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Spacelike submanifolds with parallel mean curvature vector in the de Sitter space: characterizations and gaps

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Abstract
Our purpose is to establish new gap type and characterization results concerning n-dimensional spacelike submanifolds immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index q $(1 \le q \le p)$. Initially, by applying a weak form of the Omori–Yau maximum principle, we obtain sufficient conditions which guarantee that a stochastically complete spacelike submanifold M^n immersed in $\mathbb{S}_q^{n+p}$ is either totally umbilical or isometric to a maximal isoparametric spacelike submanifold. Furthermore, by assuming that either the Hilbert–Schmidt norm of the traceless second fundamental form of M^n converges to zero at infinity or that M^n has polynomial volume growth, we provide a set of geometric hypotheses which guarantee the umbilicity of M^n.

Keywords De Sitter space · Parallel mean curvature vector · Traceless second fundamental form · Totally umbilical spacelike submanifolds · Isoparametric spacelike submanifolds

Mathematics Subject Classification Primary 53C42; Secondary 53A10 · 53C20 · 53C50

1 Introduction

The geometry of spacelike submanifolds of a semi-Riemannian space form is a classical but still fruitful thematic into the theory of isometric immersions and has gotten increasing interest motivated by their importance in problems related to Physics, such as General Relativity theory. Into this branch, Goddard [15] conjectured in his seminal paper 1977 that the unique

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complete spacelike hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ with constant mean curvature H should be the totally umbilical ones. Ten years have passed until Ramanathan [25] prove that Goddard's conjecture is true for $\mathbb{S}_1^3$ and $0 \le H \le 1$. However, for $H > 1$ he showed that the conjecture is false, as can be verified from an example due to Dajczer and Nomizu [13]. Simultaneously and independently, Akutagawa [2] also proved that Goddard's conjecture is true when either $n = 2$ and $H^2 \le 1$ or $n \ge 3$ and $H^2 < \frac{4(n-1)}{n^2}$. Moreover, he also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ with constant mean curvature H satisfying $H > 1$ and which are not totally umbilical. Next, Montiel [21] showed that Goddard's conjecture is true for compact (without boundary) spacelike hypersurfaces. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H^2 \ge 4(n-1)/n^2$ and being non totally umbilical, the so-called hyperbolic cylinders.

Related to higher codimension, Cheng [11] extended Akutagawa's result for complete spacelike submanifolds with parallel mean curvature vector in de Sitter space $\mathbb{S}_p^{n+p}$ of index p. Afterwards, Aiyama [1] studied compact spacelike submanifolds M^n in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. In the same work, she showed that compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and nonnegative sectional curvature must be also totally umbilical. Meanwhile, Alías and Romero [6] developed some integral formulas for compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ and, as application of them, they obtained a Bernstein type result for complete maximal submanifolds. Next, Li [19] showed that the result of Montiel [21] still holds for higher codimensional spacelike submanifolds in $\mathbb{S}_p^{n+p}$. Later on, Camargo, Chaves and Sousa [9] studied complete spacelike submanifolds with parallel normalized mean curvature vector (that is, the normalized mean curvature vector is parallel as a section of the normal bundle of the spacelike submanifold) and constant scalar curvature immersed in a semi-Riemannian space form $\mathbb{L}_p^{n+p}(c)$ of constant sectional curvature c and index p. In particular, they obtained characterization results concerning totaly umbilical spacelike submanifolds and hyperbolic cylinders of $\mathbb{S}_p^{n+p}$, under certain constraints on both the squared norm of the second fundamental form and on the mean curvature.

When the index of the ambient space is possibly different of the codimension of the spacelike submanifold, Mariano [20] obtained some characterization results for n-dimensional complete spacelike submanifolds with parallel mean curvature vector and locally timelike second fundamental form in $\mathbb{S}_q^{n+p}$, for $1 \le q \le p$. Later on, Yang and Li [28] applied the Omori–Yau maximum principle in order to get further characterization results concerning complete spacelike submanifolds with parallel mean curvature vector in $\mathbb{S}_q^{n+p}$, for $1 \le q \le p$. Afterwards, working in this same context, Chen, Liu and Shu [10] obtained Simons type integral inequalities and rigidity theorems related to compact spacelike submanifolds of $\mathbb{S}_q^{n+p}$. More recently, Barboza, de Lima and Velásquez [7] investigated n-dimensional spacelike submanifolds immersed with parallel mean curvature vector h in a pseudo-Riemannian space form $\mathbb{L}_q^{n+p}(c)$ of index $1 \le q \le p$ and constant sectional curvature $c \in \{-1, 0, 1\}$. Considering the cases when h is either spacelike or timelike, they proved that such a spacelike submanifold is either totally umbilical or it holds a lower estimate for the supremum of the norm of its traceless second fundamental form, occurring equality when the spacelike submanifold is pseudo-umbilical (which means that h is an umbilical direction) and its principal curvatures are constant.

Going a step further, in this work we establish new gap type and characterization results concerning n-dimensional spacelike submanifolds in the de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \le q \le p$. Initially, in Sect. 4 we apply a weak form of the Omori–Yau maximum principle to obtain sufficient conditions which guarantee that a stochastically complete spacelike submanifold M^n must be either totally umbilical or isometric to a maximal isoparametric spacelike submanifold. Afterwards, by assuming that either the Hilbert–Schmidt norm of the traceless second fundamental form of M^n converges to zero at infinity or that M^n has polynomial volume growth, in Sect. 5 we provide a set of geometric hypotheses which guarantee the umbilicity of M^n.

2 Spacelike submanifolds in a semi-Riemannian space form

Let M^n be an n-dimensional (connected) spacelike submanifold immersed in an $(n+p)$-dimensional semi-Riemannian space form $\mathbb{L}^{n+p}_q(c)$ of index $q \in \{1, \dots, p\}$ and constant sectional curvature $c \in \{-1, 0, 1\}$, where n and p are natural numbers satisfying $n \geq 2$ and $p \geq 1$. We choose a local field of semi-Riemannian orthonormal frame $e_1, \dots, e_{n+p}$ in $\mathbb{L}^{n+p}_q(c)$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n.

Along this manuscript, we will use the following convention for indices:

$$1 \leq A, B, C, \dots \leq n+p; \qquad 1 \leq i, j, k, \dots \leq n; \qquad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

We have that the semi-Riemannian metric $d\bar{s}^2$ of $\mathbb{L}^{n+p}_q(c)$ can be written as

$$d\bar{s}^2 = \sum_A \epsilon_A \omega_A^2,$$

where

$$\epsilon_A = 1, \quad 1 \leq A \leq n+p-q; \quad \epsilon_A = -1, \quad n+p-q+1 \leq A \leq n+p.$$

Denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{L}^{n+p}_q(c)$, we have that the structure equations are given by

$$d\omega_A = -\sum_B \epsilon_B\, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C\, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D K_{ABCD}\, \omega_C \wedge \omega_D, \tag{2.2}$$

and

$$K_{ABCD} = c\epsilon_A \epsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}),$$

where K_{ABCD} denote the components of indefinite curvature tensor of $\mathbb{L}^{n+p}_q(c)$.

