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Existence of solutions for a sixth-order nonlinear equation

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Abstract
In this paper, using variational methods and two critical point theorems, we prove the existence of two intervals for a parameter for which a nonlinear equation of sixth-order admits three weak solutions.

Keywords Sixth-order Sturm–Liouville equation · Multiplicity of solutions · Critical point

Mathematics Subject Classification 35J35 · 47J10 · 58E05

1 Introduction

The Sturm–Liouville equations of the $2m$th order are as follows:

$$(-1)^m\left(p_m(x)u^{(m)}\right)^{(m)} + (-1)^{m-1}\left(p_{m-1}(x)u^{(m-1)}\right)^{(m-1)}$$
$$+ \ldots + \left(p_2(x)u''\right)'' - \left(p_1(x)u'\right)' + \left(p_0(x)u\right) = \lambda f(x,u), \quad a < x < b,$$

where u satisfies in $2m$ boundary conditions at the end points a and b. Usually, the functions $p_k \in L^\infty(a,b)$, $(0 \le k \le m)$, and f is a Carathéodory function.

These problems play a crucial role in applied mathematics, nonlinear physics and engineering. They appear in the modeling and studying of many phenomena such as quantum and classical mechanics, vibrating rods and beams, hydrodynamic and magnetic hydrodynamic, a variety of fluid mechanics, etc.

In the past two decades, many researchers have studied these equations of different orders and with numerical methods such as Homotopy perturbation method [1], Lie-Group methods [16], Chebyshev method [8], Chebyshev differential matrices [20], variational iteration method [19], Matrix methods [17], shooting method [12] and Adomian method [14].

Authors have also paid attention to nonlinear analysis methods in studying Sturm–Liouville equations. In the last few years, many mathematicians, using topological degree

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theory and variational methods, have investigated Sturm–Liouville boundary value problems [2, 4, 9–11, 15, 18, 21, 22, 24, 25].

In [4, 9], the authors, employing variational method and critical point theorems, proved the existence of weak solutions for the following problem:

$$\Big(p_0(x)u''(x)\Big)'' - \Big(q_0(x)u'(x)\Big)' + \Big(r_0(x)u(x)\Big) = \lambda f_0(x, u(x)), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0 = u''(0) = u''(1),$$

where $p_0,\ q_0,\ r_0 \in L^\infty([0, 1])$ and f_0 is a Carathéodory function.

Also, in [10], the authors showed the existence of at least one non-trivial solution for the following system:

$$\Big(p_i(x)u_i''(x)\Big)'' - \Big(q_i(x)u_i'(x)\Big)' + \Big(r_i(x)u_i(x)\Big) = \lambda F_{u_i}(x, u_1, \cdots, u_n), \quad 0 < x < 1,$$
$$u_i(0) = u_i(1) = 0 = u_i''(0) = u_i''(1),$$

for $1 \le i \le n$, where $p_i,\ q_i,\ r_i \in L^\infty([0, 1])$ $(1 \le i \le n)$ and $F : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(\cdot, t_1, \cdots, t_n)$ is measurable in $[0, 1]$ for all $(t_1, \cdots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \cdots, \cdot)$ is a C^1 in $\mathbb{R}^n$ for every $x \in [0, 1]$.

In the past, Sturm–Liouville equations of sixth-order have been less examined. The purpose of this paper is to consider the following sixth-order Sturm–Liouville problem:

$$\begin{cases} -\Big(p(x)u'''(x)\Big)''' + \Big(q(x)u''(x)\Big)'' - \Big(r(x)u'(x)\Big)' + s(x)u(x) \\ \qquad\qquad = \lambda f(x, u(x)), \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1.1}$$

where $p,\ q,\ r,\ s \in L^\infty([0, 1])$ with $p^- := \operatorname{ess\,inf}_{t\in[0,1]} p(t) > 0$, λ is a positive parameter and f is a Carathéodory function. In other words, we wish to guarantee the existence of three weak solutions to the problem (1.1).

It is worth mentioning that sixth-order equations arise in studies on circular structures and appear in the literature [6, 7, 13].

The organization of the rest of the paper is as follows. Section 2 describes the basic notations and auxiliary results. In the last section, we present our main results.

2 Preliminaries and auxiliary results

First, we here recall three critical point theorems which are our main tools to prove the results. In two of these theorems, the coercivity of the functional $\Phi - \lambda\Psi$ is assumed and in the third one, a suitable sign hypothesis is considered.

Theorem 2.1 *([3, Theorem 3.1]) Assume that X be a reflexive and separable real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous and non-negative continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let there exists $x_0 \in X$ such that*

$$\Phi(x_0) = \Psi(x_0) = 0$$

and

$$\lim_{\|x\|\to+\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty$$

for all $\lambda \in [0, +\infty)$. *Suppose that there exist* $r > 0$ *and* $\bar{x} \in X$, *with* $r < \Phi(\bar{x})$ *such that*

$$\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x) < \frac{r}{r + \Phi(\bar{x})} \Psi(\bar{x}),$$

where $\overline{\Phi^{-1}(]-\infty, r[)}^w$ *denotes the closure of* $\Phi^{-1}(]-\infty, r[)$ *in the weak topology. Then, for each*

$$\lambda \in \Lambda_1 := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x}) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x)} \right[,$$

the functional $\Phi - \lambda\Psi$ *has at least three critical points in* X. *In addition, for each* $h > 1$, *there exist an open interval*

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r \frac{\Psi(\bar{x})}{\Phi(\bar{x})} - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x)} \right]$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2$, *the functional* $\Phi - \lambda\Psi$ *has at least three critical points in* X *whose norms are less than* σ.

