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# On the multifractal measures: proportionality and dimensions of Moran sets

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**Abstract**
The aim of this work is to discuss the proportionality of the multifractal measures. We will prove that the ratio of the multifractal measures is bounded. In addition, for a class of homogeneous Cantor sets, we find an explicit formula for their multifractal Hausdorff and packing function dimensions and discuss some interesting examples.

**Keywords** Multifractal analysis · Homogeneous Cantor sets · Hausdorff dimension · Packing dimension · Homogeneous Moran measures

**Mathematics Subject Classification** 28A20 · 28A75 · 28A78 · 28A80 · 49Q15

## 1 Introduction

Let $\mu$ be a probability measure on a metric space $X$. The Hausdorff multifractal spectrum function, $f_\mu$, and the packing multifractal spectrum function, $F_\mu$, of the measure $\mu$ are defined respectively by

$$f_\mu(\alpha) = \dim_H(E(\alpha)) \quad \text{and} \quad F_\mu(\alpha) = \dim_P(E(\alpha)) \quad \text{for } \alpha \geq 0,$$

where

$$E(\alpha) = \Big\{x \in \operatorname{supp}(\mu); \quad \mu\Big(B(x,r)\Big) \sim r^\alpha\Big\}.$$

During the past 25 years there has been an enormous interest in computing the multifractal spectra of measures in the mathematical literature. Particularly, the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^n$ exhibiting some degree of self-similarity

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have been computed rigorously. The reader can be referred to the paper [13], the textbooks [8, 15] and the references therein. Some heuristic arguments using techniques of statistical mechanics (see [11]) show that the singularity spectrum should be finite on a compact interval, noted by $\mathrm{Dom}(\mu)$, and is expected to be the Legendre transform conjugate of the $\tau_\mu$-function, given by

$$\tau_\mu(q) = \limsup_{r\to 0} \frac{1}{-\log r} \log \sup \left\{ \sum_i \mu(B(x_i, r))^q \right\},$$

where the supremum is taken over all centered packing $\big(B(x_i, r)\big)_i$ of $\mathrm{supp}(\mu)$. That is, for all $\alpha \in \mathrm{Dom}(\mu)$,

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}} \Big\{ \alpha q + \tau_\mu(q) \Big\} =: \tau_\mu^*(\alpha). \tag{1.1}$$

The multifractal formalism (1.1) has been proved rigorously for random and non-random self-similar measures, for self-conformal measures, for self-affine measures and for Moran measures. We notice that the proofs of the multifractal formalism (1.1) in the above-mentioned references (see for example [2, 3, 13, 14, 21, 23–25] and references therein) are all based on the same key idea. The upper bound for $f_\mu(\alpha)$ is obtained by a standard covering argument, involving Besicovitch's covering theorem or Vitali's covering theorem. However, its lower bound is usually much harder to prove and is related to the existence of an auxiliary measure (Gibbs measures, Frostman measures) which is supported by the set to be analyzed. In an attempt to develop a general theoretical framework for studying the multifractal structure of arbitrary measures, Olsen [13], Pesin [15] and Peyrière [16] suggested various ways of defining measures analogous to those of Gibbs measures in very general settings. For an arbitrary Borel probability measure $\mu$ on $\mathbb{R}^n$, they introduced two parameter families of measures,

$$\Big\{\mathscr{H}_\mu^{q,t};\ q, t \in \mathbb{R}\Big\} \quad \text{and} \quad \Big\{\mathscr{P}_\mu^{q,t};\ q, t \in \mathbb{R}\Big\},$$

based on certain generalizations of the Hausdorff measure and of the packing measure. One of the main importance of the multifractal measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$, and the corresponding dimension functions $b$ and $B$ is due to the fact that the multifractal spectra functions $f_\mu$ and $F_\mu$ are bounded above by the Legendre transforms of $b$ and $B$, respectively, i.e.,

$$\dim_H(E(\alpha)) \le b^*(\alpha) \quad \text{and} \quad \dim_P(E(\alpha)) \le B^*(\alpha) \quad \text{for all} \quad \alpha \ge 0.$$

These inequalities may be viewed as rigorous versions of the multifractal formalism. Furthermore, for many natural families of measures we have

$$\dim_H(E(\alpha)) = b^*(\alpha) \quad \text{and} \quad \dim_P(E(\alpha)) = B^*(\alpha) \quad \text{for some} \quad \alpha \ge 0,$$

see for example [2, 3, 5, 12–14, 21, 23–26]. It is clear by comparing the definitions of the measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$, and definition of the $\tau_\mu$-function which appears in the multifractal formalism that $b(q)$ and $B(q)$ are mathematically rigorous versions of $\tau_\mu(q)$, and that the one-parameter families

$$\Big\{\mathscr{H}_\mu^{q,b(q)};\ q \in \mathbb{R}\Big\} \quad \text{and} \quad \Big\{\mathscr{P}_\mu^{q,B(q)};\ q \in \mathbb{R}\Big\},$$

play the role of the auxiliary measures $\big\{\mu_q;\ q \in \mathbb{R}\big\}$. In particular, we would expect that the measures $\big\{\mathscr{H}_\mu^{q,b(q)};\ q \in \mathbb{R}\big\}$ and $\big\{\mathscr{P}_\mu^{q,B(q)};\ q \in \mathbb{R}\big\}$ have similar properties to those of the auxiliary measures $\big\{\mu_q;\ q \in \mathbb{R}\big\}$. This has been proved rigorously for self-similar, quasi self-similar, homogeneous Moran measures, self-conformal measures and for arbitrary measures.

Even though it seems rather unlikely that the multifractal Hausdorff and packing measures are proportional in general. In this paper, we will prove that the ratio of the measures $\mathscr{H}_\mu^{q,b(q)}$ and $\mathscr{P}_\mu^{q,B(q)}$ might still be bounded, i.e., there exists a number $0 < c_q < +\infty$ such that

$$\mathscr{H}_\mu^{q,b(q)}{}_{\llcorner \operatorname{supp}(\mu)} \le \mathscr{P}_\mu^{q,B(q)}{}_{\llcorner \operatorname{supp}(\mu)} \le c_q \mathscr{H}_\mu^{q,b(q)}{}_{\llcorner \operatorname{supp}(\mu)},$$

which provide a positive answer to Olsen's questions [13, Question 7.6] and [14, Question 4.1.12] in a more general framework. We give also a reasonable lower and upper bound for the multifractal Hausdorff and packing measures of homogeneous Moran sets. In particular, these results find an explicit formula for their multifractal Hausdorff and packing function dimensions. We note that our results, due to the use of the multifractal Hausdorff and packing measures introduced in [13], appear as natural multifractal generalizations of some of the main results in [1, 9, 10, 17, 18] and completely different from those found in [19].

We will now give a brief description of the organization of the paper. In the next section, we recall the definitions of the various multifractal dimensions and measures investigated in the paper. Section 2 recalls the multifractal formalism introduced in [13]. Section 3 contain our main results. The proofs are given in Sect. 4. The paper is concluded with Sect. 4 that, lists some interesting examples.

## 2 Preliminaries

We start by recalling the fine multifractal formalism introduced by Olsen in [13]. The key ideas behind the fine multifractal formalism in [13] are certain measures of Hausdorff-packing type which are tailored to see only the multifractal decomposition sets $E(\alpha)$. These measures are natural multifractal generalizations of the centered Hausdorff measure and the packing measure and are motivated by the $\tau_\mu$-function which appears in the multifractal formalism. We first recall the definition of the multifractal Hausdorff measure and the the multifractal packing measure. Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^n$. For $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define

$$\overline{\mathscr{P}}_{\mu,\delta}^{q,t}(E) = \sup\left\{\sum_i \mu\big(B(x_i, r_i)\big)^q \big(2r_i\big)^t\right\}, \quad E \neq \emptyset,$$

where the supremum is taken over all centered $\delta$-packing of $E$. Moreover we can set $\overline{\mathscr{P}}_{\mu,\delta}^{q,t}(\emptyset) = 0$. The packing pre-measure is then given by

$$\overline{\mathscr{P}}_{\mu}^{q,t}(E) = \inf_{\delta>0} \overline{\mathscr{P}}_{\mu,\delta}^{q,t}(E).$$

In a similar way, we define

$$\overline{\mathscr{H}}_{\mu,\delta}^{q,t}(E) = \inf\left\{\sum_i \mu\big(B(x_i, r_i)\big)^q \big(2r_i\big)^t\right\}, \quad E \neq \emptyset,$$

where the infimum is taken over all centered $\delta$-covering of $E$. Moreover we can set $\overline{\mathscr{H}}_{\mu,\delta}^{q,t}(\emptyset) = 0$. The Hausdorff pre-measure is defined by

$$\overline{\mathscr{H}}_{\mu}^{q,t}(E) = \sup_{\delta>0} \overline{\mathscr{H}}_{\mu,\delta}^{q,t}(E).$$

Especially, we have the conventions $0^q = \infty$ for $q \le 0$ and $0^q = 0$ for $q > 0$.

