2\beta})}{a(1)}|u|_{W_a^1}^2\\
&\le d_{a,\beta}^1\mathcal{E}_{v,u}(t), \qquad (47)
\end{aligned}$$

where $d_{a,\beta}^1$ is given in (37).

Moreover, we have

$$\begin{aligned}
\left|b\int_S^T\int_0^1 v_t\left(2xu_x+\frac{\mu_a}{2}u\right)dx\,dt\right| \le& \frac{b}{2}\int_S^T \|v_t\|_{L^2(0,1)}^2\,dt\\
&+ 2b\int_S^T \|xu_x+\frac{\mu_a}{4}u\|_{L^2(0,1)}^2\,dt.
\end{aligned}$$

From (30) and (46), we get

$$\left| b\int_S^T\int_0^1 v_t\left(2xu_x+\frac{\mu_a}{2}u\right)\,dx\,dt\right| \le \frac{b}{2}\int_S^T \|u_t\|^2_{L^2(0,1)}\,dt+\frac{\varepsilon a(1)}{4}\int_S^T v_x^2(t,1)\,dt \\ +\left(1+\frac{1}{4\varepsilon}\right)\mathcal{E}_{v,u}(S)+\frac{2b}{a(1)}\int_S^T\int_0^1 a(x)u_x^2\,dx\,dt \\ +\frac{\mu_a b}{2}\int_S^T u^2(t,1)\,dt. \tag{48}$$

Finally, inserting (47) and (48) into (44), one obtains the desired estimate (36). □

Lemma 12 *Assume* (2) *holds and that* $\beta>0$ *is given. Let* $U_0=(v_0,v_1,u_0,u_1)\in D(\mathcal{A}_a^\beta)$ *and* $U=(v,v_t,u,u_t)$ *be the solution of* (1)*. Then for all* $0\le S\le T$ *the following inequality holds:*

$$-\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt+\left(1-\frac{bC_a}{2}\right)\int_S^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt \\ -\frac{b}{2}\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt \\ \le 2\sqrt{C_a}\mathcal{E}_{v,u}(S), \tag{49}$$

where $C_a=\frac{1}{a(1)}\min\left\{4,\frac{1}{2-\mu_a}\right\}$.

Proof We multiply the first equation of (1) by v and integrate the resulting equation over $(S,T)\times(0,1)$. After suitable integrations by parts, this gives

$$0=\int_S^T\int_0^1 v(t,x)\left(v_{tt}(t,x)-(a(x)v_x(t,x))_x+bu_t(t,x)\right)\,dx\,dt \\ =\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_S^T-\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt-\int_S^T[a(x)v_x(t,x)v(t,x)]_{x=0}^{x=1}\,dt \\ +\int_S^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt+b\int_S^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt.$$

Using the fact that $a(x)v_x(t,x)v(t,x)$ vanishes at $x=1$ and, owing to (5) also at $x=0$, we get

$$-\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt+\int_S^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt \\ =-\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_S^T-b\int_S^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt. \tag{50}$$

On the other hand, using Young's inequality and the Poincaré inequality (7), we have

$$\left|\int_0^1 v(t,x)v_t(t,x)\,dx\right|\le\frac{1}{2}\int_0^1\left(\frac{1}{\sqrt{C_a}}v^2(t,x)+\sqrt{C_a}v_t^2(t,x)\right)dx \\ \le\frac{\sqrt{C_a}}{2}\int_0^1\left(a(x)v_x^2(t,x)+v_t^2(t,x)\right)dx \\ \le\sqrt{C_a}\mathcal{E}_{v,u}(t).$$

Thus

$$\left|\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_S^T\right| \le 2\sqrt{C_a}\mathcal{E}_{v,u}(S), \tag{51}$$

One can show similarly that

$$\left|b\int_S^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt\right| \le \frac{b}{2}\int_S^T\int_0^1 u_t^2(t,x)\,dx\,dt+\frac{bC_a}{2}\int_S^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt. \tag{52}$$

Then, inserting (51) and (52) into (50), we arrive at the desired inequality (49). □

