Rendiconti del Circolo Matematico di Palermo Series 2 (2024) 73:731–745
https://doi.org/10.1007/s12215-023-00947-8

Existence of solutions for parabolic variational inequalities

Farah Balaadich[1]

Received: 18 January 2023 / Accepted: 7 July 2023 / Published online: 8 September 2023

Abstract
In this paper, we are concerned with the study of parabolic variational inequality. Under appropriate assumptions on the main functions, we obtain the existence of weak solutions after the construction of the penalized Young measure by Galerkin's method and the penalty method. The passage to the limit follows relying on the theory of Young measures.

Keywords Weak solutions · Variational inequalities · Young measures

Mathematics Subject Classification 35D30 · 35K86

1 Introduction

In this paper, we are concerned with the existence of weak solutions for parabolic systems. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open domain, $p \in (2n/(n+2), \infty)$ and $0 < T < \infty$ are given constants and denote $Q = \Omega \times (0, T)$ with its boundary $\partial Q = \partial\Omega \times (0, T)$. We deal with the following variational inequality

$$\int_{Q_s} \frac{\partial u}{\partial t}(v-u)dxdt + \int_{Q_s} a(x,t,Du) : (Dv - Du)dxdt \geq \int_{Q_s} f(v-u)dxdt, \tag{1}$$

for every $v \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m))$ and $Q_s = \Omega \times (0, s)$ for all $s \in [0, T]$. Here $f \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m))$, $p' = p/(p-1)$ and $a : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a function assumed to satisfy some conditions. Here $\mathbb{M}^{m\times n}$ stands for the set of $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i=1}^m \sum_{j=1}^n \xi_{ij}\eta_{ij}$. To deal with (1), we shall find a function $u(x,t) \in K$ satisfying the previous inequality, where

$$K = \Big\{ w \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m) \cap C(0, T; L^2(\Omega; \mathbb{R}^m)), \ \frac{\partial w}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m)) : \\ 0 \leq w(x, 0) = u_0(x) \in L^2(\Omega; \mathbb{R}^m), \ w(x,t) \geq 0 \quad \text{a.e. } (x,t) \in Q \Big\}.$$

✉ Farah Balaadich
balaadich.edp@gmail.com

1 Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco

It should be noted, that the variational inequality (1) come from and is governed by the following quasilinear parabolic system

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x, t, Du) = f \quad \text{in } Q. \tag{2}$$

There is a large number of papers to consider (2). By the theory of Young measures, the author in [17] has proved the existence of weak solutions, under mild monotonicity assumptions on the function a. This theory is used to serve the existence of weak solutions, since that problem can not be treated by the classical monotone operator method developed in [10, 11, 21, 22, 25]. And this is because a does not need to satisfy the strict monotonicity condition of Leray-Lions's type. We refer the reader to [1–7] where the theory of Young measures has been applied for both elliptic and parabolic problems. The elliptic case of (1) was investigated in [8] where the authors have proved the existence of weak solutions employing the theory of Young measures and a theorem of Kinderlehrer and Stampacchia.

Variational inequalities as the development and extension of classic variational problems are a very useful tool to research partial differential equations, optimal control, and other fields. Many papers (see e.g. [15, 18, 21, 23, 24]) are interested in the solvability of the different kinds of parabolic variational inequalities, relying on the methods of time discretion, semigroup property of the corresponding differential quotient and a penalization method which transform a parabolic variational inequality into a parabolic equation with a penalty term. These works assumed the monotonicity or regularity condition of the obstacles. We find another method in [19], where only the continuity on obstacles was used. Works which are dealing with double-phase problems or multivalued problems can be found in [12, 26–29]. We point out these works are concerned with obstacle problems where the authors have used tools from the nonsmooth analysis.

Motivated by the works [17, 23, 24], we will study the existence of weak solutions to the problem (1) by using the penalty method (which transforms the inequality (1) into equality (3) below) and the theory of Young measures. To be more precise, we shall construct a Young measure $\nu^{\epsilon}_{(x,t)}$ generated by a penalized gradient sequence, with $\epsilon \in (0, 1)$, which converges to the Young measure $\nu_{(x,t)}$ as ϵ tends to zero. To the best of our knowledge, this is the first paper treating the problem (1) by such methods.

This paper is organized as follows. Section 2 is devoted to recalling some necessary properties of Young measures. In Sect. 3, we prove the existence of weak solutions by Galerkin's approximation and the theory of Young measures for (3), while Sect. 4 is concerned to show the existence of weak solutions for variational problem (1).

2 Young measures: necessary properties

Consider $C_0(\mathbb{R}^m) = \{\varphi \in C(\mathbb{R}^m) : \lim_{|\lambda|\to\infty} \varphi(\lambda) = 0\}$. Its dual is the well-known signed Radon measures $\mathcal{M}(\mathbb{R}^m)$ with finite mass. The duality of $\big(\mathcal{M}(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ is given by the following integrand

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda), \text{ where } \nu : \Omega \to \mathcal{M}(\mathbb{R}^m).$$

Lemma 1 ([14]) *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exist a subsequence (still denoted (z_k)) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such* *that for almost each* $\varphi \in C(\mathbb{R}^m)$ *we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi}(x) = \langle \nu_x, \varphi \rangle \quad \textit{weakly in } L^\infty(\Omega; \mathbb{R}^m)$$

for a.e. $x \in \Omega$.

Definition 1 The family $\nu = \{\nu_x\}_{x\in\Omega}$ is called Young measures associated with (generated by) the subsequence $(z_k)_k$.

In [9], it is shown that if for all $R > 0$

$$\limsup_{L\to\infty} \left| \{x \in \Omega \cap B_R(0) : |z_k(x)| \geq L\} \right| = 0,$$

then for any measurable $\Omega' \subset \Omega$, we have

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x, .) \rangle = \int_{\mathbb{R}^m} \varphi(x, \lambda) d\nu_x(\lambda) \quad \text{in } L^1(\Omega'),$$

for every Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that $(\varphi(x, z_k(x)))_k$ is equiintegrable.

The following lemmas are useful throughout this paper.

