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Convergence and stability of a novel iterative algorithm for weak contraction in banach spaces

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Abstract
The aim of this paper is to exhibit a novel two-step iterative algorithm named PV algorithm to determine the fixed points of weak contractions in Banach spaces. Data dependence result is also obtained. It is proved that this PV iterative algorithm converges strongly to the fixed point of weak contractions. This iteration is almost stable for weak contraction. Furthermore, it is proved that rate of convergence of the PV iterative algorithm is faster than Picard, Ishikawa, Mann, S,normal-S, Varat, and F* algorithms. Examples are also given to support the main result. The results of this paper are original and will further enrich the existing literature.

Keywords PV iteration · Weak contraction · Fixed point · Numerically stable · Data dependence · Non-linear matrix equation

Mathematics Subject Classification 47H09 · 47H10

1 Introduction

Throughout this article, we assume that Z_+ represents the set of all nonnegative integers, and we consider the mapping $H : V^* \to V^*$, where V^* is a nonempty, convex, and closed subset of a Banach space B^*. We denote by $\text{Fix}(H)$ the set of all fixed points of H.

Several nonlinear problems can be mathematically formulated using self-mappings of the form $A(x) = x$. These mappings exhibit various properties such as contraction, continuity etc. Banach's work on contraction mappings is a well-celebrated result in the literature on fixed points. However, a question arises regarding the verification of the contraction condition for self-mappings when it is relaxed.

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In response to this question, Berinde [1] introduced a new concept known as weak contraction, also referred to as almost contractions. He established that the class of weak contractions is more general than the classes of contraction mappings, Kannan mappings [2], Chatterjee mappings [3], Zamfirescu mappings [4], etc. He developed the existence and uniqueness theorem for fixed points of these weaker contractions. Due to its wide range of applications, numerous researchers have examined and proposed iterative algorithms for various classes of mappings (e.g., see [5–9]). Additionally, many researchers [10–12] have expanded the scope of this theory by obtaining several extensions of fixed point theory. The iterative algorithms listed below are known as the Picard [13], Mann [14], Ishikawa [15], S [16], normal-S [17], Varat [18], and F^* [19] algorithms, respectively, for the self- mapping H defined on V^*. Here $\{r_m\}$, $\{s_m\}$, and $\{t_m\}$ are sequences in the interval (0, 1).

$$\begin{cases} p_0 \in V^* \\ p_{m+1} = Hp_m, m \in \mathbf{Z}_+ \end{cases} \tag{1.1}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = (1 - r_m)p_m + r_m Hp_m, m \in \mathbf{Z}_+ \end{cases} \tag{1.2}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = (1 - r_m)p_m + r_m Hq_m, \\ q_m = (1 - s_m)p_m + s_m Hp_m, \quad m \in \mathbf{Z}_+ \end{cases} \tag{1.3}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = (1 - r_m)Hp_m + s_m Hp_m, \\ q_m = (1 - s_m)p_m + s_m Hp_m, \quad m \in \mathbf{Z}_+ \end{cases} \tag{1.4}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = H((1 - r_m)p_m + r_m Hp_m), \quad m \in \mathbf{Z}_+ \end{cases} \tag{1.5}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = (1 - r_m)Hz_m + r_m Hq_m, \\ z_m = (1 - t_m)p_m + t_m q_m, \\ q_m = (1 - s_m)p_m + s_m Hp_m, \quad m \in \mathbf{Z}_+ \end{cases} \tag{1.6}$$

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = Hq_m \\ q_m = (1 - r_m)p_m + r_m Hp_m, m \in \mathbf{Z}_+ \end{cases} \tag{1.7}$$

A natural question arises from the above discussion whether it is feasible to discover a two-step iterative algorithm with the rate of convergence that is more accelerated than F^* iterative algorithm (1.7) and from some other iterative algorithms?

In this paper, a novel two-step iterative algorithm, PV algorithm, is introduced which is given for a mapping $H : V^* \to V^*$ where V^* is a nonempty, closed and convex subset of a Banach space B^*, the sequence $\{p_m\}$ is defined by:

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = Hq_m, \\ q_m = H((1 - r_m)H^2 p_m + r_m Hp_m), \quad m \in Z_+, \end{cases} \tag{1.8}$$

where r_m is a sequence in (0, 1).

Now, the main results are proved using PV iterative algorithm for weak contractions which satisfy (1.10) on an arbitrary Banach space. We begin with the subsequent result on strong convergence.

Now, we recall the definition of weak contraction.

Definition 1.1 [1](Weak contraction): A map $H : B^* \to B^*$ where B^* is a Banach space is termed as a weak contraction if for some constants $\delta \in (0, 1)$ and $L \geq 0$, we have:

$$\|H(x) - H(y)\| \leq \delta\|x - y\| + L\|y - H(x)\|, \quad \forall x, y \in B^*. \tag{1.9}$$

Berinde [1] proved the subsequent theorem for the uniqueness and the existence of a fixed point in these mappings.

Theorem 1.1 [1] *Let $H : B^* \to B^*$ where B^* is a Banach space be a weak contraction with $\delta \in (0, 1)$, $L \geq 0$ and it also satisfies*

$$\|H(x) - H(y)\| \leq \delta\|x - y\| + L\|x - H(x)\|, \ \forall x, y \in B^* \tag{1.10}$$

Then, the mapping H has a unique fixed point in B^*.

Ostrowski [20] defined the notion of stability as

Definition 1.2 Let $H : B^* \to B^*$, where B^* is a Banach space with some $p \in \text{Fix}(H)$. Assume that $p_0 \in B^*$ and $p_{m+1} = g(H, p_m)$ is an iterative algorithm for some function g. For a sequence $\{p_m\}$ in B^*, let $\{y_m\}$ be an approximate sequence, and define $\alpha_m = \|y_{m+1} - g(H, y_m)\|$. Then, the iterative algorithm $p_{m+1} = g(H, p_m)$ is called H-stable if

$$\lim_{m\to\infty} \alpha_m = 0 \iff \lim_{m\to\infty} y_m = p.$$

Using Definition 1.2, Harder and Marie [21], Harder [22] proved the stability of several iterative algorithms for various types of contractive-type operators. Moreover, Osilike [23, 24] proved the stability of Ishikawa and Mann iterative schemes for operators of contractive type. Ostrowski [20] provides the subsequent definition.