$\overline{\mathscr{H}}_\mu^{q,t}$ is $\sigma$-subadditive but not increasing and $\overline{\mathscr{P}}_\mu^{q,t}$ is increasing but not $\sigma$-subadditive. That's why Olsen introduced the following modifications on the multifractal Hausdorff and packing measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$,

$$\mathscr{H}_\mu^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathscr{H}}_\mu^{q,t}(F) \quad \text{and} \quad \mathscr{P}_\mu^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}_\mu^{q,t}(E_i).$$

In follows that $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathscr{P}_\mu^{q,t} \le \overline{\mathscr{P}}_\mu^{q,t}$. Moreover, there exists an integer $\xi \in \mathbb{N}$, such that $\mathscr{H}_\mu^{q,t} \le \xi \mathscr{P}_\mu^{q,t}$. The measure $\mathscr{H}_\mu^{q,t}$ is a multifractal generalization of the centered Hausdorff measure, whereas $\mathscr{P}_\mu^{q,t}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that if $t \ge 0$, then $\mathscr{H}_\mu^{0,t} = \mathscr{H}^t$ and $\mathscr{P}_\mu^{0,t} = \mathscr{P}^t$, where $\mathscr{H}^t$ denotes the $t$-dimensional centered Hausdorff measure and $\mathscr{P}^t$ denotes the $t$-dimensional packing measure.

The measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ and the pre-measure $\overline{\mathscr{P}}_\mu^{q,t}$ assign in the usual way a multifractal dimension to each subset $E$ of $\mathbb{R}^n$. They are respectively denoted by $b_\mu^q(E)$, $B_\mu^q(E)$ and $\Lambda_\mu^q(E)$ and satisfy

$$b_\mu^q(E) = \inf\Big\{t \in \mathbb{R};\ \mathscr{H}_\mu^{q,t}(E) = 0\Big\},$$
$$B_\mu^q(E) = \inf\Big\{t \in \mathbb{R};\ \mathscr{P}_\mu^{q,t}(E) = 0\Big\}$$

and

$$\Lambda_\mu^q(E) = \inf\Big\{t \in \mathbb{R};\ \overline{\mathscr{P}}_\mu^{q,t}(E) = 0\Big\}.$$

The number $b_\mu^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim_H(E)$ of $E$ whereas $B_\mu^q(E)$ and $\Lambda_\mu^q(E)$ are obvious multifractal analogues of the packing dimension $\dim_P(E)$ and the pre-packing dimension $\Delta(E)$ of $E$ respectively. In fact, it follows immediately from the definitions that

$$\dim_H(E) = b_\mu^0(E), \quad \dim_P(E) = B_\mu^0(E) \quad \text{and} \quad \Delta(E) = \Lambda_\mu^0(E).$$

Next, for $q \in \mathbb{R}$, we define the separator functions $b_\mu$, $B_\mu$ and $\Lambda_\mu$ by

$$b_\mu(q) = b_\mu^q\big(\text{supp}(\mu)\big), \quad B_\mu(q) = B_\mu^q\big(\text{supp}(\mu)\big) \quad \text{and} \quad \Lambda_\mu(q) = \Lambda_\mu^q\big(\text{supp}(\mu)\big).$$

It is well known that the functions $b_\mu$, $B_\mu$ and $\Lambda_\mu$ are decreasing and $B_\mu$, $\Lambda_\mu$ are convex and satisfying $b_\mu \le B_\mu \le \Lambda_\mu$.

The multifractal formalism based on the measures $\mathscr{H}_\mu^{q,s}$ and $\mathscr{P}_\mu^{q,s}$ and the dimension functions $b_\mu$, $B_\mu$ and $\Lambda_\mu$ provides a natural, unifying and very general multifractal theory which includes all the hitherto introduced multifractal parameters, i.e., the multifractal spectra functions $f_\mu$ and $F_\mu$, the multifractal box dimensions. The dimension functions $b_\mu$ and $B_\mu$ are intimately related to the spectra functions $f_\mu$ and $F_\mu$, whereas the dimension function $\Lambda_\mu$ is closely related to the upper box spectrum (more precisely, to the upper multifractal box dimension function $\overline{\tau}_\mu$, see [13]).

The reader is referred to as Olsen's classical text [13] for an excellent and systematic discussion of the multifractal Hausdorff and packing measures and dimensions.

## 3 Main results

In the next, we suppose the existence of a Gibbs' measure at a state $(q, B_\mu(q))$ for the measure $\mu$, i.e., the existence of a measure $\nu_q$ on $\operatorname{supp}(\mu)$ and constants $\underline{K}, \overline{K} > 0$ and $\delta > 0$ such that for every $x \in \operatorname{supp}(\mu)$ and every $0 < r < \delta$,

$$\underline{K}\ \mu(B(x,r))^q\ (2r)^{B_\mu(q)} \le \nu_q(B(x,r)) \le \overline{K}\ \mu(B(x,r))^q\ (2r)^{B_\mu(q)}$$

to obtain the following result which provides a positive answer to Olsen's questions [13, Question 7.6] and [14, Question 4.1.12] in a more general framework.

**Theorem 1** *Let $q \in \mathbb{R}$ and we assume that there exists a Gibbs measure $\nu_q$ for $\mu$ at $(q, B_\mu(q))$, then there exists $C > 0$ such that*

$$C\ \mathscr{P}_\mu^{q,B_\mu(q)} \le \mathscr{H}_\mu^{q,b_\mu(q)} \le \xi\ \mathscr{P}_\mu^{q,B_\mu(q)} \quad \text{on} \quad supp(\mu),$$

*where $\xi$ is the constant that appears in Besicovitch's covering theorem. In addition, if $\mu$ satisfies the doubling condition, then there exists $C_1 > 0$ such that*

$$C_1\ \mathscr{P}_\mu^{q,B_\mu(q)} \le \mathscr{H}_\mu^{q,b_\mu(q)} \le \mathscr{P}_\mu^{q,B_\mu(q)} \quad on \quad supp(\mu).$$

***Example A*** Let $\mu$ be the Bernoulli measure with parameters $P_1$ and $P_2$ which is defined by the repeated subdivision of a unit mass between the basic intervals of the pre-fractals of the middle-third Cantor set $C$. Then,

$$\mathscr{H}_\mu^{q,b_\mu(q)}(C) = \mathscr{P}_\mu^{q,B_\mu(q)}(C).$$

***Example B*** To define the Bedford–McMullen carpets, we introduce a digit set

$$A \subseteq \{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\} := I \times J,$$

where $1 < m \le n$. For each $(i, j) \in A$, we define $T_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_{i,j}(x, y) = \left(\frac{x+i}{n}, \frac{y+j}{m}\right).$$

We divide the unit square into $nm$ congruent rectangles

$$R_{i,j} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{m}, \frac{j+1}{m}\right].$$

It follows immediately from the definitions that

$$T_{i,j}\Big([0, 1] \times [0, 1]\Big) = R_{i,j}, \quad \forall (i, j) \in I \times J.$$

We let $E$ be the unique non-empty compact set which satisfies

$$E = \bigcup_{(i,j)\in A} T_{i,j}(E).$$

Sets of the form $E$ are usually known as Bedford–McMullen carpets.

We introduce a positive probability vector $\mathbf{p}$ with element $p_{i,j}$ for each $(i, j) \in A$. We also define the related probability vector $\mathbf{q}$ where

$$q_i = \sum_{j,(i,j)\in A} p_{i,j}.$$

Thus, we can define a self-affine measure $\mu$ which is the unique probability measure satisfying

$$\mu = \sum_{(i,j)\in A} p_{i,j}\ \mu \circ T_{i,j}^{-1}.$$

Now, suppose that $E$ satisfies these following disjointness conditions: for all distinct ordered pairs $(i, j) \in A$, $(i', j') \in A$, we have

$$T_{i,j}\Big([0,1]\times[0,1]\Big) \cap T_{i',j'}\Big([0,1]\times[0,1]\Big) = \emptyset \quad \text{and} \quad |i-i'| \neq 1.$$

Put $\Omega$ as the set of all infinite sequences of ordered pairs belonging to $A$,

$$\Omega = \Big\{\omega = (\underline{x}_1, \underline{x}_2, \dots) \ \Big|\ \underline{x}_i \in A,\ i = 1, 2, \dots\Big\}.$$

By the disjointness conditions, we known that there is a bijection $\pi : \Omega \to E$ defined by

$$\pi(\omega) = \liminf_{n\to+\infty} T_{\underline{x}_1} \circ T_{\underline{x}_2} \circ \dots \circ T_{\underline{x}_n}(v).$$

