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General one-dimensional Clifford Fourier Transform and applications to probability theory

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Abstract
In this essay, we outline the basic characteristics of the general one-dimensional Clifford Fourier transform and its fundamental properties. In addition, we provide some applications to probability theory, Rényi and Shannon entropy. And we illustrate results with examples.

Keywords Clifford geometric algebra · Clifford Fourier transform · Rényi and Shannon entropy · Probability theory

Mathematics Subject Classification 42B10 · 30G35 · 60B11

1 Introduction

The idea of Clifford algebra was first introduced by the famous mathematician of the late 19^{th} century William Kingdon Clifford [7]. It is a mathematical stricture that extends the principles of complex numbers and vectors to higher dimensions. This concept is based on the notion of a geometric product between vectors into a field stricture, which provides a robust tool for analyzing geometric objects in higher dimensions. Its application is far-reaching, spanning fields such as physics, engineering, and computer science.

One of the significant strengths of Clifford algebra is its ability to unify and simplify many areas of mathematics and physics, including linear algebra, differential geometry, electromagnetism, and quantum mechanics. In [11], authors aim to connect Clifford algebras, manifolds and harmonic analysis, and to demonstrate the fundamental role of algebra, geometry, and differential equations in Euclidean Fourier analysis. They also combined the representation theory of Euclidean space with the representation theory of semisimple Lie groups.

Several works have explored the applications of Clifford algebra in signal processing. For instance, in [4] Bracx et al. introduces the new Clifford-Fourier transform, with a focus on the 2D case. Todd ell proposes the Fourier transform over the algebra of quaternions

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($\mathbb{H} \simeq C\ell_{0,2}$) in [10], called quaternionic Fourier transform (QFT), which he explores to analyze systems described by partial differential equations. Hitzer has also made significant contributions to the development of this theory. In his works, he examines the different forms of the quaternionic Fourier transform (QFT) and explores its application to quaternion fields, providing corresponding Plancherel theorems [13]. He also derived a new directional uncertainty principle for quaternion-valued functions using the quaternionic Fourier transform in [14], and extends it to establish similar principles in Clifford geometric algebras with quaternion subalgebras. In [15], he explains the orthogonal planes split (OPS) of quaternions based on the choice of one or two linearly independent pure unit quaternions and systematically generalizes the quaternionic Fourier transform applied to quaternion fields to conform with the OPS, establishing inverse transformations and commenting on their geometric meaning. He generalized, in his chapters [17, 19], the aforementioned split (OPS) to a freely steerable orthogonal 2D-planes split of two orthonormal and collinear pure unit quaternions. This general form of OPS allows new geometric interpretations of the action of the QFT on the signals. In their works [5, 20], P. Lounesto and Bracx et al. provide a historical review of the development and applications of quaternion and Clifford algebra wavelets.

In addition to the above, Bahri et al. introduced, in [2, 3], the one-dimensional quaternion Fourier transform and have established its properties which generalizes the Fourier transform and studied its application in probability theory. Based on the relations between the original function and the fractional Fourier transform, Authors derived, in [12], Rényi and Shannon entropic uncertainty principles. The works on Clifford algebra, therefore, have significant implications in several fields, and their continued exploration promises to unlock further insights and advancements.

2 The Clifford geometric algebra

The Clifford geometric algebra $C\ell(\mathbb{R}^{p,q}) = C\ell_{p,q}$ over the $\mathbb{R}-$linear space $\mathbb{R}^{p,q}$, is a non-commutative algebra generated by the $\mathbb{R}^{p,q}$-orthonormal vector basis $\mathscr{B} = \{e_1, \ldots, e_n\}$ (with $p + q = n$) obeying to the following associative non-commutative geometric multiplication rules (see [11, 17])

$$e_\ell e_k + e_k e_\ell = 2\delta_{\ell,k}\varepsilon_\ell \tag{1}$$

where $\delta_{\ell,k}$ is the Kronecker symbol and

$$\varepsilon_\ell = \mathbb{1}_{[\![1,p]\!]}(\ell) - \mathbb{1}_{[\![p+1,n]\!]}(\ell). \tag{2}$$

The Clifford geometric algebra $C\ell_{p,q}$ can be split into the following direct sum [17, 20]

$$C\ell_{p,q} = \bigoplus_{\ell=0}^{n} C\ell^{\ell}_{p,q} \tag{3}$$

where $C\ell^{\ell}_{p,q}$ denotes the space spanned by the ℓ-vectors family

$$\mathscr{B}_\ell = \{e_{\sigma_1}e_{\sigma_2}\cdots e_{\sigma_\ell}, \qquad 1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\ell \le n\}. \tag{4}$$