Theorem 2.2 *([5, Theorem 3.2]) Assume that* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Let there exists a constant $r > 0$ *and* $\bar{x} \in X$, *with* $\Phi(\bar{x}) > 2r$, *such that*

(a1) $\frac{\sup_{x \in \Phi^{-1}(]-\infty, r[)} \Psi(x)}{r} < \frac{2}{3} \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a2) for each $\lambda \in \left] \frac{3}{2} \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{x \in \Phi^{-1}(]-\infty, r[)} \Psi(x)} \right[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \left] \frac{3}{2} \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{x \in \Phi^{-1}(]-\infty, r[)} \Psi(x)} \right[$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points.*

Theorem 2.3 *([5, Theorem 3.1]) Suppose that* X *be a reflexive real Banach space;* $\Phi : X \to \mathbb{R}$ *be a coercive, convex and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist two positive constants $r_1, r_2 > 0$ *and* $\bar{x} \in X$, *with* $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$, *such that*

(b1) $\frac{\sup_{x \in \Phi^{-1}(]-\infty, r_1[)} \Psi(x)}{r_1} < \frac{2}{3} \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(b2) $\frac{\sup_{x \in \Phi^{-1}(]-\infty, r_2[)} \Psi(x)}{r_2} < \frac{1}{3} \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(b3) for each λ *in*

$$\Lambda^* := \left] \frac{3}{2} \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min \left\{ \frac{r_1}{\sup_{x \in \Phi^{-1}(]-\infty, r_1[)} \Psi(x)}, \frac{r_2}{2 \sup_{x \in \Phi^{-1}(]-\infty, r_2[)} \Psi(x)} \right\} \right[$$

and for every $x_1, x_2 \in X$, *which are local minima for the functional* $\Phi - \lambda\Psi$, *and such that* $\Psi(x_1) \geq 0$ *and* $\Psi(x_2) \geq 0$, *one has* $\inf_{t \in [0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda^*$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}]-\infty, r_2[$.

Let us introduce some notations. Hereafter, let

$$X := \left\{u \in H^3(0,1) \cap H_0^1(0,1) : \; u''(0) = u''(1) = 0\right\}$$

be endowed with the inner product

$$(u, v) = \int_0^1 (u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))dx \quad \forall u, v \in X,$$

which induces the norm

$$\begin{aligned}\|u\|_X &= \left(\int_0^1 \left(|u'''(x)|^2 + |u''(x)|^2 + |u'(x)|^2 + |u(x)|^2\right)dt\right)^{1/2} \\ &= (\|u'''\|_2^2 + \|u''\|_2^2 + \|u'\|_2^2 + \|u\|_2^2)^{1/2} \quad \forall u \in X,\end{aligned}$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(0,1)$.

Since X is a closed subspace of $H^3(0,1)$, $(X, \|u\|_X)$ is a Banach space.

Now, we recall the following useful Poincaré type inequalities (see [6]):

$$\|u^{(i)}\|_2^2 \le \pi^{-2(j-i)}\|u^{(j)}\|_2^2, \quad i = 0, 1, 2, \; j = 1, 2, 3, \; i < j \tag{2.1}$$

for all $u \in X$.

Take (2.1) into account, by adopting the appropriate conditions on the functions p, q, r, s, one has, the following norm

$$\|u\| := \left(\int_0^1 \left(p(x)|u'''(x)|^2 + q(x)|u''(x)|^2 + r(x)|u'(x)|^2 + s(x)|u(x)|^2\right)dx\right)^{1/2}$$

is equivalent to $\|\cdot\|_X$, that still makes X a Hilbert space.

Now, consider the following set of conditions:

(C1) $q^- \ge 0, r^- \ge 0, s^- \ge 0$,
(C2) $q^- \ge 0, r^- \ge 0, s^- < 0$ and $-\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} - \frac{s^-}{\pi^6} < p^-$,
(C3) $q^- \ge 0, r^- < 0, s^- \ge 0$ and $-\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} < p^-$,
(C4) $q^- \ge 0, r^- < 0, s^- < 0$ and $-\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} - \frac{s^-}{\pi^6} < p^-$,
(C5) $q^- < 0, r^- \ge 0, s^- \ge 0$ and $-\frac{q^-}{\pi^2} < p^-$,
(C6) $q^- < 0, r^- \ge 0, s^- < 0$ and $\max\left\{-\frac{q^-}{\pi^2}, -\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} - \frac{s^-}{\pi^6}\right\} < p^-$,
(C7) $q^- < 0, r^- < 0, s^- \ge 0$ and $-\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} < p^-$,
(C8) $q^- < 0, r^- < 0, s^- < 0$ and $-\frac{q^-}{\pi^2} - \frac{r^-}{\pi^4} - \frac{s^-}{\pi^6} < p^-$,

where

$$p^- := \operatorname{ess\,inf}_{x\in[0,1]} p(x), \qquad q^- := \operatorname{ess\,inf}_{x\in[0,1]} q(x),$$
$$r^- := \operatorname{ess\,inf}_{x\in[0,1]} r(x), \qquad s^- := \operatorname{ess\,inf}_{x\in[0,1]} s(x).$$

We can state following proposition.

Proposition 2.4 *Let* $p^- > 0$. *The condition*

(C) $\max\left\{-\frac{q^-}{\pi^2}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}-\frac{s^-}{\pi^6}\right\} < p^-$

holds if and only if one of conditions (C1)-(C8) *holds.*

Proof Suppose that one of conditions (C1)-(C8) holds. For example, we prove the following three cases.
Let (C1) holds. Then, we have

$$\max\left\{-\frac{q^-}{\pi^2}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}-\frac{s^-}{\pi^6}\right\} = -\frac{q^-}{\pi^2} < 0 < p^-.$$

Also, let (C3) holds. One has,

$$\max\left\{-\frac{q^-}{\pi^2}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}-\frac{s^-}{\pi^6}\right\} = -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4} < p^-.$$

If (C8) holds, we obtain

$$\max\left\{-\frac{q^-}{\pi^2}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}, -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}-\frac{s^-}{\pi^6}\right\} = -\frac{q^-}{\pi^2}-\frac{r^-}{\pi^4}-\frac{s^-}{\pi^6} < p^-.$$

By the same reasoning as above, readers can prove other cases. So, condition (C) is true. On the contrary, assume (C). Clearly, according to the signs of q^-, r^-, s^-, one of conditions (C1)-(C8) is immediately verified. □

In addition, setting

$$\delta = \begin{cases} p^- & \text{if (C1) holds,} \\ \min\{p^-, p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4} + \frac{s^-}{\pi^6}\} & \text{if (C2) or (C4) holds,} \\ \min\{p^-, p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}\} & \text{if (C3) holds,} \\ p^- + \frac{q^-}{\pi^2} & \text{if (C5) holds,} \\ \min\{p^- + \frac{q^-}{\pi^2}, p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}\} & \text{if (C6) holds,} \\ p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4} & \text{if (C7) holds,} \\ p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4} + \frac{s^-}{\pi^6} & \text{if (C8) holds,} \end{cases} \tag{2.2}$$

we point out the following proposition.