Definition 1.3 Let $H : B^* \to B^*$, where B^* is a Banach space, be a weak contraction, and $p \in \text{Fix}(H)$. Assume that $p_0 \in B^*$ and $p_{m+1} = g(H, p_m), m \in Z_+$, is an iterative algorithm for some function g. For a sequence $\{p_m\}$ in B^*, let $\{y_m\}$ be an approximate sequence and define $\alpha_m = \|y_{m+1} - g(H, y_m)\|$. Then, the iterative algorithm $p_{m+1} = g(H, p_m)$ is said to be almost H-stable if:

$$\sum_{m=0}^{\infty} \alpha_m < \infty \implies \lim_{m\to\infty} y_m = p.$$

To correlate the rate of convergence of two iterative algorithms, Berinde [25] provides the subsequent definitions.

Definition 1.4 Let $\{\eta_m\}$ and $\{\theta_m\}$ be two sequences in the set of positive real numbers that converge to η and θ, respectively. Suppose that:

$$l = \lim_{m\to\infty} \frac{\|\theta_m - \theta\|}{\|\eta_m - \eta\|}.$$

(i) If $l = 0$, then $\{\theta_m\}$ converges to θ faster than $\{\eta_m\}$ to η.
(ii) If $0 < l < \infty$, then $\{\theta_m\}$ and $\{\eta_m\}$ converge at the same rate of convergence.

Definition 1.5 Let $\{p_m\}$and $\{q_m\}$ be two iterative algorithms, both converging to the exact same point p with the following error estimates θ_m and η_m(best ones available) where θ_m, $\eta_m \to 0$ and satisfies

$$\|p_m - p\| \leq \theta_m \text{ and } \|q_m - p\| \leq \eta_m.$$

If $\lim_{m\to\infty} \frac{\theta_m}{\eta_m} = 0$, then $\{p_m\}$ converges faster than $\{q_m\}$.

In the context of Banach spaces, we aim to define and quantify the speed at which sequences or iterative algorithms converge to a common limit point. To achieve this, we introduce a new definition and establish its consistency with the definition 1.5.

Given $p \in X$ (Banach Space), we denote an 'ϵ' neighborhood of p as $V_\epsilon(p) = \{x \in X \mid \|x - p\| \leq \epsilon\}$.

Let $\{p_m\}$ and $\{q_m\}$ be two sequences in a Banach space X that converge to the same fixed point p. Our new definition states that the sequence $\{p_m\}$ is faster than $\{q_m\}$ if, after a certain number of steps, $\{p_m\}$ approaches p more closely than $\{q_m\}$ does. In other words, after a certain number of steps, $\{p_m\}$ always lies inside a smaller neighborhood of p compared to $\{q_m\}$.

Definition 1.6 Let $\{p_m\}$ and $\{q_m\}$ be two sequences in a Banach Space X such that both $\{p_m\}$ and $\{q_m\}$ converge to the same point p. We say that $\{p_m\}$ converges to p faster than $\{q_m\}$ if, for any positive real number $\epsilon_2 > 0$, there exists $\epsilon_1 > 0$ and $a \in \mathbb{N}$ such that $\epsilon_1 < \epsilon_2$, $\|p_m - p\| < \epsilon_1$, and $\|q_m - p\| < \epsilon_2$ for all $m \geq a$.

Now, we will demonstrate that the definition 1.6 is consistent with the definition 1.5. Consider $\{p_m\}$, $\{q_m\}$, θ_m, and η_m as in definition 1.5.

As $\lim_{m\to\infty} \frac{\theta_m}{\eta_m} = 0$, for some $0 < \epsilon_0 < 1$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, we have $\frac{\theta_m}{\eta_m} < \epsilon_0 \implies \theta_m < \epsilon_0 \eta_m$

Now since $\eta_m \to 0$, for any $\epsilon_2 > 0$, there exists $m_2 \in \mathbb{N}$ such that $0 < \eta_m < \epsilon_2$ for all $m \geq m_2$.

Consequently, as $\theta_m < \epsilon_0 \eta_m < \epsilon_0 \epsilon_2 = \epsilon_1$ (say) for all $m \geq K$ (where $K = \sup\{m_0, m_2\}$), we have $\epsilon_0 \epsilon_2 = \epsilon_1 < \epsilon_2$ (as,$0 < \epsilon_0 < 1$).

Hence, $\|p_m - p\| \leq \theta_m < \epsilon_1$ and $\|q_m - p\| \leq \eta_m < \epsilon_2$, where $\epsilon_1 < \epsilon_2$. Therefore, $\{p_m\}$ converges faster than $\{q_m\}$ as per definition 1.6.

Definition 1.7 Consider two self-operators K and H on a nonempty subset V^* of a Banach space B^*. If, for all $x \in V^*$ and for a fixed $\epsilon > 0$, we have $\|Hx - Kx\| \leq \epsilon$, then the operator K is said to be an approximate operator for H.

The subsequent lemma has an essential role in proving the major result of this paper.

Lemma 1.2 [26] *Let $\{a_m\}$ and $\{b_m\}$ be two sequences from the set of all nonnegative real numbers and $t \in [0, 1)$, such that $a_{m+1} \leq t a_m + b_m$, for all $m \geq 0$. Then,*

$$\lim_{m\to\infty} b_m = 0 \; implies \; \lim_{m\to\infty} a_m = 0.$$

Another useful lemma stated by [19]

Lemma 1.3 [19] *Let $\{\alpha_m\}$ be a sequence in $\mathbb{R}_+$ and there exists $N \in \mathbb{Z}_+$, such that for all $m \geq N$, $\{\alpha_m\}$ satisfies the following inequality:*

$$\alpha_{m+1} \leq (1 - \mu_m)\alpha_m + \mu_m \eta_m,$$

where $\mu_m \in (0, 1)$ *for all* $m \in \mathbb{Z}_+$, *such that* $\sum_{m=0}^{\infty} \mu_m = \infty$ *and* $\eta_m \geq 0$ *is a bounded sequence. Then:*

$$0 \leq \lim_{m\to\infty} \sup \alpha_m \leq \lim_{m\to\infty} \sup \eta_m.$$

2 Main results

We exhibit a novel two-step iterative algorithm, named as PV algorithm, which is given below

For a mapping $H : V^* \to V^*$,where V^* is a nonempty, closed and convex subset of a Banach space B^*, the sequence p_m is defined by:

$$\begin{cases} p_0 \in V^*, \\ p_{m+1} = Hq_m, \\ q_m = H((1-r_m)H^2 p_m + r_m H p_m), \quad m \in Z_+, \end{cases} \tag{2.1}$$

where r_m is a sequence in (0, 1).

Now, we will prove the main results using PV iterative algorithm for weak contractions which satisfy (1.10) on an arbitrary Banach space. We begin with the subsequent result on strong convergence.

Theorem 2.1 *Let* V^* *be a nonempty, closed and convex, subset of a Banach space* B^* *and* H *be a self map on* V^* *which is a weak contraction also satisfying (1.10). Then, the sequence* $\{p_m\}$ *defined by PV iterative algorithm (2.1) converges to the unique fixed point of* H.

