Rendiconti del Circolo Matematico di Palermo Series 2 (2024) 73:1255–1275
https://doi.org/10.1007/s12215-023-00978-1

Finding a common solution of variational inequality and fixed point problems using subgradient extragradient techniques

Francis O. Nwawuru[1] · **Grace N. Echezona**[1] · **Christian C. Okeke**[2]

Received: 4 February 2023 / Accepted: 29 October 2023 / Published online: 13 December 2023

Abstract

In this research paper, we propose a new iterative algorithm for finding a common solution to fixed point problems of demicontractive mapping and variational inequality problems which involves monotone and Lipschitz continuous operators in the framework of real Hilbert spaces. We incorporate a viscosity iterative technique, using subgradient extragradient method, we prove under standard assumptions that the iterative sequence generated from our algorithm strongly converges to the solution set, assuming the solution set is consistent. Furthermore, we adopt a self-adaptive stepsize that is being generated at each iteration, which is independent of the Lipschitz constant of the singled-valued operator. Our result is an improvement and an extension of many results in this direction.

Keywords Extragradient · Subgradient-extragradient · Variational inequality · Lipschitz constant · Viscosity iteration · Hilbert spaces

Mathematics Subject Classification 47H09 · 47J20 · 47J25 · 65k15

1 Introduction

Our purpose in this paper is to study an interesting combination of problems of finding a fixed point of a given nonlinear operator which turns out to solve variational inequalities in real Hilbert spaces. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, with inner

✉ Francis O. Nwawuru
fo.nwawuru@coou.edu.ng; nwawuru@gmail.com

Grace N. Echezona
gn.echezona@coou.edu.ng

Christian C. Okeke
chibueze.okeke87@yahoo.com; chibueze.okeke@wits.ac.za

1 Department of Mathematics, Faculty of Physical Science, Chukwuemeka Odumegwu Ojukwu University, Uli Campus, Anambra, Nigeria

2 School of Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

product $\langle\cdot,\cdot\rangle$ and an induced norm $\|.\|$. Let $T: C \to C$ be nonlinear. The map T has a fixed point if $Tx = x$ and the set of fixed point of T is denoted by $F(T) := \{x \in C : Tx = x\} \neq \emptyset$. The variational inequality problem (VIP) and fixed point problem (FPP)is formulated as:

$$\text{find} \quad x \in C \quad \text{such that} \quad x \in F(T) \quad \text{and} \quad \langle f(x), y - x\rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where f is a single-valued mapping defined on C.

Let the solution set of (1.1) be denoted by $\Gamma := VI(C, f) \cap F(T) \neq \emptyset$. Problem (1.1) is a generalization of many optimization problems and has been studied by many researchers in different capacities (see, [1, 4, 37, 38] and contained references). Basically, (1.1) includes two remarkable and striking problems:

1. The FPP, which can be defined as follows:

$$\text{find} \quad x \in C \quad \text{such that} \quad T(x) = x, \tag{1.2}$$

   provided the $F(T) \neq \emptyset$.
2. Another important problem embeded in problem (1.1) is the well known VIP which has following structure:

$$\text{find} \quad x \in C \quad \text{such that} \quad \langle f(x), y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

   Let $VIP(C, f)$ and $SOL\ VIP(C, f)$ denote problem (1.3) and its solution set, respectively. It is important to note that problem (1.3) is a unifying and an essential modelling tool in many field such as economics, programming, engineering mechanics and many more, for examples see [2, 5, 16] and references contained therein. This concept was introduced and studied by Stampacchia [34], for the purpose of modelling problems in mechanics.
   There are methods for solving (1.3) which include: regularization method and projection method. In what follows, our focus for solving (1.3) is the projection method. This method involves construction of an iterative algorithm of the form:

$$x_{n+1} = P_C(x_n - \lambda f x_n), n \geq 1, \tag{1.4}$$

   where λ is positive and P_C is a projection onto closed convex subset C. With this method, it is well known that the problem (1.3) is equivalent to the following fixed point problem:

$$\text{find} \quad x \in C \quad \text{such that} \quad x = P_C(x - \lambda f(x)), \tag{1.5}$$

   for an arbitrary positive constant, λ. The basic projection method for solving VIP involves the gradient method, which performs only one iteration onto the feasible set. This method requires that the operators are inverse strongly monotone or strongly monotone (see [12]) for the iterative sequence to converge to the solution set. These conditions are very strong and quite restrictive. To circumvent this challenge, Korpelevich [18] while studying sadle point problems introduced a concept called extragradient method. It was further extended to VIPs for both Euclidean and Hilbert spaces. To be precise, Korpelevich [18] constructed the following algorithm:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu A x_n), \\ x_{n+1} = P_C(x_n - \mu A y_n), \end{cases} \tag{1.6}$$

   where $\mu \in (0, \frac{1}{L})$, the singled-valued operator$A: H \to H$ is monotone and L-Lipschitz continuous and P_C is a projection onto C. He proved that the recursive sequence $\{x_n\}$ generated by (1.6) converged weakly to the solution set $SOL\ VIP(C, A)$.

Since the the introduction of (1.6), many authors have modified it in various forms (see, [8, 17, 20] and the cited references therein).

In 2006, Nadezhkina and Takahashi [26] used the concept of hybrid and shrinking projection techniques to construct an extragradient based method and obtained a strong convergence. In short, in [31], the following algorithm is presented:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu A x_n), \\ z_n = P_C(x_n - \mu A y_n), \\ C_n = \{w \in C : \|z_n - w\| \leq \|x_n - w\|\}, \\ Q_n = \{w \in C : \langle x_n - w, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.7}$$

Observe that the computation of the algorithm (1.6) requires computing two projections per iteration. It is known that projections onto closed convex set P_C has no closed form of expression. To drop the second P_C from the algorithm (1.6), Censor et al. [5] introduced a subgradient extragradient method and constructed the following iterative scheme:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu A x_n), \\ T_n = \{x \in H : \langle x_n - \mu A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \mu A y_n). \end{cases} \tag{1.8}$$

The authors of [5] considered a projection onto a half space P_{T_n} which has a closed form of expression. They proved under some mild conditions that the sequence $\{x_n\}$ generated by (1.8) converged weakly to the solution. They further modified (1.8), using a hybrid and shrinking projection method as contained in (1.7) and obtained a strong convergence(see, [6]). It is pertinent to mention that Censor et al. [40] extended [5] to Euclidean spaces. Based on this improvement in [6, 26], many researchers have used other techniques to obtain a strong convergence (see [14, 19, 23, 37]).

In recent years, there has been a tremendous interest in developing fast convergence of algorithms, especially for the inertial type extrapolation method which was first proposed by Polyak in [31]. This inertial technique is based on a discrete analogue of a second order dissipative methods. This method was not known until the Nesterov's acceleration gradient methods was published in 1983 (see, [27]) and by 2009, Beck and Teboulle [3] made it very popular. Recently, some researchers have constructed different fast iterative algorithms by means of inertial extrapolation techniques, for example, inertial Mann algorithm [25], inertial forward-backward splitting algorithm [21], inertial extragradient algorithm [22, 36], inertial projection algorithm [35], and fast iterative shrinkage-thresholding algorithm (FISTA) [3]. Recent results on the use of inertial method can be found in [8, 10, 11, 41–43] and contained references

Remark 1.1 We note that the algorithms (1.6)–(1.8) have a major drawback on the stepsize μ in the sense that, they heavily rely on the Lipschitz constant of the given operator. This dependence on the Lipschitz constant affects the efficiency of the algorithms. In many practical importance, the Lipschitz constants are not known and in several occasions, difficult to estimate.