Proposition 2.5 *Let* $p^- > 0$ *and condition* (C) *holds. Then, for every* $u \in X$,

$$\|u\|^2 \geq \delta \|u'''\|_2^2. \tag{2.3}$$

Proof Assume that (C1) holds. One has $\|u\|^2 \geq p^-\|u'''\|_2^2$ and (2.3) holds with $\delta = p^-$. Suppose that (C2) holds. In view of (2.1) one has

$$\|u\|^2 \geq p^-\|u'''\|_2^2 + q^-\|u''\|_2^2 + (r^- + \frac{s^-}{\pi^2})\|u'\|_2^2.$$

So, if $r^- + \frac{s^-}{\pi^2} \geq 0$, we conclude that the (2.3) holds with $\delta = p^-$. If $r^- + \frac{s^-}{\pi^2} < 0$, again from (2.1)

$$\|u\|^2 \geq p^-\|u'''\|_2^2 + (q^- + \frac{r^-}{\pi^2} + \frac{s^-}{\pi^4})\|u''\|_2^2.$$

Hence, if $q^- + \frac{r^-}{\pi^2} + \frac{s^-}{\pi^4} \geq 0$, we conclude that the (2.3) holds with $\delta = p^-$. If $q^- + \frac{r^-}{\pi^2} + \frac{s^-}{\pi^4} < 0$, again from (2.1)

$$\|u\|^2 \geq (p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4} + \frac{s^-}{\pi^6})\|u'''\|_2^2.$$

Therefore, always (2.3) holds with $\delta = \min\{p^-, p^- + \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4} + \frac{s^-}{\pi^6}\}$. Exploiting similar arguments shown above, readers can prove other cases. □

Proposition 2.6 *Let $p^- > 0$ and condition* (C) *holds. Then, for every $u \in X$,*

$$\|u\|_\infty \leq \frac{1}{2\pi^2\sqrt{\delta}}\|u\|. \tag{2.4}$$

Proof Since $H_0^1(0,1) \hookrightarrow C^0(0,1)$ and $\|u\|_\infty < \frac{1}{2}\|u'\|_2$, take (2.1) into account, one obtains

$$\|u\|_\infty \leq \frac{1}{2\pi^2}\|u'''\|_2.$$

So, the conclusion follows from (2.3). □

Remark 2.7 It is simple to observe that there exists $M > 0$ such that $\|u\|^2 \leq M\|u\|_X^2$. Hence, in view of (2.3), we observe that $\|\cdot\|$ defines a norm on X equivalent to $\|\cdot\|_X$.

Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory functions. We recall that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \longmapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$;
(b) the mapping $t \longmapsto f(x,t)$ is continuous for almost every $x \in [0,1]$;
(c) for every $\rho > 0$, the function $l_\rho(x) := \sup_{|t|\leq\rho}|f(x,t)| \in L^1((0,1))$.

Corresponding to the function f, we introduce the function $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi,$$

for all $(x,t) \in [0,1]\times\mathbb{R}$.
In order to study (1.1), we consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^1 F(x,u(x))\,dx$$

for every $u \in X$.
With standard arguments, we obtain that $\Phi, \Psi \in C^1(X,\mathbb{R})$ and

$$\Phi'(u)(v) = \int_0^1 \big(p(x)u'''(x)v'''(x) + q(x)u''(x)v''(x) + r(x)u'(x)v'(x) + s(x)u(x)v(x)\big)\,dx,$$

$$\Psi'(u)(v) := \int_0^1 f(x,u(x))v(x)\,dx$$

for any $v \in X$. The following proposition will be useful in the proof of the main results.

Proposition 2.8 *Let $T : X \to X^*$ be the operator defined by*

$$T(u)(v) = \int_0^1 \big(p(x)u'''(x)v'''(x) + q(x)u''(x)v''(x) + r(x)u'(x)v'(x) + s(x)u(x)v(x)\big)\,dx,$$

for every $u, v \in X$. Then, T admits a continuous inverse on X^*.

Proof First, for every $u \in X \setminus \{0\}$, one has

$$\lim_{\|u\|\to+\infty} \frac{T(u)(u)}{\|u\|} = \lim_{\|u\|\to+\infty} \frac{\|u\|^2}{\|u\|} = +\infty,$$

which shows T is coercive. Furthermore, for all $u, v \in X$, we have

$$(T(u) - T(v))(u - v) \geq \|u - v\|^2, \tag{2.5}$$

so, T is uniformly monotone. Theorem 26.A(d) of [23] ensures the existence of the inverse operator T^{-1} of T. To prove the continuity of the operator T^{-1} on X^*, Choose $g_1, g_2 \in X^*$. By (2.5), we deduce

$$\|T^{-1}(g_1) - T^{-1}(g_2)\| \leq \|g_1 - g_2\|.$$

Thus, T^{-1} is continuous. This completes the proof. □

Finally, we say that a function $u \in X$ is a *weak solution* of (1.1) if

$$\int_0^1 \big(p(x)u'''(x)v'''(x) + q(x)u''(x)v''(x) + r(x)u'(x)v'(x) + s(x)u(x)v(x)\big)\,dx$$
$$-\lambda \int_0^1 f(x, u(x))v(x)\,dx = 0$$

holds for all $v \in X$.

3 Main results

In this section, the main results are formulated. Our first existence result is the following.

Theorem 3.1 *Let there exist a function $w \in X$ and a positive constant r such that*

(A1) $\|w\|^2 > 2r$,

(A2) $\displaystyle\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx < r\frac{\int_0^1 F(x,w(x))dx}{r+\frac{\|w\|^2}{2}}$,

(A3) $\frac{2}{\delta\pi^6}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \Theta_1$ *uniformly with respect to $x \in [0,1]$ where*

$$\Theta_1 := \max\left\{\frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}{r},\right.$$
$$\left.\frac{\frac{2r}{\|w\|^2}\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}{hr}\right\}$$

with $h > 1$.