Restricting forms to M^n, we have $\omega_\alpha = 0, \alpha \in \{n+1, \dots, n+p\}$, and the induced metric ds^2 of M^n is written as $ds^2 = \sum_i w_i^2$. Since $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha$, from Cartan's Lemma we can write

$$\omega_{i\alpha} = \sum_j h^\alpha_{ij} \omega_j, \quad \text{where} \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.3}$$

The quadratic form

$$A = \sum_{i,j,\alpha} \epsilon_\alpha h^\alpha_{ij} \omega_i \omega_j e_\alpha,$$

is the second fundamental form of M^n. Denoting

$$H^\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}, \qquad \text{with} \qquad \alpha \in \{n+1, \ldots, n+p\},$$

the mean curvature vector h is expressed as

$$h = \sum_\alpha \epsilon_\alpha H^\alpha e_\alpha.$$

Moreover, we denote by H the length of h and by S the square of the length of the second fundamental form, that is,

$$H = ||h|| = \sqrt{\sum_\alpha (H^\alpha)^2} \qquad \text{and} \qquad S = |A|^2 = \sum_{i,j,\alpha} (\epsilon_\alpha h^\alpha_{ij})^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

We can express the structure equations of M^n as follows

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ji} + \omega_{ij} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M^n. Using these structure equations, we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_\alpha \epsilon_\alpha (h^\alpha_{il}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jl}). \tag{2.4}$$

In particular, the components of the Ricci tensor R_{jk} is given by

$$R_{jk} = c(n-1)\delta_{jk} + \sum_\alpha \epsilon_\alpha \left(\sum_i h^\alpha_{ii}h^\alpha_{jk} - \sum_i h^\alpha_{ik}h^\alpha_{ji}\right). \tag{2.5}$$

We define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$, by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta \epsilon_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.6}$$

and

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_m h^\alpha_{mjk}\omega_{mi} - \sum_m h^\alpha_{imk}\omega_{mj} - \sum_m h^\alpha_{ijm}\omega_{mk} - \sum_\beta \epsilon_\beta h^\beta_{ijk}\omega_{\beta\alpha}, \tag{2.7}$$

respectively. Then, by exterior differentiation of (2.3), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{2.8}$$

Furthermore, by exterior differentiation of (2.2), we get the following Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im}R_{mjkl} - \sum_m h^\alpha_{jm}R_{mikl} - \sum_\beta \epsilon_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.9}$$

The Ricci equation is quoted below

$$R_{\alpha\beta ij} = -\sum_m \left(h^\alpha_{im}h^\beta_{mj} - h^\alpha_{jm}h^\beta_{mi}\right). \tag{2.10}$$

3 A Simons type formula and some auxiliary lemmas

In what follows, we denote respectively by ∇ and Δ the gradient and the Laplacian operator in the metric of such a spacelike submanifold M^n immersed in $\mathbb{L}^{n+p}_q(c)$. The traceless second fundamental form Φ of M^n is defined as been the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \phi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \tag{3.1}$$

where

$$\phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha\delta_{ij}. \tag{3.2}$$

Considering

$$\Phi^{\alpha\beta} = \sum_{i,j} \phi^\alpha_{ij}\phi^\beta_{ij} \quad \text{and} \quad \phi^{\alpha\beta} = \sum_{i,j} h^\alpha_{ij}h^\beta_{ij}, \tag{3.3}$$

we have that the $(p \times p)$-matrix $(\Phi^{\alpha\beta})$ is symmetric and can be assumed to be diagonalized for a suitable choice of $e_{n+1}, \ldots, e_{n+p}$. Moreover, setting

$$\Phi^{\alpha\beta} = \Phi^\alpha\delta_{\alpha\beta}, \tag{3.4}$$

from a direct calculation we obtain

$$\sum_k \phi^\alpha_{kk} = 0, \qquad \Phi^{\alpha\beta} = \phi^{\alpha\beta} - nH^\alpha H^\beta \quad \text{and} \quad |\Phi|^2 = \sum_\alpha \Phi^\alpha = S - nH^2, \tag{3.5}$$

where $\Phi^\alpha = \Phi^{\alpha\alpha}$.

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.9), we obtain for any $(n+1 \le \alpha \le n+p)$,

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{k,m} h^\alpha_{km}R_{mijk} - \sum_{k,m} h^\alpha_{im}R_{mkjk} - \sum_{k,\beta} \epsilon_\beta h^\beta_{ik}R_{\beta\alpha jk}. \tag{3.6}$$

So, from (2.8) and (3.6) we obtain

$$\begin{aligned}\Delta h^\alpha_{ij} = &\sum_{i,j,\alpha,k} h^\alpha_{ij}h^\alpha_{kkij} - \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}h^\alpha_{kl}R_{lijk} - \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}h^\alpha_{li}R_{lkjk} \\ &- \sum_{\alpha,\beta}\sum_{i,j,k} \epsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}.\end{aligned} \tag{3.7}$$

Since

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij}, \tag{3.8}$$

from (3.7) and (3.8) we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha,k} h^\alpha_{ij}h^\alpha_{kkij} - \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}h^\alpha_{kl}R_{lijk}$$

$$-\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk} - \sum_{\alpha,\beta}\sum_{i,j,k}\epsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk},$$

that is,

$$\begin{aligned}\frac{1}{2}\Delta S = |\nabla h|^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha}(nH^{\alpha})_{,ij} - \sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{kl}^{\alpha}R_{lijk} \\ - \sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk} - \sum_{\alpha,\beta}\sum_{i,j,k}\epsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}.\end{aligned} \tag{3.9}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is, $N(A) = \mathrm{tr}(AA^t) = \sum_{i,j}(a_{ij})^2$. Clearly, $N(A) = N(T^tAT)$ for any orthogonal matrix T. From (2.10), we have

$$\begin{aligned}-\sum_{\alpha,\beta}\sum_{i,j,k,\beta}\epsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk} &= -\sum_{\alpha,\beta}\sum_{i,j,k,\beta}\epsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}(h_{kl}^{\beta}h_{li}^{\alpha} - h_{jl}^{\beta}h_{lk}^{\alpha}) \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\epsilon_{\beta}\left(\sum_{l}h_{kl}^{\beta}h_{lj}^{\alpha} - \sum_{l}h_{kl}^{\alpha}h_{lj}^{\beta}\right)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\epsilon_{\beta}\left(\sum_{l}\phi_{kl}^{\beta}\phi_{lj}^{\alpha} - \sum_{l}\phi_{kl}^{\alpha}\phi_{lj}^{\beta}\right)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta}\epsilon_{\beta}N(\phi_{ij}^{\alpha}\phi_{ij}^{\beta} - \phi_{ij}^{\beta}\phi_{ij}^{\alpha}) \\ &= -\frac{1}{2}\sum_{\alpha,\beta}\epsilon_{\beta}N(\Phi_{\alpha}\Phi_{\beta} - \Phi_{\beta}\Phi_{\alpha}),\end{aligned} \tag{3.10}$$

where $\Phi_{\alpha} := (\phi_{ij}^{\alpha}) = (h_{ij}^{\alpha} - H^{\alpha}\delta_{ij})$.

