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Cofiniteness of modules and local cohomology

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Abstract
Let A be a commutative noetherian ring, let $\mathfrak{a}$ be an ideal of A and let n be a non-negative integer. In this paper, we study $\mathcal{S}_n(\mathfrak{a})$, a certain class of A-modules and we find some sufficient conditions so that a module belongs to $\mathcal{S}_n(\mathfrak{a})$. Moreover, we study the cofiniteness of local cohomology modules when $\dim A/\mathfrak{a} \geq 3$.

Keywords Koszul cohomology · Cofinite module · Local cohomology

Mathematics Subject Classification 13D02 · 13D45 · 13E05

Contents

1 Introduction

Throughout this paper, assume that A is a commutative noetherian ring, $\mathfrak{a}$ is an ideal of A, M is an A-module and n is a non-negative integer. Following [5], the A-module M is said to be $\mathfrak{a}$-*cofinite* if $\text{Supp}_A(M) \subseteq V(\mathfrak{a})$ and $\text{Ext}^i_A(A/\mathfrak{a}, M)$ are finitely generated for all integers $i \geq 0$. The authors [7] studied a criterion for cofinitenss of modules. We denote by $\mathcal{S}_n(\mathfrak{a})$ the class of all A-modules M satisfying the following implication:

If $\text{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n$ and $\text{Supp}(M) \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-cofinite.

The abelianess of the category of $\mathfrak{a}$-cofinite modules is of interest to many mathematicians working in commutative algebra. This subject has been studied for small dimensions by

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various authors [1, 6, 8, 9]. As the Koszul complexes are effective tools for going down the dimensions, Sazeedeh [11] studied the cofiniteness of Koszul cohomology modules.

In Sect. 2, by means of the Koszul cohomologies of a module, we want to find out when this module belongs to $\mathcal{S}_n(\mathfrak{a})$. Let $\mathbf{x} = x_1, \ldots, x_t$ be a sequence of elements of $\mathfrak{a}$. We show that if $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n+1$ and $H^i(\mathbf{x}, M) \in \mathcal{S}_1(\mathfrak{a})$ for all $i \leq n$, then $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \leq n$ (see Proposition 2.2). Moreover, we prove the following theorem.

Theorem 1.1 *If* $H^i(\mathbf{x}, M) \in \mathcal{S}_1(\mathfrak{a})$ *for all* $i \geq 0$, *then* $M \in \mathcal{S}_{t+1}(\mathfrak{a})$.

As an application of this theorem, if $\dim H^i(\mathbf{x}, M) \leq 1$ for all $i \geq 0$, then $M \in \mathcal{S}_{t+1}(\mathfrak{a})$ (see Corollary 2.4). Let $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ be finitely generated for all $i \leq n+2$ and let $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a})$ for all $i \leq n$. Then we show that $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \leq n$ if and only if $\operatorname{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ is finitely generated for all $i \leq n+1$ (see Proposition 2.5). Moreover, we have the following theorem.

Theorem 1.2 *Let* $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a})$ *such that* $\operatorname{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *is finitely generated for all* $i \geq 0$. *Then* $M \in \mathcal{S}_{t+2}(\mathfrak{a})$.

Let $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ be finitely generated for all $i \leq n+3$ and let $H^i(\mathbf{x}, M) \in \mathcal{S}_3(\mathfrak{a})$ for all $i \leq n$. Then we prove that if $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \leq n$, then $\operatorname{Ext}^j_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ is finitely generated for all $i \leq n+1$; $j = 0, 1$; furthermore if $\operatorname{Hom}_A(A/\mathfrak{a}, H^{n+2}(\mathbf{x}, M))$ is finitely generated, then the converse also holds (see Proposition 2.8). Moreover, we have the following theorem.

Theorem 1.3 *Let* $H^i(\mathbf{x}, M) \in \mathcal{S}_3(\mathfrak{a})$ *such that* $\operatorname{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *and* $\operatorname{Ext}^1_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *are finitely generated for all* $i \geq 0$. *Then* $M \in \mathcal{S}_{t+3}(\mathfrak{a})$.

As an application of this theorem, let A be a local ring such that $\dim A/((\mathbf{x}) + xA) \leq 3$ for some $x \in \mathfrak{a}$. If $\operatorname{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M)/xH^i(\mathbf{x}, M))$ and $\operatorname{Ext}^j_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ for all $i \geq 0$ and $j = 0, 1$ are finitely generated, then $M \in \mathcal{S}_{t+3}(\mathfrak{a})$ (see Corollary 2.10).

In Sect. 3, we study the cofiniteness of local cohomology modules. Let $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ be finitely generated for all $i \geq 0$ and let $t < s$ be non-negative integers such that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \neq t, s$. Then we show that $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+s-t+1}(\mathfrak{a})$ if and only if $H^s_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$ (see Theorem 3.1). Moreover, we have the following theorem.