Then, for each $\lambda \in \Lambda_1$, where

$$\Lambda_1 := \left]\frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx},\right.$$

$$\frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\Bigg[,$$

the problem (1.1) admits at least three weak solutions in X. In addition, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq \Bigg]0, \frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\Bigg[,$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the problem (1.1) admits at least three weak solutions in X whose norms are less than σ.

Proof With the purpose of using Theorem 2.1, we consider X, Φ and Ψ as in the previous section. Owing to Proposition 2.8, Φ' admits a continuous inverse on X^*. It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, defined by

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)dx$$

for every $v \in X$, and that $\Psi' : X \to X^*$ is a compact and continuous operator. According to (A3), there exist two constant $\gamma, \tau \in \mathbb{R}$ with $\gamma < \Theta_1$ such that

$$\frac{2}{\delta\pi^6}F(x,t) \le \gamma t^2 + \tau$$

for all $x \in (0, 1)$ and all $t \in \mathbb{R}$. Fix $u \in X$. Then

$$F(x,u(x)) \le \frac{\delta\pi^6}{2}(\gamma|u(x)|^2 + \tau) \tag{3.1}$$

for all $x \in (0, 1)$.
To prove the coercivity of the functional $\Phi - \lambda\Psi$, first, we suppose that $\gamma > 0$. Thus, for any fixed $\lambda \in]0, \frac{1}{\Theta_1}]$, since

$$\|u\|_2 \le \frac{1}{\pi^3}\|u'''\|_2 \le \frac{1}{\pi^3\sqrt{\delta}}\|u\|,$$

by (3.1), we obtain

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_0^1 F(x,u(x))dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\delta\pi^6}{2\Theta_1}(\gamma\int_0^1 |u(x)|^2dx + \tau) \\ &\ge \frac{1}{2}(1 - \frac{\gamma}{\Theta_1})\|u\|^2 - \frac{\delta\pi^6}{2\Theta_1}\tau,\end{aligned}$$

from which it yilds

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

On the other hand, if $\gamma \le 0$, clearly, we have

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

Hence, in both cases the functional $\Phi - \lambda\Psi$ is coercive. Also, according to (A1), we achieve $\Phi(w) > r$. In view of $\|u\|_\infty \leq \frac{1}{2\pi^2\sqrt{\delta}}\|u\|$ for each $u \in X$ and from the definition of Φ, we get

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{u \in X;\ \ \Phi(u) < r\}\\ &\subseteq \{u \in X;\ \ \|u\| < \sqrt{2r}\}\\ &\subseteq \{u \in X;\ \ |u(x)| < \frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\} \text{ for all } x \in [0,1],\end{aligned}$$

consequently,

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) \leq \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx.$$

So, from (A2), we obtain

$$\begin{aligned}\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u) &\leq \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx\\ &< \frac{r}{r+\Phi(w)}\Psi(w).\end{aligned}$$

Now, we can use Theorem 2.1. Note for each $x \in [0,1]$,

$$\begin{aligned}&\frac{\Phi(w)}{\Psi(w) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)}\\ &\leq \frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx},\end{aligned}$$

and

$$\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)} \geq \frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}.$$

By (A2), we have

$$\begin{aligned}&\frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\\ &< \frac{\frac{1}{2}\|w\|^2}{\left(\frac{r+\frac{1}{2}\|w\|^2}{r} - 1\right)\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\\ &= \frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}.\end{aligned}$$

Also,

$$\frac{hr}{r\frac{\Phi(w)}{\Psi(w)} - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi(u)}$$

$$\leq \frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx} = \rho.$$

Take (A2) into account, one has

$$\begin{aligned} &2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx \\ &> \left(\frac{2r}{\|w\|^2} - \frac{r}{r+\frac{1}{2}\|w\|^2}\right)\int_0^1 F(x,w(x))dx \\ &\geq \left(\frac{2r}{\|w\|^2} - \frac{2r}{\|w\|^2}\right)\int_0^1 F(x,w(x))dx = 0, \end{aligned}$$

since $\int_0^1 F(x,w(x))dx \geq 0$. Owing to Theorem 2.1 with $x_0 = 0$, $\bar{x}_1 = w$, it follows that, for each $\lambda \in \Lambda_1$, problem (1.1) admits at least three weak solutions and there exist an open interval $\Lambda_2 \subset [0,\rho]$ and real positive number σ such that, for each $\lambda \in \Lambda_2$, problem (1.1) admits at least three weak solutions whose norms in X are less than σ. This completes the proof. □

Here, we present our second existence result.

Theorem 3.2 *Let there exist a positive constant r and a function $w \in X$ such that*

(B1) $\|w\|^2 > 4r$,

(B2) $\displaystyle\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx < \frac{4r}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}$,

(B3) $\displaystyle\frac{2}{\delta\pi^6}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}{r}$.

Then, for each

$$\lambda \in \left]\frac{\frac{3}{4}\|w\|^2}{\int_0^1 F(x,w(x))dx}, \frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\right[,$$

the problem (1.1) admits at least three weak solutions in X.

