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Well-posedness of generalized vector variational inequality problem via topological approach

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Abstract

In this paper, we discuss well-posedness for a generalized vector variational inequality problem (GVVIP, in short) in the framework of topological vector spaces. Unlike in the available literature, we have adopted a topological approach using admissibility and convergence of nets, instead of monotonicity and convexity etc of the function involved. We provide necessary and sufficient conditions for a GVVIP to be well-posed in generalized sense. We give a characterization for GVVIP to be well-posed in generalized sense in terms of the upper semi-continuity of the approximate solution set map. Also, we provide some necessary conditions for a GVVIP to be well-posed in generalized sense in terms of Painlevé–Kuratowski convergence.

Keywords Generalized vector variational inequality · Well-posedness · Topological vector space · Compactness

Mathematics Subject Classification 49J40 · 49K40 · 54H99

1 Introduction

The classical notion of variational inequalities (introduced by Stampacchia [26]) was extended by Gianessi [12] to vector variational inequalities (VVI, in short) in the framework of finite dimensional Euclidean spaces. Thereafter, several variants of VVI have been introduced and are used to solve vector optimization problem extensively [1, 25, 28, 30]. Ruiz-Garzon et al. [24] have provided some relations between VVI problems and optimization problems by using the condition of pseudo-invexity. Recently, in [13, 19, 20] researchers have provided the existence results for the solutions of several variants of vector variational inequality problem in the setting of topological vector spaces by using topological approach.

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The concept of well-posedness, which is closely related to stability theory, was initially introduced by Tykhonov [27] for scalar minimization problem. Due to close relationship of variational inequality problems and mathematical programming problems, the notion of well-posedness has also been generalized to variational inequalities [4, 21, 29] and further to many other problems such as fixed point problems [10], equilibrium problems [8], optimization problems with variational inequality, mixed quasi variational-like inequality and equilibrium constraints [8, 21, 22].

Crespi et al. [7] proposed the concept of well-posedness for vector optimization problem and for a VVI of the differential type and also set up a relationship between well-posedness of these two problems in the setting of Euclidean space $\mathbb{R}^n$. In [9], Fang and Huang introduced some notions of parametric well-posedness for Stampacchia and Minty type vector variational inequalities. They also, established equivalence between well-posedness of a Stampacchia VVI and parametric well-posedness of a vector optimization problem under some suitable conditions. At the subsequent time, in [15], researchers introduced various types of generalized Levitin-Polyak well-posedness for generalized variational inequality problems with functional constraints in the framework of topological spaces, equipped with norm topology. In [5], Cheng et al. discussed the well-posedness of a generalized mixed vector variational type inequality and a constrained optimization problem and proposed metric characterization of well-posedness in terms of an approximate solution set.

In [18], researchers studied the concept of parametric well-posedness for vector equilibrium problem and the concept of generalized well-posedness for an equilibrium problem with equilibrium constraints in the setting of topological vector spaces. Jayswal and Jha [16] discussed well-posedness for generalized mixed vector variational-like inequality problem and optimization problem in the framework of Banach space. Very recently, Jha et al. [17] have also discussed well-posedness for multi-time variational inequality using generalized monotonicity.

However, all the above studies have used some type of monotonicity and convexity of the function involved. In this paper, we provide an alternative method for well-posedness for the generalized vector variational inequality problem without using any such monotonicity and convexity. We use topological concepts such as closedness, compactness, upper semi-continuity, admissibility of function space topology etc to achieve the results. We have found so far that the concept of admissibility has not been much used in the existing literature to obtain such results.

In the following we recall a set valued map.

A mapping $F : X \rightrightarrows Y$ is a *set-valued* map from X to Y if for each $x \in X$, $F(x)$ is a set in Y.

The generalized vector variational inequality problem proposed by Chen and Craven [6] can be presented as follows:

Generalized vector variational inequality problem: Let X and Y be two real topological vector spaces and let K be a nonempty convex subset of X. Let $T : K \rightrightarrows \mathcal{C}(X, Y)$ be a set valued map, where $\mathcal{C}(X, Y)$ denotes the space of all continuous linear mappings from the space X to the space Y. Further, let C be a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, where $\text{int}C$ denotes the interior of C. Then the generalized vector variational inequality problem (GVVIP) is to find $x_0 \in K$ such that, there exists $t_{x_0} \in T(x_0)$ with

$$t_{x_0}(x - x_0) \notin -\text{int}C \quad \forall x \in K.$$

We consider here GVVIP in a more general framework by taking X, Y as topological vector spaces instead of real topological vector spaces:

Throughout the paper, the set of all solutions to the GVVIP is denoted by $\mathcal{S}_{GVVIP}$.