Proof Let $p \in Fix(H)$. By condition (1.10), we have:

$$\begin{aligned} \|Hp_m - p\| &= \|Hp_m - Hp\| \\ &\leq \delta\|p_m - p\| + L\|Hp - p\| \\ &\leq \delta\|p_m - p\|, \qquad \forall m \in Z_+ \end{aligned}$$

Now, by PV iterative algorithm (2.1), we have

$$\begin{aligned} \|q_m - p\| &= \|H((1-r_m)H^2 p_m + r_m.Hp_m) - p\| \\ \|q_m - p\| &= \|H((1-r_m)H^2 p_m + r_m.Hp_m) - Hp\| \\ &\leq \delta\|(1-r_m)H^2 p_m + r_m H p_m - p\| \\ &\leq \delta\|(1-r_m)H^2 p_m + r_m H p_m - (1 - r_m + r_m)p\| \\ &\text{Now as p } \in Fix(H) \text{ thus } H^2(p) = p \\ &\leq \delta(1-r_m)\|H^2 p_m - H^2 p\| + \delta r_m\|Hp_m - Hp\| \\ &\leq \delta[(1-r_m)\delta^2 \|p_m - p\| + r_m\delta \|p_m - p\|] \end{aligned}$$

Thus, we have

$$\|q_m - p\| \leq \delta^2\|p_m - p\|$$

$$\|p_{m+1} - p\| = \|Hq_m - p\| \leq \delta\|q_m - p\| \leq \delta^3\|p_m - p\|$$

Inductively, we get

$$\|p_{m+1} - p\| \leq \delta^{3(m+1)}\|p_0 - p\| \tag{2.2}$$

Now, as $0 < \delta < 1$ hence $\{p_m\}$ converges strongly to p. □

The Subsequent theorem shows the almost H-stability of the PV iterative algorithm (2.1).

Theorem 2.2 *Let H be a weak contraction from V^* to V^*, which also satisfies (1.10), where V^* is a nonempty, closed and convex subset in a Banach space B^*. Then, the PV iterative algorithm (2.1) turns out to be almost H-stable.*

Proof Let y_m be an arbitrary sequence in V^*,and the sequence constructed by PV algorithm is $p_{m+1} = g(H, p_m)$ and $\sigma_m = \|y_{m+1} - g(H, z_m)\|$,where m is in Z_+. Now, we will show:

$$\sum_{m=0}^{\infty} \sigma_m < \infty \implies \lim_{m\to\infty} y_m = p.$$

Let $\sum_{m=0}^{\infty} \sigma_m < \infty$. Then,by the PV algorithm, we have

$$\begin{aligned}
\|y_{m+1} - p\| &\le \|y_{m+1} - g(H, y_m)\| + \|g(H, y_m) - p\| \\
&\le \sigma_m + \|H(H(1 - r_m)H^2 y_m + r_m H y_m) - p\| \\
&\le \sigma_m + \delta\|H((1 - r_m)H^2 y_m + r_m H y_m) - p\| \\
&\le \sigma_m + \delta^2\|(1 - r_m)H^2 y_m + r_m H y_m - p\| \\
&\le \sigma_m + \delta^2\|(1 - r_m)(H^2 y_m - H^2 p) + r_m(H y_m - H p)\| \\
&\le \sigma_m + \delta^2(1 - r_m)\|H^2 y_m - H^2 p\| + r_m\|H y_m - H p\| \\
&\le \sigma_m + \delta^2(1 - r_m)\|H^2 y_m - H^2 p\| + r_m\|H y_m - H p\| \\
&\le \sigma_m + \delta^2((1 - r_m)\delta^2 + r_m\delta)\|y_m - p\|
\end{aligned}$$

Thus , we have,

$$\|y_{m+1} - p\| \le \sigma_m + \delta^3\|y_m - p\|$$

$u_m = \|y_m - p\|$ and $q = \delta^3$ Then, we have $u_{m+1} \le \sigma_m + q.u_m$ as $q = \delta^3$, $\delta \in (0, 1)$ thus $0 < q < 1$ and $\sum_{m=0}^{\infty} \sigma_m < \infty \implies \sigma_m \to 0$. Therefore, $u_m \to 0$ using lemma 1.2. □

In this theorem, we will demonstrate that the PV algorithm is faster than other iterative algorithms.

Theorem 2.3 *Let $H : V^* \to V^*$ be a weak contraction also satisfying (1.10), where V^* is a nonempty, closed and convex subset in a Banach space B^*. Let the sequences $\{p_{1,m}\}$, $\{p_{2,m}\}$, $\{p_{3,m}\}$, $\{p_{4,m}\}$, $\{p_{5,m}\}$, $\{p_{6,m}\}$, $\{p_{7,m}\}$ and $\{p_m\}$ be defined by Picard, Mann, Ishikawa, S, normal-S, Varat, F^* iterative algorithms, and PV, respectively, and converge to the same point $p \in Fix(H)$. Then, the PV algorithm converges more rapidly than all the algorithms mentioned above.*

Proof Because of inequality (2.2) in Theorem 2.1, we have:

$$\|p_{m+1} - p\| \le \delta^{3(m+1)}\|p_0 - p\| = \alpha_m,\ m \in Z_+.$$

As proved by [27]:

$$\|p_{1,m} - p\| \le \delta^{m+1}\|p_{1,0} - p\| = \alpha_{1,m},\ m \in Z_+.$$

Then:

$$\begin{aligned}
\frac{\alpha_m}{\alpha_{1,m}} &= \frac{\delta^{3(m+1)}\|p_0 - p\|}{\delta^{m+1}\|p_{1,0} - p\|} \\
\frac{\alpha_m}{\alpha_{1,m}} &= \delta^{2(m+1)}.\frac{\|p_0 - p\|}{\|p_{1,0} - p\|}
\end{aligned}$$

Now, as $0 < \delta < 1$ Therefore we have

$$\frac{\alpha_m}{\alpha_{1,m}} \to 0 \ as \ m \to \infty$$

Hence, the sequence p_m converges to p faster than $p_{1,m}$ Now, by normal-S algorithm as proved by ALI [19], we get:

$$\|p_{5,m} - p\| \leq \delta^{m+1}\|p_{5,0} - p\| = \alpha_{5,m}.$$
$$\frac{\alpha_m}{\alpha_{5,m}} = \frac{\delta^{3(m+1)}\|p_0 - p\|}{\delta^{m+1}\|p_{5,0} - p\|}$$
$$\frac{\alpha_m}{\alpha_{5,m}} = \delta^{2(m+1)}.\frac{\|p_0 - p\|}{\|p_{5,0} - p\|}$$

Now, as $0 < \delta < 1$ Therefore, we have

$$\frac{\alpha_m}{\alpha_{5,m}} \to 0 \ as \ m \to \infty$$

Hence, the sequence p_m converges to p faster than $p_{5,m}$.