Combining (2.4), (2.10), (3.2), (3.3), (3.5) and (3.10), we conclude that

$$\begin{aligned}-\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{li}^{\alpha}R_{lkjk}) &= nc|\Phi|^2 - \sum_{\alpha,\beta}\epsilon_{\beta}(\phi^{\alpha\beta})^2 + n\sum_{\alpha,\beta,i,j,k}\epsilon_{\beta}H^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} \\ &\quad - \sum_{\alpha,\beta,i,j,k}\epsilon_{\beta}h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk} \\ &= nc|\Phi|^2 - \sum_{\alpha,\beta}\epsilon_{\beta}(\Phi^{\alpha\beta})^2 - 2n\sum_{\alpha,\beta,i,j}\epsilon_{\beta}H^{\alpha}H^{\beta}\phi_{ij}^{\alpha}\phi_{ij}^{\beta} \\ &\quad - n^2\sum_{\alpha,\beta}\epsilon_{\beta}(H^{\alpha})^2(H^{\beta})^2 + n\sum_{\alpha,\beta,i,j,k}\epsilon_{\beta}H^{\beta}\phi_{kj}^{\beta}\phi_{ij}^{\alpha}\phi_{ik}^{\alpha} \\ &\quad + n|\Phi|^2\sum_{\beta}\epsilon_{\beta}(H^{\beta})^2 + 2n\sum_{\alpha,\beta,i,j}\epsilon_{\beta}H^{\alpha}H^{\beta}\phi_{ij}^{\alpha}\phi_{ij}^{\beta} \\ &\quad + n^2\sum_{\alpha,\beta}\epsilon_{\beta}(H^{\alpha})^2(H^{\beta})^2 - \frac{1}{2}\sum_{\alpha,\beta}\epsilon_{\beta}N(\Phi_{\alpha}\Phi_{\beta} - \Phi_{\beta}\Phi_{\alpha}).\end{aligned} \tag{3.11}$$

Hence, we arrive at

$$-\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{li}^{\alpha}R_{lkjk}) = nc|\Phi|^2 - \sum_{\alpha,\beta}\epsilon_{\beta}(\Phi^{\alpha\beta})^2 + n|\Phi|^2\sum_{\beta}\epsilon_{\beta}(H^{\beta})^2$$

$$
\begin{aligned}
&+n \sum_{\alpha,\beta,i,j,k} \epsilon_\beta H^\beta \phi^\beta_{kj}\phi^\alpha_{ij}\phi^\alpha_{ik} \\
&-\frac{1}{2}\sum_{\alpha,\beta}\epsilon_\beta N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha).
\end{aligned}
\tag{3.12}
$$

Assuming that the mean curvature vector is parallel, that is, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha}(H^\alpha_{,i})^2 = 0$, we see that $H^\alpha_{,i} = 0$ for all i, α and H^α are constant for all α, this implies that H is constant. Putting (3.10) and (3.12) into (3.9), we have the following Simons type formula.

Proposition 1 *Let M^n be an n-dimensional spacelike submanifold immersed with parallel mean curvature vector in an $(n+p)$-dimensional semi-Riemannian space form $\mathbb{L}^{n+p}_q(c)$ of index $q \in \{1, \dots, p\}$ and constant sectional curvature $c \in \{-1, 0, 1\}$. With all the notations established above, we have that the traceless second fundamental form Φ of M^n verifies*

$$
\begin{aligned}
\frac{1}{2}\Delta|\Phi|^2 = {} & |\nabla h|^2 + nc|\Phi|^2 + n|\Phi|^2\sum_\beta \epsilon_\beta (H^\beta)^2 + \sum_{\alpha,\beta}\sum_{i,j,k}\epsilon_\beta H^\beta \phi^\beta_{kj}\phi^\alpha_{ij}\phi^\alpha_{ik} \\
& - \sum_{\alpha,\beta}\epsilon_\beta N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha) - \sum_{\alpha,\beta}\epsilon_\beta(\Phi^{\alpha\beta})^2.
\end{aligned}
\tag{3.13}
$$

In order to prove our results in the next sections, we will also need the following algebraic lemmas, whose proofs can be found in Santos [26] and Li and Li [18], respectively.

Lemma 1 *Let B_1 and B_2 be symmetric $n \times n$ matrices such that $[B_1, B_2] = 0$ and $\mathrm{tr}(B_1) = \mathrm{tr}(B_2) = 0$. Then*

$$
|\mathrm{tr}\left(B_1^2 B_2\right)| \le \frac{n-2}{\sqrt{n(n-1)}}\,\mathrm{tr}\left(B_1^2\right)\sqrt{\mathrm{tr}\left(B_2^2\right)},
$$

and the equality holds if and only if $n-1$ of the eigenvalues x_i of B_1 and the corresponding eigenvalues y_i of B_2 satisfy

$$
|x_i| = \frac{\left(\mathrm{tr}\left(B_1^2\right)\right)^{1/2}}{\sqrt{n(n-1)}}, \qquad y_i = \frac{\left(\mathrm{tr}\left(B_2^2\right)\right)^{1/2}}{\sqrt{n(n-1)}} \quad \left(\textit{resp., } y_i = -\frac{\left(\mathrm{tr} B_2^2\right)^{1/2}}{\sqrt{n(n-1)}}\right).
$$

Lemma 2 *Let $B_1, \dots, B_p$, $p \ge 2$, be symmetric $n \times n$ matrices. Then*

$$
\sum_{\alpha,\beta=1}^{p}\left(\mathrm{tr}[B_\alpha, B_\beta]^2 - \mathrm{tr}\left(B_\alpha B_\beta\right)^2\right) \ge -\frac{3}{2}\left(\sum_{\alpha=1}^{p}\mathrm{tr}\left(B_\alpha^2\right)\right)^2.
$$

4 Stochastically complete spacelike submanifolds in $\mathbb{S}^{n+p}_q$

We recall that a (non necessarily complete) Riemannian manifold M^n is said to be *stochastically complete* when, for some (and, hence, for any) $(x, t) \in M^n \times (0, +\infty)$, the heat kernel $p(x, y, t)$ of the Laplace–Beltrami operator Δ satisfies the conservation property

$$
\int_M p(x, y, t)\,d\mu(y) = 1 \tag{4.1}
$$

From the probabilistic viewpoint, stochatically completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (4.1) means that the total probability of the particle to be found in the state space is constantly equal to one (see Émery [14], Grigoryan [16, 17] and Stroock [27]).

Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori–Yau maximum principle, as is expressed below (see Theorem 1.1 of Pigola, Rigoli and Setti [23] and Theorem 3.1 of Pigola, Rigoli and Setti [24]):

Lemma 3 *A Riemannian manifold M^n is stochastically complete if, and only if, for every $u \in C^2(M)$ satisfying $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\} \subset M^n$ such that*

$$\lim_{k\to\infty} u(p_k) = \sup_M u \qquad \text{and} \qquad \limsup_{k\to\infty} \Delta u(p_k) \le 0.$$

Remark 1 We also note that stochastic completeness of Riemannian manifold M^n is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only nonnegative bounded smooth solution u of $\Delta u \ge \lambda u$ on M^n is the constant $u = 0$. Moreover, it is a direct consequence of Lemma 1 jointly with the Omori [22] and Yau [29] maximum principle that complete Riemannian manifolds having Ricci curvature bounded from below are stochastically complete.

In our first result, we present a gap type theorem concerning stochastically complete spacelike submanifolds with parallel mean curvature vector.