Therefore, the set

$$\{e_\Sigma = e_{\sigma_1}e_{\sigma_2}\cdots e_{\sigma_\ell}, \qquad \Sigma \subseteq [\![1,n]\!], \quad 1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\ell \le n\} \cup \{e_\emptyset = 1\} \tag{5}$$

forms a graded (blade) basis of $C\ell_{p,q}$. The grades ℓ range from 0 for scalars, 1 for vectors, 2 for bivectors, ℓ for ℓ-vectors, up to n for pseudo-scalars. The the field $\mathbb{R}$ (resp. $\mathbb{R}$-linear space $\mathbb{R}^{p,q}$) is included in $C\ell_{p,q}$ as the subset of 0-vectors (resp. 1-vectors).
The Clifford product (1) generates a basis for $C\ell_{p,q}$ consisting of 2^n elements. A general element $\mathcal{C}$ of $C\ell_{p,q}$ (called Clifford numbers, multivectors or hypercomplex numbers) is a real linear combination of basis blades $(e_\Sigma)_\Sigma$ and can be expanded as [17]

$$\mathcal{C} = \sum_{\Sigma\subseteq[\![1,n]\!]} \mathcal{C}_\Sigma e_\Sigma = \overbrace{\mathcal{C}_\emptyset}^{\text{scalar part}} + \overbrace{\sum_{\ell\in[\![1,n]\!]} \mathcal{C}_\ell e_\ell}^{\text{vector part}} + \overbrace{\sum_{1\le\ell,k\le n} \mathcal{C}_{\ell k} e_\ell e_k}^{\text{bivector part}} + \cdots + \overbrace{\mathcal{C}_{12\cdots n} e_1 e_2 \cdots e_n}^{\text{pseudo-scalar part}} \tag{6}$$

where $\mathcal{C}_\Sigma$ are real-valued coefficients. $\mathcal{C}$ can also be written as

$$\mathcal{C} = \sum_{\ell=0}^{n} \langle\mathcal{C}\rangle_\ell = \langle\mathcal{C}\rangle_0 + \langle\mathcal{C}\rangle_1 + \cdots + \langle\mathcal{C}\rangle_n \tag{7}$$

with $\langle\mathcal{C}\rangle_\ell = \sum_{\#\Sigma=\ell} \mathcal{C}_\Sigma e_\Sigma$ denotes the ℓ-vectors part of $\mathcal{C}$. As examples, $\langle\mathcal{C}\rangle_0$ denotes the scalar part, $\langle\mathcal{C}\rangle_1$ the vector part, $\langle\mathcal{C}\rangle_2$ the bi-vector part and $\langle\mathcal{C}\rangle_n$ the pseudo-scalar part.
The principal reverse of a multi-vector $\mathcal{C} \in C\ell(p,q)$ is defined as [17, 20]

$$\widetilde{\mathcal{C}} = \sum_{\ell=0}^{n} (-1)^{\frac{\ell(\ell-1)}{2}} \overline{\langle\mathcal{C}\rangle_\ell} \tag{8}$$

where $\overline{\mathcal{C}}$ means to change in the basis decomposition of $\mathcal{C}$ the sign of every vector of negative square $\overline{e_\Sigma} = \varepsilon_{\sigma_1} e_{\sigma_1} \varepsilon_{\sigma_2} e_{\sigma_2} \cdots \varepsilon_{\sigma_\ell} e_{\sigma_\ell}$ where $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\ell \le n$ and ε_{σ_k} are given by (2).
The principal reverse is linear, involution and anti-automorphic, that is for all $\mathcal{C}, \mathcal{D} \in C\ell_{p,q}$

$$\widetilde{\mathcal{C}+\mathcal{D}} = \widetilde{\mathcal{C}} + \widetilde{\mathcal{D}}, \qquad \widetilde{\widetilde{\mathcal{C}}} = \mathcal{C}, \qquad \widetilde{\mathcal{C}\mathcal{D}} = \widetilde{\mathcal{D}}\widetilde{\mathcal{C}}. \tag{9}$$

The scalar product of $\mathcal{C}, \mathcal{D} \in C\ell_{p,q}$ can be defined by [17]

$$\mathcal{C} * \widetilde{\mathcal{D}} = \langle\mathcal{C}\widetilde{\mathcal{D}}\rangle_0 = \sum_{\Sigma\subseteq[\![1,n]\!]} \mathcal{C}_\Sigma \mathcal{D}_\Sigma. \tag{10}$$

In particular, if $\mathcal{C} = \mathcal{D}$, then the modulus of a multi-vector $\mathcal{C} \in C\ell_{p,q}$ is given by [17, 20]

$$|\mathcal{C}| = \sqrt{\langle\mathcal{C}\widetilde{\mathcal{C}}\rangle_0} = \sqrt{\sum_{\Sigma\subseteq[\![1,n]\!]} \mathcal{C}_\Sigma^2}. \tag{11}$$

For $\mathcal{C}, \mathcal{D} \in C\ell_{p,q} \quad (p+q=n\ge 3)$, the following property holds [9]

$$|\mathcal{C}\mathcal{D}| \le 2^n |\mathcal{C}||\mathcal{D}|. \tag{12}$$

Inner product on the square-integrable Clifford geometric algebra valued-function space $f, g \in L^2(\mathbb{R}, C\ell_{p,q})$ is defined as follow

$$(f,g)_{L^2(\mathbb{R},C\ell_{p,q})} = \int_{\mathbb{R}} f(x)\widetilde{g(x)}dx. \tag{13}$$