As proved by the Sintunavarat W, Pitea A [18] that

$$\|p_{6,m} - p\| \leq \delta^{m+1}\|p_{6,0} - p\| = \alpha_{6,m}.$$
$$\frac{\alpha_m}{\alpha_{6,m}} = \frac{\delta^{3(m+1)}\|p_0 - p\|}{\delta^{m+1}\|p_{6,0} - p\|}$$
$$\frac{\alpha_m}{\alpha_{6,m}} = \delta^{2(m+1)}.\frac{\|p_0 - p\|}{\|p_{6,0} - p\|}$$

Now,as $0 < \delta < 1$ therefore we have

$$\frac{\alpha_m}{\alpha_{6,m}} \to 0 \ as \ m \to \infty$$

Hence, the sequence p_m converges to p faster than $p_{6,m}$.

Now,for $p_{7,m}$ by F.Ali [19] we have that

$$\|p_{7,m} - p\| \leq \delta^{2(m+1)}\|p_{7,0} - p\| = \alpha_{7,m}.$$
$$\frac{\alpha_m}{\alpha_{7,m}} = \frac{\delta^{3(m+1)}\|p_0 - p\|}{\delta^{2(m+1)}\|p_{7,0} - p\|}$$
$$\frac{\alpha_m}{\alpha_{7,m}} = \delta^{m+1}.\frac{\|p_0 - p\|}{\|p_{7,0} - p\|}$$

Now,as $0 < \delta < 1$ Therefore we have

$$\frac{\alpha_m}{\alpha_{7,m}} \to 0 \ as \ m \to \infty$$

Hence, the sequence p_m converges to p faster than $p_{7,m}$. □

Also, F.Ali [19] showed that the F* algorithm converges quicker than Varat, Mann, Ishikawa, and normal-S algorithms for the case of weak contractions. Thus, PV iterative algorithm converges more rapidly than every iterative algorithm from (1.1) to (1.7).

Table 1 A comparison of the different iterative algorithms for Example 2.1

Iteration	PV	F*	Picard	Normal-s	Mann	Varat
1	0.4493323	0.4493323	0.4493323	0.4493323	0.4493323	0.4493323
2	0.8723104	0.8291996	1.0130704	0.9320542	0.7193926	0.9579940
3	0.8647160	0.8623066	0.7825257	0.8557003	0.8212688	0.8422444
4	0.8649333	0.8647381	0.9047664	0.8662340	0.8525708	0.8703917
5	0.8649271	0.8649136	0.8440609	0.8647430	0.8614915	0.8636068
6	0.8649273	0.8649263	0.8754487	0.8649533	0.8639767	0.8652461
7	0.8649273	0.8649272	0.8595146	0.8649236	0.8646647	0.8648503
8	0.8649273	0.8649273	0.8676839	0.8649278	0.8648548	0.8649459
9	0.8649273	0.8649273	0.8635161	0.8649272	0.8649073	0.8649228
10	0.8649273	0.8649273	0.8656478	0.8649273	0.8649218	0.8649284
11	0.8649273	0.8649273	0.8645589	0.8649273	0.8649258	0.8649270
12	0.8649273	0.8649273	0.8651155	0.8649273	0.8649269	0.8649273
13	0.8649273	0.8649273	0.8648311	0.8649273	0.8649272	0.8649273
14	0.8649273	0.8649273	0.8649764	0.8649273	0.8649272	0.8649273
15	0.8649273	0.8649273	0.8649022	0.8649273	0.8649273	0.8649273

Example 2.1 Let B^*=R Banach space with the usual norm and $V^* = [0, \frac{\pi}{2}]$. Let $f : V^* \to V^*$ be defined as $f(x) = \frac{x}{4} + \cos(x)$.Then, as f is a contraction mapping and hence a weak contraction satisfying (1.10) and has only one fixed point which is 0.865(approx). Here control sequences are taken as

$$r_m = 0.4790527832595, s_m = 0.4790527832595, \text{ and } t_m = 0.4790527832595$$

taking an initial guess of 0.44933229232998 and using python language we can see that PV iteration converges to the fixed point 0.865(approx) faster then Picard [13], Ishikawa [15], Mann [14], S [16], normal-S [17], Varat [18] and F* [19] iterative algorithms, as we can see in Table 1 and Fig. 1.

Example 2.2 Weak contraction which is not a contraction

$$f(x) = \begin{cases} \frac{\sin x}{4} & x \in [0, 0.5) \\ \frac{x}{4} & x \in [0.5, 1] \end{cases}$$

Then, as f is not continous at x=0.5 we can say that f cannot be a contraction but it can be easily verified that f is a weak contraction with $\delta = \frac{1}{2}$ and $L = 1$ and fixed point is at 0. Now, all the conditions of the Theorems 2.1 and 2.3 are satisfied.Taking control sequence r_m = 0.4790527832595, s_m =0.4790527832595, and t_m= 0.4790527832595 using python language we can show that the sequence defined by PV iterative algorithm (2.1) converges to a unique fixed point p = 0 of the mapping f faster than the algorithms (1.1) to (1.7) which is shown in Table 2 and Fig. 3.

Now, we prove a result which will be used in Application section

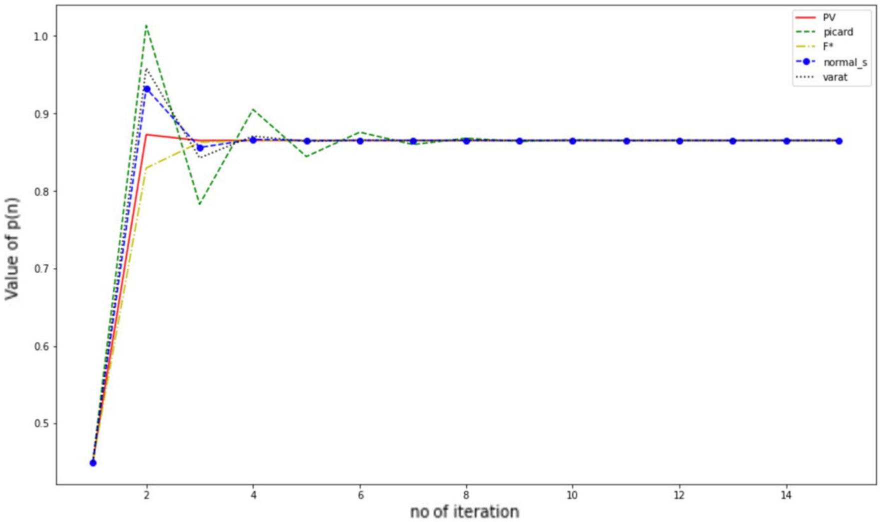

Fig. 1 Behaviour of convergence for the sequences defined by various iterative algorithms for Example 2.1

