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Controllability of retarded semilinear systems with control delay

S. Kumar[1]

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Abstract

This paper presents the controllability of a family of linear and semilinear systems with control delay in Hilbert spaces. Firstly, the approximate controllability of the linear control delay system is proved by assuming that the linear system without control delay is completely controllable. Then, Nemytskii operators have been constructed associated to control operators and the nonlinear function. The approximate controllability of the retarded semilinear system is established by using the Rothe type fixed point theorem. The applications of results are explained through examples of parabolic and hyperbolic partial differential equations.

Keywords Approximate controllability · Delay differential systems · Control delays · Complete controllability

Mathematics Subject Classification 93B05 · 93C43

1 Introduction

The real life processes observe the aftereffect in the dynamics and hence better represented by mathematical models of delay differential equations. The study of delay differential equations in control theory has adjoined real world approach to mathematical sciences more efficiently. Motivated by this, this work presents the controllability results for a class of retarded linear and semilinear systems with control delay. There are several mathematical contributions to the existence theory and controllability of linear and semilinear delay differential systems since its inception. In the pioneering work of Jeong et al. [1], and Dubey and Bahuguna [2], the existence and regularity of the solution for a class of retarded semilinear differential equations with nonlocal history conditions are obtained by fixed point theorem. Hernandez et al. [3], and with O'Regan [4] introduced the state-dependent nonlocal condition, and proved the existence and uniqueness of the solution.

Control theory of differential equations has wide range of real world applications and thus acquired significant attention from mathematicians. It is dealt as steering the system

✉ S. Kumar
suman@igntu.ac.in; ksumanrm@gmail.com

1 Department of Mathematics, IGNTU Amarkantak, Anuppur, Madhya Pradesh, India

from the initial state to the desired state under controlling factor. The appearance of delay in almost every natural phenomena has triggered the study on controllability of functional differential systems. Its applications cover engineering, physical systems, biological phenomena, ecology, finance *etc*. The significant contributions of Dauer and Mahmudov [5, 6] established sufficient conditions for the approximate and complete controllability of semilinear functional differential systems in Hilbert spaces. Sukavanam and Tomar [7] imposed the inclusion range condition of nonlinearity and control operator. Jeong et al. [8, 9] weakened the uniform boundedness of the nonlinearity and proved the inclusion of reachable sets of linear control system into the semilinear retarded functional differential equations. Henríquez and Prokopczyk [10] studied the controllability and stabilizability for a time-varying linear abstract control system with distributed delay in the state variables. Vijayakumar *et al.* [11] proved the controllability of second-order evolution impulsive control systems using the measure of noncompactness and Mönch fixed point theorem. Vijayakumar and Murugesu [12] presented the existence and controllability of second-order differential inclusions in Banach spaces by applying the weak topology and Glicksberg-Ky Fan fixed point theorem. Recently, Kim et al. [13] considered Fredholm alternative for nonlinear operators and proved approximate controllability. Sakthivel et al. [14] presented the approximate controllability of deterministic and stochastic nonlinear impulsive differential equations with resolvent operators and unbounded delay. Shukla et al. [15] applied sequence method to establish the approximate controllability of state delay semilinear system. Further, Sukavanam with coresearchers [16, 17] studied the retarded stochastic systems for approximate and complete controllability properties. Vijayakumar with co-researchers [18, 19] established the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness and fixed point theory. Nisar and Vijayakumar [20], and Kavitha et al. [21] studied Hilfer fractional differential equations with infinite delay by employing fractional calculus and fixed point theory. Hernandez et al. [22] studied the approximate controllability of a general class of first order abstract control problems with state-dependent delay. Recently, Haq and Sukavanam [23] established mild solution and approximate controllability of retarded semilinear differential equations with control delays and nonlocal conditions. Kumar and Abdal [24] applied Sadovskii's fixed point theorem to explore the approximate controllability for systems with instantaneous and non-instantaneous impulses.

In this paper, fixed point theory is applied to establish the controllability of a class of retarded semilinear systems. The novelty of this work is the consideration of time-varying control and state delays for both linear and semilinear systems. It provides feedback information to the system for further decision of control application. So, the system in consideration would represent real time observation of various practical processes. The controllability of linear control delay system is proved via sequence method. The central discussion for the semilinear systems requires the construction of Nemytskii operators. The main controllability result is established via the Rothe type fixed point theorem.

2 System description

Let X and U be Hilbert spaces of state and control with norm $||\cdot||$ and $||\cdot||_U$, respectively. The norm $||\cdot||_{op}$ denotes the operator norm between specified normed linear spaces. Define $C([-\tau, t]; X) := \{x : [-\tau, t] \to X | x \text{ is continuous}\}$ for $\tau > 0$ with the norm

$$||x||_{C([-\tau,t];X)} := \sup_{-\tau \le \xi \le t} ||x(\xi)||.$$

Let $\alpha : [0, T] \to [-\tau, T]$ be a nonexpansive continuous function satisfying $\alpha(t) \le t$ with range denoted by $\mathcal{R}(\alpha)$. The functions with the aftereffect due to α lie in $L^2(\mathcal{R}(\alpha); X)$.