Theorem 1.4 *Let* $\dim A/\mathfrak{a} = d \geq 3$ *and let* $\operatorname{depth}(\operatorname{Ann}(M), A/\mathfrak{a}) \geq d-2$. *If* $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq n+1$, *then* $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ *is finitely generated for all* $i \leq n$ *if and only if* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i < n$.

In the end of this paper we get a similar result for those rings A which $\dim A \geq 4$.

2 A criterion for cofiniteness of modules

Throughout this section, M is an A-module with $\operatorname{Supp}_A M \subseteq V(\mathfrak{a})$ and n is a non-negative integer and $\mathbf{x} = x_1, \ldots, x_t$ is a sequence of elements of $\mathfrak{a}$.

An A-module M is said to be $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_A(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ are finitely generated for all integers $i \geq 0$.

For a non-negative integer n, we denote by $\mathcal{S}_n(\mathfrak{a})$, the class of all A-modules M satisfying the following implication:

If $\mathrm{Ext}_A^i(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n$ and $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-cofinite.

Examples 2.1 (i) Assume that $\mathfrak{a}$ is an arbitrary ideal of A and M is an A-module of dimension d where $\dim M$ means the dimension of $\mathrm{Supp}_A M$; which is the length of the longest chain of prime ideals in $\mathrm{Supp}_A M$. Then $H_{\mathfrak{a}}^d(M)$ is in $\mathcal{S}_0(\mathfrak{a})$. To be more precise, if $\mathrm{Hom}_A(A/\mathfrak{a}, H_{\mathfrak{a}}^d(M))$ is a finitely generated A-module, then it follows from [10, Theorem 3.11] that $H_{\mathfrak{a}}^d(M)$ is artinian and so, since $\mathrm{Hom}_A(A/\mathfrak{a}, H_{\mathfrak{a}}^d(M))$ has finite length, according to [9, Proposition 4.1], the module $H_{\mathfrak{a}}^d(M)$ is $\mathfrak{a}$-cofinite.

(ii) Given an arbitrary ideal $\mathfrak{a}$ of A, by virtue of [1, Proposition 2.6], $M \in \mathcal{S}_1(\mathfrak{a})$ for all modules M with $\dim M \leq 1$. Especially, if $\dim A/\mathfrak{a} = 1$, then it follows from [9, Theorem 2.3] that $\mathcal{S}_1(\mathfrak{a}) = \text{Mod-}A$, the category of all A-modules. Furthermore, if $\dim A = 2$, then it follows from [10, Corollary 2.4] that $\mathcal{S}_1(\mathfrak{a}) = \text{Mod-}A$ for any ideal $\mathfrak{a}$ of A.

(iii) Let $\mathfrak{a}$ be an ideal of a local ring A with $\dim A/\mathfrak{a} = 2$. It follows from [2, Theorem 3.5] that $\mathcal{S}_2(\mathfrak{a}) = \text{Mod-}A$. Furthermore, if A is a local ring with $\dim A = 3$, then it follows from [10, Corollary 2.5] that $\mathcal{S}_2(\mathfrak{a}) = \text{Mod-}A$ for any ideal $\mathfrak{a}$ of A.

The following result shows that the Koszul cohomology modules of M are $\mathfrak{a}$-cofinite when they belong to $\mathcal{S}_1(\mathfrak{a})$.

Proposition 2.2 *Let* $\mathrm{Ext}_A^i(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \leq n+1$. *If* $H^i(\mathbf{x}, M) \in \mathcal{S}_1(\mathfrak{a})$ *for all* $i \leq n$, *then* $H^i(\mathbf{x}, M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq n$.

Proof Consider the Koszul complex

$$K^*(\mathbf{x}, M) : 0 \longrightarrow K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{t-1}} K^t \longrightarrow 0$$

and assume that $Z^i = \mathrm{Ker}\, d^i$, $B^i = \mathrm{Im}\, d^{i-1}$, $C^i = \mathrm{Coker}\, d^i$ and $H^i = H^i(\mathbf{x}, M)$ for each i. Then for each $j \geq 0$, we have an exact sequence of modules

$$0 \longrightarrow H^j \longrightarrow C^j \longrightarrow K^{j+1} \longrightarrow C^{j+1} \longrightarrow o \qquad (\dagger_j).$$

We prove by induction on i that H^i is $\mathfrak{a}$-cofinite and $\mathrm{Ext}_A^j(A/\mathfrak{a}, C^{i+1})$ is finitely generated for all $i \leq n$ and all $j \leq n - i$. Assume that $i < n$ and so the induction hypothesis implies that $\mathrm{Ext}_A^j(A/\mathfrak{a}, C^i)$ is finitely generated for all $j \leq n + 1 - i$. In view of $(\dagger_i)$, it is clear that $\mathrm{Ext}_A^j(A/\mathfrak{a}, H^i)$ is finitely generated for $i = 0, 1$ and so the fact that $H^i \in \mathcal{S}_1(\mathfrak{a})$ forces H^i is $\mathfrak{a}$-cofinite and $\mathrm{Ext}_A^j(A/\mathfrak{a}, C^{i+1})$ is finitely generated for all $j \leq n - i$. □

The following theorem provides a sufficient condition so that a module belongs to $\mathcal{S}_{t+1}(\mathfrak{a})$.