Theorem 1 *Let M^n be an n-dimensional stochastically complete spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the mean curvature H satisfies $H < 1$. Then,*

(a) *either $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold,*
(b) *or $\sup_M |\Phi| \ge \alpha^*$, where α^* is the positive root of the polynomial function*

$$P_H(x) = -a|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1}} H|\Phi| + n(1-H^2), \tag{4.2}$$

with $a = 1$ if $p - q = 1$ and $a = 3/2$ if $p - q > 1$. Moreover, when $\sup_M |\Phi| = \alpha^*$ and it is attained at some point of M^n, then M^n is isometric to a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$.

Proof We have the following:

$$\begin{aligned} n|\Phi|^2 \sum_\beta \epsilon_\beta (H^\beta)^2 &= n|\Phi|^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 - n|\Phi|^2 \sum_{\beta=n+p-q+1}^{n+p} (H^\beta)^2 \\ &= n|\Phi|^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 + n|\Phi|^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 \\ &\quad - n|\Phi|^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 - n|\Phi|^2 \sum_{\beta=n+p-q+1}^{n+p} (H^\beta)^2 \\ &= 2n|\Phi|^2 \sum_{\beta=n+1}^{n+p-q} (H^\beta)^2 - n|\Phi|^2 \sum_{\beta=n+1}^{n+p} (H^\beta)^2 \end{aligned}$$

$$\geq -n|\Phi|^2 H^2. \tag{4.3}$$

We use that

$$\sum_i (\phi_{ii}^\beta)^2 = \Phi^\beta, \quad \sum_i \phi_{ii}^\beta = 0, \quad \sum_i \mu_i^\alpha = 0 \quad \text{and} \quad \sum_i (\mu_i^\alpha)^2 = \Phi^\alpha,$$

from (3.13) we obtain

$$\begin{aligned} n\sum_{\alpha,\beta,i,j,k} \epsilon_\beta H^\beta \phi_{kj}^\beta \phi_{ij}^\alpha \phi_{ik}^\alpha &= n\sum_{\alpha,i,j,k}\sum_{\beta=n+1}^{n+p-q} H^\beta \phi_{kj}^\beta \phi_{ij}^\alpha \phi_{ik}^\alpha - n\sum_{\alpha,i,j,k}\sum_{\beta=n+p-q+1}^{n+p} H^\beta \phi_{kj}^\beta \phi_{ij}^\alpha \phi_{ik}^\alpha \\ &= n\sum_{\alpha,i}\sum_{\beta=n+1}^{n+p-q} H^\beta \phi_{ii}^\beta (\mu_i^\alpha)^2 - n\sum_{\alpha,i}\sum_{\beta=n+p-q+1}^{n+p} H^\beta \phi_{ii}^\beta (\mu_i^\alpha)^2. \end{aligned} \tag{4.4}$$

So, it follows from Lemma 1 that

$$n\sum_{\alpha,i}\sum_{\beta=n+1}^{n+p-q} H^\beta \phi_{ii}^\beta (\mu_i^\alpha)^2 \geq -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_\alpha \sum_{\beta=n+1}^{n+p-q} |H^\beta|\Phi^\alpha \sqrt{\Phi^\beta}, \tag{4.5}$$

and

$$n\sum_{\alpha,i}\sum_{\beta=n+p-q+1}^{n+p} H^\beta \phi_{ii}^\beta (\mu_i^\alpha)^2 \geq -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} |H^\beta|\Phi^\alpha \sqrt{\Phi^\beta}. \tag{4.6}$$

Hence, from (4.4), (4.5) and (4.6) we have that

$$\begin{aligned} n\sum_{\alpha,\beta,i,j,k} \epsilon_\beta H^\beta \phi_{kj}^\beta \phi_{ij}^\alpha \phi_{ik}^\alpha &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_\alpha \Phi^\alpha \left(\sum_{\beta=n+1}^{n+p-q} |H^\beta|\sqrt{\Phi^\beta} + \sum_{\beta=n+p-q+1}^{n+p} |H^\beta|\sqrt{\Phi^\beta}\right) \\ &= -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_\alpha \Phi^\alpha \sum_{\beta=n+1}^{n+p} |H^\beta|\sqrt{\Phi^\beta} \\ &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}|\Phi|^2 \left(\sqrt{\sum_\beta (H^\beta)^2 \sum_\beta \Phi^\beta}\right) \\ &= -\frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi|^3. \end{aligned} \tag{4.7}$$

On the other hand, we have

$$\begin{aligned} -\sum_{\alpha,\beta} \epsilon_\beta N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha) &= -\sum_\alpha \sum_{\beta=n+1}^{n+p-q} N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha) \\ &\quad + \sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} \epsilon_\beta N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha) \end{aligned} \tag{4.8}$$

and

$$-\sum_{\alpha,\beta} \epsilon_\beta (\Phi^{\alpha\beta})^2 = -\sum_\alpha \sum_{\beta=n+1}^{n+p-q} (\Phi^{\alpha\beta})^2 + \sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} (\Phi^{\alpha\beta})^2. \tag{4.9}$$

Thus, from (4.8) and (4.9) we get

$$
\begin{aligned}
&-\sum_{\alpha,\beta}\epsilon_\beta N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)-\sum_{\alpha,\beta}\epsilon_\beta(\Phi^{\alpha\beta})^2=-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)\\
&\quad-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}(\Phi^{\alpha\beta})^2+\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)\\
&\quad+\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}(\Phi^{\alpha\beta})^2\\
&\geq-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}(\Phi^{\alpha\beta})^2,
\end{aligned}
$$

where the inequality $N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)\geq 0$ for any α,β was used.

Now we consider two cases:

- If $p-q=1$, we have from (3.5) that

$$
-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}(\Phi^{\alpha\beta})^2=-(\Phi^{n+1n+1})^2\geq-|\Phi|^4. \tag{4.10}
$$

- If $p-q>1$, from Theorem 1 of Anmin and Jimin [3] we have

$$
\begin{aligned}
-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\Phi_\alpha\Phi_\beta-\Phi_\beta\Phi_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}(\Phi^{\alpha\beta})^2&\geq-\frac{3}{2}\left(\sum_{\alpha=n+1}^{n+p-q}(\Phi^\alpha)^2\right)\\
&\geq-\frac{3}{2}|\Phi|^4.
\end{aligned} \tag{4.11}
$$

Consequently, from (3.13), (4.3) - (4.11) we get

$$
\frac{1}{2}\Delta|\Phi|^2\geq|\nabla h|^2+|\Phi|^2\left\{n(1-H^2)-\frac{n(n-2)}{\sqrt{n(n-1}}H|\Phi|-a|\Phi|^2\right\}, \tag{4.12}
$$

where $a=1$ if $p-q=1$ and $a=3/2$ if $p-q>1$. Therefore, we have that (4.12) can be rewritten as

$$
\frac{1}{2}\Delta|\Phi|^2\geq|\Phi|^2P_H(|\Phi|), \tag{4.13}
$$

where $P_H(x)$ is the function defined by (4.2). Moreover, from the behavior of $P_H(x)$ we have that $P_H(0)>0$ and $\lim_{x\to\infty}P_H(x)=-\infty$. Now, we observe that if $\sup_M|\Phi|=+\infty$, then item (ii) is trivially satisfied. So, let us suppose that $\sup_M|\Phi|\leq+\infty$. Thus, Lemma 3 guarantees that there exists a sequence of points $\{p_k\}\subset M^n$ such that

$$
\lim_{k\to\infty}|\Phi|^2(p_k)=\sup_M|\Phi|^2\quad\text{and}\quad\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\leq 0.
$$