The value of $\pi(\omega)$ is independent of the initial value $v \in [0,1]\times[0,1]$. For $x \in E$, let us write $\omega_x = \pi^{-1}(x)$ and let us denote $\omega(n) = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)$ for all $\underline{x}_i \in A,\ i = 1, 2, \dots, n$. Let $\{a_{i,j};\ \ (i,j) \in A\}$ be a set of real numbers indexed by $A$, then write $a_{\omega(n)} = a_{\underline{x}_1} a_{\underline{x}_2} \dots a_{\underline{x}_n}$. Similarly $T_{\omega(n)}$ means that the map $T_{\underline{x}_1} \circ T_{\underline{x}_2} \circ \dots \circ T_{\underline{x}_n}$. Now, for $(x, y) \in \mathbb{R}^2$, let

$$P_H(x, y) = y \quad \text{and} \quad P_W(x, y) = x.$$

For $k \in \mathbb{N}$, we let $l = l(k) = \lfloor \sigma k \rfloor$, where $\lfloor . \rfloor$ denotes the integer part and $\sigma = \frac{\log m}{\log n}$. Then the $k$-th level *approximate square* is defined as

$$S_k(\omega) = P_W\big(r(\omega, l)\big) \times P_H\big(r(\omega, k)\big), \quad \text{where} \quad r(\omega, k) = T_{\omega(k)}\Big([0,1]\times[0,1]\Big).$$

It follows that

$$\mu(S_k(\omega)) = p_{\omega(l)}.q_{\omega(k)}.q_{\omega(l)}^{-1}.$$

Therefore, we define $\beta(q)$ ($q \in \mathbb{R}$) as the unique solution to

$$m^{-\beta(q)} \sum_{(i,j)\in A} p_{i,j}^q q_i^{(1-\sigma)q} \left(\sum_{j,(i,j)\in A} p_{i,j}^q\right)^{\sigma-1} = 1.$$

Let $\gamma_i = \sum_{j,(i,j)\in A} p_{i,j}^q$. We then define the function $\varphi_k^q(\omega)$ by

$$\varphi_k^q(\omega) = \frac{q_{\omega(l)}^q / \gamma_{\omega(l)}}{q_{\omega(k)}^{\sigma q} / \gamma_{\omega(k)}^{\sigma}}.$$

Now, we define the set $\widetilde{F}$ as a subset of $\Omega$ satisfying the following condition

$$0 < \inf_{\omega\in\widetilde{F}} \liminf_{k\to+\infty} \varphi_k^q(\omega) \le \sup_{\omega\in\widetilde{F}} \limsup_{k\to+\infty} \varphi_k^q(\omega) < +\infty. \tag{3.1}$$

Now, we write

$$P_{i,j} = p_{i,j}^q\ m^{-\beta(q)}\ q_i^{(1-\sigma)q}\ \gamma_i^{\sigma-1} \quad \text{and} \quad Q_i = \sum_{j,(i,j)\in A} P_{i,j} = m^{-\beta(q)}\ q_i^{(1-\sigma)q}\ \gamma_i^{\sigma}.$$

It follows from the definition of $\beta$ that

$$\sum_{(i,j)\in A} P_{i,j} = 1.$$

Denote by $\mu_q$ the self-affine measure generated by $P_{i,j}$ and $T_{i,j}$, then

$$\begin{aligned}
\mu_q(S_k(\omega)) &= P_{\omega(l)} \cdot Q_{\omega(k)} \cdot Q^{-1}_{\omega(l)} \\
&= \Big(p^q_{\omega(l)}\, m^{-l\beta(q)}\, q^{(1-\sigma)q}_{\omega(l)}\, \gamma^{\sigma-1}_{\omega(l)}\Big)\Big(m^{-k\beta(q)}\, q^{(1-\sigma)q}_{\omega(k)}\, \gamma^{\sigma}_{\omega(k)}\Big) \\
&\quad \Big(m^{l\beta(q)}\, q^{-(1-\sigma)q}_{\omega(l)}\, \gamma^{-\sigma}_{\omega(l)}\Big) \\
&= m^{-k\beta(q)}\Big(p_{\omega(l)}\, q_{\omega(k)}\, q^{-1}_{\omega(l)}\Big)^q \left(\frac{q^q_{\omega(l)}/\gamma_{\omega(l)}}{q^{\sigma q}_{\omega(k)}/\gamma^{\sigma}_{\omega(k)}}\right) \\
&= m^{-k\beta(q)}\mu(S_k(\omega))^q \varphi^q_k(\omega).
\end{aligned}$$

If we assume that $\mathrm{supp}(\mu) = \pi(\widetilde{F})$, $x \in \mathrm{supp}(\mu)$ and $r > 0$, then for all $\omega = \pi^{-1}(x)$, we can choose $h, k \in \mathbb{N}$ such that

$$m^{-h} < r \le m^{-h+1} \quad \text{and} \quad m^{-k} < \frac{r}{n\sqrt{2}} \le m^{-k+1}. \tag{3.2}$$

It follows from (3.2) and the disjointness conditions that

$$S_k(\omega) \subseteq B(x,r) \quad \text{and} \quad B(x,r) \cap E \subseteq S_{h-1}(\omega).$$

Now, (3.1) implies that $\mu_q$ is a Gibbs measure for $\mu$ at $(q, \beta(q))$ and

$$b_\mu(q) = \beta(q) = B_\mu(q).$$

Finally, by using Theorem 1, there exists $C_1 > 0$ such that

$$C_1\, \mathscr{P}^{q,B_\mu(q)}_\mu \le \mathscr{H}^{q,b_\mu(q)}_\mu \le \mathscr{P}^{q,B_\mu(q)}_\mu \quad \text{on} \quad \mathrm{supp}(\mu).$$

Note that this example is already discussed much more comprehensively and complexly in [14, Section 6.7]. Indeed, in [14], Olsen considers Bedford–McMullen sponges in $\mathbb{R}^d$ rather than Bedford–McMullen carpets in $\mathbb{R}^2$.

In the following, we give an example that had not already been investigated and studied for which the conditions of our main theorem are satisfied.

***Example C*** Let $p$ be an integer with $p \ge 2$. Theorem 1 applies to a family of measures supported by the full $p$-adic grid of [0, 1], namely the quasi-Bernoulli measures.

We denote $\mathscr{A}$ the set of words constructed with $\{0, 1, ..., p-1\}$ as an alphabet. Provided with concatenation, $\mathscr{A}$ is a monoid: if $a$ and $b$ are two elements of $\mathscr{A}$, we denote by $ab$ the word obtained by concatenation of $a$ and $b$. The empty word, which is the unit, is denoted by $\varepsilon$. we denote the set of words of length $n$ by $\mathscr{A}_n$. Now, we consider a sequence $\big\{\{I_a\}_{a\in\mathscr{A}_n}\big\}_{n\ge1}$ of nested finite partitions of the interval [0, 1[ in right half-open intervals: the intervals $I_{al}$, $l = 0, 1, ..., p-1$ constitute a partition of the interval $I_a$. If $x \in [0, 1[$, we denote by $I_n(x)$ the element of the $n$-th generation $\{I_a\}_{a\in\mathscr{A}}$ which contains it. The length of an interval $I$ is denoted $|I|$. We assume ($|\cdot|$ is almost multiplicative) that there is a positive constant $L$ such that

$$\forall a, b \in \mathscr{A},\; L^{-1}\,|I_a|\,|I_b| \le |I_{ab}| \le L\,|I_a|\,|I_b|\,.$$

Let us now consider the particular case where the sequence of partitions is given by the $p$-adic intervals $\left\{\{I_a\}_{a\in\mathscr{A}_n}\right\}_{n\geq 1}$:

$$\text{If} a = a_1 \ldots \cdot a_n, \quad \text{then} \quad I_a = \left[\sum_{k=1}^{n} a_k p^{-k}, \sum_{k=1}^{n} a_k p^{-k} + p^{-n}\right].$$

A probability measure on [0, 1[ is said to be quasi-Bernoulli if there exists $M > 0$ such that, for any $a$ and $b \in \mathscr{A}$,

$$M^{-1}\mu\left(I_a\right)\mu\left(I_b\right) \leq \mu\left(I_{ab}\right) \leq M\mu\left(I_a\right)\mu\left(I_b\right).$$

Let $\mu$ be a quasi-Bernoulli measure, for any $q, t \in \mathbb{R}$, we define

$$\mathscr{K}_\mu(q,t) = \limsup_{n\to+\infty} \sum_{a\in\mathscr{A}_n}^{*} \mu(I_a)^q |I_a|^t,$$

where the star $*$ means that the terms for which $\mu(I_a) = 0$ are removed (a convention valid throughout this example), and let

$$\tau_\mu(q) = \sup\left\{t \in \mathbb{R};\quad \mathscr{K}_\mu(q,t) = +\infty\right\}.$$