Let us consider the class of abstract semilinear retarded control systems as follows

$$\dot{x}(t) = Ax(t) + (B_0 u)(t) + (B_1 u)(\alpha(t)) + f(t, x(\alpha(t)), u(t)), t \in (0, T], \tag{1a}$$

$$x_0 = \phi \text{ on } [-\tau, 0], \tag{1b}$$

where $x \in C([-\tau, T]; X)$ is the trajectory of the system, $u \in L^2([0, T]; U)$ is control, $A : D(A) \subset X \to X$ is closed linear operator, $B_0 : L^2([0, T]; U) \to L^2([0, T]; X)$ and $B_1 : L^2([0, T]; U) \to L^2(\mathcal{R}(\alpha); X)$ are linear control operators, $f : [0, T] \times X \times U \to X$ is nonlinear map, and $\phi \in C([-\tau, 0]; X)$ is initial trajectory.

Let us put the following fundamental assumptions:

(A1) The operator A generates a C_0-semigroup $\{S(t)\}_{t\ge 0}$ on X.

(A2) Let $M_0 \ge 1$ be such that $||S(t)||_{op} \le M_0$ for $t \in [0, T]$.

(A3) The linear control operators B_0 and B_1 are bounded, and let

$$M_B := \max\{||B_0||_{op}, ||B_1||_{op}\}.$$

(A4) The nonlinear map f is integrable in $[0, T]$, and Lipschitz map in $X \times U$, *i.e.* there exists constant $L_f > 0$ such that

$$||f(t, x, u) - f(t, y, v)|| \le L_f (||x - y|| + ||u - v||_U)$$

for $(x, u),\ (y, v) \in X \times U$.

The theory of abstract differential equations has defined classical and mild solutions. The existence and uniqueness of the solution of semilinear systems without control have been extensively presented in [2, 25–27] and with control in [7, 22, 28], and references therein.

Definition 2.1 [7] The mild solution of (1) is a function $x \in C([-\tau, T]; X)$ given by

$$x(t) = \begin{cases} \phi(t),\ t \in [-\tau, 0], \\ S(t)\phi(0) + \int_0^t S(t-s)\,[(B_0 u)(s) + (B_1 u)(\alpha(s)) + f(s, x(\alpha(s)), u(s))]\,ds. \end{cases} \tag{2}$$

To express the dependence of the mild solution (2) on input variable u, we write $x(t) = x(t; u)$. The dependence shows the uniqueness property. The next proposition establishes that $u \mapsto x$ is a Lipschitz map.

Proposition 2.2 *Under the assumptions* $(A1) - (A4)$ *and the given initial function* $\phi \in C([-\tau, 0]; X)$*, the integral equation* (2) *satisfies*

$$||x^1 - x^2||_{C([-\tau,T];X)} \le e^{M_0 T L_f} M_0(2M_B + L_f)\sqrt{T}||u_1 - u_2||_{L^2([0,T];U)}, \tag{3}$$

where $x^i(t) = x(t; u_i)$ *is the trajectory described under the control* u_i*;* $i = 1, 2$.

Proof Let $x^1, x^2 \in C([-\tau, T]; X)$ be the mild solutions corresponding to controls $u_1, u_2 \in L^2([0, T]; U)$ for given $\phi \in C([-\tau, 0]; X)$. Then

$$x^i(t) = S(t)\phi(0) + \int_0^t S(t-s)\,[(B_0 u_i)(s) + (B_1 u_i)(\alpha(s))]\,ds + \int_0^t S(t-s) f(s, x^i(\alpha(s)), u_i(s))ds$$

implies

$$
\begin{aligned}
||x^1(t)-x^2(t)|| \le\ & M_0\int_0^t (||\,(B_0(u_1-u_2))\,(s)||+||\,(B_1(u_1-u_2))\,(\alpha(s))||)\,ds\\
&+M_0L_f\int_0^t \big(||x^1(\alpha(s))-x^2(\alpha(s))||+||u_1(s)-u_2(s)||_U\big)\,ds\\
\le\ & M_0(2M_B+L_f)\sqrt{T}||u_1-u_2||_{L^2([0,T];U)}\\
&+M_0L_f\int_0^t ||x^1(\alpha(s))-x^2(\alpha(s))||ds.
\end{aligned}
\tag{4}
$$

