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Ramified covering maps of singular curves and stability of pulled back bundles

Indranil Biswas[1] · Manish Kumar[2] · A. J. Parameswaran[3]

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Abstract

Let $f : X \longrightarrow Y$ be a generically smooth nonconstant morphism between irreducible projective curves, defined over an algebraically closed field, which is étale on an open subset of Y that contains both the singular locus of Y and the image, in Y, of the singular locus of X. We prove that the following statements are equivalent:

(1) The homomorphism of étale fundamental groups

$$f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y)$$

induced by f is surjective.

(2) There is no nontrivial étale covering $\phi : Y' \longrightarrow Y$ admitting a morphism $q : X \longrightarrow Y'$ such that $\phi \circ q = f$.

(3) The fiber product $X \times_Y X$ is connected.

(4) $\dim H^0(X, f^* f_* \mathcal{O}_X) = 1$.

(5) $\mathcal{O}_Y \subset f_* \mathcal{O}_X$ is the maximal semistable subsheaf.

(6) The pullback $f^* E$ of every stable sheaf E on Y is also stable.

Keywords Étale over singular locus · Stable bundle · Genuinely ramified map

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✉ Indranil Biswas
indranil.biswas@snu.edu.in ; indranil29@gmail.com

Manish Kumar
manish@isibang.ac.in

A. J. Parameswaran
param@math.tifr.res.in

1 Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India

2 Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India

3 School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

1 Introduction

Let k be an algebraically closed field. Let X and Y be irreducible smooth projective curves and $f : X \longrightarrow Y$ a generically smooth nonconstant map. In [1] it was proved that the following six statements are equivalent:

(1) The homomorphism between étale fundamental groups

$$f_* : \pi_1^{\mathrm{et}}(X) \longrightarrow \pi_1^{\mathrm{et}}(Y)$$

induced by f is surjective.

(2) The map f does not factor through some nontrivial étale cover of Y (in particular, f is not nontrivial étale).

(3) The fiber product $X \times_Y X$ is connected.

(4) $\dim H^0(X, f^* f_* \mathcal{O}_X) = 1$.

(5) The maximal semistable subbundle of the direct image $f_* \mathcal{O}_X$ is $\mathcal{O}_Y$.

(6) For every stable vector bundle E on Y, the pullback $f^* E$ is also stable.

Our aim here is to extend this to the context of generically smooth morphisms between singular curves. Examples show that some conditions are needed in order to be able extend the above result to the context of generically smooth morphisms between singular curves; see Sect. 4. To address this, we consider maps that are étale over singular locus (EOSL for short).

Let X and Y be reduced irreducible projective curves over k, and let

$$f : X \longrightarrow Y$$

be a generically smooth nonconstant morphism. The singular loci of X and Y are denoted by S_X and S_Y respectively. The map f is called EOSL if f is étale over a neighborhood of $S_Y \cup f(S_X)$.

Let $\widehat{X}$ and $\widehat{Y}$ be the normalizations of X and Y respectively. A map $f : X \longrightarrow Y$ produces a map $f' : \widehat{X} \longrightarrow \widehat{Y}$.

Let $f : X \longrightarrow Y$ be an EOSL map. We prove that the following seven statements are equivalent (see Theorem 3.2 and Theorem 3.4):

(1) f is genuinely ramified.

(2) The map $f' : \widehat{X} \longrightarrow \widehat{Y}$ is genuinely ramified.

(3) The homomorphism of étale fundamental groups

$$f_* : \pi_1^{\mathrm{et}}(X) \longrightarrow \pi_1^{\mathrm{et}}(Y)$$

induced by f is surjective.

(4) There is no nontrivial étale covering $\phi : Y' \longrightarrow Y$ admitting a morphism $q : X \longrightarrow Y'$ such that $\phi \circ q = f$.

(5) The fiber product $X \times_Y X$ is connected.

(6) $\dim H^0(X, f^* f_* \mathcal{O}_X) = 1$.

(7) The pullback $f^* E$ of every stable sheaf E on Y is also stable.

2 EOSL maps and semistability

The base field k is assumed to be algebraically closed. There is no assumption on its characteristic.