Based on Remark 1.1, many researchers have worked and improved the already existing results in various forms. In a recent work published by Gibali [9], he introduced an

Armijo-like search rule and remarked that, it is for local approximation of the Lipschitz constant. Although, it does not require the knowledge of Lipschitz constant but might involve additional computation of projection operator. Thong and Hieu [37] presented two parallel iterative algorithms for solving a variational inequality problem and fixed point problem for demicontractive mapping using subgradient extragradient technique. They obtained a strong convergence in both schemes under the assumption that the single-valued operator is monotone and Lipschitz continuous. But, their stepsize was dependent on the Lipschitz constant which was a heavy drawback. To improve on this result, in [36] the authors proposed another two inertial self-adaptive stepsizes for solving variational inequailties and obtained strong convergence results under the assumption that the operator is strongly pseudomonotone, which is also a stronger assumption than being monotone and Lipschitz continuous. Shehu et al. [33] studied an inertial typed subgradient extragradient method with self-adaptive stepsize. Under some mild conditions, the authors obtained a weak convergence. Furthermore, they later considered the operator A to be strongly monotone and Lipschitz continuous and obtanied a strong convergence. More so, Ogwo et al. [29] studied relaxed inertial subgradient extragradient methods for solving variational inequality problems involving quasi-monotone operator and obtained weak convergence. For more results in this direction, see for instance [1, 28–30, 38] and cited references. Motivated and inspired by the work of [9, 18, 29, 33, 37], we construct a new inertial algorithm that is simple and efficient for approximating solutions of variational inequlity problems and fixed point problems using subgradient extragradient type method.

Our contributions in this research include the following:

(a) A new inertial self-adaptive subgradient extragradient algorithm which does not require the prior knowledge of Lipschitz constant is constructed. The variable stepsize λ_n does not need the $\lim_{n\to\infty} \lambda_n = 0$, or $\sum_{n=1}^{\infty} \lambda_n = 0$ as in the case of [36]. It is more applicable than fixed stepsizes.
(b) The inertial term we use improves the rate of convergence greatly. It is quite different from the one considered by [10, 11, 19, 23, 24]. It does not also require computing norm difference between x_n and x_{n-1} before choosing the inertial factor, θ_n.
(c) We obtain that the iterative sequence $\{x_n\}$ converges strongly to the solution set. Unlike the weak convergences obtained by [5, 29, 38]. We assume that the single-valued operator A is not required to be strongly monotone as in the case of [33] or strong pseudomonotone used by [36]. These two assumptions are stronger than being monotone and Lipschitz continuous that we consider.
(d) The general class of operator called the demicontractive mapping is considered. Many important operators like nonexpansive mapping, pesudocontractivs, k-strictly pseudocontractive, quasi-nonexpansive operators among many others are all embedded in demicontractive mapping (see Remark 2.2 below). It is a more general class of operators than the ones used by [1, 4, 33, 38].
(e) Numerical examples are provided which show the general performance of our algorithm.

The rest of the paper is organized as follows: the preliminaries in Sect. 2 deal with basic definitions of the terms and related lemmas, which we state without their proofs. In Sect. 3, we state our algorithm and assumptions for our operators and control sequences. The convergence analysis is given in Sect. 4 while in Sects. 5 and 6 are devoted for numerical illustartions and conclusion, respectively.

2 Preliminaries

We list some basic concepts and lemmas which are useful for constructing and analysing the convergence of our algorithm.

Definition 2.1 Let H be a real Hilbert space and $\forall x, y \in H$, $p \in F(T)$. A map $T : H \to H$ is called:

(1) monotone on H if, $\langle T(x) - T(y), x - y\rangle \geq 0$;
(2) η-strongly monotone on H if there exists $\eta > 0$ such that $\langle T(x) - T(x), x - y\rangle \geq \eta\|x - y\|^2$;
(3) Lipschitz continuous on H if, there exists a constant $L > 0$ such that $\|T(x) - T(y)\| \leq L\|x - y\|$;
(4) a nonexpansive mapping if $\|Tx - Tx\| \leq \|x - y\|$;
(5) a quasi-nonexpansive on H if, $\|T(x) - p\| \leq \|x - p\|$;
(6) κ-strictly pseudo-contractive on H if, $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|x - y - (Tx - Ty)\|^2$, for some $\kappa \in [0, 1)$;
(7) σ-demi-contractive if, there exists $\sigma \in [0, 1)$ such that $\|Tx - p\|^2 \leq \|x - p\|^2 + \sigma\|(I - T)x\|^2$; or equivalently $\langle Tx - x, x - p\rangle \leq \frac{\sigma-1}{2}\|x - Tx\|^2$; or equivalently $\langle Tx - p, x - p\rangle \leq \|x - p\|^2 + \frac{\sigma-1}{2}\|x - Tx\|^2$.

Remark 2.2 We observe that:

(i) (7) contains as a special case, nonexpansive mapping, quasi-nonexpansive mapping, κ-strictly pseudo-contractive mapping with a nonempty fixed point.
(ii) All nonexpansive operators are κ-strictly pseudo-contractive mapping with a nonempty fixed point.
(iii) Also, all quasi-nonexpansive mappings are a subclass of 0-demi-contractive mapping.
(iv) Nonexpansive mappings are contained in quasi-nonexpansive mapping.

But the converse of all of these definitions are not necessarily true. To understand this, consider the following examples below.

Example 2.3 [13] We consider a demicontractive mapping which is not neccessarily pseudocontractive or κ-strictly pseudo-contractive. Let $H = \mathbb{R}$, $C = [-1, 1]$, let $T : C \to C$ be defined by $Tx = \frac{2}{3}xsin(\frac{1}{x})$ if $x \neq 0$ and $T0 = 0$ otherwise. Observe that the only fixed point of T is zero (0). However, for $x \in C$,

$$|Tx - 0|^2 = |Tx|^2 = |\frac{2}{3}xsin(1/x)|^2 \leq |2x/3|^2 \leq |x|^2 \leq |x - 0|^2 + \sigma|Tx - x|^2,$$

for any $\sigma \in [0, 1)$, this shows that T is a demi-contractive.

Let us now show that T is not pseudo-contractive mapping. Let $x = 2/\pi$, $y = 2/3\pi$. Then,

$$|Tx - Ty|^2 = 256/81\pi^2.$$

'It follows that

$$|x - y|^2 + |(I - T)x - (I - T)y|^2 = 160/81\pi^2.$$

Example 2.4 [38] We consider a case where T is quasi-nonexpansive but fails to be nonexpansive. Let $Tx = \frac{x}{2}sinx$, if $x \neq 0$ and $Tx = x$, then we have that $x = \frac{x}{2}sinx$, which implies that $sinx = 2$, impossible. Thus, we obtain that $x = 0$, which means that $F(T) = \{0\}$.

Now, for all $x \in H$,

$$\|Tx - 0\| = \|\frac{x}{2} sinx\| \leq \|\frac{x}{2}\| < \|x\| = \|x - 0\|,$$

this shows that T is quasi-nonexpansive. However, setting $x = 2\pi$ and $y = \frac{3\pi}{2}$, we get

$$\|Tx - Ty\| = \|\frac{2\pi}{2} sin 2\pi - \frac{3\pi}{4} sin \frac{3\pi}{2}\| = \frac{3\pi}{4} > \|2\pi - \frac{3\pi}{2}\| = \frac{\pi}{2},$$

which further means that the operator T is not a nonexpansive mapping.

Example 2.5 [7] We give an example of demi-contractive mapping that is not quasi-nonexpansive and not pseudo-contractive mapping. Let $f : [-2, 1] \to [-2, 1]$ be a real-valued function defined by $f(x) = -x^2 - x$. Then, it is demi-contractive on $[-2, 1]$ and conitnuous. It is neither quasi-nonexpansive nor pseudo-contractive mapping on $[-2, 1]$.