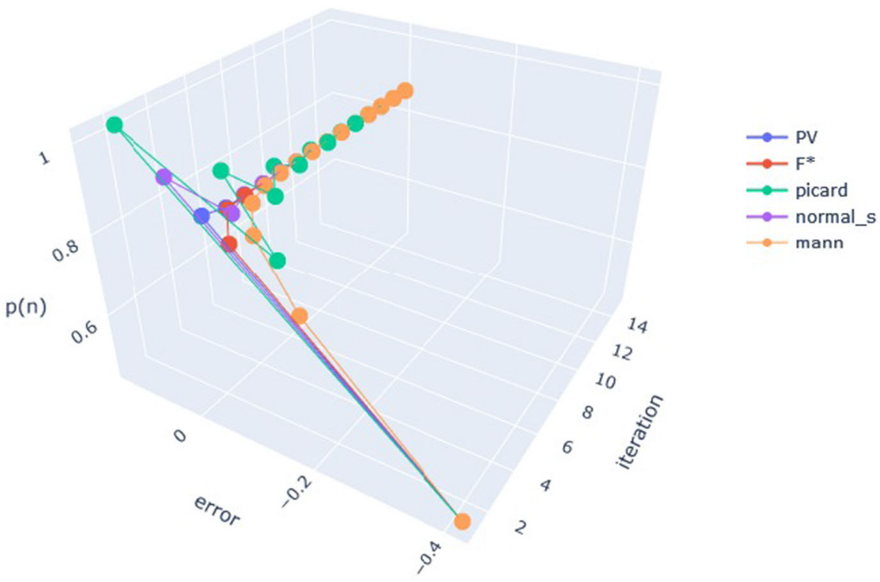

Fig. 2 Behaviour of convergence with error for the sequences defined by various iterative algorithms for Example 2.1

3 Result on data dependence

Recently, data dependence research for fixed points is a key area of fixed point theory. Noteworthy researchers who have made contributions to the data dependence field of fixed points are Markin [28], MURSEAN [29],Berinde [1, 30, 31], Soltuz [32], Soltuz and Grosan [33] and Oltainwo [34]. Now, a theorem on the data dependence of fixed points is proved.

Table 2 A comparison of the various iterative algorithms for Example 2.2

Iteration	PV	F*	Picard	Normal-S	Mann	Varat
1	0.4493323	0.4493323	0.4493323	0.4493323	0.4493323	0.4493323
2	0.0041303	0.0176236	0.1085911	0.0705530	0.2860993	0.0810321
3	0.0000393	0.0007057	0.0270944	0.0112954	0.1828412	0.0149435
4	0.0000004	0.0000283	0.0067728	0.0018093	0.1170265	0.0027578
5	0.0000000	0.0000011	0.0016932	0.0002898	0.0749481	0.0005090
6	0.0000000	0.0000000	0.0004233	0.0000464	0.0480116	0.0000939
7	0.0000000	0.0000000	0.0001058	0.0000074	0.0307593	0.0000173
8	0.0000000	0.0000000	0.0000265	0.0000012	0.0197073	0.0000032
9	0.0000000	0.0000000	0.0000066	0.0000002	0.0126265	0.0000006
10	0.0000000	0.0000000	0.0000017	0.0000000	0.0080899	0.0000001

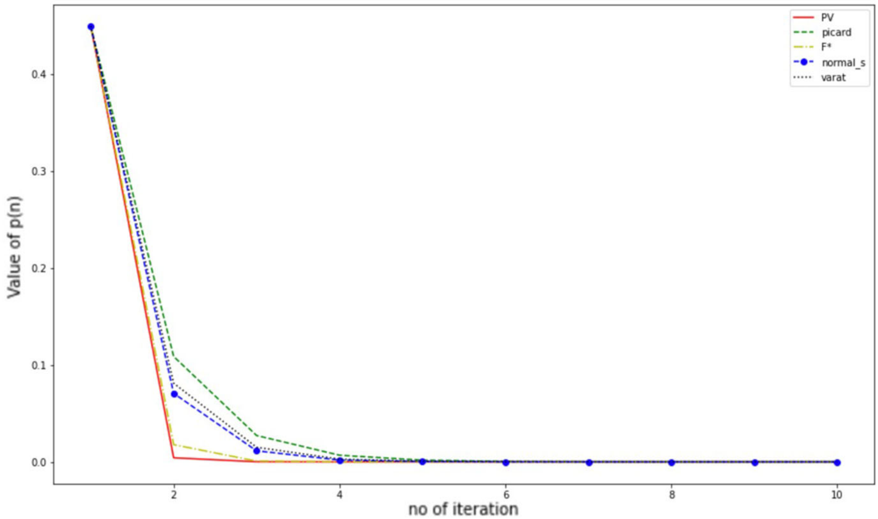

Fig. 3 Behaviour of convergence for the sequences defined by various iterative algorithms for Example 2.2

Theorem 3.1 *Let H be a weak contraction also satisfying(1.10) and let an approximate operator of H be K, p_m be a sequence generated by PV iterative algorithm (2.1) for H. Now, generate a sequence u_m for K as follows:*

$$\begin{cases} u_0 = u \in V*, \\ u_{m+1} = Kv_m, \\ v_m = K\left((1-r_m)K^2u_m + r_mKu_m\right), \quad m \in \mathbb{Z}_+, \end{cases} \tag{3.1}$$

where r_m is a sequence in (0, 1)satisfying $0.5 \leq r_m$ for all m in Z_+ and $\sum_{m=0}^{\infty} r_m < \infty$.If $Hp = p$ and $Kq = q$ such that $u_m \rightarrow q$ and $Ku_m \rightarrow q$ as $m \rightarrow \infty$, then we have:

$$\|p - q\| \leq \frac{(7 + (L+1)L)\epsilon}{1 - \delta^3}$$

where $\epsilon > 0$ is a fixed number.