Let X and Y be reduced irreducible projective curves over k, and let

$$f : X \longrightarrow Y$$

be a generically smooth nonconstant morphism. Let $B_f \subset Y$ be the branch locus of f, i.e., the finite subset of Y over which f fails to be étale. So the restriction

$$f\big|_{f^{-1}(Y\setminus B_f)} : f^{-1}(Y \setminus B_f) \longrightarrow Y \setminus B_f$$

is étale. The singular locus of X (respectively, Y) will be denoted by S_X (respectively, S_Y). The map f will be called *étale over singular locus* (EOSL for short) if

$$B_f \cap (S_Y \cup f(S_X)) = \emptyset. \tag{2.1}$$

Therefore, f is EOSL if f is étale at every point of $S_X \cup f^{-1}(S_Y)$. Moreover, when f is EOSL, then $S_X = f^{-1}(S_Y)$.

Lemma 2.1 *Let $f : X \longrightarrow Y$ be an EOSL map. Then the following two hold:*

(1) The direct image $f_*\mathcal{O}_X$ is locally free on Y.
(2) For any torsionfree sheaf E on Y, the pullback f^*E is torsionfree.

Proof The map f is flat and hence $f_*\mathcal{O}_X$ is locally free. To see this another way, note that $f_*\mathcal{O}_X$ is torsionfree and hence it is locally free on $Y \setminus S_Y$, where S_Y is the singular locus of Y. Now since f is étale over neighborhoods of points of S_Y it follows immediately that $f_*\mathcal{O}_X$ is locally free on a neighborhood of S_Y. Hence $f_*\mathcal{O}_X$ is locally free on entire Y.

The second statement is evident. □

Take an EOSL map $f : X \longrightarrow Y$. From Lemma 2.1 we know that $f_*\mathcal{O}_X$ is locally free. Let

$$F_1 \subset F_2 \subset \cdots \subset F_m = f_*\mathcal{O}_X \tag{2.2}$$

be the Harder–Narasimhan filtration of $f_*\mathcal{O}_X$ (see [4]). Note that $m = 1$ if $f_*\mathcal{O}_X$ is semistable. The subsheaf $F_1 \subset f_*\mathcal{O}_X$ in (2.2) is called the maximal semistable subsheaf of $f_*\mathcal{O}_X$, and $\frac{\text{degree}(F_1)}{\text{rank}(F_1)} \in \mathbb{Q}$ is denoted by $\mu_{\max}(f_*\mathcal{O}_X)$ [4]. In general, $\frac{\text{degree}(V)}{\text{rank}(V)} \in \mathbb{Q}$ is denoted by $\mu(V)$.

Since $f_*\mathcal{O}_X$ is locally free, the pullback $f^*f_*\mathcal{O}_X$ is locally free. In view of this, the proof of the following lemma is identical to the proof in the special case where both X and Y are smooth [1].

Lemma 2.2 *For the subsheaf F_1 in (2.2),*

$$\text{degree}(F_1) = 0.$$

Proof This is proved in [1, p. 12825, (2.7)] assuming that X and Y are smooth. Since this lemma turns out to be crucial here, we give the details of its proof.

Since $f^*\mathcal{O}_Y = \mathcal{O}_X$, it follows from the adjunction formula (see [2, p. 110]) that

$$\mathcal{O}_Y \subset f_*\mathcal{O}_X.$$

This implies that

$$\mu(F_1) = \mu_{\max}(f_*\mathcal{O}_X) \geq \mu_{\max}(\mathcal{O}_Y) = 0. \tag{2.3}$$

On the other hand, a result on general vector bundles on X says the following: Since f^*V is torsionfree for any torsionfree sheaf V on Y For any vector bundle $\mathcal{V}$ on X,

$$\mu_{\max}(f_*\mathcal{V}) \;\leq\; \frac{\mu_{\max}(\mathcal{V})}{\text{degree}(f)}$$

(see [1, p. 12824, Lemma 2.2]). Substituting $\mathcal{O}_X$ in place of $\mathcal{V}$ we conclude that

$$\mu_{\max}(f_*\mathcal{V}) \;\leq\; 0.$$

This and (2.3) together completes the proof. □

Proposition 2.3 *The subsheaf $F_1 \subset f_*\mathcal{O}_X$ in (2.2) is a subbundle, or in other words, the quotient $(f_*\mathcal{O}_X)/F_1$ is locally free.*