It follows from Bhouri [4] and Peyrière [16] that, if $\mu$ is a quasi-Bernoulli measure then there exist $K > 0$ and a measure $\nu_q$ such that for all $a \in \mathscr{A}$,

$$\frac{1}{K}\ \mu(I_a)^q\ |I_a|^{\tau_\mu(q)} \leq \nu_q(I_a) \leq K\ \mu(I_a)^q\ |I_a|^{\tau_\mu(q)}. \tag{3.3}$$

In the next, we will compare the function $\tau_\mu(q)$ to $\Lambda_\mu(q)$. For this we need the following extra condition:

$$\mu(I_a)\mu(I_b) = 0 \ \text{ whenever the intervals } I_a \text{ and } I_b \text{ are contiguous.} \tag{3.4}$$

**Lemma 1** *One has* $\tau_\mu(q) = \Lambda_\mu(q)$.

***Proof*** Let $x \in \text{supp}(\mu)$, $0 < r < \frac{1}{p}$ and $n$ such that $p^{-n-1} \leq r < p^{-n}$. To prove Lemma 1, we will prove that, there exist $a \in \mathscr{A}$ and $j \in \{0,\ldots,p-1\}$ such that

$$0 < \mu\left(I_{aj}\right) \leq \mu(B(x,r)) \leq \mu\left(I_a\right). \tag{3.5}$$

*Proof of* (3.5). Without loss of generality, we can assume that $x \in [0, 1[$. Two cases can then arise.

*Case 1.* $\mu(I_n(x)) \neq 0$.

Note that in this case, $\mu(B(x,r)) \leq \mu\left(I_n(x)\right)$. Indeed, the condition (3.4) implies that $I_n(x)$ is the only interval of the $n$-th generation, of non-zero measure, meeting the ball $B(x,r)$. Moreover, we have $I_{n+1}(x) \subset B(x,r)$ since $p^{-n-1} \leq r$. The property (3.5) is then verified if $\mu\left(I_{n+1}(x)\right) \neq 0$. It therefore remains to study the case where $\mu\left(I_{n+1}(x)\right) = 0$. Let $I_a$ be the interval $I_n(x)$, given (3.4) and the fact that $x \in \text{supp}(\mu)$, $I_{n+1}(x)$ is different from $I_{a0}$ and therefore the interval $I_{aj}$ which is just to the left of $I_{n+1}(x)$ is contained in $B(x,r)$ and has a non-zero measure. The property (3.5) is therefore satisfied in the case where $\mu\left(I_{n+1}(x)\right) = 0$.

*Case 2.* $\mu(I_n(x)) = 0$.

Since $x \in \text{supp}(\mu)$, $x$ is necessarily the left end of the interval $I_n(x)$. Consider now the interval $I_a$ of the $n$-th generation which is to the left of $I_n(x)$ and which is contiguous to it. We then have $\mu\left(I_a\right) \neq 0$, since $x \in \text{supp}(\mu)$. The real $r$ being strictly less than

$p^{-n}$, $I_a$ is therefore the only interval of order $n$, of non-zero measure, meeting $B(x,r)$. As consequence $\mu(B(x,r)) \le \mu(I_a)$. Moreover, taking into account the inequality $p^{-n-1} \le r$, we have $I_{aj} \subset B(x,r)$ for $j = p-1$. The assumption $x \in \operatorname{supp}(\mu)$ then implies that $\mu\left(I_{aj}\right) \neq 0$, which establishes the property (3.5).

The proof of Lemma 1 follows immediately from the property (3.5) (for more details see [20]). □

Let us now show that (3.3) is valid for intervals centered in the support of $\mu$, in other words $\nu_q$ is a Gibbs measure for $\mu$ at $(q, \Lambda_\mu(q))$.

We consider here only the case $q \ge 0$, the other case is treated in the same way. Let $x \in \operatorname{supp}(\mu) = \operatorname{supp}(\nu_q)$, $0 < r < p^{-1}$ and $n$ be the integer such that $p^{-n-1} \le r < p^{-n}$. It follows immediately from (3.3) that

$$\nu_q\left(I_a\right) = 0 \Leftrightarrow \mu\left(I_a\right) = 0. \tag{3.6}$$

By using (3.5) and (3.6), there is a word $a \in \mathscr{A}_n$ and a letter $j \in \{0, 1, \ldots, p-1\}$ such that

$$0 < \mu\left(I_{aj}\right) \le \mu(B(x,r)) \le \mu\left(I_a\right) \tag{3.7}$$

and

$$0 < \nu_q\left(I_{aj}\right) \le \nu_q(B(x,r)) \le \nu_q\left(I_a\right). \tag{3.8}$$

It follows from (3.3) and (3.8) that

$$\frac{1}{K}\mu\left(I_{aj}\right)^q \left|I_{aj}\right|^{\tau(q)} \le \nu_q(B(x,r)) \le K\mu\left(I_a\right)^q \left|I_a\right|^{\tau(q)}. \tag{3.9}$$

Since $\mu$ is a quasi-Bernoulli measure and $\mu\left(I_{aj}\right) \neq 0$, it results that

$$\frac{1}{K}\left(\frac{\rho}{M}\right)^q \mu\left(I_a\right)^q \left|I_{aj}\right|^{\tau(q)} \le \nu_q(B(x,r)) \le K\left(\frac{M}{\rho}\right)^q \mu\left(I_{aj}\right)^q \left|I_a\right|^{\tau(q)},$$

where $\rho = \inf\left\{\mu\left(I_b\right); b \in \{0, 1, \ldots, p-1\} \text{ and } \mu\left(I_b\right) \neq 0\right\}$. Since $r < |I_a| \le pr$, we, therefore, have by using (3.7) and (3.9),

$$\frac{1}{K}\left(\frac{\rho}{M}\right)^q \left(\frac{1}{2p}\right)^{\tau(q)} \mu(B(x,r))^q (2r)^{\tau(q)}$$
$$\le \nu_q(B(x,r)) \le K\left(\frac{M}{\rho}\right)^q (2p)^{\tau(q)} \mu(B(x,r))^q (2r)^{\tau(q)}.$$

Now, Lemma 1 implies that, there exists a constant $K_1 > 0$ such that

$$\frac{1}{K_1}\mu(B(x,r))^q (2r)^{\Lambda_\mu(q)} \le \nu_q(B(x,r)) \le K_1\mu(B(x,r))^q (2r)^{\Lambda_\mu(q)},$$

where $K_1 = \left(\frac{M}{\rho}\right)^q (2p)^{\Lambda_\mu(q)}$. Which implies that

$$b_\mu(q) = B_\mu(q) = \Lambda_\mu(q) = \tau_\mu(q)$$

and $\nu_q$ is a Gibbs measure for $\mu$ at $(q, B_\mu(q))$. Finally, it follows from Theorem 1 that, there exists $C_1 > 0$ such that

$$C_1\, \mathscr{P}_\mu^{q,B_\mu(q)} \le \mathscr{H}_\mu^{q,b_\mu(q)} \le \xi\, \mathscr{P}_\mu^{q,B_\mu(q)} \quad \text{on} \quad \operatorname{supp}(\mu),$$

where $\xi$ is the constant that appears in Besicovitch's covering theorem.

### 3.1 Moran sets

Let us recall the class of Moran sets. We denote by $\{n_k\}_{k\geq 1}$ a sequence of positive integers with $n_k \geq 2$ and $\Phi = \{\Phi_k\}_{k\geq 1}$ be a sequence of vectors satisfying

$$\Phi_k = (c_{k,1}, c_{k,2}, \ldots, c_{k,n_k}), \text{ with } 0 < c_{k,j} < 1, \ \forall k \in \mathbb{N}, \ \forall 1 \leq j \leq n_k.$$

$$D_{m,k} = \Big\{ (i_m, i_{m+1}, \ldots, i_k)\,; \quad 1 \leq i_j \leq n_j, \quad m \leq j \leq k \Big\} \quad \text{and} \quad D_k = D_{1,k}.$$

Define $D = \bigcup_{k\geq 1} D_k$.

Let $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, $\tau = (\tau_{k+1}, \ldots, \tau_m) \in D_{k+1,m}$, we denote $\sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_{k+1}, \ldots, \tau_m)$.

**Definition 1** We say that the collection $\mathscr{F} = \{J_\sigma, \sigma \in D\}$ fulfills the Moran structure if it satisfies the following conditions:

(1) For all $\sigma \in D$, $J_\sigma$ is similar to $J$, that is there exists a similarity mapping $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $S_\sigma(J) = J_\sigma$. Here we set $J_\emptyset = J$.
(2) For all $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subsets of $J_\sigma$, and satisfy that $J^\circ_{\sigma*i} \cap J^\circ_{\sigma*j} = \emptyset$ $(i \neq j)$ [we call such assumption open set condition (**OSC**)], where $A^\circ$ denotes the interior of $A$.
(3) For any $k \geq 1, \sigma \in D_{k-1}$ and $1 \leq j \leq n_k$, $c_{k,j} = \frac{|J_{\sigma*j}|}{|J_\sigma|}$, $1 \leq j \leq n_k$, where $|A|$ denotes the diameter of $A$.