Rest part of the paper is arranged as follows: In Sect. 2, we provide some basic definitions and results, which will be used in the paper. In Sect. 3, we discuss well-posedness and generalized well-posedness of a GVVIP. It is shown that GVVIP is well-posed in generalized sense if and only if the solution set of GVVIP is nonempty. We also give a characterization for the GVVIP to be well-posed in generalized sense in terms of upper semi-continuity of the approximate solution set-valued map $Q(\cdot)$. Along with, we provide an example to illustrate our result.

2 Preliminaries

In this section, we recall some definitions and basic results which will be used later to obtain the main results.

Definition 2.1 [23] Let (X, μ) be a topological space. Then

(i) A set D called a *directed set* with a partial order $\preceq$ such that for every pair α, β in D, there exists an element γ in D such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.
(ii) A function ξ from a directed set D into X, is called a *net* in X;
We usually denote $\xi(\alpha)$ by x_α and the net ξ itself is represented by $\{x_\alpha\}_{\alpha \in D}$.
(iii) A net $\{x_\alpha\}$ is said to *converge* to the point $x \in X$ if for each neighbourhood U of x, there exists some $\alpha \in D$ such that for $\alpha \preceq \beta$, we have $x_\beta \in U$.

Definition 2.2 [3] If X and Y are two topological spaces and $F : X \rightrightarrows Y$ is a set-valued mapping. Then F is called

(i) *Upper semi-continuous* at $\bar{x} \in X$ if for each open set V in Y containing $F(\bar{x})$, there is an open set U in X containing $\bar{x}$ such that $F(x) \subseteq V$, for every $x \in U$;
(ii) *Lower semi-continuous* at $\bar{x} \in X$ if for each open set V in Y with $F(\bar{x}) \bigcap V \neq \emptyset$, there is an open set U in X such that $F(x) \bigcap V \neq \emptyset$, for every $x \in U$;
(iii) Upper semi-continuous (respectively, lower semi-continuous) on X if it is upper semi-continuous (respectively, lower semi-continuous) at every point $x \in X$.

Lemma 2.3 [3] *Let X and Y be two topological spaces. Let $F : X \rightrightarrows Y$ be a set-valued mapping such that $F(x)$ is nonempty and compact for each $x \in X$. Then F is upper semi-continuous at $\bar{x}$ if and only if for any net $\{x_n\}$ in X, converging to $\bar{x}$ and $y_n \in F(x_n)$, there exists a subnet $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges to $\bar{y}$ for some $\bar{y} \in F(\bar{x})$.*

Lemma 2.4 [3] *Let X and Y be two topological spaces and let $F : X \rightrightarrows Y$ be a set-valued map. Then F is lower semi-continuous at $\bar{x}$ if and only if for any net $\{x_n\}$ in X, converging to $\bar{x}$ and for any $\bar{y} \in F(\bar{x})$, there exists a net $\{y_n\} \subseteq F(x_n)$ converging to $\bar{y}$.*

We now recall the notion of Painlevé–Kuratowski convergence [11]. For a net of sets $\{U_\alpha\}_{\alpha \in D}$ in X, we have

$$\operatorname{Li} U_\alpha = \{x \in X : x_\alpha \to x, x_\alpha \in U_\alpha, \alpha \in D\}$$
$$\operatorname{Ls} U_\alpha = \left\{x \in X : x_\beta \to x, x_\beta \in U_\beta, \beta \in D_1, D_1 \text{ is a directed subset of D}\right\}$$

The net $\{U_\alpha\}$ converges to U in the sense of Painlevé–Kuratowski if

$$\operatorname{Ls} U_\alpha \subseteq U \subseteq \operatorname{Li} U_\alpha.$$

The relation $\operatorname{Ls} U_\alpha \subseteq U$ is known as the *upper part of the convergence* and the relation $U \subseteq \operatorname{Li} U_\alpha$ is known as the *lower part of the convergence*.