Lemma 13 *Assume* (2) *holds and that* $\beta > 0$ *is given. Let* $U_0 = (v_0, v_1, u_0, u_1) \in D(\mathcal{A}_a^\beta)$ *and* $U = (v, v_t, u, u_t)$ *be the solution of* (1). *Then for all* $0 \le S \le T$ *the following inequality holds:*

$$a(1)\int_S^T v_x^2(t,1)\,dt \le \frac{4}{\min\{1,a(1)\}}\mathcal{E}_{v,u}(S) + \left(6+\frac{2b}{\min\{1,a(1)\}}\right)\int_S^T \mathcal{E}_{v,u}(t)\,dt. \tag{53}$$

Proof We multiply the second equation of (1) by $2xv_x$ and we integrate by parts over $(S,T)\times(0,1)$ as follows:

$$\begin{aligned}
0 &= \int_S^T\int_0^1 2xv_x(t,x)\left(v_{tt}(t,x)-(a(x)v_x(t,x))_x+bu_t(t,x)\right)\,dx\,dt\\
&= 2\left[\int_0^1 xv_x(t,x)v_t(t,x)\,dx\right]_{t=S}^{t=T} - 2\int_S^T\int_0^1 xv_{tx}(t,x)v_t(t,x)\,dx\,dt\\
&\quad -2\int_S^T\int_0^1 \left(xa'(x)v_x^2(t,x)+xa(x)v_x(t,x)v_{xx}(t,x)\right)\,dx\,dt\\
&\quad +b\int_S^T\int_0^1 2xv_x(t,x)u_t(t,x)\,dx\,dt\\
&= 2\left[\int_0^1 xv_x(t,x)v_t(t,x)\,dx\right]_{t=S}^{t=T} - 2\int_S^T\int_0^1 xa'v_x^2(t,x)\,dx\,dt\\
&\quad -\int_S^T\int_0^1 \left(x\left(v_t^2(t,x)\right)_x + xa(x)\left(v_x^2(t,x)\right)_x\right)\,dx\,dt + b\int_S^T\int_0^1 2xv_x(t,x)u_t\,dx\,dt.
\end{aligned} \tag{54}$$

On the other hand, by integrating by parts and owing to (3)–(4), we have

$$\int_S^T\int_0^1 x\left(v_t^2(t,x)\right)_x\,dx\,dt = -\int_S^T\int_0^1 v_t^2(t,x)\,dx\,dt$$

and

$$\int_S^T\int_0^1 xa(x)\left(v_x^2(t,x)\right)_x\,dx\,dt = a(1)\int_S^T v_x^2(t,1)\,dt - \int_S^T\int_0^1 (xa(x))'v_x^2(t,x)\,dx\,dt. \tag{55}$$

Inserting (55) into (54), we get

$$
\begin{aligned}
a(1)\int_S^T v_x^2(t,1)\,dt &= \int_S^T\int_0^1 \left\{v_t^2(t,x) + \left(a(x) - xa'(x)\right) v_x^2(t,x)\right\} dx\,dt \\
&+2\left[\int_0^1 x v_x(t,x) v_t(t,x)\,dx\right]_{t=S}^{t=T} + 2b\int_S^T\int_0^1 x v_x(t,x) u_t(t,x)\,dx\,dt.
\end{aligned}
\tag{56}
$$

Now, observe that condition (2) (ii) yields:

$$
a(x) \geq a(1)x^{\mu_a} \quad \forall x \in [0,1]. \tag{57}
$$

From Young inequality and (57), it follows that:

$$
\begin{aligned}
\left|\int_0^1 x v_x(t,x) v_t(t,x)\,dx\right| &\leq \frac{1}{2}\int_0^1 \left\{v_t^2(t,x) + x^2 v_x^2(t,x)\right\} dx \\
&\leq \frac{1}{2\min\{1,a(1)\}}\int_0^1 \left\{v_t^2(t,x) + a(x)v_x^2(t,x)\right\} dx \\
&\leq \frac{1}{\min\{1,a(1)\}}\mathcal{E}_{v,u}(t) \qquad \forall t \geq 0,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
\left|2b\int_S^T\int_0^1 x v_x(t,x) u_t(t,x)\,dx\,dt\right| &\leq b\int_S^T\int_0^1 \left\{u_t^2(t,x) + x^2 v_x^2(t,x)\right\} dx\,dt \\
&\leq \frac{b}{\min\{1,a(1)\}}\int_S^T\int_0^1 \left\{u_t^2(t,x) + a(x)v_x^2(t,x)\right\} dx\,dt \\
&\leq \frac{2b}{\min\{1,a(1)\}}\int_S^T \mathcal{E}_{v,u}(t)\,dt.
\end{aligned}
\tag{59}
$$