Consequently, taking into account the continuity of the function $P_H(x)$, from (4.13) we get

$$
\begin{aligned}
0\geq\frac{1}{2}\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)&\geq\limsup_{k\to\infty}\left(|\Phi|^2P_H(|\Phi|)\right)(p_k)=\lim_{k\to\infty}\left(|\Phi|^2P_H(|\Phi|)\right)(p_k)\\
&=\lim_{k\to\infty}|\Phi|^2(p_k)P_H(\lim_{k\to\infty}|\Phi|(p_k))=\sup_M|\Phi|^2P_H(\sup_M|\Phi|).
\end{aligned}
$$

Hence, we obtain

$$(\sup_M|\Phi|)^2 P_H(\sup_M|\Phi|) \leq 0.$$

It follows from here that either $\sup_M|\Phi|= 0$, which means that $|\Phi|= 0$ and M^n is totally umbilical, or $\sup_M|\Phi|> 0$ and then

$$P_H(\sup_M|\Phi|) \leq 0,$$

which implies that $\sup_M|\Phi|\geq \alpha^*$, where α^* is the positive root of (4.2).

To conclude the proof, let us assume that $\sup_M|\Phi|= \alpha^*$ and that it is attained at some point of M^n, then from Hopf's maximum principle we get that $|\Phi|$ is constant. Hence, from (4.12) we obtain that $|\nabla h|^2= 0$, which means that M^n is an isoparametric spacelike submanifold. On the other hand, from the fact that $|\Phi|$ is constant, we have that all the above inequalities are, in fact, equalities. Thus, from the equalities in (4.3), (4.7) and (4.10) we obtain

$$\sum_{\beta=n+1}^{n+p-q}(H^\beta)^2 = 0, \qquad \sum_\beta |H^\beta|\sqrt{\Phi^\beta} = H|\Phi|, \qquad \sum_{\alpha=n+p-q+1}^{n+p}\ \sum_{\beta=n+p-q+1}^{n+p}(\Phi^{\alpha\beta})^2.$$

This implies that $H^\beta = 0$ for $\beta = n+1, \cdots, n+p-q$, and $\Phi^\beta = 0$ for $\beta = n+p-q+1, \cdots, n+p$. Therefore, we get

$$H|\Phi|= \sum_\beta |H^\beta|\sqrt{\Phi^\beta} = 0.$$

Since we are assuming $|\Phi|\neq 0$, we have $H = 0$, that is, M^n is a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$. □

Taking into account Remark 1, from Theorem 1 we derive the following consequence.

Corollary 1 *Let M^n be an n-dimensional complete spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \dots, p\}$, such that the Ricci curvature is bounded from below and the mean curvature H satisfies $H < 1$. Then,*

(a) *either $\sup_M|\Phi|= 0$ and M^n is a totally umbilical submanifold,*
(b) *or $\sup_M|\Phi|\geq \alpha^*$, where α^* is the positive root of the polynomial function (4.2). Moreover, when $\sup_M|\Phi|= \alpha^*$ and it is attained at some point of M^n, then M^n is isometric to a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$.*

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in C^2(M)$,

$$\Delta u \geq 0 \quad \text{and} \quad \sup_M u < +\infty \quad \text{implies} \quad u = \text{constant}.$$

Since every parabolic Riemannian manifold is stochastically complete (see Pigola, Rigoli and Setti [24]), from Theorem 1 we obtain.

Corollary 2 *Let M^n be an n-dimensional spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \dots, p\}$, such that the mean curvature H satisfies $H < 1$. If M^n is parabolic, then either $|\Phi|\equiv 0$ and M^n is totally umbilical, or $\sup_M|\Phi|\geq \alpha^*$, where α^* is the positive root of the polynomial function (4.2). Moreover, when $\sup_M|\Phi|= \alpha^*$, M^n is isometric to a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$.*

Proof Since the weak Omori–Yau's maximum principle [22, 29] holds on every parabolic Riemannian manifold, if $\sup_M |\Phi|^2 < +\infty$, there is nothing to prove. On the other hand, in the case that $0 < \sup_M |\Phi|^2 \leq +\infty$, reasoning as in the first part of the proof of Theorem 1, we guarantee that $\sup_M |\Phi|^2 \geq \alpha^*$. Moreover, if $\sup_M |\Phi|^2 = \alpha^*$, then $P_H(\sup_M \Phi) \leq 0$ and, consequently, the function $|\Phi|^2$ is subharmonic on M^n. Therefore, from the parabolicity of M^n we conclude that the function $|\Phi|^2$ must be constant and equal to α^*. To close the proof, we can reason as in the last part of the proof of Theorem 1. □

In our next result, we assume a suitable hypothesis on the infimum of the sectional curvature of the submanifold.

Theorem 2 *Let M^n be an n-dimensional stochastically complete spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the infimum K of the sectional curvatures of M^n satisfies*

$$K \geq \frac{1}{n}\left(1 - \frac{1}{p-q}\right)|\Phi|^2, \tag{4.14}$$

then

(a) *either* $\sup_M |\Phi| = 0$ *and M^n is a totally umbilical submanifold,*
(b) *or* $\sup_M |\Phi| \geq \beta^*$, *where β^* is the first positive root of the polynomial function*

$$P_K(|\Phi|) = nK - \left(1 - \frac{1}{p-q}\right)|\Phi|^2. \tag{4.15}$$

Moreover, when $\sup_M |\Phi| = \beta^*$ *and it is attained at some point of M^n, then M^n is isometric to an isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$.*

Proof For a fixed α, $n+1 \leq \alpha \leq n+p$, we can take a local orthornormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Then, $\phi_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$ with $\mu_i^\alpha = \lambda_i^\alpha - H^\alpha$ and $\sum_i \mu_i^\alpha = 0$. Consequently,

$$\begin{aligned} -\sum_{\alpha,i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} - h_{li}^\alpha R_{lkjk}) &= \frac{1}{2}\sum_{\alpha,i,k}(\lambda_i^\alpha - \lambda_k^\alpha)^2 R_{kiik} \\ &\geq \frac{K}{2}\sum_{\alpha,i,k}(\lambda_i^\alpha - \lambda_k^\alpha)^2 \\ &= nK(S - nH^2) = nK|\Phi|^2. \end{aligned} \tag{4.16}$$

On the other hand, we have

$$\begin{aligned} -\frac{1}{2}\sum_{\alpha,\beta}\epsilon_\beta N(\phi_{ij}^\alpha\phi_{ij}^\beta - \phi_{ij}^\beta\phi_{ij}^\alpha) &= -\frac{1}{2}\sum_\alpha \sum_{\beta=n+1}^{n+p-q} N(\phi_{ij}^\alpha\phi_{ij}^\beta - \phi_{ij}^\beta\phi_{ij}^\alpha) \\ &\quad + \frac{1}{2}\sum_\alpha \sum_{\beta=n+p-q+1}^{n+p} N(\phi_{ij}^\alpha\phi_{ij}^\beta - \phi_{ij}^\beta\phi_{ij}^\alpha) \\ &= -\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\phi_{ij}^\alpha\phi_{ij}^\beta - \phi_{ij}^\beta\phi_{ij}^\alpha) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}N(\phi_{ij}^{\alpha}\phi_{ij}^{\beta}-\phi_{ij}^{\beta}\phi_{ij}^{\alpha})\\
&\geq -\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\phi_{ij}^{\alpha}\phi_{ij}^{\beta}-\phi_{ij}^{\beta}\phi_{ij}^{\alpha})\\
&\geq -\sum_{\alpha\neq\beta}\Phi^{\alpha}\Phi^{\beta}. \qquad (4.17)
\end{aligned}
$$