Let $\mathscr{F} = \mathscr{F}(J, \{n_k\}, \{\Phi_k\})$ be a collection having Moran structure. The set $E(\mathscr{F}) = \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} J_\sigma$ is called a Moran set determined by $\mathscr{F}$. It is convenient to denote $M(J, \{n_k\}, \{\Phi_k\})$ the collection of Moran sets determined by $J$, $\{n_k\}$ and $\{\Phi_k\}$.

***Remark 1*** If $\lim_{k\to+\infty} \sup_{\sigma\in D_k} |J_\sigma| > 0$, then E contains interior points. Thus the measure and dimension properties will be trivial. We assume therefore $\lim_{k\to+\infty} \sup_{\sigma\in D_k} |J_\sigma| = 0$.

Now, we consider a class of Moran sets $E$ which satisfy a special property called the strong separation condition (**SSC**, which is stronger than **OSC**), i.e., take $J_\sigma \in \mathscr{F}$. Let $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ be the $n_{k+1}$ basic sets of order $k+1$ contained in $J_\sigma$, then we assume that for all $1 \leq i \neq j \leq n_{k+1} - 1$, $\text{dist}(J_{\sigma*i}, J_{\sigma*j}) \geq \Delta_k |J_\sigma|$, where $(\Delta_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers, such that

$$0 < \Delta = \inf_{k\in\mathbb{N}} \Delta_k < 1.$$

Then the assumption $\lim_{k\to+\infty} \sup_{\sigma\in D_k} |J_\sigma| = 0$ follows.

If we ask $c_{k,j} = c_k$ for all $1 \leq j \leq n_k$, where $\{c_k\}_{k\geq 1}$ is a sequence of positive numbers, we can get the Moran structure and Moran sets. In this situation, we call them by homogeneous Moran structure and the collection of Moran sets, and denote by $\mathscr{F} = \mathscr{F}(J, \{n_k\}, \{c_k\})$ and $\mathscr{M} = \mathscr{M}(J, \{n_k\}, \{c_k\})$.

### 3.2 Moran measure

Let $\left\{P_{i,j}\right\}_{j=1}^{n_i}$ be probability vectors, i.e., $P_{i,j} > 0$ and $\sum_{j=1}^{n_i} P_{i,j} = 1 (i = 1, 2, \ldots)$, suppose that $P_0 = \inf\left\{P_{i,j}\right\} > 0$. Let $\mu$ be a mass distribution on $E$ such that for any $J_\sigma$ $(\sigma \in D_k)$ $\mu(J_\sigma) = P_{1,\sigma_1} P_{2,\sigma_2} \cdots P_{k,\sigma_k}$ and $\mu\left(\sum_{\sigma\in D_k} J_\sigma\right) = 1$, we call $\mu$ a Moran measure on $E$.

For $q \in \mathbb{R}$, we define the following functions

$$\beta_k(q) = \frac{\sum_{m=1}^{k} \log\left(\sum_{j=1}^{n_m} P_{m,j}^q\right)}{-\log(c_1 \cdots c_k)},$$
$$\underline{\beta}(q) = \liminf_{k\to+\infty} \beta_k(q) \quad \text{and} \quad \overline{\beta}(q) = \limsup_{k\to+\infty} \beta_k(q).$$

In the following theorem, we find an explicit formula for the multifractal Hausdorff and packing function dimensions of a homogeneous Moran set satisfying the strong separation condition.

**Theorem 2** *Let $E$ be a homogeneous Moran set satisfying (**SSC**) and $\mu$ be the Moran measure on $E$. Then for all $q \in \mathbb{R}$, we have*

$$b_\mu(q) = \underline{\beta}(q) \quad \textit{and} \quad B_\mu(q) = \overline{\beta}(q) = \Lambda_\mu(q).$$

***Remark 2*** The results developed by Beak in [1] and Feng et al. in [10] are obtained as a special case of the multifractal theorems when $q$ equals 0.

***Remark 3*** Let $E$ be a homogeneous Moran set satisfying (**SSC**) and $\mu$ be the Moran measure on $E$. If the limit $\lim_{k\to+\infty} \beta_k(q) = \beta(q)$ exists, and for all $k \geq 1$, $k(\beta(q) - \beta_k(q)) < +\infty$, then by using [24, Proposition 3.1] there exists a probability measure $\nu_q$ supported by $E$ such that for any $k \geq 1$ and $\sigma_0 \in D_k$,

$$\nu_q\left(J_{\sigma_0}\right) = \frac{\mu\left(J_{\sigma_0}\right)^q \left|J_{\sigma_0}\right|^{\beta(q)}}{\sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)}}.$$

It follows from $k(\beta(q) - \beta_k(q)) < +\infty$ for all $k \geq 1$ that

$$0 < \liminf_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} \leq \limsup_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} < +\infty.$$

Now, by using the strong separation condition, we have

$$b_\mu(q) = \beta(q) = B_\mu(q)$$

and $\nu_q$ is a Gibbs measure for $\mu$ at $(q, B_\mu(q))$ (it is the case of [6, 7]) which implies that the conditions in Theorem 1 are satisfied.

## 4 Proof of main results

### 4.1 Proof of Theorem 1

This theorem follows immediately from the following lemma.

**Lemma 2** *For any $q \in \mathbb{R}$, there exist two constants $K_1 > 0$ and $K_2 > 0$ such that*

$$\mathscr{P}_\mu^{q,B_\mu(q)} \leq K_2\, \nu_q \quad \text{and} \quad K_1\, \nu_q \leq \mathscr{H}_\mu^{q,b_\mu(q)} \quad \text{on} \quad supp(\mu).$$

***Proof*** Fix $\delta > 0$ and let $\Big(B(x_i, r_i)\Big)_{i\in\mathbb{N}}$ be a centered $\delta$-covering of $\operatorname{supp}(\mu)$. Then

$$\begin{aligned}\nu_q(\operatorname{supp}(\mu)) &\le \sum_i \nu_q\left(B(x_i, r_i)\right)\\ &\le \overline{K} \sum_i \mu\left(B(x_i, r_i)\right)^q (2r_i)^{B_\mu(q)}\\ &= \overline{K} \sum_i \mu\left(B(x_i, r_i)\right)^q (2r_i)^{b_\mu(q)}.\end{aligned}$$

Consequently

$$\frac{1}{\overline{K}}\,\nu_q(\operatorname{supp}(\mu)) \le \overline{\mathscr{H}}^{q,b_\mu(q)}_{\mu,\delta}(\operatorname{supp}(\mu)) \le \overline{\mathscr{H}}^{q,b_\mu(q)}_{\mu}(\operatorname{supp}(\mu)) \le \mathscr{H}^{q,b_\mu(q)}_{\mu}(\operatorname{supp}(\mu)).$$

Let $F$ be a closed subset of $\operatorname{supp}(\mu)$. For $\delta > 0$ write

$$B(F, \delta) = \Big\{x \in \operatorname{supp}(\mu);\quad \operatorname{dist}(x, F) \le \delta\Big\}.$$

Since $F$ is closed, $B(F, \delta) \searrow F$ for $\delta \searrow 0$. Then for all $\varepsilon > 0$, there exists $\delta_0$ satisfying

$$\nu_q\big(B(F, \delta)\big) \le \nu_q(F) + \varepsilon, \quad \forall\, 0 < \delta < \delta_0.$$

Now, fix $\delta > 0$ and let $\Big(B(x_i, r_i)\Big)_{i\in\mathbb{N}}$ be a centered $\delta$-packing of $F$. Observing that

$$\begin{aligned}\underline{K} \sum_i \mu\left(B(x_i, r_i)\right)^q (2r_i)^{B_\mu(q)} &\le \sum_i \nu_q\Big(B(x_i, r_i)\Big)\\ &\le \nu_q\big(B(F, \delta)\big) \le \nu_q(F) + \varepsilon\\ &\le \nu_q(\operatorname{supp}(\mu)) + \varepsilon.\end{aligned}$$

It results that

$$\underline{K}\, \overline{\mathscr{P}}^{q,B_\mu(q)}_{\mu}(F) \le \Big(\nu_q(\operatorname{supp}(\mu)) + \varepsilon\Big).$$

Letting $\varepsilon \downarrow 0$, now yields

$$\underline{K}\, \mathscr{P}^{q,B_\mu(q)}_{\mu}(\operatorname{supp}(\mu)) \le \overline{\mathscr{P}}^{q,B_\mu(q)}_{\mu}(\operatorname{supp}(\mu)) \le \nu_q(\operatorname{supp}(\mu))$$

which proves the desired result with $K_2 = \frac{1}{\underline{K}}$ and $K_1 = \frac{1}{\overline{K}}$. □

### 4.2 Proof of Theorem 2

We present the tools, as well as the intermediate results, which will be used in the proof of our main result. First, we express the multifractal Hausdorff and packing measures of a homogeneous Moran set as the explicit form with $n_k$, $c_k$ and $P_{i,j}$.