Lemma 2 ([16]) *If* $|\Omega| < \infty$ *and* ν_x *is the Young measure generated by the (whole) sequence* (z_k)*, then there holds*

$$z_k \longrightarrow z \textit{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x \in \Omega.$$

It should be noted that the above properties remain true when $z_k = Dw_k$, with $w_k : \Omega \to \mathbb{R}^m$ and Ω can be replaced by the cylinder Q.

Lemma 3 ([13]) *Let* $\varphi : Q \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *be a Carathéodory function and* (w_k) *be a sequence of measurable functions, where* $w_k : Q \to \mathbb{R}^m$*, such that* Dw_k *generates the Young measure* $\nu_{(x,t)}$*. Then*

$$\liminf_{k\to\infty} \int_Q \varphi(x, t, Dw_k) dxdt \geq \int_Q \int_{\mathbb{M}^{m\times n}} \varphi(x, t, \lambda) d\nu_{(x,t)}(\lambda) dxdt$$

provided that the negative part $\varphi^-(x, t, Dw_k)$ *is equiintegrable.*

The following lemma describes limits points of gradient sequences utilizing the Young measures.

Lemma 4 ([3]) *The Young measure* $\nu_{(x,t)}$ *generated by* Dw_k *in* $L^p(0, T; L^p(\Omega))$ *satisfy the following properties:*

(i) $\nu_{(x,t)}$ *is a probability measure, i.e.,* $\|\nu_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ *for a.e.* $(x, t) \in Q$.
(ii) *The weak* L^1*-limit of* Dw_k *is given by* $\langle \nu_{(x,t)}, id \rangle$.
(iii) *For a.e.* $(x, t) \in Q$, $\langle \nu_{(x,t)}, id \rangle = Dw(x, t)$.

Let $\nu^\epsilon_{(x,t)}$ be the Young measure generated by the penalized gradient sequence (Dw_ϵ).

Lemma 5 ([9]) *For every continuous function* φ,

$$\langle \nu^\epsilon_{(x,t)}, \varphi \rangle \longrightarrow \langle \nu_{(x,t)}, \varphi \rangle \quad \textit{as} \quad \epsilon \to 0 \quad \textit{for a.e. } (x, t) \in Q.$$

3 Nonlinear parabolic systems with parameter ϵ

Let Ω be a bounded open domain of $\mathbb{R}^n$, $p \in (2n/(n+2), \infty)$ and $\epsilon \in (0,1)$ be fixed. In this section, we shall consider the existence result for the following parabolic system of Dirichlet's type given in the form:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(x,t,Du) - \frac{1}{\epsilon}|u^-|^{p-2}u^- = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where $u^- = \max\{-u, 0\}$ and $a : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfy the following hypothesis:

(H_0) a is a Carathéodory function, that is measurable in $(x,t) \in Q$ for fixed $\xi \in \mathbb{M}^{m\times n}$ and continuous in ξ for fixed (x,t) in Q.

(H_1) There exist a function $l \in L^{p'}(Q)$ and a constant $\alpha_0 > 0$ such that

$$|a(x,t,\xi)| \leq l(x,t) + |\xi|^{p-1} \tag{4}$$

and

$$a(x,t,\xi) : \xi \geq \alpha_0 |\xi|^p. \tag{5}$$

(H_2) For all $\xi, \xi' \in \mathbb{M}^{m\times n}$,

$$\big(a(x,t,\xi) - a(x,t,\xi')\big) : (\xi - \xi') \geq 0.$$

Definition 2 A function $u_\epsilon \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m))$ with $\frac{\partial u_\epsilon}{\partial t} \in L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$ is called a weak solution of (3), if for all $\varphi \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m))$, it holds

$$\int_Q \frac{\partial u_\epsilon}{\partial t}\varphi\,dxdt + \int_Q a(x,t,Du_\epsilon) : D\varphi\,dxdt - \frac{1}{\epsilon}\int_Q |u_\epsilon^-|^{p-2}u_\epsilon^-\varphi\,dxdt = \int_Q f(x,t)\varphi\,dxdt.$$

Similar to that in [20], we take a sequence $\{w_j\}_{j\geq 1} \subset C_0^\infty(\Omega;\mathbb{R}^m)$, such that $C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} V_k}^{C^1(\overline{\Omega})}$, where $\{w_j\}_{j\geq 1}$ is a standard orthogonal basis in $L^2(\Omega;\mathbb{R}^m)$ and $V_k = \operatorname{span}\{w_1, ..., w_k\}$. Firstly, remark that since $u_0 \in L^2(\Omega;\mathbb{R}^m)$, there exists a sequence $\psi_k(x) \in V_k$ such that $\psi_k(x) \to u_0(x)$ in $L^2(\Omega;\mathbb{R}^m)$ as $k \to \infty$. Indeed, for $u_0 \in L^2(\Omega;\mathbb{R}^m)$, there exists a sequence v_k in $C_0^\infty(\Omega;\mathbb{R}^m)$ such that $v_k \to u_0$ in $L^2(\Omega;\mathbb{R}^m)$. Since $\{v_k\} \subset C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{N\geq 1} V_N}^{C^1(\overline{\Omega})}$, we can find a sequence $\{v_k^i\} \subset \bigcup_{N\geq 1} V_N$ such that $v_k^i \to v_k$ in $C^1(\overline{\Omega};\mathbb{R}^m)$ as $i \to \infty$. For $\frac{1}{2^k}$, there exists $i_k \geq 1$ such that $\|v_k^{i_k} - v_k\|_{C^1(\overline{\Omega})} \leq \frac{1}{2^k}$. Therefore

$$\|v_k^{i_k} - u_0\|_{L^2(\Omega)} \leq C\|v_k^{i_k} - v_k\|_{C^1(\overline{\Omega})} + \|v_k - u_0\|_{L^2(\Omega)}.$$

Hence $v_k^{i_k} \to u_0$ in $L^2(\Omega;\mathbb{R}^m)$ as $k \to \infty$. Let us denote $u_k = v_k^{i_k}$. Since $u_k \in \bigcup_{N\geq 1} V_N$, there exists V_{N_k} such that $u_k \in V_{N_k}$, without loss of generality, we assume that $V_{N_1} \subset V_{N_2}$ as $N_1 \leq N_2$. We suppose that $N_1 > 1$ and define ψ_k as follows: $\psi_k(x) = 0$, $k = 1, ..., N_1 - 1$; $\psi_k = u_1$, $k = N_1, ..., N_2 - 1$; $\psi_k = u_2$, $k = N_2, ..., N_3 - 1$;..., then we obtain the desired sequence $\{\psi_k\}$ and $\psi_k \to u_0$ in $L^2(\Omega;\mathbb{R}^m)$ as $k \to \infty$.