From Lemma 1 of Chern, do Carmo and Kobayashi [12], we have that

$$
\begin{aligned}
-\frac{1}{2}\sum_{\alpha,\beta}\epsilon_{\beta}N(\phi_{ij}^{\alpha}\phi_{ij}^{\beta}-\phi_{ij}^{\beta}\phi_{ij}^{\alpha}) &= -\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2+\sum_{\alpha=n+1}^{n+p-q}(\Phi^{\alpha})^2\\
&\geq -\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2+\frac{1}{p-q}\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2\\
&= -\left(1-\frac{1}{p-q}\right)\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2\\
&\geq -\left(1-\frac{1}{p-q}\right)|\Phi|^4. \qquad (4.18)
\end{aligned}
$$

Hence, from (3.10), (3.11), (4.16) and (4.17) we obtain

$$
\frac{1}{2}\Delta|\Phi|^2\geq |\nabla h|^2+|\Phi|^2\left(nK-\left(1-\frac{1}{p-q}\right)|\Phi|^2\right), \qquad (4.19)
$$

where K is the infimum of the sectional curvatures of M^n. Thus, we have that (4.19) can be rewritten as

$$
\frac{1}{2}\Delta|\Phi|^2\geq |\Phi|^2P_K(|\Phi|), \qquad (4.20)
$$

where $P_K(x)$ is the function defined by (4.15).

Moreover, from the behavior of $P_K(x)$ we get that $P_K(0)>0$ and $\lim_{x\to\infty}P_K(x)=-\infty$. If $\sup_M|\Phi|=+\infty$, then item (ii) is trivially satisfied. So, let us suppose that $\sup_M|\Phi|\leq+\infty$. Thus, Lemma 3 guarantees that there exists a sequence of points $\{p_k\}\subset M^n$ such that

$$
\lim_{k\to\infty}|\Phi|^2(p_k)=\sup_M|\Phi|^2 \quad \text{and} \quad \limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\leq 0.
$$

Consequently, taking into account the continuity of the function $P_K(x)$, from (4.13) we get

$$
\begin{aligned}
0\geq\frac{1}{2}\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)&\geq\limsup_{k\to\infty}\left(|\Phi|^2P_K(|\Phi|)\right)(p_k)=\lim_{k\to\infty}\left(|\Phi|^2P_K(|\Phi|)\right)(p_k)\\
&=\lim_{k\to\infty}|\Phi|^2(p_k)P_K(\lim_{k\to\infty}|\Phi|(p_k))=\sup_M|\Phi|^2P_K(\sup_M|\Phi|).
\end{aligned}
$$

Hence, we obtain

$$
(\sup_M|\Phi|)^2P_K(\sup_M|\Phi|)\leq 0. \qquad (4.21)
$$

It follows from here that either $\sup_M|\Phi|=0$, which means that $|\Phi|=0$ and M^n is totally umbilical, or $\sup_M|\Phi|>0$ and then $P_K(\sup_M|\Phi|)\leq 0$, which implies that $\sup_M|\Phi|\geq\beta^*$, where β^* is the first positive root of (4.15).

To conclude the proof, let us assume that $\sup_M|\Phi|=\beta^*$ and that it is attained at some point of M^n. Then, from Hopf's maximum principle we get that $|\Phi|$ is constant. Therefore, from (4.19) we obtain that $|\nabla h|^2=0$, which means that M^n is an isoparametric spacelike submanifold of $\mathbb{S}^{n+p}_q$. □

From Theorem 2, we car reason as in the proof of Corollary 2 to get the following result.

Corollary 3 *Let M^n be an n-dimensional parabolic spacelike submanifold immersed with parallel mean curvature vector immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}^{n+p}_q$ of index $q\in\{1,\dots,p\}$, such that the infimum K of the sectional curvatures of M^n satisfies (4.14). Then, either $\sup_M|\Phi|=0$ and M^n is a totally umbilical submanifold, or $\sup_M|\Phi|\geq\beta^*$ where β^* is the positive root of the polynomial function (4.15). Moreover, when $\sup|\Phi|=\beta^*$, then M^n is isometric to an isoparametric spacelike submanifold of $\mathbb{S}^{n+p}_q$.*

We proceed with the following gap type result.

Theorem 3 *Let M^n be an n-dimensional stochastically complete spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}^{n+p}_q$ of index $q\in\{1,\dots,p\}$, such that the infimum Q of the Ricci curvatures of M^n satisfies*

$$Q>-2+n(1+H^2)+\frac{n-2}{\sqrt{n(n-1)}}H|\Phi|+\frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right)|\Phi|^2. \tag{4.22}$$

Then,

(a) *either $\sup_M|\Phi|=0$ and M^n is a totally umbilical submanifold,*
(b) *or $\sup_M|\Phi|\geq\gamma^*$, where γ^* is the first positive root of the polynomial function*

$$P_Q(|\Phi|)=Q+2-n(1+H^2)-\frac{n-2}{\sqrt{n(n-1)}}H|\Phi|-\frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right)|\Phi|^2. \tag{4.23}$$

Moreover, when $\sup_M|\Phi|=\gamma^*$ and it is attained at some point of M^n, then M^n is isometric to a maximal isoparametric spacelike submanifold of $\mathbb{S}^{n+p}_q$.

Proof From (2.5) and (3.2) we have

$$\begin{aligned} R_{kk} &= (n-1)+(n-2)\sum_\alpha \epsilon_\alpha H^\alpha\phi^\alpha_{ik}+(n-1)\sum_{\alpha=n+1}^{n+p-q}(H^\alpha)^2\\ &\quad-(n-1)\sum_{\alpha=n+p-q+1}^{n+p}(H^\alpha)^2-\sum_{i,\alpha=n+1}^{n+p-q}(\phi^\alpha_{ik})^2+\sum_{i,\alpha=n+p-q+1}^{n+p}(\phi^\alpha_{ik})^2\\ &\leq (n-1)+(n-2)\sum_\alpha \epsilon_\alpha H^\alpha\phi^\alpha_{ik}+(n-1)H^2-\sum_{i,\alpha=n+1}^{n+p-q}(\phi^\alpha_{ik})^2+\sum_{i,\alpha=n+p-q+1}^{n+p}(\phi^\alpha_{ik})^2. \end{aligned} \tag{4.24}$$

Thus,

$$nQ\leq\sum_k R_{kk}=n(n-1)(1+H^2)-\sum_{i,\alpha=n+1}^{n+p-q}(\phi^\alpha_{ik})^2+\sum_{i,\alpha=n+p-q+1}^{n+p}(\phi^\alpha_{ik})^2.$$

From (3.3) and (3.4) we obtain

$$-\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}+\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\geq nQ-n(n-1)(1+H^2). \tag{4.25}$$