Let $t_1\in[0,T]$ be such that $-\tau\le\alpha(s)\le 0$ for $s\in[-\tau,t_1]$. Thus,

$$
\begin{aligned}
||x^1(\alpha(s))-x^2(\alpha(s))|| \le\ & \sup_{-\tau\le\xi\le t_1}||x^1(\alpha(\xi))-x^2(\alpha(\xi))||+\sup_{t_1\le\xi\le s}||x^1(\alpha(\xi))-x^2(\alpha(\xi))||\\
\le\ & \sup_{-\tau\le\eta\le s}||x^1(\eta)-x^2(\eta)||=||x^1-x^2||_{C([-\tau,s];X)}.
\end{aligned}
$$

From (4), we get

$$
\begin{aligned}
||x^1-x^2||_{C([-\tau,t];X)} \le\ & M_0(2M_B+L_f)\sqrt{T}||u_1-u_2||_{L^2([0,T];U)}\\
&+M_0L_f\int_0^t ||x^1-x^2||_{C([-\tau,s];X)}ds.
\end{aligned}
$$

Now, by Gronwall's inequality

$$
||x^1-x^2||_{C([-\tau,t];X)}\le e^{M_0TL_f}M_0(2M_B+L_f)\sqrt{T}||u_1-u_2||_{L^2([0,T];U)} \text{ for all } t>0.
$$

Hence, $||x^1-x^2||_{C([-\tau,T];X)}\le e^{M_0TL_f}M_0(2M_B+L_f)\sqrt{T}||u_1-u_2||_{L^2([0,T];U)}$. □

3 Approximate controllability

The approximate controllability of (1) and the associated linear control delay system are explicated in this section. It is assumed that the associated linear control system without delay is completely controllable.

The linear control delay system associated to (1) is

$$
\dot{y}(t)=Ay(t)+(B_0u)(t)+(B_1u)(\alpha(t)),\ \ t>0, \tag{5a}
$$

$$
y_0=\phi\ \ \text{on}\ \ [-\tau,0], \tag{5b}
$$

and the associated linear control system without delay as

$$
\dot{z}(t)=Az(t)+(B_0u)(t),\ \ t>0, \tag{6a}
$$

$$
z(0)=\phi(0). \tag{6b}
$$

The mild solution of (5) is

$$
y(t)=\begin{cases}\phi(t),\ \ t\in[-\tau,0],\\ S(t)\phi(0)+\int_0^t S(t-s)\,[(B_0u)(s)+(B_1u)(\alpha(s))]\,ds,\ t>0\end{cases} \tag{7}
$$

and of (6) is

$$
z(t)=S(t)\phi(0)+\int_0^t S(t-s)(B_0u)(s)ds. \tag{8}
$$

The approximate controllability of a control system is analogous to steering the system from an initial state to the vicinity of the desired state in finite time by the action of control.

Definition 3.1 The control system (1) is called approximately controllable on $[0, T]$ from an initial state $\phi(0)$ to any desired state $\hat{x} \in X$ if for every $\varepsilon > 0$ there exists a control $u \in L^2([0, T]; U)$ such that $x \in C([-\tau, T]; X)$ satisfies $||x(T) - \hat{x}|| < \varepsilon$.

The complete controllability is defined in many different forms, such as complete controllability at time T, complete controllability at any time, exact controllability and approximate controllability at any time. Chalishajar et al. [29] have defined as: the system (1) is said to be completely controllable on $[0, T]$ if for any $x_0, x_1 \in X$ and any fixed T, there exists a control $u \in L^2([0, T]; U)$ such that the corresponding solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$. This definition is equivalent to the exact controllability. In this article, the complete controllability of linear control system will be considered as defined by Fattorini [30].

Definition 3.2 The linear control system (6) is called completely controllable if given $\hat{z}$, $\varepsilon > 0$ there exists $u \in L^2([0, T]; U)$ such that the solution of (6) satisfies $||z(t_0) - \hat{z}|| \le \varepsilon$ for some $t_0 > 0$, depending upon u, ε.

The final time t_0 directly depends upon the given initial and desired states, however inversely on ε and control u. If t_0 is independent of u, ε then (6) is said to be completely controllable at time t_0. It is also called the approximate controllability at time t_0.

The approximate controllability of linear control systems (5) and (6) is defined similarly as Definition 3.1. The solution and controllability properties of (6) are well-explained in the books by Curtain and Zwart [31], and Zabczyk [32]. We assume that:

(A5) the linear control system (6) is completely controllable.

Theorem 3.3 *Under assumptions* $(A1) - (A3)$ *and* $(A5)$*, the linear control delay system (5) is approximately controllable.*

Proof Let $\hat{y}$ be the desired state and $\varepsilon > 0$ be given. We have to show that there exists a control $u \in L^2([0, T]; U)$ such that $||y(T; u) - \hat{y}|| < \varepsilon$.