Definition 2.6 The mapping $P_C : H \to C$ which assigns to each $v \in H$, the unique point $P_C(v)$ such that $\|P_C(v) - v\| = \inf\{\|w - v\| : w \in C\}$. This is called the projection operator.

The operator P_C satisfies the following condition:

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Also, for

$$P_C x \in H, \langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall y \in C.$$

Further implication is that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C y\|^2, \quad \forall x \in H, \forall y \in C.$$

For details on the metric projections, an interested reader is encouraged to consult [44, section 3].

Lemma 2.7 *[39] Let H be a Hilbert space and $S : H \to H$ be a nonexpansive mapping with a nonempty fixed point. If $\{x_n\}$ is a sequence in H that converges weakly to a point x^* and $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x^* = y$.*

Lemma 2.8 *Let H be a Hilbert space. Then, the following results hold for all $x, y \in H, \lambda \in \mathbb{R}$:*

(a) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2,$
(b) $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y\rangle$
(c) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$

Lemma 2.9 *[32] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\beta_n\}$ be a sequence of real numbers in $(0, 1)$ with condition $\sum_{n=1}^{\infty} \beta_n = 0$, and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n d_n, \quad n \geq 0.$$

and

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$
(b) $\limsup_{n \to} d_n \leq 0.$

Then, $\lim_{n\to\infty} a_n = 0.$

Lemma 2.10 *[24] Let $\{a_n\}$ be sequence of non-negative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1-\beta_n)a_n + \delta_n + \gamma_n, n \geq 1,$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence of real numbers. suppose that $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\delta_n \leq \beta_n M$ for some $M > 0$. Then, $\{a_n\}$ is a bounded sequence.

Lemma 2.11 *[19] Let $A : H \to H$ be a monotone and L-Lipschtiz continuous mapping on H. Let $S = P_C(I - \mu A)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in H satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$ then $q \in VI(C, A) = F(S)$.*

Lemma 2.12 *[23] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \leq a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

That is, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that $a_n \leq a_{n+1}$.

3 The proposed algorithm

We present the proposed algorithm in this section and state the standard assumptions for the control sequences and the operators.

Assumption 3.1 The conditions on the set, and operators are stated below

(1) The feasible set C is nonempty, closed and convex and H is a real Hilbert space.
(2) The operator $A : H \to H$ is monotone and Lipschitz continuous.
(3) The operator $T : H \to H$ is an $\sigma-$demi-contractive with nonempty fixed point and with demicloseness property.
(4) The mapping $f : H \to H$ is a contraction map with contraction $\rho \in (0, 1)$.
(5) The solution set $\Gamma := VI(C, A) \cap F(T) \neq \emptyset$.

Assumption 3.2 The following assumptions are considered for the control sequences and the stepsize.

(a) $\alpha_n \in (a, \tau_n^2(1-\rho))$, for some $a > 0, \rho \in (0, 1)$.
(b) $\lim_{n\to\infty} \frac{\theta_n}{\tau_n^2}\|x_n - x_{n-1}\| = 0$.
(c) $\tau_n \in (0, \frac{1}{2(1-\rho)})$, $\lim_{n\to\infty} \tau_n = 0$, $\sum_{n=1}^{\infty} \tau_n = \infty$.
(d) $\beta_n \in (a, b) \subset (0, (1-\lambda)(1-\tau_n))$ for some $a, b > 0$.
(e) $0 \leq \theta_n \leq \theta < 1$.

Algorithm 3.3 *Self-adaptive algorithm for variational inequality and fixed point problem.*

Step 0: *Choose sequences $\{\alpha_n\}, \{\beta_n\}, \{\tau_n\}$ and $\{\theta_n\}$ such that Assumption 3.2 hold. Let $\mu > 0, \alpha_1 > 0$, and $x_0, x_1 \in H$ be arbitrarily chosen.*
Iterative steps: Step 1. *Given the iterates $x_n, x_{n-1}, n \geq 1$, compute*

$$\begin{cases} w_n = \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \end{cases} \tag{3.1}$$

where

$$\lambda_{n+1} := \begin{cases} min\{\frac{\mu\|w_n-y_n\|^2}{\|Aw_n-Ay_n\|^2}, \lambda_n\} & if Aw_n \neq Ay_n \\ \lambda_n, & otherwise. \end{cases} \tag{3.2}$$

If $w_n = y_n = x_n$, *then* $x_n \in VI(C, A) \cap F(T)$. *Otherwise, construct a half-space*

$$T_n := \{w \in H : \langle w_n - \lambda_n Aw_n - y_n, w - y_n\rangle \leq 0\},$$

and compute

$$\begin{cases} z_n = P_{T_n}(w_n - \lambda_n Ay_n), \\ x_{n+1} = \tau_n f(w_n) + (1 - \tau_n)q_n, \end{cases} \tag{3.3}$$

where $q_n = \beta_n z_n + (1 - \beta_n)Tz_n$.
Set n:=n+1 *and go back to* ***Step 1.***

Remark 3.4:

- i) See that Assumption 3.1 (2) requires the operator A to be monotone and Lipschitz continuous.
- ii) The stepsize λ_n in (3.2) is self-adaptive. It is being generated at each iteration which makes our algorithm easily implemeneted without the prior knowledge of the Lipschitz constant of operator A.
- iii) The inertial term $\theta_n(x_n - x_{n-1})$ contains an extra-term like Halpern iterative scheme, this greatly improves the rate of convergence of our proposed Alogrithm.

4 Convergence analysis

To establish the main theorem of this paper, the following lemmas should be stated and proved.

Lemma 4.1 *Let* $\{x_n\}$ *be the recursive sequence generated by Algorithm 3.3 such that Assumptions 3.1 and 3.2 are satisfied, then* $\{x_n\}$ *is bounded.*

Proof Let $x^* \in \Gamma$. Then by the definition of variational inequality, we obtain that

$$\langle Ax^*, y_n - x^*\rangle \geq 0.$$

Since A is monotone, we get that

$$\langle Ay_n, y_n - x^*\rangle \geq 0.$$

Thus,

$$\langle Ay_n, y_n - x^* + z_n - z_n\rangle \geq 0.$$

Hence,

$$\langle Ay_n, x^* - z_n\rangle \leq \langle Ay_n, y_n - z_n\rangle. \tag{4.1}$$

We therefore obtain from the difinition of T_n that

$$\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle \leq 0.$$

It follows that

$$\begin{aligned}\langle w_n - \lambda_n A y_n, z_n - y_n\rangle &= \langle w_n - \lambda_n A w_n - y_n, z_n - y_n\rangle + \lambda_n \langle A w_n - A y_n, z_n - y_n\rangle \\ &\le \lambda_n \langle A w_n - A y_n, z_n - y_n\rangle. \end{aligned} \tag{4.2}$$

Now, using the definition of projection, its characterization and (4.2), we get

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle w_n - \lambda_n A y_n - y_n, z_n - y_n\rangle \\ &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n \langle A w_n - A y_n, z_n - y_n\rangle \\ &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n \|A w_n - A y_n\|.\|z_n - y_n\| \\ &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|.\|z_n - y_n\| \\ &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}}\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \frac{\lambda_n}{\lambda_{n+1}}\mu\|w_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}}\mu\|z_n - y_n\|^2 \\ &= \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right)\|w_n - y_n\|^2 - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right)\|z_n - y_n\|^2. \end{aligned} \tag{4.3}$$

Observe from (3.2) that λ_n is a monotone nonincreasing sequence. Without loss of generality, we may assume that the $\lim_{n\to\infty}\lambda_n = \lambda$. Therefore,

$$\lim_{n\to\infty}(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu) = 1 - \mu > \epsilon > 0. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\|z_n - x^*\|^2 \le \|w_n - x^*\|^2. \tag{4.5}$$