Proof Let $\beta : \widehat{Y} \longrightarrow Y$ be the normalization of Y; so $\widehat{Y}$ is an irreducible smooth projective curve. Consider the fiber product

$$\begin{array}{ccc} \widehat{X} := X \times_Y \widehat{Y} & \stackrel{\beta'}{\longrightarrow} & X \\ \downarrow f' & & \downarrow f \\ \widehat{Y} & \stackrel{\beta}{\longrightarrow} & Y \end{array} \tag{2.4}$$

The given condition that f is EOSL implies that $\widehat{X}$ in (2.4) is smooth. Indeed, since f is étale at $S_X \cup f^{-1}(S_Y)$, we have $f(S_X) = S_Y$. By base change f' is étale over $\beta^{-1}(S_Y)$. Hence all points in $(\beta')^{-1}(S_X) = (f')^{-1}(\beta^{-1}(S_Y))$ are smooth points of $\widehat{X}$.

Since $f_*\mathcal{O}_X$ is locally free, the pullback $\beta^* f_*\mathcal{O}_X$ is also locally free. We have

$$\beta^* f_*\mathcal{O}_X \;=\; f'_*\mathcal{O}_{\widehat{X}}, \tag{2.5}$$

where f' is the map in (2.4). Let $\widehat{F} \subset f'_*\mathcal{O}_{\widehat{X}}$ be the maximal semistable subsheaf (so it is the first nonzero term of the Harder–Narasimhan filtration of $f'_*\mathcal{O}_{\widehat{X}}$). We know that $\text{degree}(\widehat{F}) = 0$ [1, p. 12825, (2.7)]. Therefore, from Lemma 2.2 it follows that for the isomorphism in (2.5),

$$\beta^* f_*\mathcal{O}_X \supset \beta^* F_1 \;\subset\; \widehat{F} \subset f'_*\mathcal{O}_{\widehat{X}}. \tag{2.6}$$

Note that the algebra structure of the sheaf $\mathcal{O}_{\widehat{X}}$ makes $f'_*\mathcal{O}_{\widehat{X}}$, where f' is as in (2.4), a sheaf of algebras over $\widehat{Y}$, and the corresponding spectrum is the (ramified) covering f'. The subsheaf $\widehat{F} \subset f'_*\mathcal{O}_{\widehat{X}}$ in (2.6) turns out to be a sheaf of subalgebras (see [1, p. 12826, Lemma 2.4]). Let

$$\phi' : \widehat{Y}' \longrightarrow \widehat{Y}$$

be the (possibly ramified) covering map given by the spectrum of the sheaf of algebras $\widehat{F}$. Therefore, we have

$$\widehat{F} \;=\; \phi'_*\mathcal{O}_{\widehat{Y}'}. \tag{2.7}$$

The fact that $\widehat{F}$ is a sheaf of subalgebras of $f'_*\mathcal{O}_{\widehat{X}}$ implies that we have a morphism

$$q : \widehat{X} \longrightarrow \widehat{Y}' \tag{2.8}$$

such that $\phi' \circ q = f'$ (see [1, p. 12828, (2.11)] and the line following it). Note that this implies that $\phi'_*\mathcal{O}_{\widehat{Y}'} \subset f'_*\mathcal{O}_{\widehat{X}}$, and (2.7) implies that the two subsheaves $\phi'_*\mathcal{O}_{\widehat{Y}'}$ and $\widehat{F}$ of

$f'_*\mathcal{O}_{\widehat{X}}$ coincide. Since degree$(\widehat{F}) = 0$, using [1, p. 12825, Lemma 2.3] it follows that ϕ' is actually étale (see the lines following [1, p. 12829, (2.12)]).

The above étale covering ϕ' of $\widehat{Y}$ produces an étale covering

$$\phi : Y' \longrightarrow Y. \tag{2.9}$$

To see this, first note that the restriction of ϕ' to the complement $\widehat{Y}' \setminus (\beta \circ \phi')^{-1}(S_Y)$ produces an étale covering

$$\phi_0 : Y'_0 \longrightarrow Y \setminus S_Y \tag{2.10}$$

because the restriction

$$\beta\big|_{\widehat{Y}\setminus\beta^{-1}(S_Y)} : \widehat{Y} \setminus \beta^{-1}(S_Y) \longrightarrow Y \setminus S_Y$$

is an isomorphism. The map q in (2.8) produces a map

$$q_0 : X \setminus f^{-1}(S_Y) \longrightarrow Y'_0$$

(see (2.10)). Indeed, q_0 is simply the restriction of q to $\widehat{X} \setminus (\beta \circ f')^{-1}(S_Y)$ (note that $X\setminus f^{-1}(S_Y) = \widehat{X}\setminus(\beta \circ f')^{-1}(S_Y)$). Since $\phi' \circ q = f'$, it follows that

$$\phi_0 \circ q_0 = f\big|_{X\setminus f^{-1}(S_Y)}, \tag{2.11}$$

where ϕ_0 is the map in (2.10). Now from (2.1) and (2.11) it follows that ϕ_0 extends to an étale covering ϕ as in (2.9).