For $r \ge 1$, we get the $L^r(\mathbb{R}, C\ell_{p,q})$-norm of f as

$$\|f\|^r_{L^r(\mathbb{R},C\ell_{p,q})} = \int_{\mathbb{R}} |f(x)|^r dx. \tag{14}$$

3 The general one-dimensional Clifford Fourier transform

Let's denote $\mathbb{C}_\mu$ the 2D sub-plane of $C\ell_{p,q}$ spanned by $\{1, \mu\}$, where $\mu \in C\ell_{p,q}$ and $\mu^2 = -1$

$$\mathbb{C}_\mu = span\{1, \mu\} = \{a + b\mu \ , \quad a, b \in \mathbb{R}.\} \tag{15}$$

$\mathbb{C}_\mu$ is an algebraically closed commutative field isomorphic to the complex plane $\mathbb{C}$. Each unit hypercomplexe number $q \in \mathbb{C}_\mu$ can be written, in the polar form, as [9]

$$q = e^{\theta\mu} = (\cos\theta + \mu \sin\theta) \,, \tag{16}$$

where $\cos\theta = \mathcal{S}c(q), \quad \sin\theta = |\mathcal{V}ec(q)|$ and $\mu = \frac{\mathcal{V}ec(q)}{|\mathcal{V}ec(q)|}$.

Definition 3.1 Let $\mu \in C\ell_{p,q}$ with $\mu^2 = -1$. The general one-dimensional Clifford Fourier transform (1DCFT) of $f \in L^1(\mathbb{R}, C\ell_{p,q})$, with respect to μ is given by

$$\mathcal{F}^\mu(f)(\xi) = \int_{\mathbb{R}} f(x) e^{\mu x \xi} dx \tag{17}$$

where $x, \xi \in \mathbb{R}$.

If we use the $C\ell_{p,q}$-basis expansion; $f = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma f_\Sigma$, 1DCFT of f becomes

$$\mathcal{F}^\mu(f)(\xi) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \mathcal{F}^\mu(f_\Sigma)(\xi). \tag{18}$$

The convolution of two Clifford algebra valued functions $f, g \in L^1(\mathbb{R}, C\ell_{p,q})$ is defined by

$$f * g(y) = \int_{\mathbb{R}} f(x) g(y - x) dx. \tag{19}$$

We present in the following fundamental properties of the 1DCFT, for their proofs and more comprehensive analysis, refer to [1, 2, 16, 18],

- For all $\mathcal{C}, \mathcal{D} \in C\ell_{p,q}$ and $f, g \in L^1(\mathbb{R}, C\ell_{p,q})$ we get

$$\mathcal{F}^\mu(\mathcal{C}f + \mathcal{D}g)(\xi) = \mathcal{C}\mathcal{F}^\mu(f)(\xi) + \mathcal{D}\mathcal{F}^\mu(g)(\xi). \tag{20}$$

- For all $f \in L^1(\mathbb{R}, C\ell_{p,q})$, and $h \in \mathbb{R}$ we have

$$\mathcal{F}^\mu(\tau_h f)(\xi) = \mathcal{F}^\mu(f)(\xi) e^{-\mu h \xi} \tag{21}$$

where the translation operator is given by $\tau_h f(x) = f(x + h)$.
- For all $f, \mathcal{F}^\mu(f) \in L^1(\mathbb{R}, C\ell_{p,q})$, f is recovered from its Fourier transform as

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}^\mu(f)(\xi) e^{-\mu \xi x} d\xi. \tag{22}$$

- For all $f, \mathcal{F}^\mu(f) \in L^2(\mathbb{R}, C\ell_{p,q})$, Parseval's identity holds

$$2\pi \|f\|_{L^2(\mathbb{R}, C\ell_{p,q})} = \|\mathcal{F}^\mu(f)\|_{L^2(\mathbb{R}, C\ell_{p,q})}. \tag{23}$$

- For $f \in \mathscr{C}^m(\mathbb{R}, C\ell_{p,q})$, we have

$$\mathcal{F}^\mu\left(\frac{d^m}{dx^m} f\right)(\xi) = \mathcal{F}^\mu\,(f)\,(\xi)(-\mu\xi)^m. \tag{24}$$

- If $1 \leq p \leq 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(\mathbb{R}, C\ell_{p,q})$, then

$$\|\mathcal{F}^\mu(f)\|_{L^q(\mathbb{R},C\ell_{p,q})} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \|f\|_{L^p(\mathbb{R},C\ell_{p,q})}. \tag{25}$$