From the Algorithm 3.3, step 2, using the definition of q_n, condition (d) of Assumption 3.2 and for all $x^* \in \Gamma$, we obtain

$$\begin{aligned}\|q_n - x^*\|^2 &= \|(1-\beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*)\|^2 \\ &= (1-\beta_n)^2\|z_n - x^*\|^2 + \beta_n^2\|Tz_n - x^*\|^2 + 2\beta_n(1-\beta_n)\langle Tz_n - x^*, z_n - x^*\rangle \\ &\le (1-\beta_n)^2\|z_n - x^*\|^2 + \beta_n^2\|z_n - x^*\|^2 + \beta_n^2\sigma\|Tz_n - x^*\|^2 \\ &\quad + 2\beta_n(1-\beta_n)\left[\|z_n - x^*\|^2 - \frac{1-\sigma}{2}\|Tz_n - x^*\|^2\right] \\ &= \|z_n - x^*\|^2 + \beta_n[\sigma\beta_n - (1-\beta_n)(1-\sigma)\|Tz_n - z_n\|^2] \\ &= \|z_n - x^*\|^2 + \beta_n[\beta_n - (1-\sigma)]\|Tz_n - z_n\|^2 \\ &\le \|z_n - x^*\|^2. \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned}\|w_n - x^*\| &= \|\alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &= \|\alpha_n(x_0 - x^*) + (1-\alpha_n)(x_n - x^*) + \theta_n(x_n - x_{n-1})\| \end{aligned}$$

$$\leq \alpha_n\|x_0 - x^*\| + (1-\alpha_n)\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|$$
$$\leq \|x_n - x^*\| + \alpha_n\|x_0 - x^*\| + \theta_n\|x_n - x_{n-1}\|. \quad (4.7)$$

Furthermore

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\tau_n f(w_n) + (1-\tau_n)q_n - x^*\| \\
&= \|\tau_n(f(w_n) - x^*) + (1-\tau_n)(q_n - x^*)\| \\
&\leq \tau_n\|f(w_n) - f(x^*)\| + \tau_n\|f(x^*) - x^*\| + (1-\tau_n)\|q_n - x^*\| \\
&\leq \tau_n\rho\|w_n - x^*\| + \tau_n\|f(x^*) - x^*\| + (1-\tau_n)\|z_n - x^*\| \\
&\leq \tau_n\rho\|w_n - x^*\| + (1-\tau_n)\|w_n - x^*\| + \tau_n\|f(x^*) - x^*\| \\
&= (\tau_n\rho + 1 - \tau_n)\|w_n - x^*\| + \tau_n\|f(x^*) - x^*\| \\
&\leq (\tau_n\rho + 1 - \tau_n)\Big[\|x_n - x^*\| + \alpha_n\|x_0 - x^*\| \\
&\quad +\theta_n\|x_n - x_{n-1}\|\Big] + \tau_n\|f(x^*) - x^*\| \\
&= \Big[(\tau_n\rho + (1-\tau_n))\Big]\|x_n - x^*\| + \Big[\tau_n\rho + (1-\tau_n)\Big]\alpha_n\|x_0 - x^*\| \\
&\quad +\Big[\tau_n\rho + (1-\tau_n)\Big]\theta_n\|x_n - x_{n-1}\| + \tau_n\|f(x^*) - x^*\| \\
&\leq [1 - \tau_n(1-\rho)]\|x_n - x^*\| + [1 - \tau_n(1-\rho)]\alpha_n\|x_0 - x^*\| \\
&\quad +(1 - \tau_n(1-\rho))\theta_n\|x_n - x_{n-1}\| + \tau_n\|f(x^*) - x^*\| \\
&= [1 - \tau_n(1-\rho)]\|x_n - x^*\| + \alpha_n\|x_0 - x^*\| \\
&\quad +\theta_n\|x_n - x_{n-1}\| + \tau_n\|f(x^*) - x^*\| \\
&\leq [1 - \tau_n(1-\rho)]\|x_n - x^*\| + \tau_n(1-\rho)\|x_0 - x^*\| \\
&\quad +\theta_n\|x_n - x_{n-1}\| + \tau_n\|f(x^*) - x^*\| \\
&< [1 - \tau_n(1-\rho)]\|x_n - x^*\| + \tau_n(1-\rho)\|x_0 - x^*\| \\
&\quad +\theta_n\|x_n - x_{n-1}\| + \tau_n\|f(x^*) - x^*\| \\
&= [1 - \tau_n(1-\rho)]\|x_n - x^*\| + \tau_n(1-\rho)\Big[\|x_0 - x^*\| \\
&\quad +\tau_n\theta_n\frac{\|x_n - x_{n-1}\|}{\tau_n^2(1-\rho)} + \frac{\|f(x^*) - x^*\|}{1-\rho}\Big]. \qquad (4.8)
\end{aligned}$$

Let $\gamma_n = \tau_n(1-\rho)$ in (4.8). Since, $\lim_{n\to\infty}\frac{\theta_n}{\tau_n^2}\|x_n - x_{n-1}\| = 0$, set

$$M = Sup\left\{\|x_0 - x^*\|, \tau_n\frac{\theta_n\|x_n - x_{n-1}\|}{\tau_n^2(1-\rho)}, \frac{\|f(x^*) - x^*\|}{1-\rho}\right\}$$

and apply Lemma 2.10 in (4.8) to obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n M \\
&\;\;\vdots \\
&\leq \max\{\|x_1 - x^*\|, M\}. \qquad (4.9)
\end{aligned}$$

Thus, we conclude from (4.9) and Lemma 2.10 that the sequence, $\{x_n\}$ is bounded. Consequently, we obtain that $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{q_n\}$ are all bounded sequences and this completes the proof of Lemma 4.1. □

Lemma 4.2 *For all $x^* \in \Gamma$, we have*

$$\|x_{n+1} - x^*\|^2 \leq (1 - \zeta_n)\|x_n - x^*\|^2 + \zeta_n\left(\frac{\tau_n M_0 + \left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle}{1 - \rho}\right),$$

where $\zeta_n = \frac{2\tau_n(1-\rho)}{1-\tau_n\rho}$.

Proof Let $x^* \in \Gamma$. Then,

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|\alpha_n(x_0 - x^*) + (1 - \alpha_n)(x_n - x^*) + \theta_n(x_n - x_{n-1})\|^2 \\
&= \|\alpha_n(x_0 - x^*) + (1 - \alpha_n)(x_n - x^*)\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\theta_n\alpha_n(1 - \alpha_n)\langle x_0 - x^* + x_n - x^*, x_n - x_{n-1}\rangle \\
&= \alpha_n\|x_0 - x^*\|^2 + (1 - \alpha_n)\|(x_n - x^*)\|^2 - \alpha_n(1 - \alpha_n)\|x_n - x_0\|^2 \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\theta_n\alpha_n(1 - \alpha_n)\langle x_0 - x^* + x_n - x^*, x_n - x_{n-1}\rangle \\
&\leq \alpha_n\|x_0 - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\theta_n\alpha_n(1 - \alpha_n)\big[\langle x_0 - x^*, x_n - x_{n-1}\rangle + \langle x_n - x^*, x_n - x_{n-1}\rangle\big] \\
&\leq \alpha_n\|x_0 - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\theta_n\alpha_n(1 - \alpha_n)\Big[\|x_0 - x^*\|\|x_n - x_{n-1}\| + \|x_n - x^*\|\|x_n - x_{n-1}\|\Big] \\
&\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_0 - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +\theta_n\alpha_n(1 - \alpha_n)\big[\|x_0 - x^*\|^2 + \|x_n - x_{n-1}\|^2\big] \\
&\quad +\theta_n\alpha_n(1 - \alpha_n)[\|x_n - x^*\|^2 + \|x_n - x_{n-1}\|^2\big] \\
&\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_0 - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +\theta_n\alpha_n\|x_0 - x^*\|^2 + \theta_n\alpha_n\|x_n - x_{n-1}\|^2 \\
&\quad +\theta_n\alpha_n\|x_n - x^*\|^2 + \theta_n\alpha_n\|x_n - x_{n-1}\|^2 \\
&\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_0 - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad +\alpha_n\|x_0 - x^*\|^2 + \alpha_n\|x_n - x^*\|^2 + 2\alpha_n\theta\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\tau_n^2(1 - \rho)\|x_0 - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\
&\quad +2\tau_n^2(1 - \delta)\theta_n\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\tau_n^2\|x_0 - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + 2\tau_n^2\theta_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + 2\tau_n^2\|x_0 - x^*\|^2 + \tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n - x_{n-1}\|^2 \\
&\quad +2\tau_n^4\frac{\theta_n}{\tau_n^2}\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{4.10}$$