Theorem 1 *Let $f \in L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$. Suppose that (H_0)-(H_2) are satisfied. Then for every $\epsilon > 0$ to be fixed, there exists a weak solution of Eq. (3).*

Proof **(i) Galerkin approximation**
For each $k \in \mathbb{N}$, $k \geq 1$, we define a vector-valued function $P_k(t, v) : [0, \infty) \times \mathbb{R}^k \to \mathbb{R}^k$ as follows:

$$\big(P_k(t, v)\big)_i = \int_\Omega a\Big(x, t, \sum_{j=1}^k v_j Dw_j\Big) : Dw_i dx - \frac{1}{\epsilon}\int_\Omega |\sum_{j=1}^k v_j w_j|^{p-2}\Big(\sum_{j=1}^k v_j w_j\Big) w_i dx,$$

where $v = (v_1, ..., v_k)$. Since a is continuous, the continuity of $P_k(t, v)$ follows.

Now, we shall construct the approximate solutions of problem (3) in the form

$$u_k(x, t) = \sum_{j=1}^k (\eta_k(t))_j w_j(x),$$

where $(\eta_k(t))_k$ are unknown functions, which can be determined as solutions of the following system of ordinary differential equations

$$\begin{cases} \eta'(t) + P_k(t, \eta(t)) = F, \\ \qquad\qquad \eta(0) = U_k(0), \end{cases} \tag{6}$$

where $(F)_i = \int_\Omega f w_i dx$, $(U_k(0))_i = \int_\Omega \psi_k(x) w_i dx$, $\psi_k(x) \in V_k$, $\psi_k(x) \to u_0(x)$ in $L^2(\Omega; \mathbb{R}^m)$ as $k \to \infty$.
We multiply the Eq. (6) by $\eta(t)$, thus

$$\eta'\eta = P_k(t, \eta)\eta = F\eta.$$

By the coercivity condition in (H_1), we have

$$\begin{aligned} P_k(t, \eta)\eta = &\int_\Omega a\big(x, t, \sum_{j=1}^k \eta_j Dw_j\big) : \Big(\sum_{i=1}^k \eta_i Dw_i\Big) dx \\ &- \frac{1}{\epsilon}\int_\Omega |\Big(\sum_{j=1}^k \eta_j w_j\Big)^- |^{p-2}\Big(\sum_{j=1}^k \eta_j w_j\Big)^- \Big(\sum_{i=1}^k \eta_i w_i\Big) dx \geq 0. \end{aligned} \tag{7}$$

From (7) and Young's inequality, we arrive at

$$\frac{1}{2}\frac{\partial}{\partial t}|\eta(t)|^2 \leq |F\eta| \leq \frac{1}{2}|F|^2 + \frac{1}{2}|\eta|^2.$$

Integrating the above inequality with respect to t from 0 to t, we obtain

$$|\eta(t)| \leq c_k + \int_0^t |\eta(\tau)|^2 d\tau,$$

which implies, by Gronwall's inequality, that $|\eta(t)| \leq c_k(T)$. Consider

$$L_k = \max_{t\in[0,T]} |F - P_k(t, \eta)| \quad \text{and} \quad \tau_k = \min\Big\{T, \frac{c_k(T)}{L_k}\Big\}.$$

Since $P_k(t, \eta)$ is continuous in t and η, the Peano Theorem implies that (6) has a C^1 solution locally in $[0, \tau_k]$. Let $\tau_k = t_1$ and $\eta(t_1)$ be an initial value, then we repeat the above process and get a C^1 solution on $[t_1, t_1 + \tau_k]$. We can divide $[0, T]$ into $[(i-1)\tau_k, i\tau_k]$, $i = 1, ..., L$, where

$\frac{T}{L} \leq \tau_k$, then there exist C^1 solution $\eta_k^i(t)$ in $[(i-1)\tau_k, i\tau_k]$, $i = 1, ..., L$. Consequently, we arrive at a solution $\eta_k(t) \in C^1([0, T])$ defined by

$$\eta_k(t) = \begin{cases} \eta_k^1(t) & \text{if } t \in [0, \tau_k), \\ \eta_k^2(t) & \text{if } t \in (\tau_k, 2\tau_k], \\ \vdots & \\ \eta_k^L(t) & \text{if } t \in ((L-1)\tau_k, L\tau_k]. \end{cases}$$

Therefore, we get the approximate solutions $u_k(x,t) = \sum_{j=1}^{k} (\eta_k(t))_j w_j(x)$. From (6) it follows for $1 \leq i \leq k$, that

$$\int_\Omega \frac{\partial u_k}{\partial t} w_i dx + \int_\Omega a(x,t,Du_k) : Dw_i dx - \frac{1}{\epsilon} \int_\Omega |u_k^-|^{p-2} u_k^- w_i dx = \int_\Omega f w_i dx. \quad (8)$$

Remark by (6), that $\eta_k(t)$ should be dependent on ϵ, but for convenience we omit ϵ, and for all $\varphi \in C^1(0, T; V_j)$, $j \leq k$, there holds

$$\int_Q \frac{\partial u_k}{\partial t} \varphi dxdt + \int_Q a(x,t,Du_k) : D\varphi dxdt - \frac{1}{\epsilon} \int_Q |u_k^-|^{p-2} u_k^- \varphi dxdt = \int_Q f \varphi dxdt. \quad (9)$$

(ii) Passage to the limit

We multiply (8) by $(\eta_k(t))_i$ and sum up i from 1 to k, it holds by integrating with respect to t from 0 to τ ($\tau \in (0, T]$), that