- (1DCFT) $\mathcal{F}^\mu$ maps $L^1(\mathbb{R}, C\ell_{p,q})$ into $\mathscr{C}_0(\mathbb{R}, C\ell_{p,q})$ and it is one-to-one. Where $\mathscr{C}_0(\mathbb{R}, C\ell_{p,q})$ is the set of continuous functions vanishing at infinity.
- Let $f, g \in L^1(\mathbb{R}, C\ell_{p,q})$ such that $\mathcal{F}^\mu(f), \mathcal{F}^\mu(g) \in L^1(\mathbb{R}, C\ell_{p,q})$. We have

$$\int_{\mathbb{R}} \mathcal{F}^\mu(f)(\xi)\widetilde{g}(\xi)d\xi = \int_{\mathbb{R}} f(\xi)\widetilde{\mathcal{F}^\mu(g)}(-\xi)d\xi. \tag{26}$$

- Let $f, g \in L^1(\mathbb{R}, C\ell_{p,q})$. If we use the expansion (6) of g, we get immediately

$$\mathcal{F}^\mu(f * g)(\xi) = \sum_{\Sigma\subseteq[\![1,n]\!]} \mathcal{F}^\mu(f e_\Sigma)(\xi)\mathcal{F}^\mu(g_\Sigma)(\xi). \tag{27}$$

4 One-dimensional Clifford Fourier transform in probability theory

Definition 4.1 A Clifford algebra-valued function $f_X(x) = \sum_{\Sigma\subseteq[\![1,n]\!]} e_\Sigma (f_X)_\Sigma(x)$ is called the Clifford algebra probability density function of a real random variable X if $\forall\ \Sigma \subseteq [\![1, n]\!]$

$$\int_{\mathbb{R}} (f_X)_\Sigma(x)dx = 1 \qquad \text{and} \qquad \{(f_X)_\Sigma < 0\} = \emptyset. \tag{28}$$

Here, $(f_X)_\Sigma$ is a real probability density function. The Clifford algebra cumulative distribution function is expressed as

$$f_X(x) = \frac{d}{dx}F_X(x), \tag{29}$$

where the probability P is related to F_X given by

$$F_X(x) = P(X \leq x). \tag{30}$$

Definition 4.2 Let X be a real random variable with the Clifford Algebra probability density function f_X. The ℓ^{th} moment of X is defined as

$$m_\ell = E[X^\ell] = \int_{\mathbb{R}} x^\ell f_X(x)dx. \tag{31}$$

If we set

$$f_X(x) = \sum_{\Sigma\subseteq[\![1,n]\!]} e_\Sigma (f_X)_\Sigma(x) \qquad \text{and} \qquad \int_{\mathbb{R}} x^\ell (f_X)_\Sigma(x)dx := E[X_\Sigma^\ell] = (m_\ell)_\Sigma,$$

we get

$$m_\ell = \int_{\mathbb{R}} x^\ell \sum_{\Sigma\subseteq[\![1,n]\!]} e_\Sigma (f_X)_\Sigma(x)dx = \sum_{\Sigma\subseteq[\![1,n]\!]} e_\Sigma (m_\ell)_\Sigma. \tag{32}$$

It is easily seen that

$$|m_\ell|^2 = E[X] * \widetilde{E[X]} = \sum_{\Sigma\subseteq[\![1,n]\!]} (m_\ell)_\Sigma^2. \tag{33}$$

The variance in the Clifford Algebra setting of a real random variable X is defined by

$$\sigma^2 = m_2 - m_1^2 = E[X^2] - (E[X])^2. \tag{34}$$

Definition 4.3 Let X be a real random variable with the Clifford algebra probability density function f_X. The characteristic function of X, $\phi_X : \mathbb{R} \longrightarrow C\ell_{p,q}$, is defined by the formula (compare with (15))

$$\phi_X(t) = E[e^{\mu t X}] = \int_{\mathbb{R}} f_X(x) e^{\mu t x} dx = \mathcal{F}^{\mu}(f_X)(t). \tag{35}$$

Setting $f_X(x) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_{\Sigma} (f_X)_{\Sigma}(x)$, the characteristic function of X can be expressed as

$$\phi_X(t) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_{\Sigma} (\phi_X)_{\Sigma}(t). \tag{36}$$

By inversion formula, we get

$$f_X(x) = \mathcal{F}^{-\mu}(\phi_X)(t). \tag{37}$$

From (35), (29) and equation (24), one gets

$$\phi_X(t) = -\mathcal{F}^{\mu}\left(F_X\right)(t)\mu t.$$

Thus, for $t \neq 0$

$$\mathcal{F}^{\mu}\left(F_X\right)(t) = \frac{1}{t}\phi_X(t)\mu. \tag{38}$$

Definition 4.4 The Rènyi entropy of a Clifford algebra probability density function f_X of a real random variable X [8, 12].

$$\mathcal{H}_r(f_X) = \frac{1}{1-r} \log\left(\int_{\mathbb{R}} f_X^r(t) dt\right) \quad / \quad r \in]0, 1[\cup]1, +\infty[\tag{39}$$

Shannon entropy is given by

$$\mathcal{H}(f_X) = \lim_{r\to 1} \mathcal{H}_r(f_X) = E[\log(f_X(X))] = -\int_{\mathbb{R}} f_X(t)\log(f_X(t))dt. \tag{40}$$