It follows from (4.10), (3.3) and Lemma 2.8 (b) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - \tau_n)^2\|q_n - x^*\|^2 + 2\tau_n\rho\|w_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\tau_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \tau_n)^2\|q_n - x^*\|^2 + \tau_n\rho\|w_n - x^*\|^2 + \tau_n\rho\|x_{n+1} - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad +2\tau_n\rho\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&\leq (1-\tau_n)^2\|z_n-x^*\|^2+\tau_n\rho\|w_n-x^*\|^2+\tau_n\rho\|x_{n+1}-x^*\|^2\\
&\quad +2\tau_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&\leq (1-\tau_n)^2\|w_n-x^*\|^2+\tau_n\rho\|w_n-x^*\|^2+\tau_n\rho\|x_{n+1}-x^*\|^2\\
&\quad +2\tau_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&= [(1-\tau_n)^2+\tau_n\rho]\|w_n-x^*\|^2+\tau_n\rho\|x_{n+1}-x^*\|^2\\
&\quad +2\tau_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&\leq \frac{1}{1-\tau_n\rho}[(1-\tau_n)^2+\tau_n\rho][\|x_n-x^*\|^2+2\tau_n^2\|x_0-x^*\|^2\\
&\quad +\tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2+2\tau_n^4\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2]\\
&\quad +2\tau_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&= \frac{1}{1-\tau_n\rho}[1-2\tau_n+\tau_n\rho]\|x_n-x^*\|^2\\
&\quad +\frac{\tau_n^2}{1-\tau_n\rho}\|x_n-x^*\|^2+\frac{2\tau^2}{1-\tau_n\rho}[(1+\tau_n\rho)\|x_0-x^*\|^2\\
&\quad +\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2+2\tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2+\frac{\|x_n-x^*\|^2}{2}]\\
&\quad +\frac{2\tau_n}{1-\tau_n\rho}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\
&= (1-\frac{2\tau_n(1-\rho)}{1-\tau_n\rho})\|x_n-x^*\|^2\\
&\quad +\frac{2\tau_n(1-\rho)}{1-\tau_n\rho}\left[\frac{\tau_n\left[(1+\tau_n\rho)\|x_0-x^*\|^2+\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2\right]}{1-\rho}\right]\\
&\quad +\frac{2\tau_n(1-\rho)}{1-\tau_n\rho}\\
&\quad \left[\tau_n\left[\frac{\frac{\|x_n-x^*\|^2}{2\tau_n}+2\tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2+\langle f(x^*)-x^*, x_{n+1}-x^*\rangle}{1-\rho}\right]\right].
\end{aligned}
\tag{4.11}
$$

Letting $M_0 = Sup\Big\{(1+\tau_n\rho)\|x_0-x^*\|^2, \frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2, \frac{\|x_n-x^*\|^2}{2\tau_n}, 2\tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2; n\in \mathbb{N}\Big\}$ and in (4.11), we conclude that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \left(1-\frac{2\tau_n(1-\rho)}{1-\tau_n\rho}\right)\|x_n-x^*\|^2\\
&\quad +\frac{2\tau_n(1-\rho)}{1-\tau_n\rho}\left(\frac{\tau_n M_0+\langle f(x^*)-x^*, x_{n+1}-x^*\rangle}{1-\rho}\right).
\end{aligned}
\tag{4.12}
$$

Putting $\zeta_n = \frac{2\tau_n(1-\rho)}{1-\tau_n\rho}$ we can re-write (4.12) as follows:

$$\|x_{n+1} - x^*\|^2 \leq (1-\zeta_n)\|x_n - x^*\|^2 + \zeta_n\left(\frac{\tau_n M_0 + \langle f(x^*) - x^*, x_{n+1} - x^*\rangle}{1-\rho}\right). \tag{4.13}$$

This completes the proof of Lemma 4.2. □

Next, we establish the following important lemma.

Lemma 4.3 *We prove that*

$$\begin{aligned}
&(1-\tau_n)[(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2 + (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2]\\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \tau_n K_0.
\end{aligned}$$

Proof From the algorithm and for all $x^* \in \Gamma$, we get

$$\begin{aligned}
\|q_n - x^*\|^2 &= \|(1-\beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*)\|^2\\
&\leq (1-\beta_n)\|z_n - x^*\|^2 + \beta_n[\|z_n - x^*\|^2 + \sigma\|(I-T)z_n\|^2]\\
&\quad -\beta_n(1-\beta_n)\|(I-T)z_n\|^2\\
&\leq (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \beta_n\sigma\|(I-T)z_n\|^2\\
&= \|z_n - x^*\|^2 + \beta_n\sigma\|(I-T)z_n\|^2\\
&\leq \|w_n - x^*\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2\\
&\quad +\beta_n\sigma\|(I-T)z_n\|^2\\
&\leq \|w_n - x^*\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2\\
&\quad -\sigma(\lambda-1)(1-\tau_n)\|(I-T)z_n\|^2.
\end{aligned} \tag{4.14}$$

Furthermore from the Algorithm and for all $x^* \in \Gamma$,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \tau_n\|f(w_n) - x^*\|^2 + (1-\tau_n)\|q_n - x^*\|^2\\
&\leq \tau_n\|f(w_n) - x^*\|^2 + (1-\tau_n)\\
&\quad \left\{\|w_n - x^*\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2\right\}\\
&\leq \tau_n\|f(w_n) - x^*\|^2 + (1-\tau_n)\\
&\quad \left\{\|x_n - x^*\|^2 + \alpha_n\|x_0 - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2\right.\\
&\quad \left. -(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2\right\}\\
&= \tau_n\|f(w_n) - x^*\|^2 + (1-\tau_n)\\
&\quad \left[\|x_n - x^*\|^2 + \alpha_n\|x_0 - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|^2\right]\\
&\quad -(1-\tau_n)\left[(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n - y_n\|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \tau_n\|f(w_n)-x^*\|^2+(1-\tau_n)[\|x_n-x^*\|^2+\tau_n^2(1-\delta)\|x_0\\
&\quad -x^*\|^2+\theta_n\|x_n-x_{n-1}\|^2]\\
&\quad -(1-\tau_n)\Big[(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n-y_n\|^2+(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n-y_n\|^2\Big]\\
&\le \tau_n\|f(w_n)-x^*\|^2+(1-\tau_n)\big[\|x_n-x^*\|^2+\tau_n^2\|x_0-x^*\|^2+\theta_n\|x_n-x_{n-1}\|^2\big]\\
&\quad -(1-\tau_n)\Big[(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n-y_n\|^2+(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n-y_n\|^2\Big].
\end{aligned}
\tag{4.15}
$$

Thus, it follows from (4.15) that

$$
\begin{aligned}
&(1-\tau_n)\Big[(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|w_n-y_n\|^2+(1-\frac{\lambda_n\mu}{\lambda_{n+1}})\|z_n-y_n\|^2\Big]\\
&\quad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\tau_n K_0.
\end{aligned}
\tag{4.16}
$$

where $K_0=\sup\Big\{\|f(w_n)-x^*\|^2,\|x_n-x^*\|^2,\tau_n\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|^2,\tau_n\|x_0-x^*\|^2:n\in\mathbb{N}\Big\}$, completing lemma 4.3. □

Lemma 4.4 *For any $x^*\in\Gamma$, we obtain*

$$\sigma(\lambda-1)(1-\tau_n)\|z_n-Tz_n\|^2\le\|w_n-x^*\|^2-\|q_n-x^*\|^2.$$

Proof The result follows immediately from Lemma 4.3. □

We are now ready to establish the main theorem of this paper.