Theorem 2.3 *If* $H^i(\mathbf{x}, M) \in \mathcal{S}_1(\mathfrak{a})$ *for all* $i \geq 0$, *then* $M \in \mathcal{S}_{t+1}(\mathfrak{a})$.

Proof Assume that $\mathrm{Ext}_A^i(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq t + 1$. Then, it follows from Proposition 2.2 that $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$. Consequently, [11, Theorem 2.4] implies that M is $\mathfrak{a}$-cofinite. □

Corollary 2.4 *If* $\dim H^i(\mathbf{x}, M) \leq 1$ *for all* $i \geq 0$, *then* $M \in \mathcal{S}_{t+1}(\mathfrak{a})$.

Proof By virtue of Examples 2.1, $H^i(\mathbf{x}, M) \in \mathcal{S}_1(\mathfrak{a})$ for all $i \geq 0$ and so the result follows from Theorem 2.3. □

When the Koszul cohomology modules of an A-module belong to $\mathcal{S}_2(\mathfrak{a})$, we have the following result about the $\mathfrak{a}$-cofiniteness of these modules.

Proposition 2.5 *Let* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \leq n+2$ *and let* $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a})$ *for all* $i \leq n$. *Then the following conditions hold.*

(i) $H^i(\mathbf{x}, M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq n$.
(ii) $\mathrm{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *is finitely generated for all* $i \leq n+1$.

Proof Consider the same notation as in the proof of Proposition 2.2. (i)⇒(ii). By the assumption $\mathrm{Hom}_A(A/\mathfrak{a}, H^i)$ is finitely generated for all $i \leq n$. Thus, it is straightforward to see that $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^i)$ is finitely generated for all $i \leq n+1$ and all $j \leq n+2-i$. Since $\mathrm{Hom}_A(A/\mathfrak{a}, C^{n+1})$ is finitely generated, $(\dagger_{n+1})$ implies that $\mathrm{Hom}_A(A/\mathfrak{a}, H^{n+1})$ is finitely generated. (ii)⇒(i). We prove by induction on i that H^i is $\mathfrak{a}$-cofinite and $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^{i+1})$ is finitely generated for all $i \leq n$ and all $j \leq n+1-i$. The induction hypothesis implies that $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^i)$ is finitely generated for all $j \leq n+2-i$. Since by the assumption $\mathrm{Hom}_A(A/\mathfrak{a}, H^{i+1})$ is finitely generated, $(\dagger_{i+1})$ implies that $\mathrm{Hom}_A(A/\mathfrak{a}, C^{i+1})$ is finitely generated; and hence $(\dagger_i)$ implies that $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^i)$ is finitely generated for all $j \leq 2$. Now, since $H^i \in \mathcal{S}_2(\mathfrak{a})$, it is $\mathfrak{a}$-cofinite; furthermore $(\dagger_i)$ implies that $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^{i+1})$ is finitely generated for all $i \leq n$ and all $j \leq n+1-i$. □

The following theorem provides a sufficient condition so that a module belongs to $\mathcal{S}_{t+2}(\mathfrak{a})$.

Theorem 2.6 *Let* $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a})$ *such that* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *is finitely generated for all* $i \geq 0$. *Then* $M \in \mathcal{S}_{t+2}(\mathfrak{a})$.

Proof Assume that $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq t+2$. Then it follows from Proposition 2.5 that $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$. Consequently, [11, Theorem 2.4] implies that M is $\mathfrak{a}$-cofinite. □

Corollary 2.7 *Let* A *be a local ring such that* $\dim A/(\mathbf{x}) \leq 3$ *and let* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *be finitely generated for all* $i \geq 0$. *Then* $M \in \mathcal{S}_{t+2}(\mathfrak{a})$.

