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Hilbert squares of degeneracy loci

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Abstract
Let S be the first degeneracy locus of a morphism of vector bundles corresponding to a general matrix of linear forms in $\mathbb{P}^s$. We prove that, under certain positivity conditions, its Hilbert square $\mathrm{Hilb}^2(S)$ is isomorphic to the zero locus of a global section of an irreducible homogeneous vector bundle on a product of Grassmannians. Our construction involves a naturally associated Fano variety, and an explicit description of the isomorphism.

Keywords Hilbert schemes · Fano varieties · Grassmannians · Degeneracy loci

1 Introduction

The Hilbert scheme of 2 points $\mathrm{Hilb}^2(S)$ on a smooth variety S, called the *Hilbert square of S*, is an interesting smooth variety, whose geometry is incredibly rich, and yet not fully understood. An intriguing problem consists in finding a projective embedding of $\mathrm{Hilb}^2(S)$, for example by either writing down equations or realising it as the zero locus of a section of some vector bundle. An archetypical example is when S_g is a K3 surface of genus g, in which case $\mathrm{Hilb}^2(S_g)$ is a hyperkähler fourfold, and a projective embedding is known in a bunch of cases, including $g = 3, 5, 7, 8, 12$—the last one only up to deformations, see [1, 3, 10, 11, 16]. A few other cases are known, including the recent case of $\mathbb{P}^2$, see [15].

In this paper, we focus on the special case where $S \subset \mathbb{P}^s$ arises as the first degeneracy locus of a general morphism of vector bundles

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$$\varphi : \mathcal{O}_{\mathbb{P}^s}^{\oplus n+m} \to \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n}.$$

The case $s = 3, n = 3, m = 0$ coincides with the quartic determinantal K3 surface studied by Iliev and Manivel in [16]. Letting s, n, m vary, we find many examples of interesting varieties, including surfaces of general type.

Our idea is to study $\mathrm{Hilb}^2(S)$ via an auxiliary hypersurface $Y \subset \mathbb{P}^s \times \mathbb{P}^{n+m-1} \times \mathbb{P}^{n-1}$ naturally associated to S, and defined explicitly in Eq. 2.2. The variety Y is always a *Fano variety*, whose study was one of the initial motivations for our project. Via a modular-type construction we then pass from Y to Z, defined as

$$Z = V(\omega) \hookrightarrow \mathrm{Gr}(2, n) \times \mathrm{Gr}(2, s+1) \times \mathrm{Gr}(n+m-2, n+m),$$

where ω is a tri-tensor naturally attached to φ. As explained in Sect. 2.1, Z is the zero locus of a section of an irreducible, globally generated, homogeneous vector bundle naturally associated to Y.

Our main result proves that, in a certain infinite range, the variety Z and the Hilbert square $\mathrm{Hilb}^2(S)$ of the variety we started with are isomorphic. Namely, we have the following.

Theorem A (Theorem 6.2) *Let* $n \geq 3$, $m \geq 0$, $s \in \{m+2, \dots, 2m+3\}$. *Assume* $n > 2s - 2m - 3$. *Then, there is an isomorphism of schemes* $\vartheta : Z \xrightarrow{\sim} \mathrm{Hilb}^2(S)$.

Our proof goes via the explicit construction of the morphism ϑ. In principle, it says nothing on the cases $n \leq 2s - 2m - 3$. However, we show that for low values of m these two varieties are not even deformation equivalent—indeed, their topological Euler characteristics are different. This observation leads us to conjecture that, in fact, our bound is optimal, see Conjecture 7.6.

In Sects. 2–3 we explain the geometric setup and the main motivating ideas behind this paper; we also explicitly describe various examples in which our result applies. Sections 4–5 are the technical core of this paper: first we describe in full detail the geometry of Z (cf. Theorem 4.7) independently upon the choice of s, n, m, then we explain how the cases in which our main result does *not* work are related to the presence of some *special lines* contained in S (cf. Theorem 5.2 and Theorem 5.3). Our main result, Theorem A, is proved in Sect. 6 (cf. Theorem 6.2), whereas Sect. 7 is devoted to the study of the geometry of some interesting varieties arising as the limit cases for which our method fails, but enjoying a beautiful and rich geometry. Among these examples we include *generalised Bordiga scrolls* (cf. Example 7.3), *higher dimensional White varieties* (cf. Example 7.4), and also certain varieties containing a finite number of special lines (cf. Example 7.5 and Conjecture 7.6).