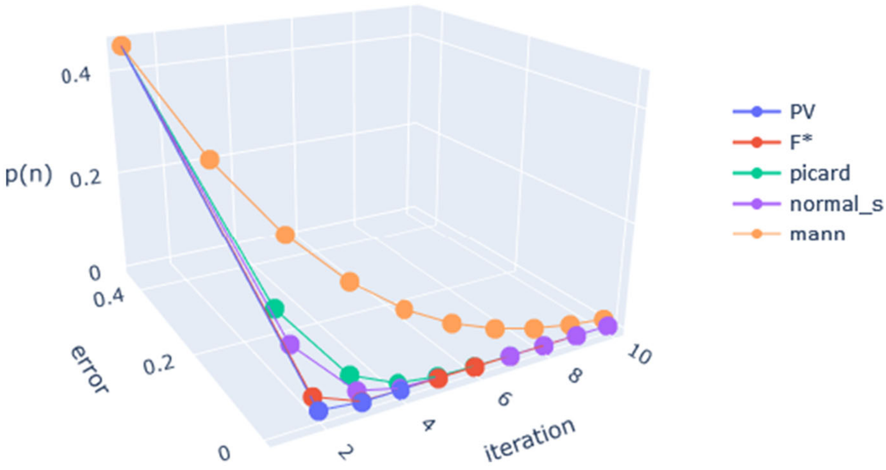

Fig. 4 Behaviour of convergence with error for the sequences defined by various iterative algorithms for Example 2.2

Proof Using (3.1),(1.10) and (2.1) we get

$$\begin{aligned}
\|q_m - v_m\| &\leq \|H((1-r_m)H^2p_m + r_nHp_m) - K((1-r_m)K^2u_m + r_mKu_m)\| \\
&\leq \|H((1-r_m)H^2p_m + r_mHp_m) - H((1-r_m)K^2u_m + r_mKu_m)\| \\
&\quad + \|H((1-r_m)K^2u_m + r_mKu_m) - K((1-r_m)K^2u_m + r_mKu_m)\| \\
&\leq \delta\{\|(1-r_m)H^2p_m + r_mHp_m - ((1-r_m)K^2u_m + r_mKu_m)\| \\
&\quad + L\|(1-r_m)H^2p_m + r_mHp_m - H((1-r_m)H^2p_m + r_mHp_m)\| + \epsilon \\
&\leq \delta\ (1-r_m)\|H^2p_m - K^2u_m\| + r_m\|Hp_m - Ku_m\| \\
&\quad + L\|(1-r_m)H^2p_m + r_mHp_m - H((1-r_m)H^2p_m + r_mHp_m)\| + \epsilon
\end{aligned} \tag{3.2}$$

One can show that

$$\begin{aligned}
&\|(1-r_m)H^2p_m + r_mHp_m - H((1-r_m)H^2p_m + r_mHp_m)\| \\
&\quad \leq (1-r_m)\delta(1-r_m)(\delta+L) \\
&\qquad + L(\delta+L)\|Hp_m - p_m\| + r_m\|Hp_m - p_m\|(\delta(1-r_m)^2(\delta+L) + r_m\delta + L)
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\|H^2p_m - K^2u_m\| \leq \delta^2\|p_m - u_m\| + L\delta\|Hp_m - p_m\| + \delta.\epsilon + \epsilon \\
&\qquad + (L+\delta)L(\|Hp_m - p_m\| + \|Hu_m - u_m\|)
\end{aligned} \tag{3.4}$$

$$\|Hp_m - Ku_m\| \leq \delta\|p_m - u_m\| + L\|Hp_m - p_m\| + \epsilon \tag{3.5}$$

Now, putting (3.3),(3.4) and (3.5) in (3.2) we get

$$\begin{aligned}
\|q_m - v_m\| \leq\ &\delta(1-r_m)(\delta^2\|p_m - u_m\| + L\delta\|Hp_m - p_m\| + \delta.\epsilon + \epsilon \\
&+ (L+\delta)L(\|Hp_m - p_m\| + \|Hu_m - u_m\|) \\
&+ \delta\ r_m(\delta\|p_m - u_m\| + L\|Hp_m - p_m\| + \epsilon)
\end{aligned}$$

$$L\Bigg((1-r_m)\Big\{\delta(1-r_m)(\delta+L)+L(\delta+L)\Big\}\|Hp_m-p_m\| \\ +r_m\|Hp_m-p_m\|\Big\{(\delta(1-r_m)^2(\delta+L)+r_m\delta\Big\}+L\Bigg)$$

Now, we consider

$$\begin{aligned}\|p_{m+1}-u_{m+1}\| = \|Hq_m-Kv_m\| &= \|Hq_m-Hv_m+Hv_m-Kv_m\| \\ &\le \|Hq_m-Hv_m\|+\|Hv_m-Kv_m\| \\ &\le \delta\|q_m-v_m\|+L\|q_m-Hq_m\|+\epsilon\end{aligned}$$

Now, using the fact that $0.5 \le r_m$ Thus $1-r_m \le r_m$ and $\delta \in (0,1)$

$$\|p_{m+1}-u_{m+1}\| \le (1-r_m(1-\delta^3))\|p_m-u_m\|+r_m(1-\delta^3) \\ \times \frac{\Big\{7\epsilon+(L+1)L\Big[\|p_m-Hp_m\|+\|u_m-Hu_m\|\Big]+\|Hp_m-p_m\|\Big(2+3L(5\delta+2L)(\delta+L)\Big)+2L\|q_m-Hq_m\|+4\epsilon\Big\}}{1-\delta^3}$$

let,

$$\begin{aligned}\theta_m &= \|p_m-u_m\| \\ \mu_m &= r_m(1-\delta^3) \\ \eta_m &= \frac{\Big\{7\epsilon+(L+1)L\Big[\|p_m-Hp_m\|+\|u_m-Hu_m\|\Big]+\|Hp_m-p_m\|\Big(2+3L+(5\delta+2L)(\delta+L)\Big)+\|q_m-Hq_m\|\Big\}}{1-\delta^3}\end{aligned}$$

Now,as $r_m \in (0,1)$, $\sum_{m=0}^{\infty} r_m < \infty$ and $\delta \in (0,1)$ therefore $\mu_m \in (0,1)$ too also $\sum_{m=0}^{\infty} \mu_m < \infty$ with θ_m, μ_m and η_m as defined above all the condition of the Lemma 1.3 are satisfied hence we have

$$0 \le \limsup_{m\to\infty} \theta_m \le \limsup_{m\to\infty} \eta_m$$
$$\implies 0 \le \limsup_{m\to\infty} \|p_m-u_m\| \le \limsup_{m\to\infty} \eta_m \tag{3.6}$$