Definition 2.5 [2, 14] Let(Y, μ_1) and (Z, μ_2) be two topological spaces. Let $\mathcal{C}(Y, Z)$ be the space of all continuous mappings from Y to Z. A topology τ on $\mathcal{C}(Y, Z)$ is called *admissible*, if the *evaluation map* $e : \mathcal{C}(Y, Z) \times Y \to Z$, defined by $e(f, y) = f(y)$, is continuous.

Lemma 2.6 [14] *A function space topology on $\mathcal{C}(X, Y)$, the collection of continuous mappings from the space X to the space Y, is admissible if and only if for any net $\{f_n\}_{n \in D_1}$ in $\mathcal{C}(X, Y)$, convergence of $\{f_n\}_{n \in D_1}$ to f implies continuous convergence of $\{f_n\}_{n \in D_1}$ to f. That is, if $\{f_n\}_{n \in D_1}$ converges to f in $\mathcal{C}(X, Y)$ and $\{x_m\}_{m \in D_2}$ is any net in X converging to $x \in X$, then $\{f_n(x_m)\}_{(n,m) \in D_1 \times D_2}$ converges to $f(x)$ in Y.*

The above characterization of admissibility remains valid for the family of continuous linear mappings from X to Y, where X and Y are topological vector spaces.

Throughout the paper 0_X and 0_Y denote the zero vectors in the space X and in the space Y respectively. All topological vector spaces considered in the paper are taken to be T_1 and hence they all are Hausdorff.

3 Well-posedness

Authors have found so far that in literature, researchers have used convexity, monotonicity, hemi-continuity etc. to discuss the well-posedness and generalized well-posedness of a variational inequality problem [16, 18] but here we adopt topological approach to discuss the generalized well-posedness of a GVVIP in the setting of topological vector spaces.

Motivated by [16, 18], below we define approximating net for GVVIP described in Sect. 1, in the framework of topological vector spaces, in the following way:

Definition 3.1 (i) A net $\{x_n\} \subseteq K$ is said to be an *approximating net* for a GVVIP if there exist a net $\{\varepsilon_n\} \subseteq \text{int}C$ converging to 0_Y and $t_{x_n} \in T(x_n)$ such that

$$t_{x_n}(y - x_n) + \varepsilon_n \notin -\text{int}C \quad \forall y \in K.$$

(ii) A GVVIP is said to be *well-posed* if there exists a unique solution $\bar{x}$ of GVVIP and every approximating net converges to $\bar{x}$.

(iii) A GVVIP is said to be *well-posed in generalized sense* if the solution set $\mathcal{S}_{GVVIP}$ of the GVVIP is nonempty and every approximating net has a subnet, which converges to some point in $\mathcal{S}_{GVVIP}$.

We define a set-valued map $Q : \text{int}C \cup \{0_Y\} \rightrightarrows K$ such that

$$Q(\varepsilon) = \{x \in K, t_x \in T(x) : t_x(y - x) + \varepsilon \notin -\text{int}C \quad \forall y \in K\}.$$

Clearly, if the net $\{\varepsilon_n\} \subseteq \text{int}C$ converges to 0_Y and $x_n \in Q(\varepsilon_n)$ for each n, then $\{x_n\}$ is an approximating net.

Following proposition provides some properties of the map Q.

Proposition 3.2 *Let $Q : \text{int}C \cup \{0_Y\} \rightrightarrows X$ be a set-valued map defined by*

$$Q(\varepsilon) = \{x \in K, t_x \in T(x) : t_x(y - x) + \varepsilon \notin -\text{int}C \quad \forall \ y \in K\}.$$

Then

(i) $\mathcal{S}_{GVVIP} \subseteq Q(\varepsilon) \quad \forall \ \varepsilon \in \text{int}C \cup \{0_Y\}$;
(ii) $Q(0_Y) = \mathcal{S}_{GVVIP}$.

Proof (i) Let $\bar{x} \in \mathcal{S}_{GVVIP}$, then there exists $t_{\bar{x}} \in T(\bar{x})$ such that $t_{\bar{x}}(y - \bar{x}) \notin -\text{int}C$, for every $y \in K$, which implies $t_{\bar{x}}(y - \bar{x}) + \varepsilon \notin -\text{int}C$, that is $\bar{x} \in Q(\varepsilon)$.