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \alpha_0 \int_{Q_s} |Du_k|^p dxdt - \frac{1}{\epsilon} \int_{Q_s} |u_k^-|^{p-2} u_k^- u_k dxdt \\ &\leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))} \|u_k\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}, \end{aligned}$$

with $Q_\tau = \Omega \times (0, \tau)$, where we have used the coercivity condition in (H_1) and Hölder's inequality. We have $u_k(x, 0) \to u_0$ in $L^2(\Omega; \mathbb{R}^m)$, this implies $\int_\Omega u_k^2(x, 0)dx \leq c$, where c is a constant independent of ϵ and k. Moreover $\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))} \leq c$. Therefore

$$\begin{aligned} &\frac{1}{2} \int_\Omega |u_k(x,\tau)|^2 dx + \alpha_0 \int_{Q_\tau} |Du_k|^p dx + \frac{1}{\epsilon} \int_{Q_\tau} |u_k^-|^p dxdt \\ &\leq c(\|u_k\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))} + 1). \end{aligned} \quad (10)$$

From this inequality, we deduce, that

$$\|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))} \leq c, \quad \int_\Omega |u_k(x,T)|^2 dx \leq c \quad \text{and} \quad \frac{1}{\epsilon} \int_Q |u_k^-|^p dxdt \leq c. \quad (11)$$

By (11) and the growth condition in (H_1), we have

$$\int_Q |a(x,t,Du_k)|^{p'} dxdt \leq c \left(\int_Q |Du_k|^p dxdt + 1 \right) \leq c, \quad (12)$$

where c is a constant independent of ϵ and k. From (11) and (12), there exist a subsequence of $(u_k)_k$ (still denoted by (u_k)), $\chi \in L^{p'}(Q; \mathbb{M}^{m\times n})$ and $g \in L^{p'}(Q; \mathbb{R}^m)$ such that

$$\begin{cases} u_k \rightharpoonup u_\epsilon \quad \text{in } L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m)), \\ u_k \rightharpoonup^* u_\epsilon \quad \text{in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^m)), \\ a(x, t, Du_k) \rightharpoonup \chi \quad \text{in } L^{p'}(Q; \mathbb{M}^{m\times n}), \\ \frac{1}{\epsilon}|u_k^-|^{p-2}u_k^- \rightharpoonup g \quad \text{in } L^{p'}(Q; \mathbb{R}^m). \end{cases}$$

By the compact embedding $W_0^{1,p}(\Omega; \mathbb{R}^m) \hookrightarrow L^p(\Omega; \mathbb{R}^m)$, one has $u_k \to u_\epsilon$ in $L^p(Q; \mathbb{R}^m)$ and almost everywhere in Q (for a subsequence). Thus, as $k \to \infty$, we have

$$\begin{cases} u_k^- \longrightarrow u_\epsilon^- \quad \text{a.e. } (x, t) \in Q, \\ \frac{1}{\epsilon}|u_k^-|^{p-2}u_k^- \longrightarrow \frac{1}{\epsilon}|u_\epsilon^-|^{p-2}u_\epsilon^- \quad \text{a.e. } (x, t) \in Q. \end{cases} \tag{13}$$

From (11) and (13), it follows that $g = \frac{1}{\epsilon}|u_\epsilon^-|^{p-2}u_\epsilon^-$. Let $\varphi \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m))$, then there exists a sequence $\varphi_k \in C^1(0, T; V_k)$ such that $\varphi_k \to \varphi$ in $L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m))$. By virtue of (9) and Hölder's inequality, we get

$$\begin{aligned} &\Big|\int_Q \frac{\partial u_k}{\partial t}\varphi_k dxdt\Big| \\ &= \Big|\int_Q f(x,t)\varphi_k dxdt - \int_Q a(x,t,Du_k) : D\varphi_k dxdt + \frac{1}{\epsilon}\int_Q |u_k^-|^{p-2}u_k^-\varphi_k dxdt\Big| \\ &\le c\|\varphi_k\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}, \end{aligned}$$

where we have used (11) and (12), and c is a constant independent of k and ϵ. Consequently, $\|\frac{\partial u_k}{\partial t}\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))} \le c$. It immediately follows the existence of a subsequence of (u_k) (still denoted as (u_k)) such that

$$\frac{\partial u_k}{\partial t} \rightharpoonup \alpha \quad \text{in } L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m)).$$

Let $\psi \in C_0^\infty(Q; \mathbb{R}^m)$, by letting $k \to \infty$ in $\int_Q \frac{\partial u_k}{\partial t}\psi dxdt = -\int_Q u_k \frac{\partial \psi}{\partial t} dxdt$, it results

$$\int_Q \alpha\psi dxdt = -\int_Q u_\epsilon \frac{\partial \psi}{\partial t} dxdt.$$

This implies $\alpha = \frac{\partial u_\epsilon}{\partial t}$. On the other hand, since $\int_\Omega |u_k(x, T)|^2 dx \le c$, there is a subsequence of $(u_k(x, T))$ (still labelled by $(u_k(x, T))$) and a function u^* in $L^2(\Omega; \mathbb{R}^m)$ such that $u_k(x, T) \rightharpoonup u^*$ in $L^2(\Omega; \mathbb{R}^m)$. To identify u^* with $u(x, T)$, we use the fact that

$$\int_Q \frac{\partial u_k}{\partial t} w_i dxdt = \int_\Omega u_k(x, T)w_i dx - \int_\Omega u_k(x, 0)w_i dx.$$

Passing k to infinity, it results by integration by parts, that $\int_\Omega u^* w_i dx = \int_\Omega u_\epsilon(x, T)w_i dx$. We conclude, by the completeness of $\{w_i\}_i$, that $u^* = u_\epsilon(x, T)$, i.e.,

$$\int_\Omega u_\epsilon^2(x, T)dx \le \liminf_{k\to\infty} \int_\Omega u_k^2(x, T)dx. \tag{14}$$

Note that, since (u_k) is bounded in $L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m))$, it follows by Lemma 1 the existence of a Young measure $\nu^{\epsilon}_{(x,t)}$ generated by (Du_k) in $L^p(Q; \mathbb{M}^{m\times n})$ and satisfying Lemma 4. Remark that the generated Young measure is labeled by ϵ, as well as Du_k, depending on it.