Proof Put $B = A/(\mathbf{x})$ and $\mathfrak{b} = \mathfrak{a}B$. We observe that $H^i(\mathbf{x}, M)$ is a B-module for each $i \geq 0$ and since $\dim B \leq 3$, it follows from Examples 2.1 (iii) that $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a}B)$ for each $i \geq 0$. Now [7, Proposition 2.15] implies that $H^i(\mathbf{x}, M) \in \mathcal{S}_2(\mathfrak{a})$; and consequently, it follows from Theorem 2.6 that $M \in \mathcal{S}_{t+2}(\mathfrak{a})$. □

When the Koszul cohomology modules of an A-module belong to $\mathcal{S}_3(\mathfrak{a})$, we have the following result about their $\mathfrak{a}$-cofiniteness

Proposition 2.8 *Let* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \leq n+3$ *and let* $H^i(\mathbf{x}, M) \in \mathcal{S}_3(\mathfrak{a})$ *for all* $i \leq n$. *Consider the following statements.*

(i) $H^i(\mathbf{x}, M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq n$.
(ii) $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *is finitely generated for* $j = 0, 1$ *and all* $i \leq n+1$.

Then (i) ⇒ (ii) *holds. Moreover, if* $\mathrm{Hom}_A(A/\mathfrak{a}, H^{n+2}(\mathbf{x}, M))$ *is finitely generated, then* (ii) ⇒ (i) *holds.*

Proof Consider the same notation as in the proof of Proposition 2.2. (i)⇒(ii). Clearly $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^i)$ is finitely generated for all $i \le n$ and $j = 0, 1$; furthermore $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^i)$ is finitely generated for all $i \le n+1$ and all $j \le n+3-i$. Since $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^{n+1})$ is finitely generated for $j = 0, 1$, the exact sequence $(\dagger_{n+1})$ implies that $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^{n+1})$ is finitely generated for $j = 0, 1$. (ii)⇒(i). We prove by induction on i that H^i is $\mathfrak{a}$-cofinite and $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^{i+1})$ is finitely generated for all $i \le n$ and all $j \le n+2-i$. The exact sequence $(\dagger_{i+2})$ implies that $\mathrm{Hom}_A(A/\mathfrak{a}, C^{i+2})$ is finitely generated and so it follows from $(\dagger_{i+1})$ that $\mathrm{Ext}^j(A/\mathfrak{a}, C^{i+1})$ is finitely generated for $j = 0, 1$. Since by the induction hypothesis, $\mathrm{Ext}^j_A(A/\mathfrak{a}, C^i)$ is finitely generated for all $j \le n+3-i$, the exact sequence $(\dagger_i)$ implies that $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^i)$ is finitely generated for all $j \le n+3-i$; and hence since $H^i \in \mathcal{S}_3(\mathfrak{a})$, we deduce that H^i is $\mathfrak{a}$-cofinite. □

The following theorem provides a sufficient condition so that a module belongs to $\mathcal{S}_{t+3}(\mathfrak{a})$.

Theorem 2.9 *Let* $H^i(\mathbf{x}, M) \in \mathcal{S}_3(\mathfrak{a})$ *such that* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *and* $\mathrm{Ext}^1_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *are finitely generated for all* $i \ge 0$. *Then* $M \in \mathcal{S}_{t+3}(\mathfrak{a})$.

Proof Assume that $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \le t+3$. Then it follows from Proposition 2.8 that $H^i(\mathbf{x}, M)$ is $\mathfrak{a}$-cofinite for all $i \ge 0$. Consequently, [11, Theorem 2.4] implies that M is $\mathfrak{a}$-cofinite. □

Corollary 2.10 *Let A be a local ring such that* $\dim A/((\mathbf{x}) + xA) \le 3$ *for some* $x \in \mathfrak{a}$. *If* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i(\mathbf{x}, M)/xH^i(\mathbf{x}, M))$ *and* $\mathrm{Ext}^j_A(A/\mathfrak{a}, H^i(\mathbf{x}, M))$ *for all* $i \ge 0$ *and* $j = 0, 1$ *are finitely generated, then* $M \in \mathcal{S}_{t+3}(\mathfrak{a})$.

Proof Set $B = A/xA$, $\mathfrak{b} = \mathfrak{a}/xA$. In view of the exact sequence

$$H^i(\mathbf{x}, x, M) \longrightarrow H^i(\mathbf{x}, M) \xrightarrow{x.} H^i(\mathbf{x}, M) \longrightarrow H^{i+1}(\mathbf{x}, x, M)$$

for $i \ge 0$, there is an exact sequence

$$0 \longrightarrow (0 :_{H^i(\mathbf{x},M)} x) \longrightarrow H^i(\mathbf{x}, M) \xrightarrow{x.} H^i(\mathbf{x}, M) \longrightarrow H^i(\mathbf{x}, M)/xH^i(\mathbf{x}, M) \longrightarrow 0.$$

As $(0 :_{H^i(\mathbf{x},M)} x)$ and $H^i(\mathbf{x}, M)/xH^i(\mathbf{x}, M)$ are B-modules, they belong to $\mathcal{S}_2(\mathfrak{b})$ by Examples 2.1 (iii) and so by virtue of [7, Proposition 2.15], they belong to $\mathcal{S}_2(\mathfrak{a})$. Thus, Theorem 2.6 implies that $H^i(\mathbf{x}, M) \in \mathcal{S}_3(\mathfrak{a})$ for each $i \ge 0$ and consequently, $M \in \mathcal{S}_{t+3}(\mathfrak{a})$ by using Theorem 2.9. □

3 Cofiniteness of local cohomology modules

Throughout this section, M is an A-module, $\mathfrak{a}$ is an ideal of A and n is a positive integer. We study the cofiniteness of local cohomology.For more details about local cohomology, we refer the reader to the textbook of Brodmann and Sharp [3]. Throughout this section n is a non-negative integer.