(ii) From (i), $\mathcal{S}_{GVVIP} \subseteq Q(0_Y)$. Let $\hat{x} \in Q(0_Y)$, then $t_{\hat{x}} \in T(\hat{x})$ with $t_{\hat{x}}(y - \hat{x}) + 0_Y \notin -\text{int}C$, for every $y \in K$, which implies $t_{\hat{x}}(y - \hat{x}) \notin -\text{int}C$, for every $y \in K$, that is, $\hat{x} \in \mathcal{S}_{GVVIP}$. Hence $Q(0_Y) \subseteq \mathcal{S}_{GVVIP}$.

□

Following theorem provides necessary and sufficient conditions for a GVVIP to be well-posed in generalized sense.

Theorem 3.3 *Let X and Y be two topological vector spaces and let $\mathcal{C}(X, Y)$ be the space of all continuous linear mappings from X to Y, equipped with an admissible topology. Let K be a nonempty closed convex compact subset of X and C be a closed convex pointed cone in Y with* $\text{int}C \neq \emptyset$. *Further, let $T : K \rightrightarrows \mathcal{C}(X, Y)$ be an upper semi-continuous set valued map such that for each $x \in K$, $T(x)$ is compact. Then the GVVIP is well-posed in generalized sense if and only if the solution set $\mathcal{S}_{GVVIP}$, of GVVIP is nonempty.*

Proof First suppose GVVIP is well-posed in generalized sense, then the solution set $\mathcal{S}_{GVVIP}$ of GVVIP is nonempty.

Conversely, suppose the solution set $\mathcal{S}_{GVVIP}$ is nonempty. We have to show GVVIP is well-posed in generalized sense. Let $\{x_n\} \subseteq K$ be an approximating net, then there exists a net $\{\varepsilon_n\} \subseteq \text{int}C$, converging to 0_Y and $t_{x_n} \in T(x_n)$ such that

$$t_{x_n}(y - x_n) + \varepsilon_n \notin -\text{int}C \quad \forall\, y \in K. \tag{1}$$

As K is compact, $\{x_n\}$ has a subnet $\{x_{n_k}\}$ converging to some $\bar{x} \in K$ and for the subnet $\{x_{n_k}\}$, (1) reduces to

$$t_{x_{n_k}}(y - x_{n_k}) + \varepsilon_{n_k} \notin -\text{int}C \quad \forall\, y \in K, \tag{2}$$

where $\{\varepsilon_{n_k}\}$ is a subnet of $\{\varepsilon_n\}$. Since T is upper semi-continuous, then for $x_{n_k} \to \bar{x}$ and $t_{x_{n_k}} \in T(x_{n_k})$ there exists a subnet $\{t_{x_{n_{k_l}}}\}$ of $\{t_{x_{n_k}}\}$, converging to $t_{\bar{x}} \in T(\bar{x})$. For $\{t_{x_{n_{k_l}}}\}$, (1) reduces to

$$t_{x_{n_{k_l}}}(y - x_{n_{k_l}}) + \varepsilon_{n_{k_l}} \notin -\text{int}C \quad \forall\, y \in K, \tag{3}$$

where $\{\varepsilon_{n_{k_l}}\}$ is a subnet of $\{\varepsilon_{n_k}\}$. Since $x_{n_{k_l}} \to \bar{x}$, $y - x_{n_{k_l}} \to y - \bar{x}$ and $t_{x_{n_{k_l}}} \to t_{\bar{x}}$, therefore by using the virtue of admissibility of function space $\mathcal{C}(X, Y)$, we have $\{t_{x_{n_{k_l}}}(y - x_{n_{k_l}})\}$ converges to $t_{\bar{x}}(y - \bar{x})$. Thus, $\{t_{x_{n_{k_l}}}(y - x_{n_{k_l}}) + \varepsilon_{n_{k_l}}\}$ converges to $t_{\bar{x}}(y - \bar{x}) + 0_Y = t_{\bar{x}}(y - \bar{x})$. If $t_{\bar{x}}(y - \bar{x}) \in -\text{int}C$, for some $y \in K$, then $t_{x_{n_{k_l}}}(y - x_{n_{k_l}}) + \varepsilon_{n_{k_l}} \in -\text{int}C$ eventually, which contradicts (3). Hence, $t_{\bar{x}}(y - \bar{x}) \notin -\text{int}C$, for any $y \in K$, that is, $\bar{x}$ is a solution of the GVVIP. □

The following example illustrates the above result.