Proof We will apply Theorem 2.2 to problem (1.1). Take the functionals $\Phi, \Psi : X \to \mathbb{R}$ as given in the previous section. By (B1), we obtain $\Phi(w) > 2r$. Bearing in mind $\|u\|_\infty \leq \frac{1}{2\pi^2\sqrt{\delta}}\|u\|$ for each $u \in X$, from the definition of Φ, we get

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{u \in X;\ \ \Phi(u) < r\}\\ &\subseteq \{u \in X;\ \ \|u\| < \sqrt{2r}\}\\ &\subseteq \{u \in X;\ \ |u(x)| < \frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\}\ \text{ for all } x \in [0,1],\end{aligned}$$

consequently,

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_0^1 F(x,u(x))dx\\ &\le \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx.\end{aligned}$$

Hence, from (B2), we deduce

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_0^1 F(x,u(x))dx}{r}\\ &\le \frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}{r}\\ &< \frac{4}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} = \frac{2\Psi(w)}{3\Phi(w)}.\end{aligned}$$

Moreover, according to (B3) there exist two constants $\eta, \vartheta \in \mathbb{R}$ with

$$\eta < \frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}{r}$$

such that

$$\frac{2}{\delta\pi^6}F(x,t) \le \eta t^2 + \vartheta$$

for all $x \in [0,1]$ and all $t \in \mathbb{R}$. Fix $u \in X$. Then

$$F(x,u(x)) \le \frac{\delta\pi^6}{2}(\eta|u(x)|^2 + \vartheta) \tag{3.2}$$

for all $x \in [0,1]$.
To prove the coercivity of the functional $\Phi - \lambda\Psi$, first, we suppose that $\eta > 0$. So, for any fixed

$$\lambda \in \left]\frac{\frac{3}{4}\|w\|^2}{\int_0^1 F(x,w(x))dx}, \frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx}\right[,$$

since

$$\|u\|_2 \le \frac{1}{\pi^3}\|u'''\|_2 \le \frac{1}{\pi^3\sqrt{\delta}}\|u\|,$$

exploiting (3.2), we get

$$\begin{aligned}\Phi(u)-\lambda\Psi(u)&=\frac{1}{2}\|u\|^2-\lambda\int_0^1 F(x,u(x))dx\\&\geq\frac{1}{2}\|u\|^2-\frac{\lambda\delta\pi^6}{2}(\eta\int_0^1|u(x)|^2dx+\vartheta)\\&\geq\frac{1}{2}(1-\eta\frac{r}{\int_0^1\sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]}F(x,t)dx})\|u\|^2-\frac{\lambda\delta\pi^6}{2}\vartheta,\end{aligned}$$

and therefore,

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty.$$

Also, if $\eta\leq 0$, clearly, we have

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty.$$

So, in both cases the functional $\Phi-\lambda\Psi$ is coercive. Hence, all the assumptions of Theorem 2.2 are verified and the conclusion follows. □

Now, we exhibit our third existence result.

Theorem 3.3 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the condition $f(x,t)\geq 0$ for all $x\in[0,1]$ and $t\geq 0$. Suppose that there exist a function $w\in X$ and two positive constants r_1 and r_2 with $4r_1<\|w\|^2<r_2$ such that*

(H1)

$$\int_0^1\sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}}\right]}F(x,t)dx<\frac{4r_1}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2},$$

(H2)

$$\int_0^1\sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}}\right]}F(x,t)dx<\frac{2r_2}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}.$$

Then, for each

$$\lambda\in\left]\frac{\frac{3}{4}\|w\|^2}{\int_0^1 F(x,w(x))dx},\Theta_2\right[,$$

where

$$\Theta_2:=\min\left\{\frac{r_1}{\int_0^1\sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}}\right]}F(x,t)dx},\right.$$
$$\left.\frac{\frac{r_2}{2}}{\int_0^1\sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}}\right]}F(x,t)dx}\right\},$$

the problem (1.1) admits at least three non-negative weak solutions v^1, v^2, v^3 *such that*

$$|v^j(x)| < \frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}}$$

for each $x \in [0, 1], j = 1, 2, 3.$

Proof We want to apply Theorem 2.3. Let us check the functional $\Phi - \lambda\Psi$ satisfies the assumption (b3) of Theorem 2.3. Assume that u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1.1). Since $f(x, t) \geq 0$ for all $(x, t) \in [0, 1] \times (\mathbb{R}^+ \cup \{0\})$, from the Weak Maximum Principle [23], we get $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for all $x \in [0, 1]$. Hence, one has $f(su_1 + (1 - s)u_2) \geq 0$ and consequently, $\Psi(su_1 + (1 - s)u_2) \geq 0$ for all $s \in [0, 1]$. Furthermore, owing to $4r_1 < \|w\|^2 < r_2$, we have $2r_1 < \Phi(w) < \frac{r_2}{2}$. Thanks to assumption (H1), it follows

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int_0^1 F(x,u(x))dx}{r_1}\\ &\frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_1}{2}}\right]} F(x,t)dx}{r}\\ &\frac{4}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} = \frac{2\Psi(w)}{3\Phi(w)}.\end{aligned}$$

As above, recalling (H2), we achieve

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\int_0^1 F(x,u(x))dx}{r_2}\\ &\frac{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r_2}{2}}\right]} F(x,t)dx}{r_2}\\ &\frac{2}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} = \frac{2\Psi(w)}{3\Phi(w)}.\end{aligned}$$

Hence, all hypotheses of Theorem 2.3 are satisfied. So, problem (1.1) admits at least three distinct weak solutions in X. This concludes the proof. □

In sequel, we present the Corollaries 3.4–3.6 which are special cases of Theorems 3.1–3.3, respectively, for a fixed test function w.
Put

$$k := \|p\|_\infty + \frac{1}{\pi^2}\|q\|_\infty + \frac{1}{\pi^4}\|r\|_\infty + \frac{1}{\pi^6}\|s\|_\infty. \tag{3.3}$$

It is easy to see that $k > 0$ and $\delta < k$.

Corollary 3.4 *Assume that there exist two positive constants c and d with* $c < \frac{25\sqrt{15}}{2\pi^2}d$ *such that*

(A4) $F(x, t) \geq 0$ *for a.e.* $x \in [0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$, *and* $t \in [0, d]$,

(A5)

$$\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx < (\pi^2 c\sqrt{\delta})^2 \frac{\int_{\frac{2}{5}}^{\frac{3}{5}} F(x,d)dx}{(\pi^2 c\sqrt{\delta})^2 + \frac{9375}{4}kd^2},$$

(A6) $\frac{2}{\delta\pi^6} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \Theta_3$ *uniformly with respect to* $x \in [0,1]$ *where*

$$\Theta_3 := \max\left\{ \frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\pi^2 c\sqrt{\delta})^2}, \frac{(\pi^2 c\sqrt{\delta})^2 \frac{\int_{\frac{2}{5}}^{\frac{3}{5}} F(x,d)dx}{\frac{9375}{4}kd^2} - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2h(\pi^2 c\sqrt{\delta})^2} \right\}$$

with $h > 1$.