Finally, combining (56), (58), (59) and the inequality $x|a'(x)| < 2a(x)$, one obtains (53). □

Lemma 14 *Assume (2) holds and that $\beta > 0$ is given. Let $U_0 = (v_0, v_1, u_0, u_1) \in D(\mathcal{A}_a^\beta)$ and $U = (v, v_t, u, u_t)$ be the solution of (1). Then for all $0 \leq S \leq T$ and all $\delta > 0$ the following inequality holds:*

$$
\begin{aligned}
a(1)\int_S^T u^2(t,1)\,dt &\leq \delta\left(1 + bC_a + \frac{1}{\beta^3}\right)\int_S^T \mathcal{E}_{v,u}(t)\,dt + \frac{2+2b\sqrt{C_a}}{\beta\sqrt{c_{a,\beta}}}\mathcal{E}_{v,u}(S) \\
&+\frac{1}{2\delta}\left(1 + \frac{1+b}{\beta c_{a,\beta}}\right)\mathcal{E}_{v,u}(S).
\end{aligned}
\tag{60}
$$

Proof Set $\lambda = u(t,1)$ and let z be the solution of the degenerate elliptic problem (27). We multiply the second equation of (1) by z and integrate by parts the resulting equation over $(S,T) \times (0,1)$, to obtain

$$
\begin{aligned}
0 &= \int_S^T\int_0^1 z(t,x)\left(u_{tt}(t,x)-(a(x)u_x(t,x))_x - bv_t(t,x)\right)\,dx\,dt\\
&= \left[\int_0^1 z(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T}\\
&\quad - \int_S^T\int_0^1 z_t(t,x)u_t(t,x)\,dx\,dt - \left[\int_S^T a(x)u_x(t,x)z(t,x)\,dt\right]_{x=0}^{x=1}\\
&\quad + \left[\int_S^T a(x)z_x(t,x)u(t,x)\,dt\right]_{x=0}^{x=1}\\
&\quad - \int_S^T\int_0^1 (a(x)z_x(t,x))_x\, u(t,x)dx\,dt - b\left[\int_0^1 z(t,x)v(t,x)\,dx\right]_{t=S}^{t=T}\\
&\quad + b\int_S^T\int_0^1 z_t(t,x)v(t,x)\,dx\,dt.
\end{aligned}
$$

By using the boundary conditions at $x=0$ together with (5), this gives

$$
\begin{aligned}
0 &= \left[\int_0^1 z(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} - b\left[\int_0^1 z(t,x)v(t,x)\,dx\right]_{t=S}^{t=T}\\
&\quad - \int_S^T\int_0^1 z_t(t,x)u_t(t,x)\,dx\,dt\\
&\quad + b\int_S^T\int_0^1 z_t(t,x)v(t,x)\,dx\,dt - a(1)\int_S^T z(t,1)u_x(t,1)\,dt + a(1)\\
&\quad \int_S^T z_x(t,1)u(t,1)\,dt.
\end{aligned}
$$

Then, from the boundary conditions at $x=1$ in both systems (1) and (27), we get

$$
\begin{aligned}
0 &= \left[\int_0^1 z(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} - b\left[\int_0^1 z(t,x)v(t,x)\,dx\right]_{t=S}^{t=T}\\
&\quad - \int_S^T\int_0^1 z_t(t,x)u_t(t,x)\,dx\,dt\\
&\quad + b\int_S^T\int_0^1 z_t(t,x)v(t,x)\,dx\,dt + a(1)\int_S^T z(t,1)u_t(t,1)\,dt + a(1)\int_S^T u^2(t,1)\,dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
a(1)\int_S^T u^2(t,1)\,dt &= \int_S^T\int_0^1 z_t(t,x)u_t(t,x)\,dx\,dt - b\int_S^T\int_0^1 z_t(t,x)v(t,x)\,dx\,dt\\
&\quad - \left[\int_0^1 z(t,x)u_t(t,x)\,dx\right]_{t=S}^{t=T} + b\left[\int_0^1 z(t,x)v(t,x)\,dx\right]_{t=S}^{t=T}\\
&\quad - a(1)\int_S^T z(t,1)u_t(t,1)\,dt.
\end{aligned}
\tag{61}
$$