Theorem 4.5 (Rènyi and Shannon entropy uncertainty principle) *Let* $f \in L^r(\mathbb{R}, C\ell_{p,q})$, $1 \leq r < 2$ *and* $\frac{1}{r} + \frac{1}{s} = 1$, *then*

$$\frac{1}{r-2}\log(r) + \frac{1}{s-2}\log(s) \leq \mathcal{H}_{\frac{r}{2}}(|f_X|^2) + \mathcal{H}_{\frac{s}{2}}(|\phi_X|^2). \tag{41}$$

And

$$\log(4e) \leq \mathcal{H}\left(|f_X|^2\right) + \mathcal{H}\left(|\phi_X|^2\right). \tag{42}$$

Proof Young-Hausdorff type inequality (25) becomes

$$r^{-\frac{1}{2r}} s^{\frac{1}{2s}} \leq \left(\int_{\mathbb{R}} |f_X(x)|^r dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}} |\phi_X(\xi)|^s d\xi\right)^{\frac{-1}{s}}. \tag{43}$$

Since $\frac{2r}{2-r} = \frac{2s}{s-2}$, then

$$r^{\frac{1}{r-2}} s^{\frac{1}{s-2}} \leq \left(\int_{\mathbb{R}} |f_X(x)|^r dx \right)^{\frac{2}{2-r}} \left(\int_{\mathbb{R}} |\phi_X(\xi)|^s d\xi \right)^{\frac{2}{2-s}}. \tag{44}$$

Taking log on both sides, one gets

$$\frac{1}{r-2}\log(r) + \frac{1}{s-2}\log(s) \leq \frac{2}{2-r}\log\left(\int_{\mathbb{R}} |f_X(x)|^r dx \right) + \frac{2}{2-s}\log\left(\int_{\mathbb{R}} |\phi_X(\xi)|^s d\xi \right). \tag{45}$$

Which means

$$\frac{1}{r-2}\log(r) + \frac{1}{s-2}\log(s) \leq \mathcal{H}_{\frac{r}{2}}(|f_X|^2) + \mathcal{H}_{\frac{s}{2}}(|\phi_X|^2). \tag{46}$$

We have

$$\frac{\log(r)}{r-2} + \frac{\log(s)}{s-2} = \frac{r+s-4}{(r-2)(s-2)}\log(2) + \frac{1}{r-2}\log\left(\frac{r}{2}\right) + \frac{1}{s-2}\log\left(\frac{s}{2}\right) \tag{47}$$

and

$$\lim_{r\to 2} \frac{1}{r-2}\log\left(\frac{r}{2}\right) = \frac{1}{2}, \qquad \text{and} \qquad 2 \leq \frac{r+s-4}{(r-2)(s-2)}. \tag{48}$$

In the limit when $(r, s) \longrightarrow (2, 2)$, we get

$$\log(4e) \leq \mathcal{H}\left(|f_X|^2\right) + \mathcal{H}\left(|\phi_X|^2\right) \tag{49}$$

□

Theorem 4.6 *Let X be a real random variable with the Clifford algebra probability density function f_X^2 ($f_X \in \mathscr{C}^1(\mathbb{R}, C\ell_{p,q})$), then*

$$|f_X(x)| \leq 2^n \left\| \frac{df_X}{dx} \right\|_{L^2(\mathbb{R}, C\ell_{p,q})}. \tag{50}$$

Proof ***i-*** $\frac{df_X}{dx} \notin L^2(\mathbb{R}, C\ell_{p,q})$, the aforementioned inequality obviously holds.
ii- $\frac{df_X}{dx} \in L^2(\mathbb{R}, C\ell_{p,q})$. By (12) and (22), one gets

$$|f_X(x)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} \mathcal{F}^{\mu}(f_X)(\xi) e^{\mu\xi x} d\xi \right| \leq \frac{2^{n-1}}{\pi} \int_{\mathbb{R}} \left| \mathcal{F}^{\mu}(f_X)(\xi) \right| d\xi. \tag{51}$$

Cauchy-Schwartz inequality, (23) and (24) give, for $\rho > 0$,

$$\begin{aligned}
|f_X(x)| &\leq \frac{2^{n-1}}{\pi} \int_{\mathbb{R}} (\rho + \xi^2) \left| \mathcal{F}^{\mu}(f_X)(\xi) \right|^2 d\xi \int_{\mathbb{R}} \frac{1}{\rho + \xi^2} d\xi \\
&\leq \frac{2^{n-1}}{\sqrt{\rho}} \int_{\mathbb{R}} (\rho + \xi^2) \left| \mathcal{F}^{\mu}(f_X)(\xi) \right|^2 d\xi \\
&\leq \frac{2^{n-1}}{\sqrt{\rho}} \left[\int_{\mathbb{R}} \rho \left| \mathcal{F}^{\mu}(f_X)(\xi) \right|^2 d\xi + \int_{\mathbb{R}} \xi^2 \left| \mathcal{F}^{\mu}(f_X)(\xi) \right|^2 d\xi \right] \\
&\leq \frac{2^{n-1}}{\sqrt{\rho}} \left[2\pi\rho + 2\pi \int_{\mathbb{R}} \left| \frac{df_X}{dx}(x) \right|^2 dx \right] \\
&\leq 2^{n-1}\sqrt{\rho} + \frac{2^{n-1}}{\sqrt{\rho}} \left\| \frac{df_X}{dx} \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})}.
\end{aligned}$$