Then, for each $\lambda \in \Lambda_1'$, *where*

$$\Lambda_1' := \left] \frac{\frac{9375}{2}kd^2}{\int_{\frac{2}{5}}^{\frac{3}{5}} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}, \frac{2(\pi^2 c\sqrt{\delta})^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \right[,$$

the problem (1.1) admits at least three weak solutions in X. In addition, for each $h > 1$, *there exist an open interval*

$$\Lambda_2' \subseteq \left]0, \frac{2h(\pi^2 c\sqrt{\delta})^2}{(\pi^2 c\sqrt{\delta})^2 \frac{\int_{\frac{2}{5}}^{\frac{3}{5}} F(x,d)dx}{\frac{9375}{4}kd^2} - \int_0^1 \sup_{t\in\left[-c,c\right]} F(x,t)dx}\right[,$$

and a positive real constant σ *such that, for each* $\lambda \in \Lambda_2'$, *the problem (1.1) admits at least three weak solutions in X whose norms are less than* σ.

Proof We claim that all the hypotheses of Theorem 3.1 are fulfilled with $r = 2(\sqrt{\delta}\pi^2 c)^2$ and

$$w(x) = \begin{cases} \varrho(x)d, & x \in [0, \frac{2}{5}[, \\ d, & x \in [\frac{2}{5}, \frac{3}{5}], \\ \varrho(1-x)d, & x \in]\frac{3}{5}, 1]. \end{cases} \tag{3.4}$$

where $\varrho(x) = \left(\frac{5}{2}\right)^4 x^4 - 2\left(\frac{5}{2}\right)^3 x^3 + 5x$ for all $x \in [0, 2/5[$. A simple computation shows that $w \in X$, and in particular,

$$9375\,\delta d^2 \le \|w\|^2 \le 9375\,kd^2.$$

So, bearing $c < \frac{25\sqrt{15}}{2\pi^2}d$ in mind, we get $\|w\|^2 > 2r$. Since, $0 \le w(x) \le d$ for each $x \in [0,1]$ the assumption (A4) assures that

$$\int_0^{2/5} F(x, w(x))dx + \int_{3/5}^1 F(x, w(x))dx \ge 0,$$

thanks to (A5), we deduce

$$\begin{aligned}\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx &\leq (\sqrt{\delta}\pi^2 c)^2 \frac{\int_{2/5}^{3/5} F(x,d)dx}{(\sqrt{\delta}\pi^2 c)^2 + \frac{9375}{4}kd^2} \\ &= 2(\sqrt{\delta}\pi^2 c)^2 \frac{\int_{2/5}^{3/5} F(x,d)dx}{2(\sqrt{\delta}\pi^2 c)^2 + \frac{9375}{2}kd^2} \\ &\leq 2(\delta\pi^2 c)^2 \frac{\int_0^1 F(x,d)dx}{2(\sqrt{\delta}\pi^2 c)^2 + \frac{9375}{2}kd^2} \\ &\leq r\frac{\int_0^1 F(x,w(x))dx}{r+\frac{1}{2}\|w\|^2},\end{aligned}$$

thus, (A2) holds (note $c^2 = \frac{r}{2\delta\pi^4}$). Next, notice that

$$\begin{aligned}&\frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx} \\ &\leq \frac{\frac{9375}{2}kd^2}{\left(\int_{2/5}^{3/5} F(x,d)dx\right) - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\end{aligned}$$

and

$$\frac{r}{\int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx} = \frac{2(\sqrt{\delta}\pi^2 c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.$$

Additionally,

$$\begin{aligned}&\frac{\frac{9375}{2}kd^2}{\int_{2/5}^{3/5} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \\ &< \frac{\frac{9375}{2}kd^2}{\left(\frac{2(\sqrt{\delta}\pi^2 c)^2 + \frac{9375}{2}kd^2}{2(\sqrt{\delta}\pi^2 c)^2} - 1\right)\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \\ &= \frac{2(\sqrt{\delta}\pi^2 c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.\end{aligned}$$

Finally, note that

$$\begin{aligned}&\frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in\left[-\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}},\frac{1}{\pi^2\sqrt{\delta}}\sqrt{\frac{r}{2}}\right]} F(x,t)dx} \\ &\leq \frac{2(\sqrt{\delta}\pi^2 c)^2 h}{2(\sqrt{\delta}\pi^2 c)^2 \frac{\int_{2/5}^{3/5} F(x,d)dx}{\frac{9375}{2}kd^2} - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx},\end{aligned}$$

and owing to $\Lambda_1' \subseteq \Lambda_1$ and $\Lambda_2' \subseteq \Lambda_2$, our conclusion follows by Theorem 3.1. □

Corollary 3.5 *Suppose that there exist two positive constants c and d with* $c < \frac{25\sqrt{15}}{2\pi^2\sqrt{2}}d$ *such that the assumption (A4) in Corollary 3.4 holds. Assume further that*

(B4) $\frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c)^2} < \frac{4}{3}\frac{\int_{2/5}^{3/5} F(x,d)dx}{9375kd^2}$,

(B5) $\frac{2}{\delta\pi^6}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c)^2}$.

Then, for each

$$\lambda \in \left]\frac{3}{4}\frac{9375kd^2}{\int_{2/5}^{3/5} F(x,d)dx}, \frac{2(\sqrt{\delta}\pi^2 c)^2}{\int_0^1 \sup_{t\in\left[-c,c\right]} F(x,t)dx}\right[,$$

the problem (1.1) admits at least three weak solutions.