Theorem 4.5 *Let C be a nonempty closed convex subset of a real Hilbert space, H. Let A be a monotone and Lipschitz continuous defined on H and $T:H\to H$, be a σ-demi-contractive mapping. Let f be a contraction map defined on H. Assume the solution set $\Gamma\ne\emptyset$. If the Assumptions 3.1 and 3.2 are satisfied, then the sequence $\{x_n\}$ converges strongly to the solution set.*

Proof Since $P_\Gamma f$ is a contraction on H, there exists $q\in\Gamma$ such that $q=P_\Gamma f(q)$. Thus, we prove that the iterative sequence $\{x_n\}$ converges strongly to $q=P_\Gamma f(q)$. In order to establish this result, we consider the following two cases.

Case 1: Suppose there is $n_0\in\mathbb{N}$ such that $\{\|x_n-q\|\}_{n=n_0}^\infty$ is nonincreasing. Then, $\lim_{n\to\infty}\{\|x_n-q\|\}$ exists. It follows from this fact that

$$\lim_{n\to\infty}(\|x_n-q\|-\|x_{n+1}-q\|)=0.$$

From the Algorithm 3.3, we get

$$
\begin{aligned}
\|w_n-x_n\|&\le\|\alpha_n(x_0-x_n)\|+\|\theta_n(x_n-x_{n-1})\|\\
&=\alpha_n\|x_0-x_n\|+\theta_n\|x_n-x_{n-1}\|\\
&\le\tau_n^2(1-\rho)\|x_0-x_n\|+\tau_n^2\frac{\theta_n}{\tau_n^2}\|x_n-x_{n-1}\|.
\end{aligned}
\tag{4.17}
$$

It follows from (4.17) and the Assumption 3.2(b–c) that

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{4.18}$$

From Lemma 4.3, we obtain that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0 = \|z_n - y_n\|. \tag{4.19}$$

It follows from (4.19) that

$$\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - z_n\|.$$

Subsequently,

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{4.20}$$

Since the $\lim_{n\to\infty}\{\|x_n - q\|\}$ exists,it follows that

$$\lim_{n\to\infty} (\|w_n - p\|^2 - \|q_n - p\|^2) = 0.$$

That is,

$$\lim_{n\to\infty} \|w_n - p\| = \lim_{n\to\infty} \|q_n - p\|. \tag{4.21}$$

And consequently from the Lemma 4.4, we get

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{4.22}$$

Using the condition on τ_n and definition of q_n, we estimate that

$$\begin{aligned}\|q_n - y_n\| &\leq (1-\beta_n)\|z_n - y_n\| + \beta_n\|Tz_n - y_n\| \\ &\leq \|z_n - y_n\| + \beta_n\|Tz_n - y_n\| \\ &= \|z_n - y_n\| + \tau_n \frac{\beta_n}{\tau_n}\|Tz_n - y_n\| \to 0. \end{aligned} \tag{4.23}$$

It follows from (4.23) and (4.19) that

$$\lim_{n\to\infty} \|q_n - y_n\| = 0. \tag{4.24}$$

Now, using the definition of q_n once again, (4.18), (4.19) and (4.24), we get

$$\|q_n - x_n\| \leq \|q_n - y_n\| + \|y_n - w_n\| + \|w_n - x_n\|. \tag{4.25}$$

And consequently from (4.25), we conclude that

$$\lim_{n\to\infty} \|q_n - x_n\| = 0. \tag{4.26}$$

Applying the condition on α_n, the Assumption 3.2(b) and the estimate (4.26), we obtain

$$\begin{aligned}\|w_n - q_n\| &\leq \alpha_n\|x_0 - q_n\| + (1-\alpha_n)\|x_n - q_n\| + \theta_n\|x_n - x_{n-1}\| \\ &\leq \tau_n^2(1-\rho)\|x_0 - q_n\| + \|x_n - q_n\| + \theta_n\|x_n - x_{n-1}\| \\ &\leq \tau_n^2\|x_0 - q_n\| + \|x_n - q_n\| + \theta_n\|x_n - x_{n-1}\| \\ &= \tau_n^2\|x_0 - x^*\| + \|x_n - q_n\| + \frac{\tau^2\theta_n}{\tau_n^2}\|x_n - x_{n-1}\| \to 0. \end{aligned} \tag{4.27}$$

Thus, we obtain from (4.27) that

$$\lim_{n\to\infty} \|w_n - q_n\| = 0. \tag{4.28}$$

Finally, applying the condition of τ_n in the inequality below, using (4.26), we get that

$$\|x_{n+1} - x_n\| \leq \tau_n \|f(w_n) - x_n\| + (1 - \tau_n)\|q_n - x_n\| \to 0. \tag{4.29}$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.30}$$

Because $\{x_n\}$ is bounded (see Lemma 4.1), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to some q as $k \to \infty$ and

$$\limsup_{n\to\infty}\langle x_n - x^*, f(x^*) - x^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - x^*, f(x^*) - x^*\rangle = \langle q - x^*, f(x^*) - x^*\rangle. \tag{4.31}$$

Since $x_{n_k} \rightharpoonup q$ and $\|w_n - y_n\| = \|w_n - P_C(w_n - \lambda_n A w_n)\| \to 0$ by Lemma 2.11, we obtain $q \in VI(C, A)$. Furthermore, since $\|z_n - x_n\| \to 0$, $z_{n_k} \rightharpoonup q$ and $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$, we get from these facts that $q \in F(T)$. Therefore, $q \in VI(C, A) \cap F(T)$, that is, $q \in \Gamma$. Also, by $p = P_{VI(C,A)\cap F(T)}(0)$, we get

$$\limsup_{n\to\infty}\langle x_n - x^*, f(x^*) - x^*\rangle = \langle q - x^*, f(x^*) - x^*\rangle \leq 0.$$

By our estimate that $\|x_{n+1} - x_n\| \to 0$, we obtain

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \leq 0.$$

Therefore, using (4.12), (4.13), (4.31) and Lemma 2.9, we find that the sequence $\{x_n\}$ converges strongly to q.
Case 2: There exists a subsequence $\{\|x_{n_k} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_k} - p\|^2 < \|x_{n_k+1} - p\|^2$ for all $k \in \mathbb{N}$. It follows from Lemma 2.12 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2 \text{ and } \|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2. \tag{4.32}$$

From Lemma 4.3, we get that

$$\begin{aligned}
&(1 - \tau_{m_k})[(1 - \frac{\lambda_{m_k}\mu}{\lambda_{m_k+1}})\|w_{m_k} - y_{m_k}\|^2 + (1 - \frac{\lambda_{m_k}\mu}{\lambda_{m_k+1}})\|z_{m_k} - y_{m_k}]\\
&\leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \tau_{m_k} K_0\\
&\leq \tau_{m_k} K_0.
\end{aligned} \tag{4.33}$$

Using (4.33) together with assumptions in $\{\lambda_n\}$ and $\{\tau_n\}$, it follows that

$$\lim_{k\to\infty} \|w_{m_k} - y_{m_k}\| = 0, \lim_{k\to\infty} \|z_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k\to\infty} \|w_{m_k} - z_{m_k}\| = 0. \tag{4.34}$$