Theorem 3.1 *Let* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \ge 0$ *and let* $t < s$ *be non-negative integers such that* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \ne t, s$. *Then* $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+s-t+1}(\mathfrak{a})$ *if and only if* $H^s_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$.

Proof We first assume $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+s-t+1}(\mathfrak{a})$ and that $\operatorname{Ext}^p_A(A/\mathfrak{a}, H^s_{\mathfrak{a}}(M)) \in \mathcal{S}_n(\mathfrak{a})$ is finitely generated for all $p \leq n$. There is the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_A(A/\mathfrak{a}, H^q_{\mathfrak{a}}(M)) \Rightarrow \operatorname{Ext}^{p+q}_A(A/\mathfrak{a}, M).$$

For each $r \geq 3$, consider the sequence $E_{r-1}^{p-r+1,t+r-2} \overset{d_{r-1}^{p-r+1,t+r+2}}{\longrightarrow} E_{r-1}^{p,t} \overset{d_{r-1}^{p,t}}{\longrightarrow} E_{r-1}^{p+r-1,t-r+2}$ and so $E_r^{p,t} = \operatorname{Ker} d_{r-1}^{p,t} / \operatorname{Im} d_{r-1}^{p-r+1,t+r-2}$. Considering $p \leq n+s-t+1$, we show that $E_2^{p,t}$ is finitely generated. If $r = s-t+2$, then $p-r+1 \leq n$ and so $E_{r-1}^{p-r+1,t+r-2}$ is finitely generated by the argument in the beginning of proof. If $r \neq s-t+2$, then the assumption implies that $E_{r-1}^{p-r+1,t+r-2}$ is finitely generated for all $p \geq 0$ (we observe that $t+r-2 \neq t$). Consequently, $\operatorname{Im} d_{r-1}^{p-r+1,t+r-2}$ is finitely generated for all $r \geq 3$ and $p \leq n+s-t+1$. But there is a finite filtration

$$0 = \Phi^{p+t+1}H^{p+t} \subset \Phi^{p+t}H^{p+t} \subset \cdots \subset \Phi^1 H^{p+t} \subset \Phi^0 H^{p+t} \subset H^{p+t}$$

such that $\Phi^p H^{p+t}/\Phi^{p+1}H^{p+t} \cong E_\infty^{p,t}$ for all $p \geq 0$. In view of the assumption, $H^{p+t} = \operatorname{Ext}^{p+t}_A(A/\mathfrak{a}, M)$ is finitely generated and so $E_\infty^{p,t}$ is finitely generated for all $p \geq 0$. On the other hand, $E_r^{p,t} = E_\infty^{p,t}$ for sufficiently large r and so $E_r^{p,t}$ is finitely generated for all $p \geq 0$. The previous argument implies that $\operatorname{Ker} d_{r-1}^{p,t}$ is finitely generated for all $p \leq n+s-t+1$; moreover since $r \geq 3$, we have $t-r+2 \leq t-1$ and so the assumption implies that $E_{r-1}^{p+r-1,t-r+2}$ is finitely generated, and hence $E_{r-1}^{p,t}$ is finitely generated for all $p \leq n+s-t+1$. Continuing this way, we deduce that $E_2^{p,t}$ is finitely generated for all $p \leq n+s-t+1$, and since $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+s-t+1}(\mathfrak{a})$, we deduce that $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. Therefore, $E_2^{p,q}$ is finitely generated for all $q \neq s$ and all $p \geq 0$. By a similar argument, we have $E_\infty^{p,s} = E_r^{p,s}$ for sufficiently large r and all $p \geq 0$ and since $E_\infty^{p,s}$ is a subquotient of $\operatorname{Ext}^{p+s}_A(A/\mathfrak{a}, M)$, it is finitely generated so that $E_r^{p,s}$ is finitely generated. In view of the sequence $E_{r-1}^{p-r+1,s+r-2} \overset{d_{r-1}^{p-r+1,s+r+2}}{\longrightarrow} E_{r-1}^{p,s} \overset{d_{r-1}^{p,s}}{\longrightarrow} E_{r-1}^{p+r-1,s-r+2}$, since $E_{r-1}^{p-r+1,s+r-2}$ is finitely generated and $E_r^{p,s} = \operatorname{Ker} d_{r-1}^{p,s}/\operatorname{Im} d_{r-1}^{p-r+1,s+r-2}$, we conclude that $\operatorname{Ker} d_{r-1}^{p,s}$ is finitely generated; and hence $E_{r-1}^{p,s}$ is finitely generated. Continuing this way, we deduce that $E_3^{p,s}$ is finitely generated and so is $\operatorname{Ker} d_2^{p,s}$ for all $p \geq 0$. Now the exact sequence $0 \longrightarrow \operatorname{Ker} d_2^{p,s} \longrightarrow E_2^{p,s} \longrightarrow E_2^{p+2,s+1}$ implies that $E_2^{p,s}$ is finitely generated for all $p \geq 0$; and consequently $H^s_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. A similar proof gets the converse. □