It only remains to estimate in a suitable way the terms on the right-hand side of the previous inequality as follows. First, using Young's inequality and thanks to the second inequality in

(26), we have

$$\left|\int_0^1 z(t,x)u_t(t,x)\,dx\right| \le \frac{1}{2\beta\sqrt{c_{a,\beta}}}\int_0^1 u_t^2(t,x)\,dx + \frac{\beta\sqrt{c_{a,\beta}}}{2}\int_0^1 z^2(t,x)\,dx$$
$$\le \frac{1}{\beta\sqrt{c_{a,\beta}}}\left(\frac{1}{2}\int_0^1 u_t^2(t,x)\,dx + \frac{\beta a(1)}{2}u^2(t,1)\right)$$
$$\le \frac{1}{\beta\sqrt{c_{a,\beta}}}\mathcal{E}_{v,u}(t) \quad \forall t\in[S,T]. \tag{62}$$

Similarly, by Young's inequality and the Poincaré inequality (7), we have

$$\left|\int_0^1 bz(t,x)v(t,x)\,dx\right| \le b\left(\frac{1}{2\beta\sqrt{c_{a,\beta}C_a}}\int_0^1 v^2(t,x)\,dx + \frac{\beta\sqrt{c_{a,\beta}C_a}}{2}\int_0^1 z^2(t,x)\,dx\right)$$
$$\le \frac{b\sqrt{C_a}}{\beta\sqrt{c_{a,\beta}}}\left(\frac{1}{2}\int_0^1 a(x)v_x^2(t,x)\,dx + \frac{\beta a(1)}{2}u^2(t,1)\right)$$
$$\le \frac{b\sqrt{C_a}}{\beta\sqrt{c_{a,\beta}}}\mathcal{E}_{v,u}(t) \quad \forall t\in[S,T]. \tag{63}$$

On the other hand, keeping in mind the second inequality in (26), we have

$$\|z_t\|^2_{L^2(0,1)} \le \frac{a(1)}{\beta c_{a,\beta}}u_t^2(t,1). \tag{64}$$

Furthermore, thanks to the first inequality in (26) and the definition of $|\cdot|_{W_a^1(0,1)}$, we have

$$\beta a(1)z^2(t,1) \le |z|^2_{W_a^1(0,1)} \le \frac{a(1)}{\beta}u^2(t,1),$$

so that

$$z^2(t,1) \le \frac{1}{\beta^2}u^2(t,1) \le \frac{2}{\beta^3 a(1)}\mathcal{E}_{v,u}(t). \tag{65}$$

Hence, by using the Young's inequality

$$ab \le \frac{\delta a^2}{2} + \frac{b^2}{2\delta} \quad \forall\delta>0, \forall a,b\in\mathbb{R}$$

and taking into account the estimates (62)–(65) in (61), we deduce that

$$a(1)\int_S^T u^2(t,1)\,dt \le \delta\left(1+bC_a+\frac{1}{\beta^3}\right)\int_S^T \mathcal{E}_{v,u}(t)\,dt + \frac{2+2b\sqrt{C_a}}{\beta\sqrt{c_{a,\beta}}}\mathcal{E}_{v,u}(S)$$
$$+\frac{1}{2\delta}\left(1+\frac{1+b}{\beta c_{a,\beta}}\right)\int_S^T a(1)u_t^2(t,1)\,dt. \tag{66}$$

Finally, using (22) in the above estimate, one obtains (60). □

Proof of Theorem 9 As usual, let us assume that $U=(v,v_t,u,u_t)$ is a regular solution of (1) (the general case can be recovered by an approximation argument). We start by combining (49) multiplied by $\frac{2-\mu_a}{2}$ and (36) multiplied by 2, to obtain that, for all $0\le S\le T$ and all