However, from Lemma 4.4, we obtain

$$\sigma(\lambda - 1)(1 - \tau_{m_k})\|z_{m_k} - Tz_{m_k}\| \leq \|w_{m_k} - p\|^2 - \|q_{m_k} - p\|^2. \tag{4.35}$$

Using Lemma 4.1, it follows from (4.35) that

$$\lim_{k\to\infty} \|z_{m_k} - Tz_{m_k}\| = 0.$$

Furthermore, by the same argument in case 1, we have

$$\limsup_{k\to\infty}\langle x_{m_k+1} - x^*, f(x^*) - x^*\rangle \leq 0.$$

It follows from (4.12) and (4.32) that

$$\|x_{m_k+1} - x^*\|^2 \leq (1-\zeta_{m_k})\|x_{m_k} - x^*\|^2 + \zeta_{m_k}\left(\frac{\tau_{m_k+1}M_0 + \langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-\rho}\right)$$

and hence

$$\zeta_{m_k}\|x_{m_k+1} - x^*\|^2 \leq \zeta_{m_k}\left(\frac{\tau_{m_k+1}M_0 + \langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-\rho}\right).$$

Since $\zeta_{m_k} > 0$ and using (4.32) we get

$$\|x_k - x^*\|^2 \leq \|x_{m_k+1} - x^*\|^2 \leq \left(\frac{\tau_{m_k+1}M_0 + \langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle}{1-\rho}\right).$$

Taking the limit in the above inequality as $k \to \infty$, we conclude that x_k converges strongly to $q = P_\Gamma f(q)$. This completes the proof of Theorem 4.5. □

5 Numerical illustrations

Let $H = \big(l_2(\mathbb{R}),\ ||.||_{l_2}\big)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \ldots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \big(\sum_{i=1}^{\infty} |x_i|^2\big)^{\frac{1}{2}}$, $\forall x \in l_2(\mathbb{R})$. Let $C = \{x \in l_2(\mathbb{R}) : ||x - a||_{l_2} \leq r\}$, where $a = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $r = 3$. Then C is a nonempty closed and convex subset of $l_2(\mathbb{R})$. Thus,

$$P_C(x) = \begin{cases} x, & \text{if } x \in ||x - a||_{l_2} \leq r, \\ \frac{x-a}{||x-a||_{l_2}} r + a, & \text{otherwise.} \end{cases}$$

Now, define the operators $A, f, T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Aa := \left(\frac{a_1 + |a_1|}{2}, \frac{a_2 + |a_2|}{2}, \ldots, \frac{a_i + |a_i|}{2}, \ldots\right), \quad f(a) = \frac{1}{3}a, \quad T(a) = \frac{-11}{2}a \ \ \forall a \in l_2.$$

Then A is Lipschitz continuous and monotone with Lipschitz constant $L = 1$, f is a contraction with $\rho = \frac{1}{3}$ and T is a demi-contractive mapping with $\sigma = \frac{117}{169}$.

Furthermore, we choose $\lambda_1 = 1$, $\mu = 0.5$, $\tau_n = \frac{1}{n+1}$, $\alpha_n = \frac{2n+3}{50n+100}$, $\beta_n = \frac{n+1}{100n+2}$ and

$$\theta_n := \begin{cases} \min\left\{\theta, \frac{e_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise,} \end{cases}$$

where $e_n = \frac{1}{n^2}$ and $\theta = 0.3$. Then, we consider the following cases for the numerical experiments.

Case 1: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.
Case 2: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ and $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case 3: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.
The stopping criterion is $\text{TOL}_n < 10^{-5}$, where $\text{TOL}_n = 0.5\|x_n - T(PC(x_n - x_n))\|^2$. Note that $\text{TOL}_n = 0$ implies that x_n is a solution.

Remark 5.1 The numerical illustration 5 is provided to check the general performance of our Algorithm 3.3. The Table 1 and Fig. 1 show the time it takes and number of iterations it takes for the recursive sequence to converge in the cases 1 to 3 stated above.

Table 1 Numerical results summarizing the performance of our Algorithm

		Algorithm 3.3	
case 1	CPU time (s) No. of iteration	0.2331	1251
case 2	CPU time (s) No. of iteration	0.1947	1250
case 3	CPU time (s) No. of iteration	2.1289	8072

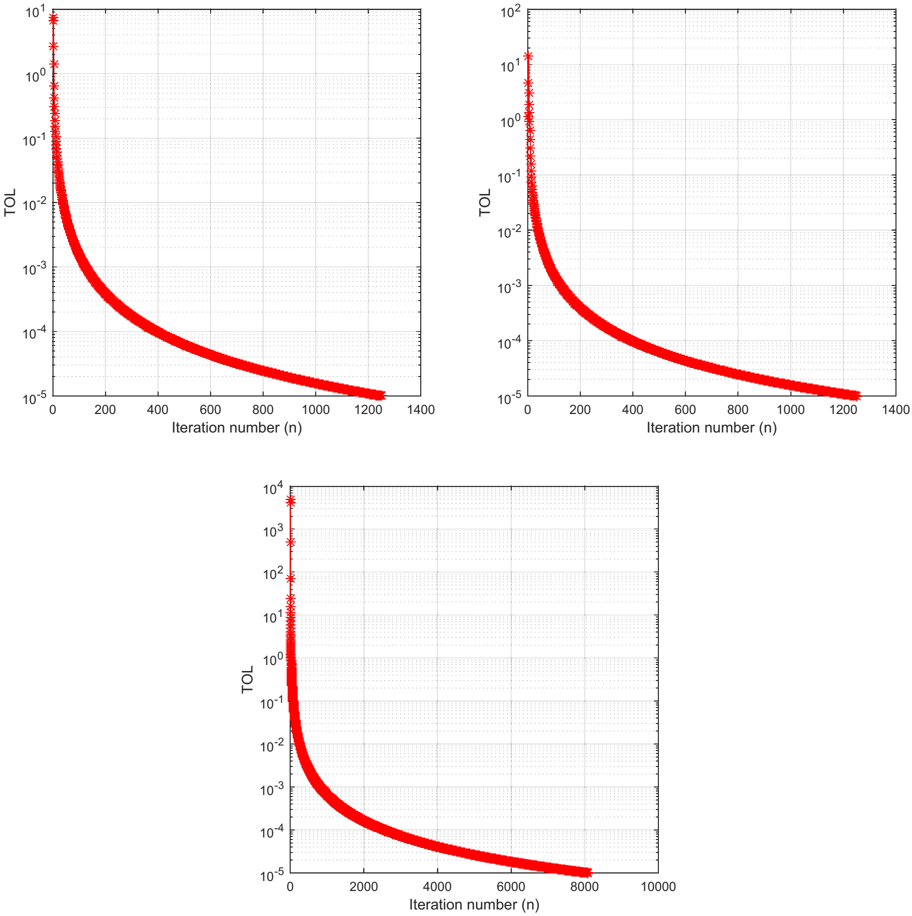

Fig. 1 The behavior of TOL_n with $\varepsilon = 10^{-5}$: Top Left: Case 1; Top Right: Case 2; Bottom Right: Case 3

6 Conclusion

In the real Hilbert space setting, we have proposed subgradient extragradient methods for approximating a common solution for variational inequality problems and fixed point problems. We obtained a strong convergence under some mild assumptions, that is, the associated single-valued operator A is monotone and Lipschitz continuous. Furthermore, we have adopted self-adaptive stepsize which is generated at each iteration. An appropriate computational experiment is provided to support our theoritical argument. Our algorithm is an improvement to the recent work of [33, 37, 38] among other already announced results in this direction.