The identification of $\widehat{Y}'\setminus(\beta \circ \phi')^{-1}(S_Y)$ with Y'_0 extends to a map

$$\beta_1 : \widehat{Y}' \longrightarrow Y'$$

because $\widehat{Y}'$ is smooth. Since the diagram

$$\begin{array}{ccc} \widehat{Y}' & \xrightarrow{\beta_1} & Y' \\ \downarrow \phi' & & \downarrow \phi \\ \widehat{Y} & \xrightarrow{\beta} & Y \end{array}$$

is Cartesian, we conclude that

$$\phi'_*\mathcal{O}_{\widehat{Y}'} = \beta^*(\phi_*\mathcal{O}_{Y'}). \tag{2.12}$$

We have degree$(\phi_*\mathcal{O}_{Y'}) = 0$ because ϕ is étale (see [1, p. 13825, Lemma 2.3]). Hence from Lemma 2.2 it follows that

$$\phi_*\mathcal{O}_{Y'} \subset F_1.$$

Consequently, (2.12) implies that $\phi'_*\mathcal{O}_{\widehat{Y}'} \subset \beta^*F_1$. This and (2.7) together give that $\widehat{F} \subset \beta^*F_1$. From this and (2.6) we conclude that

$$\beta^*F_1 = \widehat{F} \tag{2.13}$$

as subsheaves of $\beta^*f_*\mathcal{O}_X = f'_*\mathcal{O}_{\widehat{X}}$ (see (2.5)). On the other hand, from (2.7) and (2.12) we have $\widehat{F} = \beta^*(\phi_*\mathcal{O}_{Y'})$. Combining this with (2.13) it is deduced that $F_1 = \phi_*\mathcal{O}_{Y'}$. Now observe that $\phi_*\mathcal{O}_{Y'}$ is subbundle of $f_*\mathcal{O}_X$ because ϕ is étale and f is étale over a neighborhood of $S_Y \cup f(S_X)$. This completes the proof. □

Corollary 2.4 *The notation of the proof of Proposition 2.3 is used.*

(1) $\mathrm{rank}(F_1) = \mathrm{rank}(\widehat{F})$.
(2) $F_1 = \mathcal{O}_Y$ *if and only if* $\widehat{F} = \mathcal{O}_{\widehat{Y}}$.

Proof The first statement follows immediately from (2.13).

Since $\mathcal{O}_Y \subset f_* \mathcal{O}_X$ and $\mathcal{O}_{\widehat{Y}} \subset f'_* \mathcal{O}_{\widehat{X}}$, from Lemma 2.2 it follows that $\mathcal{O}_Y \subset F_1$ and $\mathcal{O}_{\widehat{Y}} \subset \widehat{F}$. Therefore, $F_1 = \mathcal{O}_Y$ (respectively, $\widehat{F} = \mathcal{O}_{\widehat{Y}}$) if and only if $\mathrm{rank}(F_1) = 1$ (respectively, $\mathrm{rank}(\widehat{F}) = 1$). Now the second statement follows from the first statement. □

Proposition 2.5 *Let $f : X \longrightarrow Y$ be an EOSL map. For any semistable vector bundle E on Y the pullback f^*E is also semistable.*