From equations (7) and (8), we have $y(t; u) = z(t; u) + \xi_t$, where

$$\xi_t = \int_0^t S(t - s)(B_1 u)(\alpha(s))ds, \ t > 0.$$

By the property of α, there is $\hat{t} \ge 0$ such that $\alpha(s) \le 0$ for $s \in [-\tau, \hat{t}]$. Take a sequence $0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$ such that $\alpha(s) \le t_i$ for $s \in (0, t_{i+1}]$ and $i = 0, 1, ..., n$. Let $\hat{y}_1, ..., \hat{y}_{n+1} = \hat{y} \in X$ be given and take $\hat{z}_1 = \hat{y}_1$. Then, by complete controllability of (6), we get a control $u_1 \in L^2([0, t_1]; U)$ such that $||z(t_1, u_1) - \hat{z}_1|| < \varepsilon$. Let us take $w_1(t) = u_1(t), \ t \in [0, t_1]$. Since $y(t) \equiv z(t)$ as $\xi_t \equiv 0$ for $t \in (0, t_1]$, therefore

$$||y(t_1; w_1) - \hat{y}_1|| = ||z(t_1; u_1) - \hat{z}_1|| < \varepsilon.$$

Now, we set $\hat{z}_i = \hat{y}_i - \xi_{t_i}$ for $i = 2, ..., n + 1$; where $\xi_{t_i} = \int_0^{t_i} S(t_i - s)(B_1 w_{i-1})(\alpha(s))ds$ and define $w_i \in L^2([0, t_i]; U)$ as

$$w_i(t) = \begin{cases} w_{i-1}(t), & t \in [0, t_{i-1}] \\ u_i(t), & t \in (t_{i-1}, t_i], \end{cases} \qquad \text{for } \ u_i \in L^2([t_{i-1}, t_i]; U).$$

For $i = 2$, by approximate controllability of linear control system, there exists a control $u_2 \in L^2([t_1, t_2]; U)$ such that (8) satisfies $||z(t_2; u_2) - \hat{z}_2|| < \varepsilon$. Further, the mild solution

(7) of linear control delay system gives $y(t; w_2) = z(t; u_2) + \xi_t$ for $t \in (t_1, t_2]$. Thus

$$||y(t_2; w_2) - \hat{y}_2|| = ||z(t_2; u_2) - (\hat{y}_2 - \xi_{t_2})|| < \varepsilon.$$

Proceeding similarly, finally at $i = n + 1$, there exists $u_{n+1} \in L^2([t_n, t_{n+1}]; U)$ such that $||z(t_{n+1}; u_{n+1}) - \hat{z}_{n+1}|| < \varepsilon$. Then, from (7), we get

$$||y(t_{n+1}; w_{n+1}) - \hat{y}_{n+1}|| = ||z(t_{n+1}; u_{n+1}) - (\hat{y}_{n+1} - \xi_{t_{n+1}})|| < \varepsilon.$$

Thus, we get $u = w_{n+1} \in L^2([0, T]; U)$ satisfying $||y(T; w_{n+1}) - \hat{y}|| < \varepsilon$. This completes the proof of theorem. □

To express the trajectory from starting time to current time under some control $u \in L^2([0, T]; U)$, we shall denote the mild solution $x(t)$ as $x(0, t; u)$ for the upcoming discussion.

If we write $\mathcal{B}(t, u(t)) = (B_0u)(t) + (B_1u)(\alpha(t))$, then we can define a Nemytskii operator $\mathfrak{B}_\alpha : L^2([0, T]; U) \to L^2([0, T]; X)$ as $(\mathfrak{B}_\alpha u)(\cdot) = \mathcal{B}(\cdot, u(\cdot)) = (B_0u)(\cdot) + (B_1u)(\alpha(\cdot))$. Define a Nemytskii operator $\mathcal{F} : C([-\tau, T]; X) \times L^2([0, T]; U) \to L^1([0, T]; X)$ by $\mathcal{F}(x, u)(\cdot) = f(\cdot, x(\alpha(\cdot)), u(\cdot)) \in L^1([0, T]; X)$. Then, the mild solution (2) of semilinear retarded control system is written as

$$x(0, t; u) = S(t)\phi(0) + \int_0^t S(t - s)\,[(\mathfrak{B}_\alpha u)(s) + \mathcal{F}(x, u)(s)]\,ds$$

and the mild solution (7) of linear control delay system is written as

$$y(0, t; u) = S(t)\phi(0) + \int_0^t S(t - s)(\mathfrak{B}_\alpha u)(s)ds.$$

Proposition 3.4 *Under assumptions $(A3)$ and $(A4)$, the Nemytskii operators $\mathfrak{B}_\alpha$ and $\mathcal{F}$, respectively, satisfy*

(i) $\quad ||(\mathfrak{B}_\alpha u)||_{L^2([0,T];X)} \le 2M_B||u||_{L^2([0,T];U)}$.