The following lemma extends [7, Proposition 2.6].

Lemma 3.2 *Let* $\dim A/\mathfrak{a} = 3$ *and* $\operatorname{depth}(\operatorname{Ann}(M), A/\mathfrak{a}) > 0$. *If* $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq n+1$, *then the following conditions are equivalent.*

(i) $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ *is finitely generated for all* $i \leq n$.
(ii) $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i < n$.

Proof Since $\operatorname{depth}(\operatorname{Ann}(M), A/\mathfrak{a}) > 0$, there exists an element $x \in \operatorname{Ann}(M)$ such that x is a non-zerodivisor on $A/\mathfrak{a}$. Taking $\mathfrak{b} = \mathfrak{a} + xA$, we have $\dim A/b = 2$ and it follows from [4, Proposition 1] that $\operatorname{Ext}^i_A(A/\mathfrak{b}, M)$ is finitely generated for all $i \leq n+1$. Moreover, we have $\Gamma_{xA}(M) = \Gamma_{x\mathfrak{a}}(M) = M$ and $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M)$. Thus the ideals $\mathfrak{a}$ and xA of A provides the following Mayer-Vietoris exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(M) \oplus M \longrightarrow M \longrightarrow H^1_{\mathfrak{b}}(M) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and the isomorphism $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{b}}(M)$ for each $i \geq 2$. (i)⇒(ii). The case $n = 1$ follows from [7, Proposition 2.6]. For $n \geq 2$, $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. Then $\Gamma_{\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{b}$-cofinite too. Moreover, it is clear that $\mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n$. Applying the functor $\mathrm{Hom}_A(A/\mathfrak{b}, -)$ to the above exact sequence, we deduce that $\mathrm{Hom}_A(A/\mathfrak{b}, \Gamma_{\mathfrak{b}}(M))$ and $\mathrm{Hom}_A(A/\mathfrak{b}, H^1_{\mathfrak{b}}(M))$ are finitely generated. Furthermore $\mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{b}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is finitely generated for each $2 \leq i \leq n$. Now, [10, Theorem 3.7] implies that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite and consequently $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < n$ using [4, Proposition 2]. (ii)⇒(i) Since $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < n$, by the previous argument, $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite for all $i < n$; and hence it follows from [10, Theorem 3.7] that $\mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{b}}(M))$ is finitely generated for all $i \leq n$. We observe that $\mathrm{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{b}}(M)) = \mathrm{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) = \mathrm{Hom}_A(A/\mathfrak{a}, M)$ is finitely generated. Furthermore, $\mathrm{Hom}_A(A/\mathfrak{a}, H^1_{\mathfrak{b}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{a}, \mathrm{Hom}_A(A/xA, H^1_{\mathfrak{b}}(M))) \cong \mathrm{Hom}_A(A/\mathfrak{b}, H^1_{\mathfrak{b}}(M))$ is finitely generated. Now, since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, applying the functor $\mathrm{Hom}_A(A/\mathfrak{a}, -)$ to the above exact sequence, we conclude that $\mathrm{Hom}_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated. By the argument mentioned in the beginning of the proof $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{b}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{b}}(M))$ is finitely generated for all $2 \leq i \geq n$. □

It was proved in [10, Theorems 3.3, 3.7] that if $\dim A/\mathfrak{a} \leq 2$, then the conditions in Lemma 3.2 are equivalent. In the following theorem, we extend this result for $\dim A/\mathfrak{a} \geq 3$, but by an additional assumption.