Proof All the hypotheses of Theorem 3.2 are fulfilled by choosing w as given in (3.4) and $2(\sqrt{\delta}\pi^2 c)^2$. So, in light of Theorem 3.2, we achieve the conclusion. □

Corollary 3.6 *Suppose that* $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies the condition* $f(x,t)\geq 0$ *for all* $x\in[0,1]$ *and* $t\geq 0$. *Also, let there exist three positive constants* c_1, c_2 *and d with* $c_1 < \frac{25\sqrt{15}}{2\sqrt{2}\pi^2}d$ *and* $\frac{25\sqrt{15}}{\pi^2\sqrt{2}}\sqrt{\frac{k}{\delta}}d < c_2$ *such that*

(H3)

$$\frac{\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c_1)^2} < \frac{4}{3}\frac{\int_{2/5}^{3/5} F(x,d)dx}{9375kd^2},$$

(H4)

$$\frac{\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c_2)^2} < \frac{2}{3}\frac{\int_{2/5}^{3/5} F(x,d)dx}{9375kd^2}.$$

Then, for each

$$\lambda \in \left]\frac{\frac{3}{4}9375kd^2}{\int_{2/5}^{3/5} F(x,d)dx}, \Theta_4\right[,$$

where

$$\Theta_4 := \min\left\{\frac{2(\sqrt{\delta}\pi^2 c_1)^2}{\int_0^1 \sup_{t\in\left[-c_1,c_1\right]} F(x,t)dx}, \frac{(\sqrt{\delta}\pi^2 c_2)^2}{\int_0^1 \sup_{t\in\left[-c_2,c_2\right]} F(x,t)dx}\right\},$$

problem (1.1) admits at least three non-negative weak solutions v^1, v^2, v^3 *such that*

$$|v^j(x)| < c_2$$

for each $x\in[0,1]$, $j=1,2,3$.

Proof We conclude the stated assertion by exploiting Theorem 3.3 with w as defined in (3.4), $r_1 = 2(\sqrt{\delta}\pi^2 c_1)^2$ and $r_2 = 2(\sqrt{\delta}\pi^2 c_2)^2$. □

Now, we exhibit the following corollaries which are particular cases of Corollaries 3.4–3.6 in the autonomous case. Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and set $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$.

Corollary 3.7 *Suppose that there exist two positive constants c and d with $c < \frac{25\sqrt{15}}{2\pi^2} d$ such that*

(A7) $f(t) \geq 0$ *for all* $t \in [0, d]$

(A8) $\max\limits_{t\in[-c,c]} F(t) < (\pi^2 c\sqrt{\delta})^2 \dfrac{\frac{1}{5}F(d)}{(\pi^2 c\sqrt{\delta})^2 + \frac{9375}{4}kd^2}$,

(A9) $\frac{2}{\delta\pi^6} \limsup_{|t|\to+\infty} \dfrac{F(t)}{t^2} < \Theta_5$

where

$$\Theta_5 := \max\left\{\frac{\max_{t\in[-c,c]} F(t)}{2(\pi^2 c\sqrt{\delta})^2}, \frac{\frac{4}{3}25^{-3}(\pi^2 c\sqrt{\delta})^2 \frac{F(d)}{kd^2} - \max_{t\in[-c,c]} F(t)}{2h(\pi^2 c\sqrt{\delta})^2}\right\}$$

with $h > 1$.

Then, for each $\lambda \in \Lambda_1''$, where

$$\Lambda_1'' := \left]\frac{\frac{9375}{2}kd^2}{\frac{1}{5}F(d) - \max_{t\in[-c,c]} F(t)}, \frac{2(\pi^2 c\sqrt{\delta})^2}{\max_{t\in[-c,c]} F(t)}\right[,$$

the problem

$$\begin{cases} -\Big(p(t)u'''(t)\Big)''' + \Big(q(t)u''(t)\Big)'' - \Big(r(t)u'(t)\Big)' + s(t)u(t) = \lambda f(u), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.5}$$

admits at least three weak solutions in X. In addition, for each $h > 1$, there exist an open interval

$$\Lambda_2'' \subseteq \left]0, \frac{2h(\pi^2 c\sqrt{\delta})^2}{\frac{4}{3}25^{-3}(\pi^2 c\sqrt{\delta})^2 \frac{F(d)}{kd^2} - \max_{t\in\left[-c,c\right]} F(t)}\right[,$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2''$, the problem (3.5) admits at least three weak solutions in X whose norms are less than σ.

Corollary 3.8 *Let there exist two positive constants c and d with $c < \frac{25\sqrt{15}}{2\pi^2\sqrt{2}} d$ such that the assumption (A7) in Corollary 3.7 holds. Suppose further that*

(B6)

$$\frac{\max_{t\in[-c,c]} F(t)}{2(\sqrt{\delta}\pi^2 c)^2} < \left(\frac{2}{375}\right)^2 \frac{F(d)}{kd^2},$$

(B7)

$$\frac{2}{\delta\pi^6} \limsup_{|t|\to+\infty} \frac{F(t)}{t^2} < \frac{\max_{t\in[-c,c]} F(t)}{2(\sqrt{\delta}\pi^2 c)^2}.$$

Then, for each

$$\lambda \in \left]\left(\frac{375}{2}\right)^2 \frac{kd^2}{F(d)}, \frac{2(\sqrt{\delta}\pi^2 c)^2}{\max_{t\in[-c,c]} F(t)}\right[,$$

the problem (3.5) admits at least three weak solutions.

Corollary 3.9 *Let $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(t) \geq 0$ for all $t \geq 0$. Suppose that there exist three positive constants c_1, c_2 and d with $c_1 < \frac{25\sqrt{15}}{2\pi^2\sqrt{2}} d$ and $\frac{25\sqrt{15}}{\pi^2\sqrt{2}}\sqrt{\frac{k}{\delta}}\, d < c_2$ such that*

(H5) $\frac{\max_{t\in[-c_1,c_1]} F(t)}{2(\sqrt{\delta}\pi^2 c_1)^2} < \left(\frac{2}{375}\right)^2 \frac{F(d)}{kd^2}$,

(H6) $\frac{\max_{t\in[-c_2,c_2]} F(t)}{(\sqrt{\delta}\pi^2 c_2)^2} < \left(\frac{2}{375}\right)^2 \frac{F(d)}{kd^2}$.