Setting $\rho = \left\| \frac{df_X}{dx} \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})}$, we obtain

$$|f(x)| \leq 2^n \left\| \frac{df_X}{dx} \right\|_{L^2(\mathbb{R}, C\ell_{p,q})}. \tag{52}$$

□

Theorem 4.7 *Let X be a real random variable with the Clifford algebra probability density function $f_X = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma f_\Sigma$ where $\{f_\Sigma \leq 0\} = \emptyset$. Then $\forall \quad \Sigma \subseteq [\![1, n]\!]$*

$$1 \leq \left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \|\xi(\phi_X)_\Sigma\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} (m_2)_\Sigma. \tag{53}$$

Proof Let

$$f_X = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma f_\Sigma = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma g_\Sigma^2. \tag{54}$$

The Heisenberg uncertainty principle [6] gives

$$\frac{\|g\|^4_{L^2(\mathbb{R}, C\ell_{p,q})}}{4} \leq \|\xi \mathcal{F}^\mu(g_\Sigma)\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \|x g_\Sigma\|^2_{L^2(\mathbb{R}, C\ell_{p,q})}. \tag{55}$$

We have

$$\|f\|_{L^1(\mathbb{R}, C\ell_{p,q})} = \|g\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} = 1, \tag{56}$$

and

$$\|x g_\Sigma\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} = \int_{\mathbb{R}} x^2 g_\Sigma^2(x) dx = \int_{\mathbb{R}} x^2 f_\Sigma(x) dx = (m_2)_\Sigma. \tag{57}$$

By Parseval identity (23)

$$\begin{aligned}
\|\xi \mathcal{F}^\mu(g_\Sigma)\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} &= \left\| \mathcal{F}^\mu \left(\frac{\partial}{\partial x} g_\Sigma \right) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \\
&= 2\pi \left\| \frac{\partial}{\partial x} g_\Sigma \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \\
&= \frac{\pi}{2} \left\| \frac{1}{\sqrt{f_\Sigma}} \frac{\partial}{\partial x} f_\Sigma \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \\
&= \frac{\pi}{2} \left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \frac{\partial}{\partial x} f_\Sigma \right\|_{L^1(\mathbb{R}, C\ell_{p,q})}.
\end{aligned}$$

Since

$$\begin{aligned}
\left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \frac{\partial}{\partial x} f_\Sigma \right\|_{L^1(\mathbb{R}, C\ell_{p,q})} &\leq \left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \left\| \frac{\partial}{\partial x} f_\Sigma \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \\
&\leq \frac{1}{2\pi} \left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \left\| \mathcal{F}^\mu \left(\frac{\partial}{\partial x} f_\Sigma \right) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})}.
\end{aligned}$$

Then (55) becomes

$$1 \leq \left\| \frac{\partial}{\partial x} \ln(f_\Sigma) \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} \left\| \xi(\phi_X)_\Sigma \right\|^2_{L^2(\mathbb{R}, C\ell_{p,q})} (m_2)_\Sigma . \tag{58}$$

□

Theorem 4.8 *Let ϕ_X and ψ_X be two Clifford Algebra characteristic functions of the random variable X, given by*

$$\phi_X(t) = \int_{\mathbb{R}} f_X(x) e^{\mu t x} dx \qquad \textit{and} \qquad \psi_X(x) = \int_{\mathbb{R}} g_X(t) e^{\mu t x} dt, \tag{59}$$

then

$$\int_{\mathbb{R}} g_X(t)\phi_X(t) e^{-\mu t y} dt = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma f_X * (\check{\psi}_X)_\Sigma(y), \tag{60}$$

with $\quad (\check{\psi}_X)_\Sigma(x) = (\psi_X)_\Sigma(-x)$

Proof Let's expand g_X on $C\ell_{p,q}$-basis; $\quad g_X(x) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma (g_X)_\Sigma(x)$. Forward calculations yield

$$\begin{aligned}
\int_{\mathbb{R}} g_X(t)\phi_X(t) e^{\mu t x} dt &= \int_{\mathbb{R}} g_X(t) \left(\int_{\mathbb{R}} f_X(x) e^{\mu t x} dx \right) e^{-\mu t y} dt \\
&= \int_{\mathbb{R}} g_X(t) \left(\int_{\mathbb{R}} f_X(x) e^{\mu t (x-y)} dx \right) dt \\
&= \int_{\mathbb{R}} \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma (g_X)_\Sigma(t) \left(\int_{\mathbb{R}} f_X(x) e^{\mu t (x-y)} dx \right) dt \\
&= \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \int_{\mathbb{R}} f_X(x) \int_{\mathbb{R}} (g_X)_\Sigma(t) e^{\mu t (x-y)} dt dx \\
&= \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \int_{\mathbb{R}} f_X(x) (\psi_X)_\Sigma(x-y) dx. \\
&= \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma f_X * (\check{\psi}_X)_\Sigma(y).
\end{aligned}$$