Theorem 3.3 *Let* $\dim A/\mathfrak{a} = d \geq 3$ *and let* $\mathrm{depth}(\mathrm{Ann}(M), A/\mathfrak{a}) \geq d-2$. *If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq n+1$, *then the conditions in Lemma 3.2 are equivalent.*

Proof We proceed by induction on d. The case $d = 3$ follows from Lemma 3.2 and so we may assume that $d \geq 4$. Since $\mathrm{depth}(\mathrm{Ann}(M), A/\mathfrak{a}) > 0$, there exists an element $x \in \mathrm{Ann}_R M$ which is a non-zerodivisor on $A/\mathfrak{a}$. Taking $\mathfrak{b} = \mathfrak{a} + xA$, we have $\dim A/\mathfrak{b} = d - 1$. Since $\mathrm{Supp}_A A/\mathfrak{b} \subseteq \mathrm{Supp}_A A/\mathfrak{a}$, it follows $\mathrm{Ext}^i_A(A/\mathfrak{b}, M)$ is finitely generated for all $i \leq n+1$. If $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for each $i < n$, then $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite for each $i < n$. The induction hypothesis implies that $\mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{b}}(M))$ is finitely generated for all $i \leq n$. By the same reasoning in the proof of Lemma 3.2, we have $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n$. Conversely, assume that $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n$ and so $\mathrm{Hom}_A(A/\mathfrak{b}, H^i_{\mathfrak{b}}(M))$ is finitely generated for all $i \leq n$. The induction hypothesis implies that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite for each $i < n$ and so $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for each $i < n$. □

Corollary 3.4 *Let* $\mathfrak{b}$ *an ideal of* A *and* $\mathfrak{a} = \Gamma_{\mathfrak{b}}(A)$ *such that* $\dim A/\mathfrak{a} = 3$. *Then* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A/\mathfrak{b}))$ *is finitely generated for all* $i \leq n$ *if and only if* $H^i_{\mathfrak{a}}(A/\mathfrak{b})$ *is* $\mathfrak{a}$*-cofinite for all* $i < n$.

Proof As $\Gamma_{\mathfrak{b}}(A/\mathfrak{a}) = 0$, we have $\mathrm{depth}(\mathfrak{b}, A/\mathfrak{a}) > 0$. Now the assertion is obtained by using Lemma 3.2. □

Corollary 3.5 *Let* $\mathfrak{b}$ *an ideal of* A *and* $\mathfrak{a} = \Gamma_{\mathfrak{b}}(A)$ *such that* $\dim A/\mathfrak{a} = 3$. *Then* $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A))$ *is finitely generated for all* $i \leq n$ *if and only if* $H^i_{\mathfrak{a}}(A)$ *is* $\mathfrak{a}$*-cofinite for all* $i < n$. *In particular,* $\mathrm{Hom}_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(A))$ *is finitely generated.*

Proof There exists a positive integer t such that $\mathfrak{b}^t\mathfrak{a} = 0$ and so $\Gamma_{\mathfrak{a}}(\mathfrak{b}^t) = \mathfrak{b}^t$. Thus applying the functor $\Gamma_{\mathfrak{a}}(-)$ to the exact sequence $0 \longrightarrow \mathfrak{b}^t \longrightarrow A \longrightarrow A/\mathfrak{b}^t \longrightarrow 0$, we deduce that $H^i_{\mathfrak{a}}(A) \cong H^i_{\mathfrak{a}}(A/\mathfrak{b}^t)$ for each $i > 0$. Since $\Gamma_{\mathfrak{b}}(A/\mathfrak{a}) = 0$, we have $\operatorname{depth}(\mathfrak{b}^t, A/\mathfrak{a}) = \operatorname{depth}(\mathfrak{b}, A/\mathfrak{a}) > 0$. If $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A))$ is finitely generated for all $i \leq n$, then $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A/\mathfrak{b}^t))$ is finitely generated for all $i \leq n$. Now Lemma 3.2 implies that $H^i_{\mathfrak{a}}(A/\mathfrak{b}^t)$ is $\mathfrak{a}$-cofinite for all $i < n$; and hence $H^i_{\mathfrak{a}}(A)$ is $\mathfrak{a}$-cofinite for all $i < n$. Conversely, if $H^i_{\mathfrak{a}}(A)$ is $\mathfrak{a}$-cofinite for all $i < n$, then $H^i_{\mathfrak{a}}(A/\mathfrak{b}^t)$ is $\mathfrak{a}$-cofinite for all $i < n$ and so using again Lemma 3.2, $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A/\mathfrak{b}^t))$ is finitely generated for all $i \leq n$ so that $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(A))$ is finitely generated for all $i \leq n$ □

Corollary 3.6 *Let $\mathfrak{p}$ be a prime ideal of A with $\dim A/\mathfrak{p} = 3$ and let $\mathfrak{b}$ be an ideal of A such that $\mathfrak{b} \not\subseteq \mathfrak{p}$. Then $\operatorname{Hom}_A(A/\mathfrak{p}, H^i_{\mathfrak{p}}(A/\mathfrak{b}))$ is finitely generated for all $i \leq n$ if and only if $H^i_{\mathfrak{p}}(A/\mathfrak{b})$ is $\mathfrak{p}$-cofinite for all $i < n$.*

Proof Since $\mathfrak{b} \not\subseteq \mathfrak{p}$, we have $\operatorname{depth}(\mathfrak{b}, A/\mathfrak{p}) > 0$ and so the the result follows from Theorem 3.3. □

If $\dim A \geq 4$, then we have the following result.