$\varepsilon > 0$,

$$
\begin{aligned}
&-\frac{2-\mu_a}{2}\int_0^T\int_0^1 v_t^2(t,x)\,dx\,dt+\frac{2-\mu_a}{2}\left(1-\frac{bC_a}{2}\right)\int_0^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt\\
&\quad-\frac{2-\mu_a}{2}\frac{b}{2}\int_0^T\int_0^1 u_t^2(t,x)\,dx\,dt+2\left(\frac{2-\mu_a}{2}-\frac{b}{2}\right)\int_0^T\int_0^1 u_t^2(t,x)\,dx\,dt\\
&\quad+2\left(\frac{2-\mu_a}{2}-\frac{2b}{a(1)}\right)\int_0^T\int_0^1 a(x)u_x^2(t,x)\,dx\,dt+(2-\mu_a)\beta a(1)\int_S^T u^2(t,1)\,dt\\
&\leq\left(2C_\varepsilon^1+(2-\mu_a)\sqrt{C_a}\right)\mathcal{E}_{v,u}(S)+\frac{\varepsilon}{2}a(1)\int_S^T v_x^2(t,1)\,dt+2\int_S^T h(t)\,dt,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\frac{2-\mu_a}{4}\int_0^T\int_0^1 v_t^2\,dx\,dt+\frac{2-\mu_a}{2}\left(1-\frac{bC_a}{2}\right)\int_0^T\int_0^1 a(x)v_x^2\,dx\,dt\\
&\quad+\frac{2-\mu_a}{4}\left(1-\frac{6-\mu_a}{2-\mu_a}b\right)\int_0^T\int_0^1 u_t^2\,dx\,dt\\
&\quad+2\left(\frac{2-\mu_a}{2}-\frac{2b}{a(1)}\right)\int_S^T\int_0^1 a(x)u_x^2\,dx\ dt\\
&\quad+(2-\mu_a)\beta a(1)\int_S^T u^2(t,1)\,dt+\frac{3(2-\mu_a)}{4}\int_S^T\int_0^1\left(u_t^2-v_t^2\right)\,dx\,dt\\
&\leq\left(2C_\varepsilon^1+(2-\mu_a)\sqrt{C_a}\right)\mathcal{E}_{v,u}(S)+\frac{\varepsilon}{2}a(1)\int_S^T v_x^2(t,1)\,dt+2\int_S^T h(t)\,dt.
\end{aligned}
\tag{67}
$$

Moreover, from (30), we can see that

$$
\begin{aligned}
\frac{3(2-\mu_a)}{4}\int_S^T\int_0^1\left(u_t^2-v_t^2\right)\,dx\,dt\geq&-\frac{3(2-\mu_a)}{8b}\varepsilon a(1)\int_S^T v_x^2(t,1)\,dt\\
&-\frac{3(2-\mu_a)}{4b}\left(2+\frac{1}{2\varepsilon}\right)\mathcal{E}_{v,u}(S).
\end{aligned}
$$

Inserting this last inequality into (67) and using the definition of $M_{a,b}$ (see (28)), then we have

$$
\begin{aligned}
(2-\mu_a)M_{a,b}\int_S^T\mathcal{E}_{v,u}(t)\,dt\leq&\left(2C_\varepsilon^1+(2-\mu_a)\sqrt{C_a}+\frac{3(2-\mu_a)}{4b}\left(2+\frac{1}{2\varepsilon}\right)\right)\mathcal{E}_{v,u}(S)\\
&+\left(\frac{1}{2}+\frac{3(2-\mu_a)}{8b}\right)\varepsilon a(1)\int_S^T v_x^2(t,1)\,dt+2\int_S^T h(t)\,dt.
\end{aligned}
\tag{68}
$$

We now choose $\varepsilon = \varepsilon_{a,b} := \frac{(2-\mu_a)M_{a,b}}{4\left(\frac{1}{2}+\frac{3(2-\mu_a)}{8b}\right)(6+\frac{2b}{\min\{1,a(1)\}})}$ and use (53), then (68) becomes

$$\begin{aligned}\frac{3(2-\mu_a)}{4}M_{a,b}\int_S^T \mathcal{E}_{v,u}(t)\,dt \le{}& \left(2C^1_{\varepsilon_{a,b}} + (2-\mu_a)\sqrt{C_a}\right.\\ &\left.+\frac{3(2-\mu_a)}{4b}\left(2+\frac{1}{2\varepsilon_{a,b}}\right)\right)\mathcal{E}_{v,u}(S)\\ &+\frac{4\varepsilon_{a,b}}{\min\{1,a(1)\}}\left(\frac{1}{2}+\frac{3(2-\mu_a)}{8b}\right)\mathcal{E}_{v,u}(S)\\ &+2\int_S^T h(t)\,dt. \qquad (69)\end{aligned}$$