□

Theorem 4.9 *If X is a real random variable, then there exists ℓ^{th} derivatives for the Clifford Algebra characteristic function ϕ_X which is given by the formula*

$$\frac{d^\ell}{dt^\ell}\phi_X(t) = \int_{\mathbb{R}} x^\ell f_X(x) e^{\mu t x} dx \mu^\ell. \tag{61}$$

Moreover

$$m_\ell = E[X^\ell] = \frac{d^\ell}{dt^\ell}\phi_X(0)(-\mu)^\ell. \tag{62}$$

Proof For $\ell = 1$, direct computations reveal that

$$\begin{aligned}\frac{d}{dt}\phi_X(t) &= \frac{d}{dt}\left(\int_{\mathbb{R}} f_X(x)e^{\mu tx}dx\right)\\ &= \int_{\mathbb{R}} f_X(x)\frac{d}{dt}\left(e^{\mu tx}\right)dx\\ &= \int_{\mathbb{R}} f_X(x)e^{\mu tx}x dx\mu.\end{aligned}$$

Suppose that

$$\frac{d^{\ell-1}}{dt^{\ell-1}}\phi_X(t) = \int_{\mathbb{R}} x^{\ell-1} f_X(x)e^{\mu tx}dx\mu^{\ell-1}. \tag{63}$$

We have

$$\begin{aligned}\frac{d^{\ell}}{dt^{\ell}}\phi_X(t) &= \frac{d}{dt}\left(\frac{d^{\ell-1}}{dt^{\ell-1}}\phi_X(t)\right)\\ &= \frac{d}{dt}\left(\int_{\mathbb{R}} x^{\ell-1} f_X(x)e^{\mu tx}dx\mu^{\ell-1}\right)\\ &= \int_{\mathbb{R}} x^{\ell-1} f_X(x)\frac{d}{dt}\left(e^{\mu tx}\right)dx\mu^{\ell-1}\\ &= \int_{\mathbb{R}} x^{\ell} f_X(x)e^{\mu tx}dx\mu^{\ell}.\end{aligned}$$

Hence

$$\frac{d^{\ell}}{dt^{\ell}}\phi_X(t)(-\mu)^{\ell} = \int_{\mathbb{R}} x^{\ell} f_X(x)e^{\mu tx}dx. \tag{64}$$

Then

$$m_{\ell} = E[X^{\ell}] = \frac{d^{\ell}}{dt^{\ell}}\phi_X(0)(-\mu)^{\ell}. \tag{65}$$

□

By (65), the variance σ of X in terms of the Clifford Algebra characteristic function can be expressed as

$$\sigma^2 = \frac{d^2}{dt^2}\phi_X(0)(-\mu)^2 - \left[\frac{d}{dt}\phi_X(0)(-\mu)\right]^2 = \left[\frac{d}{dt}\phi_X(0)\right]^2 - \frac{d^2}{dt^2}\phi_X(0). \tag{66}$$

5 Examples

i- Consider a real random variable X that can occur according to a Clifford algebra uniform law

$$f_X(x) = \sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\mathbb{1}_{[\alpha_{\Sigma},\beta_{\Sigma}]}. \tag{67}$$

We have

$$\mathcal{F}^{\mu}\left(\mathbb{1}_{[\alpha_{\Sigma},\beta_{\Sigma}]}\right)(t)=\begin{cases}\beta_{\Sigma}-\alpha_{\Sigma} & \text{if} \quad t=0\\ \dfrac{2}{t}\sin\left((\beta_{\Sigma}-\alpha_{\Sigma})\dfrac{t}{2}\right)e^{\mu(\beta_{\Sigma}+\alpha_{\Sigma})\frac{t}{2}} & \text{if} \quad t\neq 0.\end{cases}$$

It follows from (35) that

$$\phi_X(t)=\mathcal{F}^{\mu}\left(f_X\right)(t)=\begin{cases}\displaystyle\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\beta_{\Sigma}-\alpha_{\Sigma} & \text{if} \quad t=0\\ \displaystyle\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\frac{2}{t}\sin\left((\beta_{\Sigma}-\alpha_{\Sigma})\frac{t}{2}\right)e^{\mu(\beta_{\Sigma}+\alpha_{\Sigma})\frac{t}{2}} & \text{if} \quad t\neq 0.\end{cases}$$

The first and second derivatives of each real-valued coefficient of ϕ_X are given by

$$\frac{d}{dt}(\phi_X)_{\Sigma}(0)=\mu\frac{\beta_{\Sigma}^2-\alpha_{\Sigma}^2}{2},\qquad \text{and} \qquad \frac{d^2}{dt^2}(\phi_X)_{\Sigma}(0)=\frac{\alpha_{\Sigma}^3-\beta_{\Sigma}^3}{3}. \tag{68}$$