Proposition 3.7 *Let $\dim A = d \geq 4$ with $\operatorname{depth}(\operatorname{Ann}(M), A/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(A))) \geq d - 3$ and let $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ be finitely generated for all $i \leq n + 1$. Then the conditions in Lemma 3.2 are equivalent.*

Proof We can choose an integer t such that $(0 :_A \mathfrak{a}^t) = \Gamma_{\mathfrak{a}}(A)$. Put $\overline{A} = A/\Gamma_{\mathfrak{a}}(A)$ and $\overline{M} = M/(0 :_M \mathfrak{a}^t)$ which is an $\overline{A}$-module. Taking $\overline{\mathfrak{a}}$ as the image of $\mathfrak{a}$ in $\overline{A}$, we have $\Gamma_{\overline{\mathfrak{a}}}(\overline{A}) = 0$. Thus $\overline{\mathfrak{a}}$ contains an $\overline{A}$-regular element so that $\dim A/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(A)) = \dim \overline{A}/\overline{\mathfrak{a}} \leq d - 1$. The assumption on M together with the fact that $\operatorname{Supp}_A(A/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(A))) \subset \operatorname{Supp}_A(A/\mathfrak{a})$ and [4, Proposition 1] imply that $\operatorname{Ext}^i_A(A/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(A)), M)$ is finitely generated for all $i \leq n$. Since by the assumption $(0 :_M \mathfrak{a})$ is finitely generated, it is clear that $(0 :_M \mathfrak{a}^t)$ is finitely generated and we have an exact sequence

$$0 \longrightarrow (0 :_M \mathfrak{a}^t) \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(\overline{M}) \longrightarrow 0$$

and the isomorphism $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(\overline{M})$ for all $i > 0$. In order to prove (i)⇒(ii), assume that $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n$. Then in view of the previous argument and the independence theorem for local cohomology $\operatorname{Hom}_A(A/(\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)), H^i_{\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)}(\overline{M}))$ is finitely generated for all $i \leq n$. It now follows from Theorem 3.3 that $H_{\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)}(\overline{M})$ is $\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)$-cofinite for all $i < n$; and hence using the change of ring principle [4, Proposition 2], $H^i_{\mathfrak{a}}(\overline{M})$ is $\mathfrak{a}$-cofinite for all $i < n$. Consequently, the previous argument implies that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < n$. (ii)⇒(i). Assume that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < n$. By the same reasoning as mentioned before, we deduce that $H^i_{\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)}(\overline{M})$ is $\mathfrak{a} + \Gamma_{\mathfrak{a}}(A)$-cofinite for all $i < n$. Now, using again Theorem 3.3, we deduce that $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(\overline{M})) \cong \operatorname{Hom}_A(A/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(A)), H^i_{\mathfrak{a}+\Gamma_{\mathfrak{a}}(A)}(\overline{M}))$ is finitely generated for all $i \leq n$ and consequently the previous argument yields that $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n$. □

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Declarations

Conflict of interest The authors have no conflict of interest.

References

1. Bahmanpour, K., Naghipour, R., Sedghi, M.: Cofiniteness with respect to ideals of small dimension. Algebras Represent. Theory **18**(2), 369–379 (2014)
2. Bahmanpour, K., Naghipour, R., Sedghi, M.: On the category of cofinite modules which is abelian. Proc. Am. Math. Soc **142**(4), 1101–1107 (2014)
3. Brodmann, M., Sharp, R.Y.: Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge University Press, Cambridge (1998)
4. Delfino, D., Marley, T.: Cofinite modules of local cohomology. J. Pure. Appl. Algebra **121**(1), 45–52 (1997)
5. Hartshorne, R.: Affine duality and cofiniteness. Invent. Math. **9**, 145–164 (1970)
6. Kawasaki, K.-I.: On a category of cofinite modules which is abelian. Math. Z. **269**(1–2), 587–608 (2011)
7. Khazaei, M., Sazeedeh, R.: A criterion for cofiniteness of modules. Rend. Sem. Mat. Univ. Padova (2023). https://doi.org/10.4171/RSMUP/128
8. Melkersson, L.: Modules cofinite with respect to an ideal. J. Algebra **285**, 649–668 (2005)
9. Melkersson, L.: Cofiniteness with respect to ideals of dimesnion one. J. Algebra **372**, 459–462 (2012)
10. Nazari, M., Sazeedeh, R.: Cofiniteness with respect to two ideals and local cohomology. Algebras Represent. Theory **22**, 375–385 (2019)
11. Sazeedeh, R.: Cofiniteness of Koszul cohomology modules. arXiv:math.AC/4690830 (2023)