We now estimate the last term on the right-hand side of this inequality where we recall that the function h is given in (38). We have that

$$h(t) \le \eta_1 u_t^2(t,1) + \eta_2 a(1)u^2(t,1), \quad \forall t \in (S,T),$$

where

$$\eta_1 = \left(1+\frac{3}{2}a(1)\right) \text{ and } \eta_2 = \beta^2+\beta-\mu_a\beta+\frac{\mu_a}{2a(1)}b+\frac{1}{2}\left(2\beta-\frac{\mu_a}{2}\right)^2.$$

Therefore, by using (60) with $\delta = \delta_{a,b,\beta} := \frac{(2-\mu_a)M_{a,b}}{8\eta_2\left(1+bC_a+\frac{1}{\beta^3}\right)}$ and keeping in mind the dissipation relation (22), we get

$$\begin{aligned}2\int_S^T h(t)\,dt &\le 2\eta_1\int_S^T u_t^2(t,1)\,dt + 2\eta_2\int_S^T a(1)u^2(t,1)\,dt\\ &\le \frac{2\eta_1}{a(1)}\mathcal{E}_{v,u}(S) + \frac{(2-\mu_a)\,M_{a,b}}{4}\int_S^T \mathcal{E}_{v,u}(t)\,dt + \frac{2\eta_2\left(2+2b\sqrt{C_a}\right)}{\beta\sqrt{c_{a,\beta}}}\mathcal{E}_{v,u}(S)\\ &\quad+\frac{\eta_2}{\delta_{a,b,\beta}}\left(1+\frac{1+b}{\beta c_{a,\beta}}\right)\mathcal{E}_{v,u}(S). \qquad (70)\end{aligned}$$

Using (70) in (69), it results that

$$\int_S^T \mathcal{E}_{v,u}(t)\,dt \le M_{a,b,\beta}\mathcal{E}_{v,u}(S), \qquad (71)$$

where

$$\begin{aligned}M_{a,b,\beta} = \frac{2}{(2-\mu_a)\,M_{a,b}}&\left[2C^1_{\varepsilon_{a,b}} + (2-\mu_a)\sqrt{C_a} + \frac{3(2-\mu_a)}{4b}\left(2+\frac{1}{2\varepsilon_{a,b}}\right)\right.\\ &+\frac{4\varepsilon_{a,b}}{\min\{1,a(1)\}}\left(\frac{1}{2}+\frac{3(2-\mu_a)}{8b}\right)+\frac{2\eta_1}{a(1)}+\frac{2\eta_2\left(2+2b\sqrt{C_a}\right)}{\beta\sqrt{c_{a,\beta}}}\\ &\left.+\frac{\eta_2}{\delta_{a,b,\beta}}\left(1+\frac{1+b}{\beta c_{a,\beta}}\right)\right]. \qquad (72)\end{aligned}$$

Finally, by [9, Theorem 8.1] (that has also been used before in [41]), we conclude that the energy of the system (1) satisfies the exponential decay estimate (29). The proof is thus complete. □

Remark 2 We observe that the constant $M_{a,b,\beta}$, which is the reciprocal of the exponential decay rate, satisfies

$$\lim_{b \longrightarrow 0^+} M_{a,b,\beta} = +\infty.$$

In this case, the decay estimate will be weaker: there is no exponential energy decay. In our opinion, this is quite natural due to the lack of coupling effects.

Moreover, concerning the influence of the parameter β on the decay rate, we also have

$$\lim_{\beta \longrightarrow 0^+} M_{a,b,\beta} = +\infty,$$

as for the case of a single degenerate wave equation (see [37]). Consequently, if $\beta = 0$, then exponential stability of the system (1) is still an open problem.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of the paper.

Research involving human participants and/or animals Not applicable.

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