**Proposition 1** *Let $E$ be a homogeneous Moran set satisfying (**SSC**) and $\mu$ be the Moran measure on $E$. Then for all $q, t \in \mathbb{R}$,*

(1) *there exists $A > 0$ such that*

$$A \liminf_{k\to\infty} \prod_{m=1}^{k} \sum_{j=1}^{n_m} P^q_{m,j} c^t_m \le \mathscr{H}^{q,t}_{\mu}(E) \le \liminf_{k\to\infty} \prod_{m=1}^{k} \sum_{j=1}^{n_m} P^q_{m,j} c^t_m.$$

(2) *If* $\limsup_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t = 0$ *or* $+\infty$ *then* $\mathscr{P}_\mu^{q,t}(E) = 0$ *or* $+\infty$ *which implies that* $\overline{\mathscr{P}}_\mu^{q,t}(E) = 0$ *or* $+\infty$.
(3) *There exist* $C$ *and* $c_p > 1$ *such that*

$$\max\Big(1, C^{2q}\Big)c_p^{-1}\limsup_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t \le \mathscr{P}_\mu^{q,t}(E)$$
$$\le \overline{\mathscr{P}}_\mu^{q,t}(E) \le c_p \limsup_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t.$$

***Proof*** The verification of this proposition now follows routinely from the theory described by Wu et al. [24, Propositions 3.3 and 3.4] and [22, Theorem 1]. □

***Remark 4*** For any $k \ge 1$ and $\sigma \in D_{k-1}$, $J_\sigma, J_{\sigma*1}, \ldots, J_{\sigma*n_k}$ are arranged from the left to the right, $J_{\sigma*1}$ and $J_\sigma$ have the same left endpoint, $J_{\sigma*n_k}$ and $J_\sigma$ have the same right endpoint, and the lengths of the gaps between any two consecutive sub-intervals are equal. We denote the length of one of the gaps by $y_k$. Motivated by some results developed in [17, 18], we conjecture that if $y_{k+1} \le y_k$ (or $y_{k+2} \le y_k$ and $y_{k+3} \le y_k$) for all $k \ge 1$ then

$$\mathscr{H}_\mu^{q,b_\mu(q)}(E) = \liminf_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^{b_\mu(q)}.$$

Now, we define the following auxiliary dimensions

$$\underline{\varphi}(q) = \inf\Big\{t \in \mathbb{R};\quad \liminf_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t = 0\Big\}$$
$$= \sup\Big\{t \in \mathbb{R};\quad \liminf_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t = +\infty\Big\}$$

and

$$\overline{\varphi}(q) = \inf\Big\{t \in \mathbb{R};\quad \limsup_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t = 0\Big\}$$
$$= \sup\Big\{t \in \mathbb{R};\quad \limsup_{k\to\infty}\prod_{m=1}^{k}\sum_{j=1}^{n_m} P_{m,j}^q c_m^t = +\infty\Big\}.$$

It follows from Proposition 1 that, for all $q \in \mathbb{R}$

$$b_\mu(q) = \underline{\varphi}(q) \quad \text{and} \quad B_\mu(q) = \overline{\varphi}(q) = \Lambda_\mu(q).$$

Theorem 2 is a consequence of the following proposition.

**Proposition 2** *Let* $E$ *be a homogeneous Moran set satisfying* (**SSC**) *and* $\mu$ *be the Moran measure on* $E$. *Then for all* $q \in \mathbb{R}$, *we have*

$$\underline{\varphi}(q) = \underline{\beta}(q) \quad \text{and} \quad \overline{\varphi}(q) = \overline{\beta}(q).$$

***Proof*** Let $t > \underline{\beta}(q)$, then there exists a subsequence $(k_i)_i$ such that

$$t > \frac{\sum_{m=1}^{k_i} \log\left(\sum_{j=1}^{n_m} P_{m,j}^q\right)}{-\log(c_1 \cdots c_{k_i})}.$$

Which implies that

$$\prod_{m=1}^{k_i} \sum_{j=1}^{n_m} P_{m,j}^q c_m^t \le 1 \quad \text{and} \quad \liminf_{k\to\infty} \prod_{m=1}^{k} \sum_{j=1}^{n_m} P_{m,j}^q c_m^t \le 1$$

which clearly implies that $\underline{\varphi}(q) \le \underline{\beta}(q)$. The proof of the other inequality is identical to the proof of the first statement and is therefore omitted.

We will prove the second assertion. Let $t > \overline{\beta}(q)$, there exists $N \in \mathbb{N}$ such that for all $k \ge N$ we have

$$t > \frac{\sum_{m=1}^{k} \log\left(\sum_{j=1}^{n_m} P_{m,j}^q\right)}{-\log(c_1 \cdots c_k)}.$$

Which clearly implies that

$$\prod_{m=1}^{k} \sum_{j=1}^{n_m} P_{m,j}^q c_m^t \le 1 \quad \text{and} \quad \limsup_{k\to\infty} \prod_{m=1}^{k} \sum_{j=1}^{n_m} P_{m,j}^q c_m^t \le 1.$$

It follows that $\overline{\varphi}(q) \le \overline{\beta}(q)$. The proof of the other inequality is identical to the proof of the first statement and is therefore omitted which yields the desired result. □

## 5 Some examples

In this section, more motivations and examples related to these concepts, will be discussed. In particular, some examples show that the two main results are completely related.

### 5.1 Example 1

If $J = [0, 1]$, $n_k = 2$ and $c_k = \frac{1}{3}$ for all $k \ge 1$ then the set $E$ is the middle-third Cantor set and $\mu$ is the Bernoulli measure with parameters $P_1 = P_{1,1}$ and $P_2 = P_{1,2}$. Also, Theorem 2 implies that

$$b_\mu(q) = B_\mu(q) = \frac{\log\left(P_1^q + P_2^q\right)}{\log 3}.$$

### 5.2 Example 2

Let $A = \{a, b\}$ be a two-letter alphabet, and $A^*$ the free monoid generated by $A$. Let $F$ be the homomorphism on $A^*$, defined by $F(a) = ab$ and $F(b) = a$. It is easy to see that $F^n(a) = F^{n-1}(a)F^{n-2}(a)$. We denote by $|F^n(a)|$ the length of the word $F^n(a)$, thus

$$F^n(a) = s_1 s_2 \cdots s_{|F^n(a)|}, \quad s_i \in A.$$

Therefore, as $n \to \infty$, we get the infinite sequence

$$\omega = \lim_{n \to +\infty} F^n(a) = s_1 s_2 s_3 \cdots s_n \cdots \in \{a, b\}^{\mathbb{N}}$$

which is called the Fibonacci sequence. For any $n \geq 1$, write $\omega_n = \omega|_n = s_1 s_2 \cdots s_n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter $a$ in $\omega_n$, and $|\omega_n|_b$ the number of occurrence of $b$. Then $|\omega_n|_a + |\omega_n|_b = n$. It follows from Wu [23] that $\lim_{n \to +\infty} \frac{|\omega_n|_a}{n} = \eta$, where $\eta^2 + \eta = 1$.

Let $0 < r_a < \frac{1}{2}, 0 < r_b < \frac{1}{3}, r_a, r_b \in \mathbb{R}$. In the above Moran construction, let

$$|J| = 1, \quad n_k = \begin{cases} 2, & \text{if } s_k = a \\ 3, & \text{if } s_k = b \end{cases}$$

and

$$c_k = \begin{cases} r_a, & \text{if } s_k = a \\ \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \leq j \leq n_k.$$

Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E := E(\omega) = (J, \{n_k\}, \{c_k\})$. By the construction of $E$, we have

$$|J_\sigma| = r_a^{|\omega_k|_a} r_b^{|\omega_k|_b}, \quad \forall \sigma \in D_k.$$

Let $P_a = \left(P_{a_1}, P_{a_2}\right), P_b = \left(P_{b_1}, P_{b_2}, P_{b_3}\right)$ be probability vectors, i.e.,

$$P_{a_i} > 0, \quad P_{b_i} > 0, \quad \text{and} \quad \sum_{i=1}^{2} P_{a_i} = 1, \quad \sum_{i=1}^{3} P_{b_i} = 1.$$

For any $k \geq 1$ and any $\sigma \in D_k$, we know $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ where

$$\sigma_k \in \begin{cases} \{1, 2\}, & \text{if } s_k = a \\ \{1, 2, 3\}, & \text{if } s_k = b. \end{cases}$$

For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$, we define $\sigma(a)$ as follows: let $\omega_k = s_1 s_2 \cdots s_k$ and $e_1 < e_2 < \cdots < e_{|\omega_k|_a}$ be the occurrences of the letter a in $\omega_k$, then $\sigma(a) = \sigma_{e_1} \sigma_{e_2} \cdots \sigma_{e_{|\omega_k|_a}}$. Similarly, let $\delta_1 < \delta_2 < \cdots < \delta_{|\omega_k|_b}$ be the occurrences of the letter b in $\omega_k$, then $\sigma(b) = \sigma_{\delta_1} \sigma_{\delta_2} \cdots \sigma_{\delta_{|\omega_k|_b}}$.