1.1 Notation

We work over the field of complex numbers $\mathbb{C}$. For an arbitrary positive integer d we let V_d be a d-dimensional $\mathbb{C}$-vector space, which we also identify with the d-dimensional affine space $\mathbb{A}^d$.

We denote by $\mathrm{Gr}(k, n)$ the Grassmannian of k-dimensional subspaces in V_n. We denote with $\mathcal{U}$ the rank k tautological vector bundle over it, with anti-ample determinant. We write $X = (G, \mathcal{F})$ to denote the zero locus $X = V(\sigma) \subset G$, for a general section $\sigma \in \mathrm{H}^0(G, \mathcal{F})$ of a vector bundle $\mathcal{F}$ on a variety G. Sometimes we will need to work with a specific σ, and we will specify it accordingly.

2 Setup, motivation and some toy cases

2.1 Degeneracy loci, Fano varieties, and Hilbert schemes

We start by considering a very simple construction from linear algebra. We consider a general $n \times (n+m)$ matrix of homogeneous linear forms

$$M = \begin{pmatrix} f_1^1 & \dots & f_{n+m}^1 \\ f_1^2 & \dots & f_{n+m}^2 \\ \vdots & \ddots & \vdots \\ f_1^n & \dots & f_{n+m}^n \end{pmatrix}$$

on an ambient projective space $\mathbb{P}^s = \mathbb{P}(V_{s+1})$. If we ask M to have non-maximal rank, we have to consider the locus where all the $\binom{n+m}{n}$ maximal minors vanish. This is of course equivalent to the existence of some linear relations between the rows of M. We can therefore consider two strictly related loci: the first one is $S_{n,s,m} \subset \mathbb{P}^s$,

given by the vanishing of the maximal minors of M—i.e. the *first degeneracy locus* $D_{n-1}(\varphi)$—where we implicitly identify the matrix with the morphism $\varphi : \mathcal{O}_{\mathbb{P}^s}^{\oplus n+m} \to \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n}$ defining it. Sometimes we will shorten $S_{n,s,m}$ with S, when the subscripts are clear from the context. In other words,

$$S_{n,s,m} = \left\{[v] \in \mathbb{P}^s \mid \operatorname{rank}(M_v) \leq n-1\right\},$$

where $M_v \in \mathsf{Mat}_{n,n+m}(\mathbb{C})$ is the evaluation of M at $v \in V_{s+1}$.

The second relevant locus is a subvariety $X_{n,s,m} \subset \mathbb{P}^s \times \mathbb{P}^{n+m-1}$, given by n bihomogeneous linear polynomials of bi-degree $(1, 1)$, i.e. by a section of $\mathcal{O}(1,1)^{\oplus n}$.

What is the relation between S and X? First of all, assume S to be smooth with $\dim(S) > 0$. Under our generality assumption, this will be equivalent to requiring $m + 2 \leq s \leq 2m + 3$, where the second inequality ensures that the further degeneracy loci will be empty.

Now, $X_{n,s,m}$ is constructed in a tautological way as follows: if $y_1, \dots, y_{n+m}$ are chosen coordinates on $\mathbb{P}^{n+m-1}$, and $F_i = (f_1^i, \dots, f_{n+m}^i)$ is the i-th row of our matrix M, we will have

$$X_{n,s,m} = V(F_1 \cdot \underline{y}, \dots, F_n \cdot \underline{y}) \subset \mathbb{P}^s \times \mathbb{P}^{n+m-1}.$$

Using our notation,

$$X_{n,s,m} = \left(\mathbb{P}^s \times \mathbb{P}^{n+m-1}, \mathcal{O}(1,1)^{\oplus n}\right). \tag{2.1}$$

The fibres of the projection $\pi : X_{n,s,m} \hookrightarrow \mathbb{P}^s \times \mathbb{P}^{n+m-1} \to \mathbb{P}^s$ are generically cut out by n linear equations, or $n-1$ exactly where there is a linear dependence relation in M (and that is all that can happen, since by hypothesis there are no further degenerations): in other words, we have proved the following lemma.