Example 3.4 Consider $X = Y = \mathbb{R}$, $K = [0, 2]$. Clearly, K is closed convex and compact. Let $C = \mathbb{R}^+ \cup \{0\}$, then C is a closed convex pointed cone with $\text{int}C \neq \emptyset$, and $-\text{int}C = (-\infty, 0)$. Further, let $T : K \rightrightarrows \mathcal{C}(X, Y)$ be a set valued map defined by $T(x) = \{t'_x, t''_x\}$, where $t'_x(u) = \langle x, u\rangle$ and $t''_x(u) = -\langle x, u\rangle$ for $x \in K$ and $u \in X$. That the induced topology of $\mathcal{C}(X, Y)$ is admissible can be verified by the fact that if $\{x_m\}$ converges to x in X and $\{\xi_n\}$

converges to ξ in $\mathcal{C}(X, Y)$, then we have

$$\begin{aligned}\|\xi_n(x_m) - \xi(x)\| &= \|\xi_n(x_m) - \xi_n(x) + \xi_n(x) - \xi(x)\| \\ &\leq \|\xi_n(x_m) - \xi_n(x)\| + \|\xi_n(x) - \xi(x)\| \\ &\leq \|\xi_n\|\|x_m - x\| + \|\xi_n(x) - \xi(x)\|.\end{aligned}$$

Hence, $\xi_n(x_m) \to \xi(x)$.

We take $x_0 = 0$. Then for any y in K, we have $t^{''}_{x_0}(y - x_0) = -\langle x_0, y - x_0\rangle = 0$. So, $t^{''}_{x_0}(y - x_0)) \notin -\text{int}C$ and so x_0 is a solution for the GVVIP. Similarly, $x_0 = 2$ is also a solution for the GVVIP. Thus $\mathcal{S}_{GVVIP} \neq \emptyset$.

Now we show that the GVVIP is well-posed in generalized sense, that is, every approximating net has a convergent subnet, which converges to some point in $\mathcal{S}_{GVVIP}$. Let $\{x_n\}$ be an approximating net in $[0, 2]$. Then there exists a net $\{\varepsilon_n\}$ in $\text{int}C = (0, \infty)$ where $\varepsilon_n \to 0$ and $t^{''}_{x_n}(y - x_n) + \varepsilon_n = -\langle x_n, y - x_n\rangle + \varepsilon_n = -x_n(y - x_n) + \varepsilon_n = x_n^2 - x_n y + \varepsilon_n \notin (-\infty, 0)$, for all $y \in [0, 2]$. Since $[0, 2]$ is compact, therefore $\{x_n\}$ has a subnet $\{x_{n_k}\}$, which converges to some $\hat{x} \in [0, 2]$. Also,

$$t^{''}_{n_k}(y - x_{n_k}) + \varepsilon_{n_k} \to \hat{x}^2 - \hat{x}y.$$

Clearly, the inequality $x_0^2 - x_0 y \geq 0 \quad \forall y \in [0, 2]$, is satisfied by $x_0 = 0$. If $\hat{x} \neq 0$, then we have $\hat{x} - y \geq 0 \;\; \forall y \in [0, 2]$. This implies $\hat{x} = 2$. It is already shown above that $2 \in \mathcal{S}_{GVVIP}$. Thus, $\{x_n\}$ has a convergent subnet which converges to some point in $\mathcal{S}_{GVVIP}$. Hence the GVVIP is well-posed in generalized sense.

Next theorem gives necessary and sufficient condition for a GVVIP to be well-posed in generalized sense in terms of upper semi-continuity of the map $Q(\cdot)$.

Theorem 3.5 *GVVIP is well-posed in generalized sense if and only if the map $Q(\cdot)$ is upper semicontinuous at 0_Y and $\mathcal{S}_{GVVIP}$ is compact.*

Proof Suppose $Q : \text{int}C \cup \{0_Y\} \rightrightarrows X$ is not upper semicontinuous at $\varepsilon = 0_Y$, then there exists an open set W containing $Q(0_Y)$ such that for every neighbourhood V of 0_Y in $\text{int}C \cup \{0_Y\}$, there exists $\varepsilon_V \in V$ such that $Q(\varepsilon_V) \nsubseteq W$. Clearly, $\varepsilon_V \in \text{int}C$. Let $\mathcal{U}$ denote the family of neighbourhoods of 0_Y in the subspace $\text{int}C \cup \{0_Y\}$. Then it can be shown that $(\mathcal{U}, \geq)$ is a directed set, where "$\geq$" denotes the inverse set inclusion. Hence $\{\varepsilon_V : V \in \mathcal{U}\}$ is a net. For the sake of simplicity, we denote $\{\varepsilon_V\}_{V \in \mathcal{U}}$ by $\{\varepsilon_n\}_{n \in D}$, where $D = \mathcal{U}$. It can be shown that $\{\varepsilon_n\}$ converges to 0_Y.