Let

$$P_{\sigma(a)} = P_{\sigma_{e_1}} P_{\sigma_{e_2}} \cdots P_{\sigma_{e|\omega_k|_a}} \quad \text{and} \quad P_{\sigma(b)} = P_{\sigma_{\delta_1}} P_{\sigma_{\delta_2}} \cdots P_{\sigma_{\delta|\omega_k|_b}}.$$

Obviously

$$\sum_{\sigma \in D_k} P_{\sigma(a)} P_{\sigma(b)} = 1.$$

Let $\mu$ be a mass distribution on E, such that for any $\sigma \in D_k$,

$$\mu\left(J_\sigma\right) = P_{\sigma(a)} P_{\sigma(b)}.$$

It follows that

$$\beta_k(q) = -\frac{|\omega_k|_a \log\left(\sum_{i=1}^{2} P_{a_i}^q\right) + |\omega_k|_b \log\left(\sum_{i=1}^{3} P_{b_i}^q\right)}{|\omega_k|_a \log r_a + |\omega_k|_b \log r_b}.$$

By using Theorem 2 we have

$$b_\mu(q) = \lim_{k\to+\infty} \beta_k(q) = -\frac{\log\left(\sum_{i=1}^{2} P_{a_i}^q\right) + \eta \log\left(\sum_{j=1}^{3} P_{b_j}^q\right)}{\log r_a + \eta \log r_b} = B_\mu(q),$$

where $\eta^2 + \eta = 1$.

Given $q \in \mathbb{R}$, it follows from Wu [23, Proposition 3.1] that there exists a probability measure $\nu_q$ supported by $E$ such that for any $k \geq 1$ and $\sigma_0 \in D_k$,

$$\nu_q\left(J_{\sigma_0}\right) = \frac{\mu\left(J_{\sigma_0}\right)^q \left|J_{\sigma_0}\right|^{\beta(q)}}{\sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)}},$$

where

$$b_\mu(q) = \lim_{k\to+\infty} \beta_k(q) = B_\mu(q) = \beta(q).$$

By a simple calculation, we have

$$\sum_{\sigma\in D_k} \left(P_{\sigma(a)} P_{\sigma(b)}\right)^q \left|J_\sigma\right|^{\beta_k(q)} = 1$$

which implies that $\beta(q) - \beta_k(q) = O(\frac{1}{k})$ and

$$\sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} = \left|J_\sigma\right|^{\beta(q)-\beta_k(q)} \geq \left(\min\left\{r_a, r_b\right\}\right)^{k(\beta(q)-\beta_k(q))},$$

which gives that

$$\liminf_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} > 0.$$

By a similar way, we obtain

$$\limsup_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} < +\infty.$$

This implies that

$$0 < \liminf_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} \leq \limsup_{k\to+\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} < +\infty. \tag{5.1}$$

Now, (5.1) gives that $\nu_q$ is a Gibbs measure for $\mu$ at $(q, B_\mu(q))$ and then the conditions of Theorem 1 are satisfied.

### 5.3 Example 3

A particular homogeneous Moran set $E$ satisfying (**SSC**) and a Moran measure $\mu$ on $E$ may now be defined as follows: Let

$$n_k = \begin{cases} 2, & k \text{ is odd number,} \\ 3, & k \text{ is even number.} \end{cases}$$

and

$$c_k = \begin{cases} r_1, \; k \;\text{ is odd number,} \\ \\ r_2, \; k \;\text{ is even number,} \end{cases}$$

where $0 < r_1 < \frac{1}{2}$ and $0 < r_2 < \frac{1}{3}$. Put

$$P_{k,j} = \begin{cases} P_{1,j}, \; k \;\text{ is odd number, } \; 1 \le j \le 2, \\ \\ P_{2,j}, \; k \;\text{ is even number, } 1 \le j \le 3, \end{cases}$$

where

$$\sum_{j=1}^{2} P_{1,j} = 1 \quad \text{and} \quad \sum_{j=1}^{3} P_{2,j} = 1.$$

We conclude that

$$\beta_k(q) = \begin{cases} -\dfrac{(k+1)\log\sum_{j=1}^{2} P_{1,j}^q + (k-1)\log\sum_{j=1}^{3} P_{2,j}^q}{(k+1)\log r_1 + (k-1)\log r_2}, \; k \;\text{ is odd number,} \\ \\ -\dfrac{\log\sum_{j=1}^{2} P_{1,j}^q + \log\sum_{j=1}^{3} P_{2,j}^q}{\log r_1 + \log r_2}, \qquad k \;\text{ is even number,} \end{cases}$$

It follows from Theorem 2 that

$$b_\mu(q) = \lim_{k\to+\infty} \beta_k(q) = -\frac{\log\sum_{j=1}^{2} P_{1,j}^q + \log\sum_{j=1}^{3} P_{2,j}^q}{\log r_1 + \log r_2} = B_\mu(q).$$

It is obvious that $k(\beta(q) - \beta_k(q)) < +\infty$ for all $k \ge 1$ which implies, by using similar techniques of Sect. 5.2, that

$$0 < \liminf_{k\to+\infty} \sum_{\sigma\in D_k} \mu\,(J_\sigma)^q\,|J_\sigma|^{\beta(q)} \le \limsup_{k\to+\infty} \sum_{\sigma\in D_k} \mu\,(J_\sigma)^q\,|J_\sigma|^{\beta(q)} < +\infty.$$

Wu and Xiao [24, Proposition 3.1] implies that, there exists a probability measure $\nu_q$ supported by $E$ such that for any $k \ge 1$ and $\sigma_0 \in D_k$,

$$\nu_q\left(J_{\sigma_0}\right) = \frac{\mu\left(J_{\sigma_0}\right)^q \left|J_{\sigma_0}\right|^{\beta(q)}}{\sum_{\sigma\in D_k} \mu\,(J_\sigma)^q\,|J_\sigma|^{\beta(q)}}.$$

Now, it follows from Theorem 2 that $\nu_q$ is a Gibbs measure for $\mu$ at $(q, B_\mu(q))$ which gives that the conditions in Theorem 1 are satisfied.

Next, we give concrete interesting examples related to our main result and we obtain the multifractal Hausdorff and packing dimension functions of Moran measure associated with homogeneous Moran fractals for which $b_\mu(q)$ and $B_\mu(q)$ differ for all $q \ne 1$.

### 5.4 Example 4

Let $(t_k)_k$ be a sequence of integers such that

$$t_1 = 1, \quad t_2 = 3 \text{ and } t_{k+1} = 2t_k, \quad \forall\, k \geq 3.$$

In the Moran construction described in Definition 1, we define the family of parameters $n_i$, $c_i$ and $p_{i,j}$ as follows:

$$n_1 = 2, \quad n_i = \begin{cases} 3, & \text{if } t_{2k-1} \leq i < t_{2k}, \\ 2, & \text{if } t_{2k} \leq i < t_{2k+1}. \end{cases}$$

For $0 < r_a < \frac{1}{2}$ and $0 < r_b < \frac{1}{3}$, let

$$c_1 = r_a, \quad c_i = \begin{cases} r_b, & \text{if } t_{2k-1} \leq i < t_{2k}, \\ r_a, & \text{if } t_{2k} \leq i < t_{2k+1}. \end{cases}$$

Let $(P_{a,j})_{j=1}^2$ and $(P_{b,j})_{j=1}^3$ be two probability vectors. Define

$$P_{1,j} = P_{a,j}, \text{ for all } 1 \leq j \leq 2,$$

and

$$P_{i,j} = \begin{cases} P_{b,j}, & \text{if } t_{2k-1} \leq i < t_{2k}, \quad 1 \leq j \leq 3, \\ P_{a,j}, & \text{if } t_{2k} \leq i < t_{2k+1}, \quad 1 \leq j \leq 2. \end{cases}$$