(ii) $\quad$ *$\mathcal{F}$ is Lipschitz in $C([-\tau, T]; X) \times L^2([0, T]; U)$ and*

$$\begin{aligned}&||\mathcal{F}(x, u) - \mathcal{F}(y, v)||_{L^1([0,T];X)}\\ &\quad \le L_f(e^{M_0TL_f}M_0T(2M_B + L_f) + 1)\sqrt{T}||u - v||_{L^2([0,T];U)}.\end{aligned}$$

Proof The proof of (i) is obvious from assumption $(A3)$. We shall prove (ii):

$$\begin{aligned}||\mathcal{F}(x, u) &- \mathcal{F}(y, v)||_{L^1([0,T];X)}\\ &= \int_0^T ||\mathcal{F}(x, u)(t) - \mathcal{F}(y, v)(t)||dt\\ &= \int_0^T ||f(t, x(\alpha(t)), u(t)) - f(t, y(\alpha(t)), v(t))||dt\\ &\le L_f \int_0^T (||x(\alpha(t)) - y(\alpha(t))|| + ||u(t) - v(t)||_U)dt\\ &\le L_f(T||x - y||_{C([-\tau,T];X)} + \sqrt{T}||u - v||_{L^2([0,T];U)}).\end{aligned} \tag{9}$$

Thus, $\mathcal{F}$ is Lipschitz in $C([-\tau, T]; X) \times L^2([0, T]; U)$. By using (3) in (9), we get

$$\begin{aligned}
&||\mathcal{F}(x, u) - \mathcal{F}(y, v)||_{L^1([0,T];X)} \\
&\quad \leq L_f(e^{M_0 T L_f} M_0 T(2M_B + L_f) + 1)\sqrt{T}||u - v||_{L^2([0,T];U)}.
\end{aligned}$$

□

Let us impose the following hypotheses for further discussion:

(A6) There is a function $q \in L^1[0, T]$ such that

$$||\mathcal{F}(x, u)||_{L^1([0,T];X)} \leq ||q||_{L^1[0,T]} \quad \forall\ (x, u) \in C([-\tau, T]; X) \times L^2([0, T]; U).$$

Let us consider the controllability map $\mathcal{W}^T : L^2([0, T]; U) \to X$ defined by

$$\mathcal{W}^T u = \int_0^T S(T - s)(\mathfrak{B}_\alpha u)(s)ds$$

and the controllability grammian $\mathcal{G}_{\mathfrak{B}}^T : X \to X$ by

$$\mathcal{G}_{\mathfrak{B}}^T x = \int_0^T S(T - s)\mathfrak{B}_\alpha \mathfrak{B}_\alpha^* S^*(T - s)xds,$$

where $\mathfrak{B}_\alpha^*$ and $S^*(t)$ denote the adjoint of $\mathfrak{B}_\alpha$ and $S(t)$, respectively.

Clearly, $\mathcal{G}_{\mathfrak{B}}^T = \mathcal{W}^T(\mathcal{W}^T)^*$, which is a monotone positive operator on X and for all $\lambda > 0$, $\lambda I + \mathcal{G}_{\mathfrak{B}}^T$ is invertible with $||(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1}|| \leq \frac{1}{\lambda}$. From the well established results [5, 6, 33], $\lambda(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1} \to 0$ as $\lambda \to 0^+$ in the strong operator topology is equivalent to the approximate controllability of the system (5). The treatment of the semilinear system is inspired by [5, 6].

Theorem 3.5 *The semilinear retarded control system (1) is approximately controllable on* $[0, T]$ *under assumptions* $(A1) - (A6)$.

Proof Let $\hat{x} \in X$ be the desired state and $\lambda > 0$ be arbitrary. We would find a control $u^\lambda \in L^2([0, T]; U)$ such that $x^\lambda(0, T; u^\lambda) \to \hat{x}$ as $\lambda \to 0^+$.

Define an operator $H^\lambda : C([-\tau, T]; X) \times L^2([0, T]; U) \to C([-\tau, T]; X) \times L^2([0, T]; U)$ as $H^\lambda(x, u) = (y, v)$, where

$$\begin{aligned}
v(t) &= \mathfrak{B}_\alpha^* S^*(T - t)(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1} p(x, u), \\
y(t) &= S(t)\phi(0) + \int_0^t S(t - s)[(\mathfrak{B}_\alpha v)(s) + \mathcal{F}(x, u)(s)]ds, \\
y_0(\theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \\
p(x, u) &= \hat{x} - S(T)\phi(0) - \int_0^T S(T - s)\mathcal{F}(x, u)(s)ds. \qquad (10)
\end{aligned}$$