As $Q(\varepsilon_n) \nsubseteq W$, there exists $x_n \in Q(\varepsilon_n)$ for each $n \in D$ such that $x_n \notin W$. Clearly, $\{x_n\}$ is an approximating net. Since the GVVIP is well-posed in generalized sense, there exists a subnet $\{x_{n_k}\}$ of $\{x_n\}$, converging to some $\bar{x} \in \mathcal{S}_{GVVIP}$. Now, $\mathcal{S}_{GVVIP} = Q(0_Y)$ by Proposition 3.2 and $Q(0_Y) \subseteq W$. Thus, $\bar{x} \in W$ and hence $\{x_n\}$ is eventually contained in W. This contradicts the fact that $x_{n_k} \in Q(\varepsilon_{n_k}) \nsubseteq W$. Hence, our assumption is wrong and hence $Q(\cdot)$ is upper semicontinuous at $\varepsilon = 0_Y$.

We now show that $\mathcal{S}_{GVVIP}$ is compact. Let $\{u_n\}$ be a net in $\mathcal{S}_{GVVIP} = Q(0_Y)$, then there exist $t_{u_n} \in T(u_n)$ such that

$$t_{u_n}(y - u_n) + 0_Y \notin -\text{int}C \;\; \forall\, y \in K,$$

which gives

$$t_{u_n}(y - u_n) + \varepsilon_n \notin -\text{int}C \;\; \forall\, y \in K,$$

which implies $\{u_n\}$ is an approximating net. Since the GVVIP is well-posed in generalized sense, therefore there exists a subnet $\{u_{n_k}\}$ of $\{u_n\}$, converging to some $\bar{u} \in \mathcal{S}_{GVVIP}$.

Conversely, let $\{x_n\}$ be an approximating net, then there exists a net $\{\varepsilon_n\} \subseteq \text{int}C$ converging to 0_Y such that $x_n \in Q(\varepsilon_n)$ for every n. Since $Q(\cdot)$ is upper semi-continuous at 0_Y and $Q(0_Y) = \mathcal{S}_{GVVIP}$ is compact, then by Lemma 2.3, for every net $\{\varepsilon_n\} \subseteq \text{int}C$, converging to 0_Y and $\{x_n\} \subseteq Q(\varepsilon_n)$, there exists a subnet $\{x_{n_k}\}$ of $\{x_n\}$, $x_{n_k} \in Q(\varepsilon_{n_k})$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in Q(0_Y) = \mathcal{S}_{GVVIP}$. Hence, GVVIP is well-posed in generalized sense. □

In the next result, we give necessary condition for a GVVIP to be well-posed in terms of lower semi-continuity of the map $Q(\cdot)$.

Theorem 3.6 *If GVVIP is well-posed, then the map $Q(\cdot)$ is lower semi-continuous at 0_Y.*

Proof Let $\bar{x}$ be the unique solution of GVVIP, that is, $\mathcal{S}_{GVVIP} = \{\bar{x}\}$ such that every approximating net converges to $\bar{x}$. We shall show that the map $Q(\cdot)$ is lower semi-continuous at 0_Y. Let $\{\varepsilon_n\} \subseteq \text{int}C$ be a net which converges to 0_Y. Let $y_0 \in Q(0_Y) = \{x \in K, t_x \in T(x) : t_x(y - x) + 0_Y \notin -\text{int}C, \quad \forall\, y \in K\} = \mathcal{S}_{GVVIP} = \{\bar{x}\}$. Thus $y_0 = \bar{x}$. Let $\{y_n\}$ be a net such that $y_n \in Q(\varepsilon_n)$, for each n, then $\{y_n\}$ is an approximating net. Since GVVIP is well-posed, $\{y_n\}$ converges to $\bar{x} = y_0$. Hence $Q(\cdot)$ is lower semi-continuous at 0_Y. □

From Theorem 3.5 and 3.6, we can conclude the following:

Corollary 3.7 *If the GVVIP is well-posed then the map $Q(\cdot)$ is continuous at 0_Y.*

In next couple of theorems we provide necessary conditions for a GVVIP to be well-posed in generalized sense in terms of convergence.