If $N_k$ is the number of integers $i \leq k$ such that $P_{i,j} = P_{a,j}$, then

$$\liminf_{k \to +\infty} \frac{N_k}{k} = \frac{1}{3}, \quad \limsup_{k \to +\infty} \frac{N_k}{k} = \frac{2}{3}$$

and

$$\beta_k(q) = -\frac{\frac{N_k}{k} \log\left(\sum_{j=1}^2 P_{a,j}^q\right) + \left(1 - \frac{N_k}{k}\right) \log\left(\sum_{j=1}^3 P_{b,j}^q\right)}{\frac{N_k}{k} \log r_a + \left(1 - \frac{N_k}{k}\right) \log r_b}.$$

We can then conclude from Theorem 2 that

$$b_\mu(q) = \min\left\{-\frac{\frac{1}{3}\log\sum_{j=1}^2 P_{a,j}^q + \frac{2}{3}\log\sum_{j=1}^3 P_{b,j}^q}{\frac{1}{3}\log r_a + \frac{2}{3}\log r_b}, -\frac{\frac{2}{3}\log\sum_{j=1}^2 P_{a,j}^q + \frac{1}{3}\log\sum_{j=1}^3 P_{b,j}^q}{\frac{2}{3}\log r_a + \frac{1}{3}\log r_b}\right\}$$

and

$$B_\mu(q) = \max\left\{-\frac{\frac{1}{3}\log\sum_{j=1}^2 P_{a,j}^q + \frac{2}{3}\log\sum_{j=1}^3 P_{b,j}^q}{\frac{1}{3}\log r_a + \frac{2}{3}\log r_b}, -\frac{\frac{2}{3}\log\sum_{j=1}^2 P_{a,j}^q + \frac{1}{3}\log\sum_{j=1}^3 P_{b,j}^q}{\frac{2}{3}\log r_a + \frac{1}{3}\log r_b}\right\}.$$

### 5.5 Example 5

Let $A = \{a, b\}$ be a two-letter alphabet, $\omega = s_1 s_2 ... s_k ....$ be a sequence over A, $s_i \in A$. For any $n \geq 1$, write $\omega_n = \omega|_n = s_1 s_2 \cdots s_n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter $a$ in $\omega_n$, and $|\omega_n|_b$ the number of occurrence of $b$. Then $|\omega_n|_a + |\omega_n|_b = n$. In the above Moran construction, we take

$$J = (a, b), \quad n_k = \begin{cases} 2, & \text{if } s_k = a \\ 3, & \text{if } s_k = b, \end{cases}$$

$$c_{k_j} = c_k = \begin{cases} r_a, & \text{if } s_k = a \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \leq j \leq n_k.$$

where $0 < r_a < \frac{1}{2}, 0 < r_b < \frac{1}{3}$. Let $P_a = \left(P_{a_1}, P_{a_2}\right), P_b = \left(P_{b_1}, P_{b_2}, P_{b_3}\right)$ be probability vectors such that

$$P_{a_1} \geq P_{b_1} \geq P_{b_2} \geq P_{b_3} \geq P_{a_2} \quad \text{and} \quad P_{b_1} \geq \frac{1}{e}. \tag{5.2}$$

By a simple calculation, we get

$$\begin{aligned} \beta_k(q) &= -\frac{\log\left(\sum_1^2 P_{a_i}^q\right) + \frac{k-|\omega_k|_a}{|\omega_k|_a} \log\left(\sum_i^3 P_{b_i}^q\right)}{\log r_a + \frac{k-|\omega_k|_a}{|\omega_k|_a} \log r_b} \\ &= \frac{\tau_k(a)\left(\log\left(\sum_i^3 P_{b_j}^q\right) - \log\left(\sum_i^2 P_{a_i}^q\right)\right) - \log\left(\sum_1^3 P_{b_j}^q\right)}{\tau_k(b)\left(\log r_a - \log r_b\right) + \log r_b}, \end{aligned}$$

where $\tau_k(a) = \frac{|\omega_k|_a}{k}$. Write $\underline{\tau}(a) = \liminf_{k\to\infty} \tau_k(a)$ and $\overline{\tau}(a) = \limsup_{k\to\infty} \tau_k(a)$. Using (5.2), we have

(1) if $q < 1$, then

$$\log(r_a) \log\left(\sum_i^3 P_{b_j}^q\right) - \log(r_b) \log\left(\sum_i^2 P_{a_i}^q\right) \geq 0,$$

$$b_\mu(q) = \frac{\underline{\tau}(a)\left(\log\left(\sum_i^3 P_{b_j}^q\right) - \log\left(\sum_i^2 P_{a_i}^q\right)\right) - \log\left(\sum_1^3 P_{b_j}^q\right)}{\underline{\tau}(a)\left(\log r_a - \log r_b\right) + \log r_b}$$

and

$$B_\mu(q) = \frac{\overline{\tau}(a)\left(\log\left(\sum_i^3 P_{b_j}^q\right) - \log\left(\sum_i^2 P_{a_i}^q\right)\right) - \log\left(\sum_1^3 P_{b_j}^q\right)}{\overline{\tau}(a)\left(\log r_a - \log r_b\right) + \log r_b}.$$

(2) If $q \geq 1$, then

$$\log(r_a) \log\left(\sum_i^3 P_{b_j}^q\right) - \log(r_b) \log\left(\sum_i^2 P_{a_i}^q\right) \leq 0,$$

$$b_\mu(q) = \frac{\overline{\tau}(a)\left(\log\left(\sum_i^3 P_{b_j}^q\right) - \log\left(\sum_i^2 P_{a_i}^q\right)\right) - \log\left(\sum_1^3 P_{b_j}^q\right)}{\overline{\tau}(a)\left(\log r_a - \log r_b\right) + \log r_b}$$

and

$$B_\mu(q) = \frac{\underline{\tau}(a)\left(\log\left(\sum_i^3 P_{b_j}^q\right) - \log\left(\sum_i^2 P_{a_i}^q\right)\right) - \log\left(\sum_1^3 P_{b_j}^q\right)}{\underline{\tau}(a)\left(\log r_a - \log r_b\right) + \log r_b}.$$

### 5.6 Example 6

Let $J = [0, 1]$, $n_i = 2$ and $\mathcal{N} := \{N_k\}_{k\in\mathbb{N}}$ be an increasing sequence of integers with $N_0 = 0$ and $\lim_{k\to+\infty} \frac{N_{k+1}}{N_k} = +\infty$. Fix four real numbers $A, B, p, \widetilde{p}$ with $A > B > 2$ and $0 < p, \widetilde{p} \le 1/2$. Now for every $i \in \mathbb{N}$, we define $c_i$ and $\{P_{i,j}\}_{1\le j\le n_i}$ as follows:

$$c_i = \begin{cases} 1/A, \text{ if } N_{2k} < i \le N_{2k+1}, \\ 1/B, \text{ if } N_{2k+1} < i \le N_{2k+2}. \end{cases} \quad \text{and}$$

$$P_{i,j} = \begin{cases} p, & \text{if } N_{2k} < i \le N_{2k+1} \text{ and } j = 1, \\ 1-p, & \text{if } N_{2k} < i \le N_{2k+1} \text{ and } j = 2, \\ \widetilde{p}, & \text{if } N_{2k+1} < i \le N_{2k+2} \text{ and } j = 1, \\ 1-\widetilde{p}, & \text{if } N_{2k+1} < i \le N_{2k+2} \text{ and } j = 2. \end{cases}$$

Now, we can define a homogeneous Moran set $E$ satisfying (**SSC**) and a Moran measure $\mu$ on it. Define the functions

$$\begin{aligned} \beta_1 : \mathbb{R} &\to \mathbb{R} \\ q &\mapsto \frac{\log(p^q + (1-p)^q)}{\log A}, \end{aligned}$$

and

$$\begin{aligned} \beta_2 : \mathbb{R} &\to \mathbb{R} \\ q &\mapsto \frac{\log(\widetilde{p}^q + (1-\widetilde{p})^q)}{\log B}. \end{aligned}$$

We can conclude that

$$b_\mu(q) = \min\{\beta_1(q), \beta_2(q)\} \text{ and } B_\mu(q) = \max\{\beta_1(q), \beta_2(q)\}.$$

If $-\frac{\log(1-\widetilde{p})}{\log B} < -\frac{\log p}{\log A}$, the method in [2, 26] can follows that:

$$\text{for all } \alpha \in \left[-\frac{\log(1-\widetilde{p})}{\log B}, \min\{-\frac{\log p}{\log A}, -\frac{\log \widetilde{p}}{\log B}\}\right] \text{ we have } f_\mu(\alpha) = b_\mu^*(\alpha),$$

and

$$F_\mu(\alpha) = B_\mu^*(\alpha), \quad \text{when } \alpha \in \{B'_\mu(q) : q \in \mathbb{R} \text{ and } B_\mu \text{ is differentiable at } q\}.$$

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