Proof As before, $B_f \subset Y$ is the finite subset over which f fails to be étale. Let $Y^o = Y \backslash B_f$. Consider the étale cover

$$f' := f\big|_{X^o} : X^o \longrightarrow Y^o$$

obtained by restricting f to the complement $X^o := X \backslash f^{-1}(B_f)$. Let

$$f'' : Z^o \longrightarrow Y^o$$

be the Galois closure of f' with the Galois group $\mathrm{Gal}(f'')$ being denoted by G. Since the map f is EOSL, it can be shown that the above map f'' extends to a ramified G–Galois cover

$$\widetilde{f} : Z \longrightarrow Y,$$

where Z is a projective curve containing Z^o. To prove this, let $y_1, \cdots, y_r$ be the singular points of Y. Let $\widehat{Y}_i$ be the formal neighborhood of y_i in Y and $\widehat{Y}_i^o = Y^o \times_Y \widehat{Y}_i$ for $1 \le i \le r$. Note that for each $i \in \{1, \cdots, r\}$, the map f is étale over y_i and f'' is the Galois closure of f' which is the restriction of f. Hence the pullback of f'' along $\widehat{Y}_i^o \longrightarrow Y^o$ gives an isomorphism of G–Galois covers

$$Z^o \times_{Y^o} \widehat{Y}_i^o \longrightarrow \widehat{Y}_i^o$$

with the G–Galois cover $\mathrm{Ind}_{\{e\}}^G \widehat{Y}_i^o \longrightarrow \widehat{Y}_i^o$ induced from the trivial cover defined by the identity map on $\widehat{Y}_i^o$. These isomorphisms allow us to patch G-Galois covers

$$\bigcup_{i=1}^{r} \mathrm{Ind}_{\{e\}}^G \widehat{Y}_i \longrightarrow \bigcup_{i=1}^{r} \widehat{Y}_i$$

and $Z^o \longrightarrow Y^o$ along $\bigcup_{i=1}^r \mathrm{Ind}_{\{e\}}^G \widehat{Y}_i^o \longrightarrow \bigcup_{i=1}^r \widehat{Y}_i^o$ to obtain the G-cover $\widetilde{f} : Z \longrightarrow Y$ (using [3, Theorem 3.1.9] with the category of modules replaced by category of G-covers as in [3, Theorem 3.2.4]).

Note that $\widetilde{f}$ is étale on a neighborhood of S_Y. So $\widetilde{f}$ is flat. Let

$$\phi : Z \longrightarrow X$$

be the map such that $\widetilde{f} = f \circ \phi$.

Take any semistable vector bundle E on Y. Assume that f^*E is not semistable. Let $V \subset f^*E$ be a subsheaf that destabilizes f^*E. Then ϕ^*V destabilizes $\phi^*f^*E = \widetilde{f}^*E$.

Let $W \subset \widetilde{f}^*E$ be the maximal semistable subsheaf of $\widetilde{f}^*E$ (in other words, it is the first nonzero term in the Harder–Narasimhan filtration of $\widetilde{f}^*E$). Note that the natural action of the Galois group $\mathrm{Gal}(\widetilde{f})$ on $\widetilde{f}^*E$ preserves the above subsheaf W. Indeed, this follows immediately from the uniqueness of the Harder–Narasimhan filtration.

The restriction of W to $Z \setminus \widetilde{f}^{-1}(S_Y)$ descends $Y \setminus S_Y$. On the other hand, the map f is étale over a neighborhood U of S_Y, so the restriction of W to $f^{-1}(U)$ descends to U. Consequently, W descends to a subsheaf of E. Since W destabilizes $\widetilde{f}^*E$, it follows immediately that this descend of W to a subsheaf of E destabilizes E. But E is semistable. In view of this contradiction we conclude that f^*E is semistable. □

3 Genuinely ramified maps

Let $f : X \longrightarrow Y$ be an EOSL map. Consider the maximal semistable subsheaf $F_1 \subset f_*\mathcal{O}_X$. From Proposition 2.3 we know that F_1 is a subbundle of $f_*\mathcal{O}_X$.

Definition 3.1 An EOSL map $f : X \longrightarrow Y$ will be called *genuinely ramified* if $F_1 = \mathcal{O}_Y$.

From Corollary 2.4(2) we know that f is genuinely ramified if and only if the map $f' : \widehat{X} \longrightarrow \widehat{Y}$ in (2.4) is genuinely ramified. From the proof the Proposition 2.3 it follows that the homomorphism of étale fundamental groups

$$f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y)$$

induced by f is surjective if and only if the homomorphism of étale fundamental groups

$$f'_* : \pi_1^{\text{et}}(\widehat{X}) \longrightarrow \pi_1^{\text{et}}(\widehat{Y})$$

induced by f' in (2.4) is surjective.