Then,

$$\begin{aligned}
||p(x, u)|| &\leq ||S(T)\phi(0) - \hat{x}|| + M_0 \int_0^T ||\mathcal{F}(x, u)(s)||ds \\
&\leq ||S(T)\phi(0) - \hat{x}|| + M_0||q||_{L^1[0,T]}
\end{aligned}$$

so that

$$\begin{aligned}||v(t)||_U &\le ||\mathfrak{B}_\alpha^*||\,||S^*(T-s)||\,\frac{1}{\lambda}\,||p(x,u)|| \\ &\le \frac{2M_BM_0}{\lambda}\left(||S(T)\phi(0)-\hat{x}||+M_0||q||_{L^1[0,T]}\right), \\ ||v||_{L^2([0,T];U)} &\le \frac{4M_B^2M_0^2T}{\lambda^2}\left(||S(T)\phi(0)-\hat{x}||+M_0||q||_{L^1[0,T]}\right)^2\end{aligned}$$

and

$$\begin{aligned}||y(t)|| &\le ||S(t)\phi(0)||+M_0\int_0^t(||(\mathfrak{B}_\alpha v)(s)||+||\mathcal{F}(x,u)(s)||)ds,\ \ t>0 \\ &\le M_0||\phi(0)||+M_0\left(2M_B\sqrt{T}||v||_{L^2([0,T];U)}+||q||_{L^1[0,T]}\right), \\ ||y||_{C([0,T];X)} &\le M_0||\phi(0)||+M_0\left(\frac{8M_B^3M_0^2T^{\frac{3}{2}}}{\lambda^2}\left(||S(T)\phi(0)-\hat{x}||\right.\right. \\ &\quad \left.\left.+M_0||q||_{L^1[0,T]}\right)^2+||q||_{L^1[0,T]}\right).\end{aligned}$$

Let

$$\begin{aligned}\frac{R_\lambda}{2}=\max\Bigg\{&\frac{4M_B^2M_0^2T}{\lambda^2}\left(||S(T)\phi(0)-\hat{x}||+M_0||q||_{L^1[0,T]}\right)^2, \\ &M_0||\phi(0)||+M_0\left(\frac{8M_B^3M_0^2T^{\frac{3}{2}}}{\lambda^2}\left(||S(T)\phi(0)-\hat{x}||\right.\right. \\ &\left.\left.+M_0||q||_{L^1[0,T]}\right)^2+||q||_{L^1[0,T]}\right)\Bigg\}.\end{aligned}$$

Then, the operator H^λ maps the sphere $\partial\mathfrak{S}=\{(x,u)\in C([-\tau,T];X)\times L^2([0,T];U): ||(x,u)||=R_\lambda\}$ into the ball $\mathfrak{S}=\{(y,v)\in C([-\tau,T];X)\times L^2([0,T];U):||(y,v)||\le R_\lambda\}$. Thus, by the Rothe type fixed point theorem, for all $\lambda>0$, H^λ has a fixed point in the ball $\mathfrak{S}$.

Let us denote the fixed point of H^λ by $(x^\lambda,u^\lambda)\in C([-\tau,T];X)\times L^2([0,T];U)$. It is a mild solution of the system (1) satisfying

$$u^\lambda(t)=\mathfrak{B}_\alpha^*S^*(T-t)(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda) \tag{11}$$

and from (10),

$$\begin{aligned}x^\lambda(0,T;u^\lambda) &= S(T)\phi(0)+\int_0^T S(T-s)(\mathfrak{B}_\alpha u^\lambda)(s)ds+\int_0^T S(T-s)\mathcal{F}(x^\lambda,u^\lambda)(s)ds \\ &= \hat{x}-p(x^\lambda,u^\lambda)+\mathcal{W}^T(\mathcal{W}^T)^*(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda) \\ &= \hat{x}-p(x^\lambda,u^\lambda)-\lambda(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda) \\ &\quad +\lambda(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda)+\mathcal{G}_{\mathfrak{B}}^T(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda) \\ &\quad +(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda) \\ &= \hat{x}-\lambda(\lambda I+\mathcal{G}_{\mathfrak{B}}^T)^{-1}p(x^\lambda,u^\lambda).\end{aligned}$$

Now, consider the sequence $\{\mathcal{F}(x^\lambda,u^\lambda)\}_{\lambda>0}$ in $L^1([0,T];X)$ given as $f_n(s,x^\lambda(\alpha(s)),u^\lambda(s))=\mathcal{F}(x^\lambda,u^\lambda)(s)$, $\lambda=\frac{1}{n}$. Since $||\mathcal{F}(x^\lambda,u^\lambda)||_{L^1([0,T];X)}\le||q||_{L^1[0,T]}$ for all $\lambda>0$, therefore

$\{f_n\}$ converges pointwise to some function $\tilde{F} \in L^1([0,T];X)$ so that

$$\lim_{n\to\infty} f_n(s, x^\lambda(\alpha(s)), u^\lambda(s)) = \lim_{\lambda\to 0^+} \mathcal{F}(x^\lambda, u^\lambda)(s) = \tilde{F}(s).$$

Hence, by Lebesgue Dominated Convergence Theorem, we get

$$\lim_{\lambda\to 0^+} \int_0^t ||\mathcal{F}(x^\lambda, u^\lambda)(s) - \tilde{F}(s)||ds = 0.$$