Then

$$m_1=\frac{d}{dt}\phi_X(0)(-\mu)=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\frac{\beta_{\Sigma}^2-\alpha_{\Sigma}^2}{2}, \tag{69}$$

and

$$m_2=\frac{d^2}{dt^2}\phi_X(0)(-\mu)^2=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\frac{\beta_{\Sigma}^3-\alpha_{\Sigma}^3}{3}. \tag{70}$$

By (66), we get

$$\sigma^2=m_2-m_1^2=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\frac{\beta_{\Sigma}^3-\alpha_{\Sigma}^3}{3}-\left(\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\frac{\beta_{\Sigma}^2-\alpha_{\Sigma}^2}{2}\right)^2. \tag{71}$$

ii- Let Y be a real random variable that has the probability density function

$$g_Y(x)=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\sqrt{\frac{\lambda_{\Sigma}}{\pi}}e^{-\lambda_{\Sigma}x^2}, \tag{72}$$

where $\left(\lambda_{\Sigma}\right)_{\Sigma\subseteq[\![1,n]\!]}$ is a finite sequence of strictly positive real numbers.
It follows from (35) that

$$\phi_Y(t)=\mathcal{F}^{\mu}\left(g_Y\right)(t)=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}\sqrt{\frac{\lambda_{\Sigma}}{\pi}}\int_{\mathbb{R}} e^{-\lambda_{\Sigma}x^2}e^{\mu t x}dx=\sum_{\Sigma\subseteq[\![1,n]\!]} e_{\Sigma}e^{-\frac{t^2}{4\lambda_{\Sigma}}}. \tag{73}$$

The first and second derivatives of each real-valued coefficient of ϕ_Y are given as

$$\frac{d}{dt}(\phi_Y)_{\Sigma}(t)=-\frac{t}{2\lambda_{\Sigma}}e^{-\frac{t^2}{4\lambda_{\Sigma}}}, \tag{74}$$

and

$$\frac{d^2}{dt^2}(\phi_Y)_{\Sigma}(t)=\left(\frac{t^2}{4\lambda_{\Sigma}^2}-\frac{1}{2\lambda_{\Sigma}}\right)e^{-\frac{t^2}{2\lambda_{\Sigma}}}. \tag{75}$$

Then

$$m_1 = \frac{d}{dt}\phi_Y(0)(-\mu) = 0, \tag{76}$$

and

$$m_2 = \frac{d^2}{dt^2}\phi_Y(0)(-\mu)^2 = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{1}{2\lambda_\Sigma}. \tag{77}$$

From (74,75), we conclude

$$\sigma^2 = m_2 - m_1^2 = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{1}{2\lambda_\Sigma}. \tag{78}$$

iii- Let Z be a real random variable that has the probability density function

$$h_Z(x) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \lambda_\Sigma e^{-\lambda_\Sigma x} \mathbb{1}_{[0,+\infty[}(x), \tag{79}$$

where $\left(\lambda_\Sigma\right)_{\Sigma \subseteq [\![1,n]\!]}$ is a finite sequence of strictly positive real numbers.
It follows from (35) that

$$\phi_Z(t) = \mathcal{F}^\mu\left(h_Z\right)(t) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \int_{\mathbb{R}} \lambda_\Sigma e^{-\lambda_\Sigma x} \mathbb{1}_{[0,+\infty[}(x) e^{\mu t x} dx = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{\lambda_\Sigma}{\lambda_\Sigma - \mu t}. \tag{80}$$

The first and second derivatives of each real-valued coefficient of ϕ_Z are given as

$$\frac{d}{dt}(\phi_Z)_\Sigma(t) = \frac{\lambda_\Sigma \mu}{(\lambda_\Sigma - \mu t)^2}, \tag{81}$$

and

$$\frac{d^2}{dt^2}(\phi_Z)_\Sigma(t) = \frac{2\mu\lambda_\Sigma t - 2\lambda_\Sigma^2}{(\lambda_\Sigma - \mu t)^4}. \tag{82}$$

Then

$$m_1 = \frac{d}{dt}\phi_Z(0)(-\mu) = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{1}{\lambda_\Sigma}, \tag{83}$$

and

$$m_2 = \frac{d^2}{dt^2}\phi_Z(0)(-\mu)^2 = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{2}{\lambda_\Sigma^2}. \tag{84}$$

$$\sigma^2 = m_2 - m_1^2 = \sum_{\Sigma \subseteq [\![1,n]\!]} e_\Sigma \frac{1}{\lambda_\Sigma^2}. \tag{85}$$

Conclusion

This article introduces and explores the properties of the one-dimensional Clifford Fourier transform (1DCFT), and showcases its practical application in deriving a related inequalities. The effectiveness of 1DCFT in probability theory is demonstrated by examining in detail the expected value, characteristic function, and variance within the framework of Clifford algebra. These results represent an important step forward in the development of probability theory using Clifford algebra. The study recommends future research into uncertainty principles concerning the Clifford Algebra probability density function and its characteristic function.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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