Theorem 3.2 *Let $f : X \longrightarrow Y$ be an EOSL map between projective curves. Then the following six statements are equivalent:*

(1) f is genuinely ramified.
(2) The map $f' : \widehat{X} \longrightarrow \widehat{Y}$ in (2.4) is genuinely ramified.
(3) The homomorphism of étale fundamental groups

$$f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y)$$

induced by f is surjective.
(4) There is no nontrivial étale covering $\phi : Y' \longrightarrow Y$ admitting a morphism $q : X \longrightarrow Y'$ such that $\phi \circ q = f$.
(5) The fiber product $X \times_Y X$ is connected.
(6) $\dim H^0(X,\ f^*f_*\mathcal{O}_X) = 1$.

Proof It was shown that the first two statements are equivalent. The third and fourth statements are clearly equivalent.

To show that the fifth and sixth statements are equivalent, consider the fiber product

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\varphi} & X \\ \big\downarrow \beta & & \big\downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

We have $\beta_*\mathcal{O}_{X\times_Y X} = f^*f_*\mathcal{O}_X$. Since the above diagram is Cartesian, we have

$$f^*f_*\mathcal{O}_X = \beta_*\varphi^*\mathcal{O}_X = \beta_*\mathcal{O}_{X\times_Y X},$$

and hence

$$H^0(X,\ f^*f_*\mathcal{O}_X) = H^0(X,\ \beta_*\mathcal{O}_{X\times_Y X}) = H^0(X\times_Y X,\ \mathcal{O}_{X\times_Y X}). \tag{3.1}$$

Since $X\times_Y X$ is connected if and only if $\dim H^0(X\times_Y X,\ \mathcal{O}_{X\times_Y X}) = 1$, from (3.1) it follows that the fifth and sixth statements are equivalent.

To show that the second and third statements are equivalent, recall that the third statement holds if and only if the homomorphism of étale fundamental groups

$$f'_* : \pi_1^{\mathrm{et}}(\widehat{X}) \longrightarrow \pi_1^{\mathrm{et}}(\widehat{Y})$$

induced by f' in (2.4) is surjective. But this homomorphism f'_* is surjective if and only if f' is genuinely ramified [1, p. 12828, Proposition 2.6]. So the second and third statements are equivalent.

We will now show that the first statement implies the sixth statement. From Proposition 2.5 we conclude that for any vector bundle V on Y, the Harder–Narasimhan filtration of f^*V is simply the pullback, by f, of the Harder–Narasimhan filtration of V.

Assume that f is genuinely ramified. This implies that the Harder–Narasimhan filtration of $f_*\mathcal{O}_X$ in (2.2) is of the form

$$\mathcal{O}_Y = F_1 \subset F_2 \subset \cdots \subset F_m = f_*\mathcal{O}_X,$$

where $\mathrm{degree}(F_j/F_{j-1}) < 0$ for all $2 \leq j \leq m$. Consequently, the Harder–Narasimhan filtration of $f^*f_*\mathcal{O}_X$ is the following:

$$\mathcal{O}_X = f^*F_1 \subset f^*F_2 \subset \cdots \subset f^*F_m = f^*f_*\mathcal{O}_X, \tag{3.2}$$

Since $\mathrm{degree}(f^*F_j/f^*F_{j-1}) = \mathrm{degree}(f)\cdot\mathrm{degree}(F_j/F_{j-1}) < 0$, we have

$$H^0(X,\ (f^*F_j)/(f^*F_{j-1})) = 0$$

for all $2 \leq j \leq m$. In view of this, from (3.2) it follows that

$$H^0(X,\ f^*f_*\mathcal{O}_X) = H^0(X,\ f^*F_1) = H^0(X,\ \mathcal{O}_X).$$

Hence the sixth statement holds.

Finally, we will show that the sixth statement implies the fourth statement. To prove this by contradiction, let $\phi : Y' \longrightarrow Y$ be a nontrivial étale covering, and $q : X \longrightarrow Y'$ a morphism, such that $\phi\circ q = f$. Then we have

$$\phi_*\mathcal{O}_{Y'} \subset f_*\mathcal{O}_X,$$

and hence

$$q^*\phi^*\phi_*\mathcal{O}_{Y'} = f^*\phi_*\mathcal{O}_{Y'} \subset f^*f_*\mathcal{O}_X.$$

This implies that

$$\dim H^0(X,\ f^*f_*\mathcal{O}_X) \geq \dim H^0(Y',\ \phi^*\phi_*\mathcal{O}_{Y'}). \tag{3.3}$$