Putting the value of η_m in (3.6) above and using the fact that $\|Hp_m-p_m\| \to 0$ and $\|Hu_m-u_m\| \le \epsilon$ we get

$$\limsup_{m\to\infty} \|p_m-u_m\| \le \frac{(7+(L+1)L)\epsilon}{1-\delta^3}.$$

□

The following example supports the above Theorem

Example 3.1 Consider

$$L(t) = \begin{cases} \frac{4\sin\frac{4t}{5}}{5} & t \in [-1,0] \\ \frac{-4\sin\frac{4t}{5}}{5} & t \in (0,1] \end{cases}$$

One can easily shown that L is a weak contraction with $\delta = \frac{16}{25}$. Now,consider

$$H(t) = \begin{cases} 0.127+0.631(t-0.2)-0.04(t-0.2)^2 & t \in [-1,0] \\ -0.127-0.631(t-0.2)+0.04(t-0.2)^2 & t \in (0,1] \end{cases}$$
$$\max_{t\in[-1,1]} |H(t)-L(t)| = 0.032$$

Table 3 Approximated fixed point of operator K by using the Iterative algorithm 3.7

It no	Iter. algorithm (3.7)
1	0.320537
2	0.066663
3	0.012948
4	001116
5	−0.00151
6	−0.0021
7	−0.00223
8	−0.00226
9	−0.00227
10	−0.00227
11	−0.00227

thus here $\epsilon = 0.032$.

Fixed point of the function H is q=−0.002 and $u_m \to q$ also at −0.002. H is continuous thus $u_m \to H(q) = q$.

Let us take $K(t) = 0.127+0.0631(t-0.2)-0.04(t-0.2)^2$ *and* $r_m = 0.49$, $u_0 = u \in Y$

$$\begin{cases} u_{m+1} & = K(v_m) = 0.127 + 0.0631(v_m - 0.2) - 0.04(v_m - 0.2)^2 \\ v_m & = K\Big((1-0.49)K^2 u_m + 0.49 K u_m\Big) \end{cases} \tag{3.7}$$

From the Table 3 we can see that u_m converges to the fixed point $q = -0.0022$ of K Now, using the Theorem 3.1 we have

$$\|p - q\| \leq \frac{((7 + (L+1)L)\epsilon}{1 - \delta^3}$$

for this example we have $L = 0$ and $\delta = \frac{16}{25}$ thus we get

$$\|p - q\| \leq \frac{7\epsilon}{1 - \delta^3}$$

Putting the value of ϵ=0.0032 ,$\delta = \frac{16}{25}$ we have

$$\|p - q\| \leq 0.30356$$

Thus from the theorem we have $\|p - q\| \leq 0.30356$ and we have actually $\|p - q\| = 0.0022$.

4 Application

In this section, we will show an application of our results to solve nonlinear Matrix equations, these kinds of applications can be seen in papers such as [35] and [36]. Therefore, we introduce the following terminology.

$\|.\|_{tr}$ represents the trace norm. $\|A\|_{tr}$ also written as $tr(A)$ is obtained by adding singular values of A where singular values of A are the square roots of the eigenvalues of A^*A.

$\|.\|$ is representation of spectral norm.

$\|A\| = \sqrt{\lambda^{+} A^{*}A}$ where $\lambda^{+}(A^{*}A)$ is the largest eigenvalue of $A^{*}A$.
M_k represents set of $k \times k$ matrices.
H_k represents set of $k \times k$ hermitian matrices.
P_k represents set of $k \times k$ positive semi definite matrices.
$X_1 \geq 0$ means $X_1 \in P_k$.
$X_1 > X_2$ means $X_1 - X_2 > 0$.
$X_1 \geq X_2$ means $X_1 - X_2 \geq 0$.

Remark [37]$P_k \subseteq H_k \subseteq M_k$ and $(H_k, \leq)$ is a partially ordered set then H_k with trace norm is a complete metric space and hence a Banach Space.

Lemma 4.1 [38] *If* $X_2 \geq 0$ *and* $X_1 \geq 0$ *then* $0 \leq tr(X_2 X_1) \leq \|X_1\| tr(X_2)$.

Consider the following non-linear matrix equation

$$X = Q_1 + \sum_{i=1}^{m} A_i^{*} F(X) A_i \tag{4.1}$$

where each A_i is an arbitrary $k \times k$ matrix for each $i = 1, 2, ., m$. Q_1 is a positive definite hermitian matrix. F is an order-preserving continuous map from P_k into P_k such that H_k, endowed with trace norm is a normed Banach Space. Hence, it is a complete metric space. Let $G : P_k \to P_k$ be a continuous order preserving self map such that

$$G(X) = Q_1 + \sum_{i=1}^{m} A_i^{*} F(X) A_i$$

for all $X \in P_k$. Clearly, a fixed point of G is a solution of the above equation.

Define $C = \{t Q_1 + (1-t) X_0 \, \forall \, t \in [0, 1]\}$.

Lemma 4.2 *If we have* G *as defined above such that* $G(Q_1)$ *and* $G(X_0) \in C$ *for some* X_0*; and* F *satisfies* $F(tX + (1-t)Y) = tF(X) + (1-t)F(Y)$ *for all* $X, Y \in C$.

Then G *is a mapping from* C *to* C.

Proof Let $A \in C$ Then $A = tQ_1 + (1-t)X_0$ for some $t \in [0, 1]$

Now,

$$G(A) = Q_1 + \sum_{i=1}^{m} A_i^{*} F(A) A_i$$

Putting value of A in above we get

$$G(A) = Q_1 + \sum_{i=1}^{m} A_i^{*} F(tQ_1 + (1-t)X_0) A_i$$

$$G(A) = Q_1 + \sum_{i=1}^{m} A_i^{*} (tF(Q_1) + (1-t)F(X_0) A_i$$

$$G(A) = Q_1 + \sum_{i=1}^{m} A_i^{*} (tF(Q_1) + (1-t)F(X_0)) A_i$$

$$G(A) = tQ_1 + (1-t)Q_1 + \sum_{i=1}^{m} A_i^{*} (tF(Q_1) + (1-t)F(X_0)) A_i$$

$$G(A) = t(Q_1 + \sum_{i=1}^{m} A_i^* F(Q_1)A_i) + (1-t)(Q_1 + \sum_{i=1}^{m} A_i^* F(X_0)A_i)$$
$$G(A) = tG(Q_1) + (1-t)G(X_0)$$

Now, as $G(Q_1)$ and $G(X_0) \in C$ then so is $G(A) = tG(Q_1) + (1-t)G(X_0) \in C$ as C is a convex set. □

Theorem 4.3 *Let (4.1) be the nonlinear matrix equation given above. Now consider*

$$G(X) = Q_1 + \sum_{i=1}^{m} A_i^* F(X)A_i$$

assume $\exists\, X_0$ *such that* $G(X_0) \leq X_0$. *Let* $C = \{tQ_1 + (1-t)X_0 \mid t \in [0,1]\}$. *F is a nonlinear function* $F : P_k \longrightarrow P_k$, *and* $F(tX_0 + (1-t)Q_1) = tF(X_0) + (1-t)F(Q_1)\, \forall\, t \in [0,1]$ *and* $G(X_0), G(Q_1) \in C$. *Using Lemma 4.2, we have* $G : C \to C$, *where C is a closed and convex set of* H_k *under trace norm, which is a Banach space. Further, let us also have the following conditions.*

(i) $\|F(X_1) - F(Y_1)\|_{tr} \leq \beta(\|X_1 - G(X_1)\|_{tr} + \|Y_1 - G(Y_1)\|_{tr})$. *Where,* $\beta \in [1, \frac{3}{2}]$.
(ii) $\| \sum_{i=1}^{m} A_i^* A_i \| \leq \alpha$. *Where,* $\alpha \in [0, \frac{1}{4}]$.