To identify χ with $a(x, t, Du_\epsilon)$, we will need the following inequality:

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(a(x,t,\lambda) - a(x,t,Du_\epsilon)\big) : (\lambda - Du_\epsilon) d\nu^{\epsilon}_{(x,t)}(\lambda) dxdt \leq 0. \tag{15}$$

To see this, consider the sequence

$$\begin{aligned} Y_k &= \big(a(x,t,Du_k) - a(x,t,Du_\epsilon)\big) : (Du_k - Du_\epsilon) \\ &= Y_{k,1} - Y_{k,2}, \end{aligned}$$

where $Y_{k,1} = (a(x,t,Du_k) : (Du_k - Du_\epsilon)$ and $Y_{k,2} = a(x,t,Du_\epsilon) : (Du_k - Du_\epsilon)$. As in (12), it follows that $a(x,t,Du_\epsilon) \in L^{p'}(Q; \mathbb{M}^{m\times n})$. On the one hand, because of the weak limit defined in Lemma 4, we obtain

$$\begin{aligned} \liminf_{k\to\infty} \int_Q Y_{k,2} dxdt &= \int_Q \int_{\mathbb{M}^{m\times n}} a(x,t,Du_\epsilon) : (\lambda - Du_\epsilon) dxdt \\ &= \int_Q a(x,t,Du_\epsilon) \Big(\underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu^{\epsilon}_{(x,t)}(\lambda)}_{:=Du_\epsilon(x,t)} - Du_\epsilon \Big) dxdt = 0. \end{aligned} \tag{16}$$

On the other hand, since $(a(x,t,Du_k) : (Du_k - Du_\epsilon))$ is equiintegrable (by (11), (12) and Hölder's inequality), Lemma 3 implies

$$\begin{aligned} &\liminf_{k\to\infty} \int_Q a(x,t,Du_k) : (Du_k - Du_\epsilon) dxdt \\ &\geq \int_Q \int_{\mathbb{M}^{m\times n}} a(x,t,\lambda) : (\lambda - Du_\epsilon) d\nu^{\epsilon}_{(x,t)}(\lambda) dxdt. \end{aligned} \tag{17}$$

The next step is to show, that the left-hand side of the above inequality is ≤ 0. We have

$$\int_Q \frac{\partial u_k}{\partial t} u_k dxdt + \int_Q a(x,t,Du_k) : Du_k dxdt - \frac{1}{\epsilon} \int_Q |u_k^-|^{p-2} u_k^- u_k dxdt = \int_Q f u_k dxdt.$$

By using this equation, it results

$$\begin{aligned} Y &:= \liminf_{k\to\infty} \int_Q a(x,t,Du_k) : (Du_k - Du_\epsilon) dxdt \\ &= \liminf_{k\to\infty} \left(\int_Q f u_k dxdt - \int_Q \frac{\partial u_k}{\partial t} u_k dxdt + \frac{1}{\epsilon} \int_Q |u_k^-|^{p-2} u_k^- u_k dxdt \right. \\ &\left. - \int_Q a(x,t,Du_k) : Du_\epsilon dxdt \right). \end{aligned} \tag{18}$$

Passing to the limit as $k \to \infty$ in (9), we then have the following energy equality:

$$\int_Q \frac{\partial u_\epsilon}{\partial t}\varphi dxdt + \int_Q \chi : D\varphi dxdt - \frac{1}{\epsilon}\int_Q |u_\epsilon^-|^{p-2}u_\epsilon^- \varphi dxdt = \int_Q f\varphi dxdt.$$

Passing to the limit as $k \to \infty$ on the right hand-side of (18), we get

$$\begin{aligned} Y \le \int_Q f u_\epsilon dxdt - \frac{1}{2}\int_\Omega u_\epsilon^2(x,T)dx + \frac{1}{2}\int_\Omega u_\epsilon^2(x,0)dx \\ - \frac{1}{\epsilon}\int_Q |u_\epsilon^-|^p dxdt - \int_Q \chi : Du_\epsilon dxdt. \end{aligned}$$

By taking $\varphi = u_\epsilon$ in the energy equality and plugging it in the right-hand side of the above inequality, we arrive at $Y \le 0$ as desired. From this and (16), the Eq. (15) follows. In virtue of the monotonicity of the function a, we conclude the following localization of the support of $\nu^\epsilon_{(x,t)}$:

$$\big(a(x,t,\lambda) - a(x,t,Du_\epsilon)\big) : (\lambda - Du_\epsilon) = 0 \quad \text{on supp}\, \nu^\epsilon_{(x,t)}. \tag{19}$$

Now, we identify χ with $a(x,t,Du_\epsilon)$ as follows:
From the monotonicity assumption, we can write for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{M}^{m\times n}$

$$\begin{aligned} 0 &\le \big(a(x,t,\lambda) - a(x,t,Du_\epsilon + \tau\xi)\big) : (\lambda - Du_\epsilon - \tau\xi) \\ &= a(x,t,\lambda) : (\lambda - Du_\epsilon) - a(x,t,\lambda) : \tau\xi - a(x,t,Du_\epsilon + \tau\xi) : (\lambda - Du_\epsilon - \tau\xi), \end{aligned} \tag{20}$$

which implies by (19)

$$-a(x,t,\lambda) : \tau\xi \ge -a(x,t,Du_\epsilon) : (\lambda - Du_\epsilon) + a(x,t,Du_\epsilon + \tau\xi) : (\lambda - Du_\epsilon - \tau\xi).$$