Let $h = S(T)\phi(0) + \int_0^T S(T-s)\tilde{F}(s)ds - \hat{x}$. Then

$$\begin{aligned}||p(x^\lambda, u^\lambda) - h|| &= || \int_0^T S(T-s)\Big(\mathcal{F}(x^\lambda, u^\lambda)(s) - \tilde{F}(s)||ds \\ &\le M_0 \int_0^t ||\mathcal{F}(x^\lambda, u^\lambda)(s) - \tilde{F}(s)||ds \to 0\end{aligned}$$

as $\lambda \to 0^+$. Thus,

$$\begin{aligned}||x^\lambda(0,T;u^\lambda) - \hat{x}|| &= ||\lambda(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1} p(x^\lambda, u^\lambda)|| \\ &\le ||\lambda(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1} h|| + ||\lambda(\lambda I + \mathcal{G}_{\mathfrak{B}}^T)^{-1}(p(x^\lambda, u^\lambda) - h)|| \to 0\end{aligned}$$

as $\lambda \to 0^+$ in the strong operator topology. This completes the proof. □

4 Application

Example 4.1 Consider the following parabolic partial differential equation describing the diffusion process:

$$\begin{aligned}\frac{\partial y}{\partial t}(t,x) &= \frac{\partial^2 y}{\partial x^2}(t,x) + b(x)\int_0^t u(s,x)ds + c(x)u(\alpha(t),x) \\ &\quad + f(t, y(\alpha(t),x), u(t,x)),\ \ t \in [0,T],\ x \in [0,\pi], \qquad (12a)\\ \frac{\partial y}{\partial x}(t,0) &= \frac{\partial y}{\partial x}(t,\pi) = 0,\ \ t \in [0,T], \qquad (12b)\\ y_0(\theta,x) &= \phi(\theta,x),\ \ \theta \in \left[-\frac{1}{T}, 0\right],\ x \in [0,\pi], \qquad (12c)\end{aligned}$$

where $y(t,x)$ is the density at time t and at point x; $b, c \in L^\infty([0,\pi];\mathbb{R}^+)$ are weights for the linear action of control u, and $\alpha(t) = \dfrac{t^2-1}{T}$. Clearly, α satisfies the delay property $\alpha(t) \le t$ and $\mathcal{R}(\alpha) = [-\frac{1}{T}, T - \frac{1}{T}]$.

We transform the equation (12) into the abstract form (1) by constructing suitable spaces. Let $X = L^2[0,\pi]$ be the state space and $y(t,\cdot)$ be the state. Define $Ay = \frac{d^2y}{dx^2}$ with $D(A) = H^2[0,\pi] \cap H_0^1[0,\pi]$. Then, A generates a C_0-semigroup $\{S(t)\}_{t\ge 0}$. Further, $\{\psi_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx) : 0 \le x \le \pi\}$ forms an orthonormal basis for X associated to the eigenspectrum $\{\lambda_n = -n^2\}$, $n \in \{0\} \cup \mathbb{N}$, of operator A. Then, $S(t)y = \sum_{n=0}^\infty e^{-n^2 t}\langle y, \psi_n\rangle\psi_n$, where $y = \sum_{n=0}^\infty \langle y, \psi_n\rangle\psi_n$ and $||S(t)||_{op} \le 1 = M_0$.

Let us take $U = L^2[0, \pi]$ and $V = L^2([0, T]; U)$ as the control space. Define control operators B_0 and B_1 as

$$(B_0u)(t, x) = b(x) \int_0^t u(s, x)ds \quad \text{and} \quad (B_1u)(\alpha(t), x) = c(x)u(\alpha(t), x).$$

Thus, B_0 and B_1 are bounded linear operators with $M_B = \text{ess sup}_{x\in[0,\pi]}\{b(x), c(x)\}$. Consider the nonlinear function f given as

$$f(t, y(\alpha(t)), u(t)) = \frac{t||y(\alpha(t))||}{1 + ||y(\alpha(t))||}\psi_n(x) + \frac{t^2||u(t)||_U}{1 + ||u(t)||_U}\psi_{n+k}(x).$$

Then, the parabolic control system (12) resembles the abstract form (1). Now, we need to verify that the appropriate operators satisfy assumptions for controllability. For $y^1(\alpha(t))$, $y^2(\alpha(t)) \in X$ corresponding to controls $u_1, u_2 \in L^2([0, T]; U)$, we have

$$\begin{aligned}&||f(t, y^1(\alpha(t)), u_1(t)) - f(t, y^2(\alpha(t)), u_2(t))||\\ &\quad \le \max\{t, t^2\}(||y^1(\alpha(t)) - y^2(\alpha(t))|| + ||u_1(t) - u_2(t)||_U).\end{aligned}$$

This satisfies the Lipschitz condition. The control operator is $\mathfrak{B}_\alpha u = (B_0u)(\cdot) + (B_1u)(\alpha(\cdot))$ and the Nemytskii operator is $\mathcal{F}(y, u) = f(\cdot, y(\alpha(\cdot)), u(\cdot))$. For $q(t) = t^2(1+t)^2$, $\mathcal{F}$ satisfies condition $(A6)$. Hence, (12) is approximately controllable.