Hence, from (4.25) we see that

$$\begin{aligned}\sum_{\alpha,\beta}\epsilon_{\beta}(\Phi^{\alpha\beta})^2&=-\sum_{\alpha}\epsilon_{\beta}(\Phi^{\alpha})^2=-\sum_{\alpha=n+1}^{n+p-q}(\Phi^{\alpha})^2+\sum_{\alpha=n+p-q+1}^{n+p}(\Phi^{\alpha})^2\\&\geq-\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2+\frac{1}{q}\left(\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\right)^2\\&=-\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}\right)^2+\left(\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\right)^2+\left(\frac{1}{q}-1\right)\left(\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\right)^2\\&\geq\left(-\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}+\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\right)\left(\sum_{\alpha=n+1}^{n+p-q}\Phi^{\alpha}+\sum_{\alpha=n+p-q+1}^{n+p}\Phi^{\alpha}\right)-\left(1-\frac{1}{q}\right)|\Phi|^4\\&\geq(nQ-n(n-1)(1+H^2))|\Phi|^2-\left(1-\frac{1}{q}\right)|\Phi|^4. \end{aligned}\tag{4.26}$$

From (4.18) we have

$$\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\Phi_{\alpha}\Phi_{\beta}-\Phi_{\beta}\Phi_{\alpha})\geq-2\left(1-\frac{1}{p-q}\right)|\Phi|^4. \tag{4.27}$$

Thus, from (3.13), (4.3), (4.7), (4.26) and (4.27) we get

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2&\geq|\nabla h|^2+n|\Phi|^2-n|\Phi|^2H^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3-2\left(1-\frac{1}{p-q}\right)|\Phi|^4\\&\quad+(nQ-n(n-1)(1+H^2))|\Phi|^2-\left(1-\frac{1}{q}\right)|\Phi|^4\\&=|\nabla h|^2+n|\Phi|^2\left\{Q+2-n(1+H^2)-\frac{n-2}{\sqrt{n(n-1)}}H|\Phi|-\frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right)|\Phi|^2\right\},\end{aligned}\tag{4.28}$$

where Q is the infimum of the Ricci curvatures of M^n. Hence, we have that (4.28) can be rewritten as

$$\frac{1}{2}\Delta|\Phi|^2\geq|\nabla h|^2+n|\Phi|^2P_Q(|\Phi|)\geq n|\Phi|^2P_Q(|\Phi|), \tag{4.29}$$

where $P_Q(x)$ is the function defined by (4.23).

Moreover, from the behavior of $P_Q(x)$ we get that $P_Q(0)>0$ and $\lim_{x\to\infty}P_Q(x)=-\infty$. If $\sup_M|\Phi|=+\infty$, then item (ii) is trivially satisfied. So, let us suppose that $\sup_M|\Phi|\leq+\infty$. Thus, Lemma 3 guarantees that there exists a sequence of points $\{p_k\}\subset M^n$ such that

$$\lim_{k\to\infty}|\Phi|^2(p_k)=\sup_M|\Phi|^2\quad\text{and}\quad\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\leq 0.$$

Consequently, taking into account the continuity of the function $P_Q(x)$, from (4.13) we get

$$0\geq\frac{1}{2}\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\geq\limsup_{k\to\infty}\left(|\Phi|^2P_Q(|\Phi|)\right)(p_k)=\lim_{k\to\infty}\left(|\Phi|^2P_Q(|\Phi|)\right)(p_k)$$

$$= \lim_{k\to\infty} |\Phi|^2(p_k) P_Q(\lim_{k\to\infty} |\Phi|(p_k)) = \sup_M |\Phi|^2 P_Q(\sup_M |\Phi|).$$

Hence, we obtain

$$(\sup_M |\Phi|)^2 P_Q(\sup_M |\Phi|) \leq 0. \tag{4.30}$$

It follows from here that either $\sup|\Phi|= 0$, which means that $|\Phi|= 0$ and M^n is totally umbilical, or $\sup_M |\Phi|> 0$ and then

$$P_Q(\sup_M |\Phi|) \leq 0,$$

which implies that $\sup_M |\Phi|\geq \gamma^*$, where γ^* is the first positive root of (4.23).

To conclude the proof, let us assume that $\sup_M |\Phi|= \gamma^*$ and that it is attained at some point of M^n. From Hopf's maximum principle we get that $|\Phi|$ is constant. Hence, from (4.28) we obtain that $|\nabla h|^2= 0$, which means that M^n is an isoparametric spacelike submanifold. On the order hand, in the latter case, we see that the equalities in (4.3) and (4.24) hold. Thus, we have

$$\sum_{\alpha=n+1}^{n+p-q} (H^\alpha)^2 = 0, \qquad \sum_{\alpha=n+p-q+1}^{n+p} (H^\alpha)^2 = 0, \tag{4.31}$$

which imply that $H^\alpha = 0$ for $\alpha = n+1, \cdots, n+p$ and $H = 0$, that is, M^n must be a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$. □

From Theorem 4, we obtain the following consequence.

Corollary 4 *Let M^n be an n-dimensional parabolic spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the infimum Q of the Ricci curvatures of M^n satisfies* (4.22)*. Then, either* $\sup_M |\Phi|= 0$ *and M^n is a totally umbilical submanifold, or* $\sup_M |\Phi|\geq \gamma^*$ *where γ^* is the positive root of the polynomial function* (4.23)*. Moreover, when* $\sup_M |\Phi|= \gamma^*$*, then M^n is isometric to a maximal isoparametric spacelike submanifold of $\mathbb{S}_q^{n+p}$.*

5 Umbilicity of complete spacelike submanifolds in $\mathbb{S}_q^{n+p}$

This section is devoted to study the umbilicity of a complete spacelike submanifold M^n immersed with parallel mean curvature vector in $\mathbb{S}_q^{n+p}$, by assuming that either the Hilbert–Schmidt norm of the traceless second fundamental form of M^n converges to zero at infinity or that M^n has polynomial volume growth.

5.1 Umbilicity via a maximum principle at infinity

In this subsection, our approach will be based on a suitable maximum principle at infinity for complete noncompact Riemannian manifolds due to Alías, Caminha and Nascimento [4]. To quote it, we need to recall the following concept established in the beginning of Section 2 of Alías, Caminha and Nascimento [4]: Let M^n be a complete noncompact Riemannian manifold and let $d(\cdot, o) : M^n \to [0, +\infty)$ denote the Riemannian distance of M^n, measured from a fixed point $o \in M^n$. We say that a smooth function $f \in C^\infty(M)$ *converges to zero at infinity*, when it satisfies the following condition

$$\lim_{d(x,o)\to+\infty} f(x) = 0. \tag{5.1}$$

Keeping in mind this concept, the following maximum principle at infinity corresponds to item (a) of Theorem 2.2 of Alías, Caminha and Nascimento [4].

Lemma 4 *Let M^n be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a vector field on M^n. Assume that there exists a nonnegative, non-identically vanishing function $f \in C^\infty(M)$ which converges to zero at infinity and such that $\langle \nabla f, X\rangle \geq 0$. If $\operatorname{div} X \geq 0$ on M^n, then $\langle \nabla f, X\rangle \equiv 0$ on M^n.*

So, our purpose is to apply Lemma 4 jointly with Proposition 1 in order to obtain our next three characterization results of totally umbilical spacelike submanifolds of $\mathbb{S}_q^{n+p}$.