Note that

$$\begin{aligned} &a(x,t,Du_\epsilon + \tau\xi) : (\lambda - Du_\epsilon - \tau\xi) \\ &\quad = a(x,t,Du_\epsilon + \tau\xi) : (\lambda - Du_\epsilon) - a(x,t,Du_\epsilon + \tau\xi) : \tau\xi \\ &\quad = a(x,t,Du_\epsilon) : (\lambda - Du_\epsilon) \\ &\qquad + \tau\Big(\big(\nabla a(x,t,Du_\epsilon)\xi\big) : (\lambda - Du_\epsilon) - a(x,t,Du_\epsilon) : \xi\Big) + o(\tau), \end{aligned}$$

where ∇ is the derivative of a with respect to its third variable. Therefore

$$-a(x,t,\lambda) : \tau\xi \ge \tau\Big(\big(\nabla a(x,t,Du_\epsilon)\xi\big) : (\lambda - Du_\epsilon) - a(x,t,Du_\epsilon) : \xi\Big) + o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, we get

$$a(x,t,\lambda) : \xi = a(x,t,Du_\epsilon) : \xi + \big(\nabla a(x,t,Du_\epsilon)\xi\big) : (Du_\epsilon - \lambda) \tag{21}$$

holds on the support of $\nu^{\epsilon}_{(x,t)}$. The equiintegrability of $a(x,t,Du_k)$ implies that its weak L^1-limit $\overline{a}_\epsilon$ is given by

$$\begin{aligned}\overline{a}_\epsilon(x,t) &:= \int_{\mathbb{M}^{m\times n}} a(x,t,\lambda)d\nu^{\epsilon}_{(x,t)}(\lambda)\\ &= \int_{\operatorname{supp}\nu^{\epsilon}_{(x,t)}} \Big(a(x,t,Du_\epsilon) + \big(\nabla a(x,t,Du_\epsilon)\big) : (Du_\epsilon - \lambda)\Big)d\nu^{\epsilon}_{(x,t)}(\lambda) \quad \text{(by (21))}\\ &= a(x,t,Du_\epsilon),\end{aligned} \tag{22}$$

where we used

$$\|\nu^{\epsilon}_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1 \quad \text{and} \quad \big(\nabla a(x,t,Du_\epsilon)\big)\int_{\operatorname{supp}\nu^{\epsilon}_{(x,t)}} (Du_\epsilon - \lambda)d\nu^{\epsilon}_{(x,t)}(\lambda) = 0.$$

Consequently

$$a(x,t,Du_k) \rightharpoonup \chi = a(x,t,Du_\epsilon) \quad \text{in } L^{p'}(Q;\mathbb{M}^{m\times n}).$$

In view of (9), for all $\varphi \in C^1(0,T;V_j)$ with $j \leq k$, letting $k \to \infty$ it holds

$$\int_Q \frac{\partial u_\epsilon}{\partial t}\varphi dxdt + \int_Q a(x,t,Du_\epsilon) : D\varphi dxdt - \frac{1}{\epsilon}\int_Q |u_\epsilon^-|^{p-2}u_\epsilon^-\varphi dxdt = \int_Q f\varphi dxdt, \tag{23}$$

and since $C^1(0,T;\bigcup_{j\geq 1} V_j)$ is dense in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, the Eq. (23) holds for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$. □

4 Variational inequality

We shall prove the main result of this paper. Denote

$$K = \Big\{w \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap C(0,T;L^2(\Omega;\mathbb{R}^m)),\ \frac{\partial w}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)) : 0 \leq w(x,0) = u_0(x) \in L^2(\Omega;\mathbb{R}^m),\ w(x,t) \geq 0 \quad \text{a.e. } (x,t) \in Q\Big\}.$$

The main theorem can be stated as follows:

Theorem 2 *Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and suppose that (H_0)-(H_2) are satisfied. Then there exists a function $u(x,t) \in K$ such that for all $v \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ with $v(x,t) \geq 0$ for a.e. $(x,t) \in Q$, there holds*

$$\int_{Q_s} \frac{\partial u}{\partial t}(v-u)dxdt + \int_{Q_s} a(x,t,Du) : (Dv - Du)dxdt \geq \int_{Q_s} f(v-u)dxdt,$$

for almost every $s \in [0,T]$.

Proof *(i) A prior estimates*
Let us take $\varphi = u_\epsilon.\chi_{(0,t)}$ as a test function in Definition 2 (where $\chi_{(0,t)}$ is the characteristic function of $(0,t)$), $t \in (0,T]$, thus

$$\int_{Q_t} \frac{\partial u_\epsilon}{\partial t} u_\epsilon dxdt + \int_{Q_t} a(x,t,Du_\epsilon) : Du_\epsilon dxdt - \frac{1}{\epsilon}\int_{Q_t} |u_\epsilon^-|^{p-2}u_\epsilon^- u_\epsilon dxdt = \int_{Q_t} f(x,t)u_\epsilon dxdt,$$

where $Q_t = \Omega \times (0,t)$. Integrate the first term, it follows by the coercivity condition in (H_1) and Hölder's inequality, that

$$\frac{1}{2}\int_\Omega |u_\epsilon(x,t)|^2 dx + \alpha_0 \int_{Q_t} |Du_\epsilon|^p dxdt + \frac{1}{\epsilon}\int_{Q_t} |u_\epsilon^-|^p dxdt \le c(1 + \|u_\epsilon\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}), \tag{24}$$

where c is a constant independent of ϵ and t. Consequently

$$(u_\epsilon)_\epsilon \quad \text{is bounded in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)) \cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \quad \text{and} \quad \frac{1}{\epsilon}\int_Q |u_\epsilon^-|^p dxdt \le c. \tag{25}$$

Similar to that in (12), one has

$$\|a(x,t,Du_\epsilon)\|_{L^{p'}(Q)} \le c \quad \text{and} \quad \left\|\frac{1}{\epsilon}|u_\epsilon^-|^{p-2}u_\epsilon^-\right\|_{L^{p'}(Q)} \le c. \tag{26}$$

Using (25) and (26), we deduce from Definition 2 that for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$

$$\left\|\frac{\partial u_\epsilon}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))} = \sup_{\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}\le 1} \left|\int_Q \frac{\partial u_\epsilon}{\partial t}\varphi dxdt\right| \le c. \tag{27}$$

(ii) Passage to the limit
As in the previous section, from (25 to 27) there exists a subsequence of (u_ϵ) (still labeled by (u_ϵ)), such that

$$\begin{cases} u_\epsilon \rightharpoonup u \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)), \\ u_\epsilon \rightharpoonup^* u \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)), \\ a(x,t,Du_\epsilon) \rightharpoonup \sigma \quad \text{in } L^{p'}(Q;\mathbb{M}^{m\times n}), \\ u_\epsilon^- \longrightarrow 0 \quad \text{in } L^p(Q;\mathbb{R}^m), \\ \frac{\partial u_\epsilon}{\partial t} \rightharpoonup \alpha \quad \text{in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)). \end{cases} \tag{28}$$