Acknowledgements The authors sincerely thank the editor and the anonymous reviewers for their careful reading, constructive comments and fruitful suggestions that substantially improved the manuscript.

Author Contributions The first author conceptualize, design and carry out the computations with the second author while the third author handles the numerical part and equally oversees the correctness of the entire manuscript and gives it a beautiful look.

Funding There is no funding for this project.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval Not applicable.

References

1. Abbas, M., Iqbal, H.: Two inertial extragradient viscosity algorithms for solving variational inequality and fixed point problems. J. Nonlinear Var. Anal. **4**, 377–398 (2020)
2. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities. Applications to Free Boundary Problems. Wiley, New York (1984)
3. Beck, A., Teboulle, M.: A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J. Imaging Sci. **2**, 183–202 (2009)
4. Ceng, L.-C., Shang, M.: Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. Optimization (2019). https://doi.org/10.1080/02331934.2019.1647203
5. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
6. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Methods Softw. **26**, 827–845 (2011)
7. Chidume, C.E., Măruşter, Ş: Iterative methods for the computation of fixed points of demicontractive mappings. J. Comput. Appl. Math. **234**(3), 861–882 (2010). https://doi.org/10.1016/j.cam.2010.01.050
8. Gibali, A., Jolaoso, L.O., Mewomo, O.T., Taiwo, A.: Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces. Results Math. **75**, Art. No. 179, 36 pp (2020)
9. Gibali, A.: A new non-Lipschitzian projection method for solving variational inequalities in Euclidean spaces. J. Nonlinear Anal. Optim. **6**, 41–51 (2015)
10. Godwin, E.C., Izuchukwu, C., Mewomo, O.T.: An inertial extrapolation method for solving generalized split feasibility problems in real Hilbert spaces. Boll. Unione Mat. Ital. **14**(2), 379–401 (2021)
11. Godwin, E.C., Izuchukwu, C., Mewomo, O.T.: Image restoration using a modified relaxed inertial method for generalized split feasibility problems Math. Methods Appl. Sci. **46**(5), 5521–5544 (2023)
12. He, B.S., Liao, L.Z.: Improvements of some projection methods for monotone nonlinear variational inequalities. J. Optim. Theory Appl. **112**, 111–128 (2002)
13. Hicks, T.L., Kubicek, J.D.: On the Mann iteration process in a Hilbert space. J. Math. Anal. Appl. **59**(3), 498–504 (1977). https://doi.org/10.1016/0022-247x(77)90076-2
14. Izuchukwu, C, Mebawondu, A.A., Mewomo, O.T.: A new method for solving split variational inequality problems without co-coerciveness. J. Fixed Point Theory Appl. **22**(4), Art. No. 98 (2020)
15. Izuchukwu, C., Ogwo, G.N., Mewomo, O.T.: An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions. Optimization (2020). https://doi.org/10.1080/02331934.1808648
16. Jolaoso, L.O., Taiwo, A., Alakoya, O.T., Mewomo, O.T.: Strong convergence theorem for solving pseudomonotone variational inequality problem using projection method in a reflexive Banach space. J. Optim. Theory Appl. **185**(3), 744–766 (2020)

17. Khan, S.H., Alakoya, T.O., Mewomo, O.T.: Relaxed projection methods with self-adaptive step size for solving variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings in Banach Spaces. Math. Comput. Appl. **25**, Art. 54 (2020)
18. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekonomikai Matematicheskie Metody **12**, 747–756 (1976)
19. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
20. Liu, H., Yang, J.: Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput. Optim. Appl. **77**(2), 491–508 (2020)
21. Lorenz, D., Pock, T.: An inertial forward–backward algorithm for monotone inclusions. J. Math. Imaging Vis. **51**, 311–325 (2015)
22. Luo, Y.L., Tan, B.: A self-adaptive inertial extragradient algorithm for solving pseudo-monotone variational inequality in Hilbert spaces. J. Nonlinear Convex Anal. (in press) (2020)
23. Maingé, P.E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. SIAM J. Control Optim. **47**, 1499–1515 (2008)
24. Mainge, P.E.: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)
25. Maingé, P.E.: Convergence theorems for inertial KM-type algorithms. J. Comput. Appl. Math. **219**, 223–236 (2008)
26. Nadezhkina, N., Takahashi, W.: Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. SIAM J. Optim. **16**, 1230–1241 (2006)
27. Nesterov, Y.: A method of solving a convex programming problem with convergence rate O(1/k2). Sov. Math. Dokl. **27**, 372–376 (1983)
28. Ogwo, G.N., Izuchukwu, C., Mewomo, O.T.: Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity. Numer. Algorithms **88**(3), 1419–1456 (2022)
29. Ogwo, G.N., Izuchukwu, C., Shehu, Y., Mewomo, O.T.: Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems. J. Sci. Comput. (2022). https://doi.org/10.1007/s10915-021-01670-1
30. Ogwo, G.N., Izuchukwu, C., Mewomo, O.T.: Relaxed inertial methods for solving split variational inequality problems without product space formulation. Acta Math. Sci. Ser. B (Engl. Ed.) **42**, 1701–1733 (2022)
31. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. Comput. Math. Math. Phys. **4**, 1–17 (1964)
32. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75**, 742–750 (2012)
33. Shehu, Y., Iyiola, O.S., Reich, S.: A modified inertial subgradient extragradient method for solving variational inequalities. Optim. Eng. **23**, 421–449 (2022). https://doi.org/10.1007/s11081-020-09593-w
34. Stampacchia, G.: Variational inequalities. In: Theory and Applications of Monotone Operators. Proceedings of the NATO Advanced Study Institute, Venice, Italy, Edizioni Odersi, Gubbio, Italy, pp. 102–192 (1968)
35. Tan, B., Xu, S.S., Li, S.: Inertial shrinking projection algorithms for solving hierarchical variational inequality problems. J. Nonlinear Convex Anal. (in press) (2020)
36. Thong, D.V., Hieu, D.V.: Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. J. Comput. Appl. Math. **341**, 80–98 (2018)
37. Thong, D.V., Hieu, D.V.: Modified subgradient extragradient algorithms for variational inequality problems and fixed point problems. Optimization **67**(1), 83–102 (2017). https://doi.org/10.1080/02331934.2017.1377199
38. Tian, M., Tong, M.: Self-adaptive subgradient extragradient method with inertial modification for solving monotone variational inequality problems and quasi-nonexpansive fixed point problems. J. Inequal. Appl. (2019). https://doi.org/10.1186/s13660-019-1958-1
39. Xu, H.K.: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279–291 (2004)
40. Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optim. J. Math. Program. Oper. Res. **61**(9), 1119–1132 (2012). https://doi.org/10.1080/02331934.2010.539689
41. Ezeora, J.N., Francis, O.: Nwawuru: an inertial-based hybrid and shrinking projection methods for solving split common fixed point problems in real reflexive spaces. Int. J. Nonlinear Anal. Appl. **14**(1), 2541–2556 (2023). https://doi.org/10.22075/ijnaa.2022.24912.2852
42. Nwawuru, F.O., Ezeora, J.N.: Inertial-based extragradient algorithm for approximating a common solution of split-equilibrium problems and fixed-point problems of nonexpansive semigroups. J. Inequal. Appl. **2023**, 22 (2023). https://doi.org/10.1186/s13660-023-02923-3

43. Ezeora, J.N., Enyi, C.D., Nwawuru, F.O., Richard, C.: Ogbonna. An algorithm for split equilibrium and fixed-point problems using inertial extragradient techniques. Comput. Appl. Math. **42**, 103 (2023). https://doi.org/10.1007/s40314-023-02244-7
44. Goebel, K., Reich, S.: Uniform Convexity. Hyperbolic Geometry, and Nonexpansive Mappings, Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83 (1984)