Theorem 4 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \dots, p\}$, such that the mean curvature H satisfies $H < 1$. If $|\Phi|$ converges to zero at infinity and $\sup_M |\Phi| \leq \alpha^*$, where α^* is the positive root of the polynomial function (4.2), then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Proof Following the same steps of the proof of Theorem 1, we deduce inequality (4.13). Let us suppose by the contradiction that M^n is not totally umbilical or, equivalently, that $f = |\Phi|^2$ is a non-identically vanishing smooth function on M^n. Thus, considering on M^n the tangent vector field $X = \nabla|\Phi|^2$, we have that

$$\langle \nabla f, X\rangle = |\nabla|\Phi|^2|^2 \geq 0.$$

Moreover, since $\sup_M |\Phi| \leq \alpha^*$, from (4.13) we obtain

$$\operatorname{div} X = \Delta|\Phi|^2 \geq 0.$$

Hence, since we are assuming that $|\Phi|$ converges to zero at infinity, we can apply Lemma 4 to conclude that $|\nabla|\Phi|^2|^2 \equiv 0$, that is, $|\Phi|$ is constant on M^n. But, taking into account once more that $|\Phi|$ converges to zero at infinity, we have that $|\Phi|$ must be identically zero on M^n and we arrived at a contradiction. □

Theorem 5 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \dots, p\}$, such that the infimum K of the sectional curvatures of M^n satisfies (4.14). If $|\Phi|$ converges to zero at infinity and $\sup_M |\Phi| \leq \beta^*$, where β^* is the positive root of the polynomial function (4.15), then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Proof Reasoning as in the proof of Theorem 2, we obtain inequality (4.20). Let us suppose by the contradiction that M^n is not totally umbilical or, equivalently, that $f = |\Phi|^2$ is a non-identically vanishing smooth function on M^n. So, considering on M^n the tangent vector field $X = \nabla|\Phi|^2$, we have that

$$\langle \nabla f, X\rangle = |\nabla|\Phi|^2|^2 \geq 0.$$

Moreover, since $\sup_M |\Phi| \leq \alpha^*$, from (4.20) we obtain

$$\operatorname{div} X = \Delta|\Phi|^2 \geq 0.$$

Hence, since we are assuming that $|\Phi|$ converges to zero at infinity, we can apply Lemma 4 to conclude that $|\nabla|\Phi|^2|^2 \equiv 0$, that is, $|\Phi|$ is constant on M^n. But, using once more that $|\Phi|$ converges to zero at infinity, we have that $|\Phi|$ must be identically zero on M^n and we arrived at a contradiction. □

Theorem 6 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the infimum Q of the Ricci curvature satisfies (4.22). If $|\Phi|$ converges to zero at infinity and $\sup_M |\Phi| \leq \gamma^*$, where γ^* is the positive root of the polynomial function (4.23), then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Proof Proceeding as in the proof of Theorem 3, we get inequality (4.29). Let us suppose by the contradiction that M^n is not totally umbilical or, equivalently, that $f = |\Phi|^2$ is a non-identically vanishing smooth function on M^n. Considering on M^n the tangent vector field $X = \nabla|\Phi|^2$, we have that

$$\langle \nabla f, X\rangle = |\nabla|\Phi|^2|^2 \geq 0.$$

Moreover, since $\sup_M |\Phi| \leq \gamma^*$, from (4.29) we obtain

$$\mathrm{div} X = \Delta|\Phi|^2 \geq 0.$$

Hence, since we are assuming that $|\Phi|$ converges to zero at infinity, we can apply Lemma 4 to conclude that $|\nabla|\Phi|^2|^2 \equiv 0$, that is, $|\Phi|$ is constant on M^n. But, since $|\Phi|$ converges to zero at infinity, we have that $|\Phi|$ must be identically zero on M^n and we arrived at a contradiction. □

5.2 Umbilicity via polynomial volume growth

We start quoting the maximum principle that will be used to prove our results in this last subsection. For this, let M^n be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p, t)$ the geodesic ball centered at p and with radius t. Given a polynomial function $\sigma : (0, +\infty) \longrightarrow (0, +\infty)$, we say that M^n has polynomial volume growth like $\sigma(t)$ if there exists $p \in M^n$ such that

$$\mathrm{vol}(B(p, t)) = \mathcal{O}(\sigma(t)),$$

as $t \longrightarrow +\infty$, where vol denotes the Riemannian volume.

If $p, q \in M^n$ are at distance d from each other, it is straightforward to check that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

Hence, the choice of p in the notion of volume growth is immaterial, so that, henceforth, we will simply say that M^n has polynomial volume growth.

In this context, we have the following maximum principle, which is derived from Theorem 2.1 of Alías, Caminha and Nascimento [5].

Lemma 5 *Let M^n be a connected, oriented, complete noncompact Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$ be a nonnegative smooth function such that $\Delta f \geq af$ on M^n, for some positive constant $a \in \mathbb{R}$. If M^n has polynomial volume growth and $|\nabla f|$ is bounded on M^n, then f vanishes identically on M^n.*

We will also need of Lemma 1 of Barros, Brasil and Sousa [8], which is stated below in our context.

Lemma 6 *Let M^n be a spacelike submanifold immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$. Then, the traceless second fundamental form Φ of M^n defined in (3.1) verifies the following inequality*

$$|\nabla|\Phi|^2|^2 \leq \frac{4n}{n+2}|\Phi|^2|\nabla\Phi|^2.$$

In this setting, we obtain our the following result.

Theorem 7 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the mean curvature H satisfies $H < 1$ and the traceless second fundamental form Φ verifies $|\nabla\Phi| < +\infty$ and $\sup_M |\Phi| \leq \alpha^*$, where α^* is the first positive root of the polynomial function (4.2). If M^n has polynomial volume growth, then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Proof From the proof of Theorem 1, we get inequality (4.13). Thus, since $\sup_M|\Phi| \leq \alpha^*$, from the behavior of $P_H(x)$ we obtain

$$\Delta|\Phi|^2 \geq a|\Phi|^2,$$

where $a = P_H(\alpha) > 0$. Moreover, taking into account that $\sup_M|\Phi| \leq \alpha < \alpha^*$ and $|\nabla\Phi|$ is bounded, Lemma 6 guarantees that $|\nabla|\Phi|^2|$ is also bounded. Therefore, we can apply Lemma 5 to conclude that $\sup_M|\Phi| = 0$ and, hence, M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$. □

Using inequalities (4.20) and (4.29), respectively, we can reason as in the proof of Theorem 7 to establish our last two characterization results of totally umbilical spacelike submanifolds of $\mathbb{S}_q^{n+p}$.

Theorem 8 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the infimum K of the sectional curvature of M^n satisfies (4.14) and the traceless second fundamental form Φ verifies $|\nabla\Phi| < +\infty$ and $\sup_M |\Phi| \leq \beta^*$, where β^* is the positive root of the polynomial function (4.15). If M^n has polynomial volume growth, then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Theorem 9 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with parallel mean curvature vector immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $q \in \{1, \ldots, p\}$, such that the infimum Q of the Ricci curvatures of M^n satisfies (4.22) and the traceless second fundamental form Φ of M^n verifies $|\nabla\Phi| < +\infty$ and $\sup_M |\Phi| \leq \gamma^*$, where γ^* is the positive root of the polynomial function (4.23). If M^n has polynomial volume growth, then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

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Declarations

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