Let $\varphi \in C_0^\infty(Q;\mathbb{R}^m)$, we have $\int_Q \frac{\partial u_\epsilon}{\partial t}\varphi dxdt = -\int_Q u_\epsilon \frac{\partial \varphi}{\partial t} dxdt$. Passing to the limit and using (28), there holds $\int_Q \alpha\varphi dxdt = -\int_Q u\frac{\partial\varphi}{\partial t}dxdt$, and therefore $\alpha = \frac{\partial u}{\partial t}$. By virtue of (28), there exists a subsequence, still denoted as (u_ϵ), such that $u_\epsilon \to u$ in $L^p(Q;\mathbb{R}^m)$ and almost everywhere, thus $u_\epsilon^- \to u^-$ a.e. $(x,t) \in Q$. Moreover, from (13) we have $u^- = 0$ for a.e. $(x,t) \in Q$, that is to say $u(x,t) \ge 0$ for a.e. $(x,t) \in Q$. Since $u_\epsilon \in$

$L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, for all $s\in[0,T]$, we have $u_\epsilon(x,s)\rightharpoonup u^*$ in $L^2(\Omega;\mathbb{R}^m)$. Let $\varphi\in C_0^\infty(\Omega;\mathbb{R}^m)$ and $\eta(t)\in C([0,s])$. By passing to the limit in

$$\int_{Q_s}\frac{\partial u_\epsilon}{\partial t}\eta(t)\varphi(x)dxdt$$
$$=\int_\Omega u_\epsilon(x,s)\eta(s)\varphi(x)dx-\int_\Omega u_0(x)\eta(0)\varphi(x)dx-\int_{Q_s}u_\epsilon\frac{\partial\eta}{\partial t}\varphi dxdt,$$

it follows by the integration by parts, that

$$\int_\Omega\Big(\big(u^*-u(x,s)\big)\eta(s)\varphi(x)-\big(u(x,0)-u_0(x)\big)\eta(0)\varphi(x)\Big)dx=0.$$

If we choose $\eta(s)=1$ and $\eta(0)=0$, or $\eta(s)=0$ and $\eta(0)=1$, we then get $u^*=u(x,s)$ and $u(x,0)=u_0(x)$ (by the density of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $L^2(\Omega;\mathbb{R}^m)$).
By (25), there exists a Young measure $\nu_{(x,t)}$ generated by Du_ϵ in $L^p(Q;\mathbb{M}^{m\times n})$ and verify the properties of Lemma 4. The next step has for goal to identify σ with $a(x,t,Du)$. To do this, we consider the sequence

$$I_\epsilon=\big(a(x,t,Du_\epsilon)-a(x,t,Du)\big):(Du_\epsilon-Du)$$

According to the weak limit in Lemma 4, we have

$$\lim_{\epsilon\to0}\int_Q a(x,t,Du):(Du_\epsilon-Du)dxdt$$
$$=\int_Q a(x,t,Du):\Big(\underbrace{\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)}_{:=Du(x,t)}-Du\Big)dxdt=0.$$

This and Lemma 3 implies

$$\liminf_{\epsilon\to0}\int_Q I_\epsilon dxdt\ge\int_Q\int_{\mathbb{M}^{m\times n}}a(x,t,\lambda):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt.$$

Similar to the previous section, there holds

$$\big(a(x,t,\lambda)-a(x,t,Du)\big):(\lambda-Du)=0\quad\text{on supp}\,\nu_{(x,t)}.\tag{29}$$

By the same procedure from (20) to (22) and equiintegrability of $(a(x,t,Du_\epsilon))$, it follows that the weak L^1-limit of $a(x,t,Du_\epsilon)$ is $a(x,t,Du)$. Therefore $\sigma=a(x,t,Du)$. □

Remark 1 Note that, since $a(x,t,Du_\epsilon) = \int_{\mathbb{M}^{m\times n}} a(x,t,\lambda)d\nu^\epsilon_{(x,t)}(\lambda)$, thus one can directly pass to the limit using Lemma 5 and (21) as follows:

$$\begin{aligned} a(x,t,Du_\epsilon) &= \int_{\operatorname{supp}\nu^\epsilon_{(x,t)}} a(x,t,\lambda)d\nu^\epsilon_{(x,t)}(\lambda) \\ &\rightharpoonup \int_{\operatorname{supp}\nu_{(x,t)}} a(x,t,\lambda)d\nu_{(x,t)}(\lambda) \\ &= \int_{\operatorname{supp}\nu_{(x,t)}} \Big(a(x,t,Du) + \big(\nabla a(x,t,Du)\big) : (Du-\lambda)\Big)d\nu_{(x,t)}(\lambda) \\ &= a(x,t,Du). \end{aligned}$$

Proof **(iii) Existence of weak solutions**
Let $v \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, $v \geq 0$. By taking $\varphi = v - u_\epsilon$ as a test function in Definition 2, we get

$$\begin{aligned} &\int_{Q_s} \frac{\partial u_\epsilon}{\partial t} v dxdt + \int_{Q_s} a(x,t,Du_\epsilon) : (Dv - Du_\epsilon)dxdt - \int_{Q_s} f(v-u_\epsilon)dxdt \\ &\quad= \int_{Q_s} \frac{\partial u_\epsilon}{\partial t} u_\epsilon dxdt + \frac{1}{\epsilon}\int_{Q_s} |u_\epsilon^-|^{p-2}u_\epsilon^-(v-u_\epsilon)dxdt \\ &\quad\geq \frac{1}{2}\int_\Omega |u_\epsilon(x,s)|^2dx - \frac{1}{2}\int_\Omega |u_\epsilon(x,0)|^2dx, \end{aligned}$$

i.e.,

$$\begin{aligned} &\int_{Q_s} \frac{\partial u_\epsilon}{\partial t} v dxdt + \int_{Q_s} a(x,t,Du_\epsilon) : Dv dxdt - \int_{Q_s} f(v-u_\epsilon)dxdt \\ &\quad\geq \frac{1}{2}\int_\Omega |u_\epsilon(x,s)|^2dx - \frac{1}{2}\int_\Omega |u_\epsilon(x,0)|^2dx + \int_{Q_s} a(x,t,Du_\epsilon) : Du_\epsilon dxdt. \end{aligned} \tag{30}$$