Proof Let $X_1, Y_1 \in C$ assume without loss of generality assume that $X_1 \geq Y_1$ as all elements in C are comparable.

Now,

$$\|G(X_1) - G(Y_1)\|_{tr} = \| \sum_{i=1}^{m} A_i^* F(X_1)A_i - \sum_{i=1}^{m} A_i^* F(Y_1)A_i \|_{tr}$$
$$\|G(X_1) - G(Y_1)\|_{tr} = \| \sum_{i=1}^{m} A_i^* (F(X_1) - F(Y_1))A_i \|_{tr}$$

Now,since F is an order-preserving map then $X_1 \geq Y_1 \implies F(X_1) \geq F(Y_1)$ thus $A_i^*(F(X_1) - F(Y_1))A_i \geq 0$; and hence $\sum_{i=1}^{m} A_i^*(F(X_1) - F(Y_1))A_i \geq 0$

$$\|G(X_1) - G(Y_1)\|_{tr} = tr \sum_{i=1}^{m} A_i^*(F(X_1) - F(Y_1))A_i$$
$$\|G(X_1) - G(Y_1)\|_{tr} = \sum_{i=1}^{m} tr A_i^*(F(X_1) - F(Y_1)A_i$$
$$\|G(X_1) - G(Y_1)\|_{tr} = tr \sum_{i=1}^{m} A_i^* A_i (F(X_1) - F(Y_1))$$
$$\|G(X_1) - G(Y_1)\|_{tr} = tr \big(\sum_{i=1}^{m} A_i^* A_i\big)(F(X_1) - F(Y_1))$$

Now applying the Lemma 4.1 we get

$$\|G(X_1) - G(Y_1)\|_{tr} = \|\sum_{i=1}^{m} A_i^* A_i\| \|(F(X_1) - F(Y_1))\|_{tr}$$
$$\|G(X_1) - G(Y_1)\|_{tr} \leq \alpha\beta(\|X_1 - G(X_1)\|_{tr} + \|Y_1 - G(Y_1)\|_{tr})$$
$$\|G(X_1) - G(Y_1)\|_{tr} \leq \frac{1}{2}(\|X_1 - G(X_1)\|_{tr} + \|Y_1 - G(Y_1)\|_{tr})$$

Thus, G is a Kannan map from C to C. Now, as a Kannan mapping implies weak contraction, we can apply Theorem 2.1 to obtain the fixed point of G, which will be the solution of the equation (4.1). □

Now, an example to support above result

Example 4.1 Consider the matrix difference equation

$$G(X) = Q_1 + \sum_{i=1}^{m} A_i^* F(X) A_i$$

Let m=2,$C = \{tQ_1 + (1-t)X_0 \, \forall t \in [0, 1]\}$, $Q_1 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$,$F(X) = X + Q, A_1 = \begin{pmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & \frac{\sqrt{3}}{4} \end{pmatrix}$ and $A_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.Then one can easily show that $F : C \to C$ satisfy

$$F(tX + (1-t)X) = tF(X) + (1-t)F(Y) \, \forall X, Y \in C$$

. Also, we have $\|A_1^* A_1 + A_2^* A_2\| = \frac{1}{4}$ and $G(X) = Q_1 + \frac{1}{4}(X + Q_1)$.

Now, consider

$$\begin{aligned}\|F(X) - F(Y)\|_{tr} &= \|X + Q - (Y + Q)\|_{tr} = \|X - Y\|_{tr} \\ &\leq \|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr} + \|G(Y) - G(X)\|_{tr}\end{aligned}$$

Using the formula for G(X) and G(Y)

$$\begin{aligned}&\leq \|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr} + \frac{1}{4}\|X - Y\|_{tr} \\ &\leq \|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr} + \frac{1}{4}\|X - Q - (Y - Q)\|_{tr} \\ \|F(X) - F(Y)\|_{tr} &\leq \|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr} + \frac{1}{4}\|F(X) - F(Y)\|_{tr} \\ \frac{3}{4}\|F(X) - F(Y)\|_{tr} &\leq \|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr} \\ \|F(X) - F(Y)\|_{tr} &\leq \frac{4}{3}(\|X - G(X)\|_{tr} + \|Y - G(Y)\|_{tr})\end{aligned}$$

Thus, all the conditions of the Theorem 4.3 are satisfied hence we can apply the Theorem 2.1 and 4.3 to obtain the solution of the matrix difference equation.

5 Conclusion

In this research paper, we have presented a novel and advanced two-step iterative algorithm for determining fixed points of weak contractions in Banach spaces. This algorithm is more effective and converges faster than some major iterative algorithms, as demonstrated by Theorem 2.3. Additionally, in Theorem 2.2, we have proved that the PV iterative algorithm is almost H-stable. Our claims are validated by Examples 2.1 and 2.2. Furthermore, we have obtained a result regarding data dependence, and an example illustrates the validity of this result. Lastly, we approximate the solution of a nonlinear matrix difference equation. However, a few natural questions arise in this field which can be further proved in the coming years:

(Q1) Is it possible to define an iterative technique whose convergence rate is faster than that of the PV iterative procedure for the class of weak contractions in a Banach space?
(Q2) Does the PV iteration strongly converge to the fixed point of weak contractions in spaces with weaker conditions than a Banach space, such as a metric space or quasi-Banach space?
(Q3) Does the PV iterative algorithm converge for other classes of mappings, such as enriched contractions or quasi-nonexpansive mappings?

Author Contributions Both the authors contributed equally and significantly. Both the authors have read and approved the final manuscript.

Availability of data The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest Both the authors declare that they do not have a conflict of interests.

Consent to participate Not applicable.

Consent for publication Both the authors give their consent to the publisher to publish their research findings.

Ethics approval This research doesn't contain any studies performed on humans or animals as participants.

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