Since ϕ is étale, Y' is a connected component of $Y'\times_Y Y'$ using the diagonal map. So $Y'\times_Y Y'$ is not connected. Hence setting $f = \phi$ in (3.1) we conclude that

$$\dim H^0(Y',\ \phi^*\phi_*\mathcal{O}_{Y'}) \geq 2.$$

Therefore, (3.3) contradicts the sixth statement. So the sixth statement implies the fourth statement. This completes the proof. □

Lemma 3.3 *Let $f : X \longrightarrow Y$ be an EOSL map of degree d between projective curves. Assume that there is a finite group Γ acting faithfully on X such that $Y = X/\Gamma$. If f is genuinely ramified, then*

$$f^*((f_*\mathcal{O}_X)/\mathcal{O}_Y) = (f^*f_*\mathcal{O}_X)/\mathcal{O}_X \hookrightarrow \bigoplus_{i=1}^{d-1} \mathcal{L}_i,$$

where each $\mathcal{L}_i$ is a line bundle on X of negative degree.

Lemma 3.3 was proved in [1] under the assumption that X is smooth (see [1, p. 12837, Proposition 3.5]). The same proof works here.

Theorem 3.4 *Let $f : X \longrightarrow Y$ be an EOSL map of degree d between projective curves. Then the following two statements are equivalent:*

(1) f is genuinely ramified;
(2) f^*E is stable for every stable vector sheaf E on Y.

Theorem 3.4 is proved exactly as Theorem 5.3 of [1, p. 12850] is proved.

4 Some examples

4.1 Example 1

Let Y an irreducible nodal projective curve of arithmetic genus at least two. Let $f : X \longrightarrow Y$ be the normalization. Then f satisfies (4) and (5) of Theorem 3.2 but does not satisfy (1) and (3) of Theorem 3.2. Note that f is not an EOSL map.

4.2 Example 2

Consider the map

$$\phi : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1, \quad z \longrightarrow z^2.$$

Let $\psi_1 : \mathbb{CP}^1 \longrightarrow X$ be the rational nodal curve of arithmetic genus one obtained by identifying 1 and $\sqrt{2}$. Let $\psi_2 : \mathbb{CP}^1 \longrightarrow Y$ be the rational nodal curve of arithmetic genus one obtained by identifying 1 and 2. The map $\psi_2 \circ \phi$ factors ψ_1. In other words, there is a unique map

$$f : X \longrightarrow Y$$

such that $\psi_2 \circ \phi = f \circ \psi_1$. This map f is clearly not EOSL. Note that the homomorphism of étale fundamental groups

$$f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y)$$

induced by f is surjective. So statement (3) of Theorem 3.2 holds. We will show that there is a stable vector bundle on Y whose pullback to X is not stable.

Let $\beta : Z \longrightarrow Y$ be the unique étale covering of degree two (note that $\pi_1(Y) = \mathbb{Z}$). Let L be a holomorphic line bundle on Z of degree one. Then the direct image β_*L is a vector bundle of rank two and degree one. To prove that $\beta_* L$ is semistable, take any rank one subsheaf $F \subset \beta_* L$. Then we have a nonzero homomorphism $\beta^* F \longrightarrow L$ because

$$H^0(Y,\, \mathrm{Hom}(F,\, \beta_* L)) \cong H^0(Z,\, \mathrm{Hom}(\beta^* F,\, L))$$

(see [2, p. 110]). Since there is a nonzero homomorphism $\beta^* F \longrightarrow L$, we conclude that

$$2 \cdot \mathrm{degree}(F) = \mathrm{degree}(\beta^* F) \leq \mathrm{degree}(L) = 1.$$

Therefore, it follows that $\beta_* L$ is semistable. Since $\mathrm{degree}(\beta_* L) = 1$, this implies that $\beta_* L$ is stable. So $f^* \beta_* L$ is a vector bundle on X of rank two and degree two.