Since $\frac{\partial u_\epsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $a(x,t,Du_\epsilon) \rightharpoonup \sigma = a(x,t,Du)$ in $L^{p'}(Q;\mathbb{M}^{m\times n})$ and $Du_\epsilon \rightharpoonup \langle \nu_{(x,t)}, id\rangle = Du(x,t)$ in $L^p(Q;\mathbb{M}^{m\times n})$, we conclude as $\epsilon \to 0$, that

$$\int_{Q_s} \frac{\partial u}{\partial t}(v-u)dxdt + \int_{Q_s} a(x,t,Du) : (Dv - Du)dxdt \geq \int_{Q_s} f(v-u)dxdt,$$

for almost every $s \in [0,T]$. Remark that, since $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, $\frac{\partial u}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and $\big\{u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) : \frac{\partial u}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))\big\}$ is continuously embedded in $C(0,T;L^2(\Omega;\mathbb{R}^m))$, thus $u \in C(0,T;L^2(\Omega;\mathbb{R}^m))$ and the proof is complete. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Azroul, E., Balaadich, F.: Quasilinear elliptic systems in perturbed form. Int. J. Nonlinear Anal. Appl. **10**(2), 255–266 (2019)
2. Azroul, E., Balaadich, F.: A weak solution to quasilinear elliptic problems with perturbed gradient. Rend. Circ. Mat. Palermo. (2020). https://doi.org/10.1007/s12215-020-00488-4
3. Azroul, E., Balaadich, F.: Strongly quasilinear parabolic systems in divergence form with weak monotonicity. Khayyam J. Math. **6**(1), 57–72 (2020)
4. Azroul, E., Balaadich, F.: On strongly quasilinear elliptic systems with weak monotonicity. J. Appl. Anal. (2021). https://doi.org/10.1515/jaa-2020-2041
5. Azroul, E., Balaadich, F.: Existence of solutions for a class of Kirchhoff-type equation via Young measures. Numer. Funct. Anal. Optim. **42**, 460–473 (2021)
6. Balaadich, F.: On p-Kirchhoff-type parabolic problems. Rend. Circ. Mat. Palermo II. Ser **72**, 1005–1016 (2023). https://doi.org/10.1007/s12215-021-00705-8
7. Balaadich, F., Azroul, E.: A note on quasilinear elliptic systems with L^∞-data. Eurasian Math. J. **14**(1), 16–24 (2023)
8. Balaadich, F., Azroul, E.: Weak solutions for obstacle problems with weak monotonicity. Stud. Sci. Math. Hungar. **58**, 171–181 (2021)
9. Ball, J.M.: A version of the fundamental theorem for Young measures. In: PDEs and Continuum Models of Phase Transitions (Nice, 1988). Lecture Notes in Phys, vol. 344, 207–215 (1989)
10. Brézis, H.: Operateurs Maximaux Monotones et Semigroups de Contractions dans les Espaces de Hilbert. North-Holland, Amsterdam (1973)
11. Browder, F.E.: Existence theorems for nonlinear partial differential equations, in Global Analysis (Berkeley, Calif), Proc. Sympos. Pure Math. 16. Am. Math. Soc. Providence **1970**, 1–60 (1968)
12. Cen, J., Khan, A.A., Motreanu, D., Zeng, S.: Inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems. Inverse Probl. **38**, 065006 (2022)
13. Dolzmann, G., Hungerühler, N., Muller, S.: Nonlinear elliptic systems with measure-valued right hand side. Math. Z. **226**, 545–574 (1997)
14. Evans, L.C.: Weak convergence methods for nonlinear partial differential equations, Number 74 (1990)
15. Friedman, A.: Variational Principles and Free Boundary Value Problems. Wiley Interscience, New York (1983)
16. Hungerbühler, N.: A refinement of Ball's theorem on Young measures. N.Y. J. Math. **3**, 48–53 (1997)
17. Hungerbühler, N.: Quasilinear parabolic systems in divergence form with weak monotonicity. Duke Math. J. **107**(3), 497–519 (2000)
18. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities. Acad. Press, New York (1980)
19. Korte, R., Kuusi, T., Siljander, J.: Obstacle problem for nonlinear parabolic equations. J. Differ. Equ. **246**, 3668–3680 (2009)
20. Landes, R.: On the existence of weak solutions for quasilinear parabolic boundary problems. Proc. R. Soc. Edinburgh Sect. A **89**, 217–237 (1981)
21. Lions, J.L.: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. Dunod, Gauthier-Villars, Paris (1969)
22. Minty, G.: Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29**, 341–346 (1962)
23. Rudd, M., Schmitt, K.: Variational inequalities of elliptic and parabolic type. Taiwan. J. Math. **6**, 287–322 (2002)
24. Shahgholian, H.: Analysis of the free boundary for the p-parabolic variational problem ($p \geq 2$). Rev. Mat. Iberoamericana **19**, 797–812 (2003)
25. Visik, M.L.: On general boundary problems for elliptic differential equations. Am. Math. Soc. Transl. **24**(2), 107–172 (1963)
26. Zeng, S., Bai, Y., Gasinski, L., Winkert, P.: Existence results for double phase implicit obstacle problems involving multivalued operators. Calc. Var. **59**, 176 (2020)
27. Zeng, S., Rădulescu, V.D., Winkert, P.: Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions. SIAM J. Math. Anal. **54**, 1898–1926 (2022)

28. Zeng, S., Migórski, S., Liu, Z.: Well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 2829–2862 (2021)
29. Zeng, S., Migórski, S., Khan, A.A.: Nonlinear quasi-hemivariational inequalities: existence and optimal control. SIAM J. Control Optim. **59**, 1246–1274 (2021)