Theorem 3.8 *If a GVVIP is well-posed in generalized sense and $\{\varepsilon_n\}$ is a net in* int$C$ *such that $\{\varepsilon_n\}$ converges to 0_Y. Then*

$$Ls\ Q(\varepsilon_n) \subseteq Q(0_Y).$$

Proof Let $\{x_n\}$ be a net in K with $x_n \in Q(\varepsilon_n)$. Let $x \in \text{Ls}\ Q(\varepsilon_n)$, then there exists a subnet $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in Q(\varepsilon_{n_k})$ and $\{x_{n_k}\}$ converges to x, where $\{\varepsilon_{n_k}\}$ is a subnet of $\{\varepsilon_n\}$. Clearly, $\{x_{n_k}\}$ is an approximating net for the GVVIP. Since the GVVIP is well-posed in generalized sense, therefore $\{x_{n_k}\}$ has a subnet that converges to some point in $\mathcal{S}_{GVVIP}$. As $\{x_{n_k}\}$ converges to x, every subnet of $\{x_{n_k}\}$ converges to x. Hence, $x \in \mathcal{S}_{GVVIP} = Q(0_Y)$. Painlevé–Kuratowski □

Theorem 3.9 *Suppose K is compact, the GVVIP is well-posed in generalized sense, $Q(0_Y)$ is singleton and $\{\varepsilon_n\}$ is a net in* int$C$ *such that $\{\varepsilon_n\}$ converges to 0_Y. Then*

$$Q(0_Y) \subseteq \text{Li}\ Q(\varepsilon_n).$$

Proof Let $Q(0_Y) = \{\bar{x}\}$. Let $\{\varepsilon_n\} \subseteq \text{int}C$ be a net, which converges to 0_Y and let $\{x_n\}$ be a net in K such that $x_n \in Q(\varepsilon_n)$, then there exist $t_{x_n} \in T(x_n)$ with

$$t_{x_n}(y - x_n) + \varepsilon_n \notin -\text{int}C \ \ \forall\, y \in K.$$

Since K is compact, therefore $\{x_n\}$ has a convergent subnet $\{x_{n_k}\}$ converging to some member of K. Then $\{x_{n_k}\}$ is an approximating net as $x_{n_k} \in Q(\varepsilon_{n_k})$, where $\{\varepsilon_{n_k}\}$ is a subnet of $\{\varepsilon_n\}$, converging to 0_Y. As the GVVIP is well-posed in generalized sense, $\{x_{n_k}\}$ has a convergent subnet $\{x_{n_{k_l}}\}$ converging to some $\hat{x} \in \mathcal{S}_{GVVIP}$. Since $\mathcal{S}_{GVVIP} = Q(0_Y)$, by Proposition 3.2 and $Q(0_Y)$ is singleton, therefore $\hat{x} = \bar{x}$. Hence $\{x_{n_{k_l}}\}$ converges to $\bar{x}$. Thus, there exists a net $\{x_{n_{k_l}}\}$ in $Q(\varepsilon_n)$ converging to $\bar{x}$. Therefore $\bar{x} \in \text{Li}\ Q(\varepsilon_n)$. □

From Theorem 3.8 and 3.9 one can conclude the following result:

Theorem 3.10 *Let K be compact and the GVVIP be well-posed in generalized sense. Let $Q(0_Y)$ be singleton, then whenever the net $\{\varepsilon_n\} \subseteq \text{int}C \cup 0_Y$ converges to 0_Y, the image net $\{Q(\varepsilon_n)\}$ converges to $Q(0_Y)$ in Painlevé–Kuratowski sense.*

4 Conclusion

In this paper, well-posedness is discussed for a GVVIP. We have obtained the results following a new technique, addopting a topological approach. No convexity and monotonicity conditions are used unlike in the available literature. Topological concepts including admissibility of the function space topology, convergence of nets etc are used to obtain our results. Further investigation is required to check whether this approach can also work for studying other aspects of variational inequality problem.

Declarations

Conflict of interest The authors have no conflict of interest to declare.

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