It can be shown that there is no stable vector bundle of rank two and degree two on X. Indeed, if W is a vector bundle on X of rank two and degree two, then

$$\dim H^0(X,\, W) \geq \dim H^0(X,\, W) - \dim H^1(X,\, W) = 2.$$

Take two linearly independent sections s and t on W, and consider the evaluation homomorphism

$$\eta : \mathcal{O}_X \oplus \mathcal{O}_X \longrightarrow W$$

that sends any $(c_1,\, c_2) \in \mathcal{O}_x \oplus \mathcal{O}_x = \mathbb{C}^2, x \in X$, to $c_1 \cdot s(x) + c_2 \cdot t(x) \in W_x$. This η is not an isomorphism over X, because $\mathrm{degree}(W) = 2 > \mathrm{degree}(\mathcal{O}_X \oplus \mathcal{O}_X)$. Therefore, there is $(a,\, b) \neq (0,\, 0)$ such that $as + bt$ vanishes at some point of X. The degree of the rank one subsheaf of W generated by $as + bt$ is at least one. Hence W is not stable. In particular, $f^* \beta_* L$ is not stable.

4.3 Example 3

Let

$$\gamma : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \tag{4.1}$$

be the morphism defined by $z \longrightarrow z^d$, with $d \geq 5$. Denote by X the nodal curve of arithmetic genus 1 obtained by identifying $1 \in \mathbb{CP}^1$ with $-1 \in \mathbb{CP}^1$. Let Y be the nodal curve of arithmetic genus 1 obtained by identifying $1 \in \mathbb{CP}^1$ with $\exp(\pi\sqrt{-1}/d) \in \mathbb{CP}^1$. The map γ in (4.1) produces a map

$$f : Y \longrightarrow X. \tag{4.2}$$

Note that $\pi_1(Y,\, y_0) = \mathbb{Z} = \pi_1(X,\, f(y_0))$, and the induced homomorphism

$$f_* : \pi_1(Y,\, y_0) \longrightarrow \pi_1(X,\, f(y_0)) \tag{4.3}$$

is an isomorphism.

Consider the vector bundle $\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ on $\mathbb{CP}^1$ of rank two and degree one. Take any isomorphism of fibers over 1 and -1

$$I := (\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_1 \longrightarrow (\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_{-1} \tag{4.4}$$

such that $I(\mathcal{O}_{\mathbb{CP}^1}(1)_1) = (\mathcal{O}_{\mathbb{CP}^1})_{-1}$. Identifying the fibers of $\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ over -1 and -1 using I we obtain a vector bundle E on X of rank two and degree 1. It is straightforward to check that E is stable.

Now consider the vector bundle

$$\gamma^*(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1)) \cong \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(d)$$

on $\mathbb{CP}^1$, where γ is the map in (4.1). Consider the following isomorphism of its fibers over 1 and $\exp(\pi\sqrt{-1}/d)$:

$$\gamma^*(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_1 = (\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_1 \xrightarrow{I} (\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_{-1}$$
$$= \gamma^*(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))_{\exp(\pi\sqrt{-1}/d)},$$

where I is the isomorphism in (4.4). Identifying the fibers of $\gamma^*(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))$ over -1 and $\exp(\pi\sqrt{-1}/d)$ using this isomorphism we obtain a vector bundle V on Y of rank two and degree d. Note that we have

$$f^*E = V, \tag{4.5}$$

where f is the map in (4.2).

We will construct a subsheaf of f^*E of rank one and degree $d-2$. Consider the subsheaf

$$\mathcal{O}_{\mathbb{CP}^1}(d-2) \cong (\gamma^*(\mathcal{O}_{\mathbb{CP}^1}(1))) \otimes \mathcal{O}_{\mathbb{CP}^1}(-1-\exp(\pi\sqrt{-1}/d))$$
$$\subset \gamma^*(\mathcal{O}_{\mathbb{CP}^1}(1)) \subset \gamma^*(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1)).$$

It produces a subsheaf of V of rank 1 and degree $d-2$. Now using the isomorphism in (4.5) this subsheaf produces a subsheaf of f^*E of rank 1 and degree $d-2$. Consequently, the vector bundle f^*E is not stable (recall that $d \geq 5$), although E is stable and the homomorphism in (4.3) is an isomorphism.

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References

1. Biswas, I., Parameswaran, A.J.: Ramified covering maps and stability of pulled back bundles. Int. Math. Res. Not. **17**, 12821–12851 (2022)
2. Hartshorne, R.: *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, (1977)
3. Harbater, D.: Patching and Galois theory. In: *Galois groups and fundamental groups*, 313–424. Math. Sci. Res. Inst. Publ., 41 Cambridge University Press, Cambridge, (2003)
4. Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig (1997)