Then, for each

$$\lambda \in \left]\left(\frac{375}{2}\right)^2 \frac{kd^2}{F(d)}, \Theta_6\right[,$$

where

$$\Theta_6 := \min\left\{\frac{2(\sqrt{\delta}\pi^2 c_1)^2}{\max_{t\in[-c_1,c_1]} F(t)}, \frac{(\sqrt{\delta}\pi^2 c_2)^2}{\max_{t\in[-c_2,c_2]} F(t)}\right\},$$

problem (3.5) admits at least three non-negative weak solutions $v^1,\ v^2,\ v^3$ such that

$$|v^j(x)| < c_2$$

for each $x \in [0,1]$, $j = 1, 2, 3$.

In support of our theoretical conclusions, we present an example that is entirely consistent with them.

Example 3.10 Assume that $p(x) = 4$, $q(x) = \frac{(\pi x)^2}{3}$, $r(x) = \frac{(\pi x)^4}{3}$ and $s(x) = \frac{(\pi x)^6}{3}$ for all $x \in [0,1]$. Clearly, we have $p^- = 4$, $q^- = r^- = s^- = 0$. In view of (2.2) and (3.3), one has $\delta = 4$ and $k = 5$. Additionally, define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $f(x,t) = xf^*(t)$ where

$$f^*(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 2112t - 2111 & \text{if } 1 < t \leq 2, \\ -2112t + 6337 & \text{if } 2 < t \leq 3, \\ 1 & \text{if } 3 < t \leq 70, \\ f^{**}(t) & \text{if } t > 70, \end{cases}$$

where $f^{**} :]70, +\infty[\to \mathbb{R}$ can be any arbitrary function. A simple calculation shows that

$$F^*(t) = \begin{cases} t & \text{if } t \leq 1, \\ 1056t^2 - 2111t + 1056 & \text{if } 1 < t \leq 2, \\ -1056t^2 + 6337t - 7392 & \text{if } 2 < t \leq 3, \\ t + 2112 & \text{if } 3 < t \leq 70, \\ 2182 + \int_{70}^{t} f^{**}(\xi)d\xi & \text{if } t > 70, \end{cases}$$

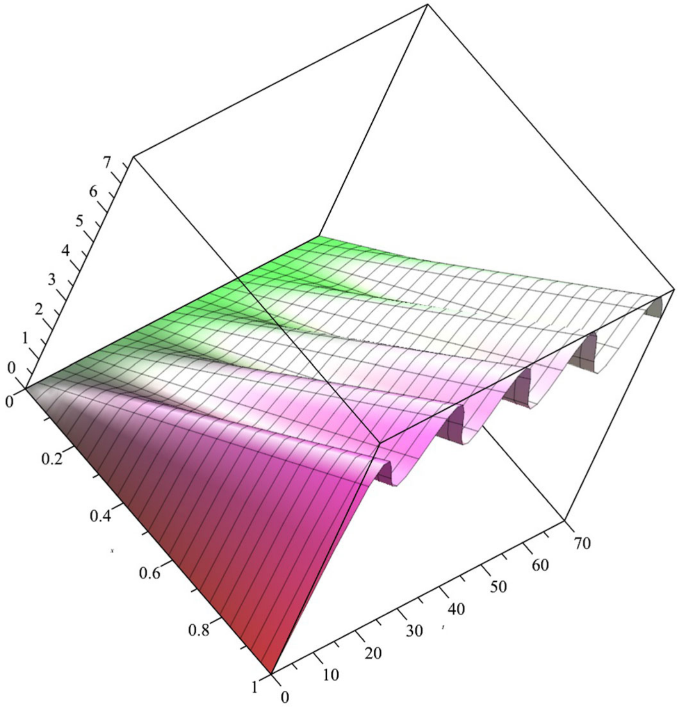

Fig. 1 Graph of $xF^{\blacklozenge}(t)$.

where $F^*(t) = \int_0^t f^*(\xi)d\xi$ for all $t \in \mathbb{R}$. We now verify that the assumptions of Corollary 3.6 are hold. Considering, for example, $c_1 = 1$, $d = 3$ and $c_2 = 70$, we find that $c_1 = 1 < \frac{25\sqrt{15}}{2\pi^2\sqrt{2}}d \approx 10.40$ and $\frac{25\sqrt{15}}{\pi^2\sqrt{2}}\sqrt{\frac{k}{\delta}}d \approx 23.26 < c_2 = 70$. Moreover,

$$\frac{\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c_1)^2} \approx 64 \times 10^{-5} < \frac{4}{3}\frac{\int_{2/5}^{3/5} F(x,d)dx}{9375kd^2} \approx 133 \times 10^{-5}, \quad (3.6)$$

$$\frac{\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx}{2(\sqrt{\delta}\pi^2 c_2)^2} \approx 28 \times 10^{-5} < \frac{2}{3}\frac{\int_{2/5}^{3/5} F(x,d)dx}{9375kd^2} \approx 66 \times 10^{-5}. \quad (3.7)$$

From (3.6) and (3.7), we get the conditions (H3) and (H4) of Corollary 3.6 are verified. Hence, it follows that for each $\lambda \in [749, 1558]$, the problem

$$\begin{cases} -4u^{(6)}(x) + \left(\frac{(\pi x)^2}{3}u''(x)\right)'' - \left(\frac{(\pi x)^4}{3}u'(x)\right)' + \frac{(\pi x)^6}{3}u(x) \\ \qquad = \lambda x f^*(u(x)), \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \quad (3.8)$$

admits at least three non-negative weak solutions $v^1,\ v^2,\ v^3$ such that

$$|v^j(x)| < 70$$

for each $x \in [0, 1],\ j = 1, 2, 3$.

Remark 3.11 In the above example, we can place many functions instead of p, q, r, s and f to study various other problems. For instance, consider the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, t) = x f^{\blacklozenge}(t)$ where

$$f^{\blacklozenge}(t) = \begin{cases} 5t^4 & \text{if } t \leq 1, \\ \frac{5}{t^2} + \frac{2}{3\pi}\cos(\frac{3\pi}{2}t) & \text{if } t > 1. \end{cases}$$

Clearly, $F(x, t) = x F^{\blacklozenge}(t) = x \int_0^t f^{\blacklozenge}(\xi) d\xi$ for all $(x, t) \in [0, 1] \times \mathbb{R}$, see Fig. 1. The problem (3.8) can be studied again with the new potential term f.

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