Example 4.2 Consider the following hyperbolic partial differential equation representing the wave propagation:

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2}(t, x) = &\frac{\partial^2 y}{\partial x^2}(t, x) + \beta(x)[u(t) + u(\alpha(t))]\\ &+ f(t, y(\alpha(t), x), u(t, x)), \;\; t \in [0, T],\\ &x \in [0, 1], y(t, 0) = y(t, 1) = 0, \;\; t \in [0, T], \end{aligned} \tag{13a}$$

$$y_0(\theta, x) = \phi(\theta, x), \;\; \theta \in \left[-\frac{T}{2}, 0\right], \; x \in [0, 1], \tag{13b}$$

where $y(t, x)$ is the intensity at time t and point x; and $\alpha(t) = \frac{3t-T}{2}$. Clearly, α satisfies the delay property $\alpha(t) \le t$, and $\mathcal{R}(\alpha) = [-\frac{T}{2}, T]$.

The function $\beta \in L^\infty([0, 1]; \mathbb{R}^+)$ is the shaping function given by $\beta(x) = \frac{1}{2\varepsilon}\chi_{[x_0-\varepsilon,x_0+\varepsilon]}(x)$, where χ denotes the characteristic function of $[x_0 - \varepsilon, x_0 + \varepsilon]$.

Let us take $X = L^2[0, 1]$ and define A_0 by $A_0 y = \frac{d^2y}{dx^2}$ with domain

$$\begin{aligned}D(A_0) = \{y \in L^2[0, 1] : \; y, \frac{dy}{dx} &\text{ are absolutely continuous,}\\ &\frac{d^2y}{dx^2} \in L^2[0, 1] \text{ and } y(0) = 0 = y(1)\}.\end{aligned}$$

It is well-explained in [31] that the operator $A = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}$, $D(A) = D(A_0) \oplus D(A_0^{\frac{1}{2}})$, is the infinitesimal generator of a C_0-semigroup on the state space $\mathfrak{X} = D(A_0^{\frac{1}{2}}) \oplus L^2[0, 1]$.

Let $U = L^2[0, 1]$ and $V = L^2([0, T]; U)$ as the control space and the control operator $\mathfrak{B}_\alpha = \begin{pmatrix} 0 \\ \beta(x) \end{pmatrix}$. The nonlinear function f is Lipschitz in y and u. Suppose that the associated

Nemytskii operator $\mathcal{F}(z,u)(t) = f(t, z(\alpha(t)), u(t))$ is dominated by a Bochner integrable function $q \in L^1([0,T];\mathbb{R}^+)$.

Thus, the hyperbolic system (13) takes the abstract form

$$\dot{z}(t) = Az(t) + (\mathfrak{B}_\alpha u)(t) + \mathcal{F}(z,u)(t).$$

The eigenvalues of the operator A are $\{\lambda_n = n\pi : n = \pm 1, \pm 2, \dots\}$ and the corresponding eigenfunctions are $\{\psi_n(x) = \frac{1}{\lambda_n}\begin{pmatrix} \sin(n\pi x) \\ \lambda_n \sin(n\pi x)\end{pmatrix} : n = \pm 1, \pm 2, \dots\}$. This is a Riesz basis for the state space $\mathfrak{X}$. It is described in Example 4.2.5 of Curtain and Zwart [31] that the linear hyperbolic partial differential equation of (13) is approximately controllable if and only if

$$\int_0^1 \beta(x)\sin(n\pi x)dx = \frac{1}{n\pi\varepsilon}\sin(n\pi x_0)\sin(n\pi\varepsilon) \neq 0, \qquad \text{for } n \geq 1.$$

Hence, by Theorem 3.5, the hyperbolic system (13) is approximately controllable.

5 Conclusion

The approximate controllability of linear and retarded semilinear systems with control delay is presented in this work under general assumptions on the system operator, the control operator and the nonlinearity. The required results have been presented with the fixed point theory and the Nemytskii operators by sequential approach. The analytical discussion is motivated by the works of Dauer and Mahmudov [5, 6]. However, the uniform boundedness and the growth conditions in [5] have been relaxed by assumption $(A6)$. Further, this paper aims to present the concept of generalized time-varying control delay. It appears in mathematical representation of various real life processes: medicines, epidemiology, finance *etc.* Future works will extend this idea for stochastic control problems with nonlocal condition and impulsive control systems.

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Declarations

Conflict of Interest This research work is an output of major revision of the preprint available online on the Research Square with DOI https://doi.org/10.21203/rs.3.rs-686705/v1 of the same author. The author and the Research Square have no conflict of interest to publish this revised manuscript in a journal.

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