Lemma 2.1 *In the setup above, $\pi : X_{n,s,m} \to \mathbb{P}^s$ is generically a $\mathbb{P}^{m-1}$-bundle jumping to a $\mathbb{P}^m$-bundle exactly over $S_{n,s,m}$.*

We call $X = X_{n,s,m}$ a generalised $(m-1, m)$ blow up of $S = S_{n,s,m}$. This construction is sometimes referred to as *Cayley trick*. This is in fact a generalisation of the blow up formula, and it implies that the vanishing cohomologies of X and S are isomorphic, and also that $D^b(X)$, the bounded derived category of coherent sheaves over X, contains a copy of $D^b(S)$. References for this fact can be found in [18, Theorem 2.4] and [4, Proposition 46].

We could have built yet another natural variety starting from the matrix M (or better, its transpose). If we take the transpose M^t of the matrix M, and we apply it to a vector $\underline{z} = (z_1, \dots, z_n)^t$ we can consider the locus $\Gamma_{n,s,m} \subset \mathbb{P}^s \times \mathbb{P}^{n-1}$, given by $M^t \cdot \underline{z} = 0$. In other words, if we write $F_i^t = (f_i^1, \dots, f_i^n)$, we have then

$$\Gamma_{n,s,m} = V(F_1^t \cdot \underline{z}, \dots, F_{n+m}^t \cdot \underline{z}) \subset \mathbb{P}^s \times \mathbb{P}^{n-1}$$

and again, in our notation,

$$\Gamma_{n,s,m} = (\mathbb{P}^s \times \mathbb{P}^{n-1}, \mathcal{O}(1,1)^{\oplus n+m}).$$

Consider, this time, the restricted projection $\Gamma_{n,s,m} \hookrightarrow \mathbb{P}^s \times \mathbb{P}^{n-1} \to \mathbb{P}^s$. This time the fibre is generically empty, and it becomes a point exactly where the rank drops, i.e. on S. In other words, one has the following lemma.

Lemma 2.2 *The projection $\mathbb{P}^s \times \mathbb{P}^{n-1} \to \mathbb{P}^s$ restricts to an isomorphism $\Gamma_{n,s,m} \xrightarrow{\sim} S_{n,s,m}$.*

This implies that the Picard group of $S_{n,s,m}$ is $\mathbb{Z}^2$ (at least generically), and the line bundles $\mathcal{O}(1,1)$ and $\mathcal{O}(1,0)$ (restricted from $\mathbb{P}^s \times \mathbb{P}^{n-1}$) are both very ample. In what follows, we will study as well the morphism induced by $\mathcal{O}(0,1)$, showing that it will be very ample in a certain range (namely $n > 2s - 2m - 3$) as well.

Consider now two triples (n_1, s_1, m_1) and (n_2, s_2, m_2): if we set $n_2 = s_1 + 1, s_2 = n_1 - 1, m_2 = n_1 + m_1 - s_1 - 1$, then Γ_{n_1,s_1,m_1} and Γ_{n_2,s_2,m_2} are both $(n_1 + m_1)$-codimensional linear sections of $\mathbb{P}^{s_1} \times \mathbb{P}^{n_1-1}$, with the role of the two projective spaces exchanged, hence they belong to the same deformation family. When the triples satisfy such a relation, we call them *associated*.

If we are in the correct range for the first triple, i.e. $m_1 + 2 \leq s_1 \leq 2m_1 + 3$, $n_1 \geq 3$ and $n_1 > 2s_1 - 2m_1 - 3$, then the second triple will be in the correct range as well (in fact $n_2 > 2s_2 - 2m_2 - 3$ reduces exactly to $s_1 \leq 2m_1 + 3$).

In this range both projections to $\mathbb{P}^{s_1}$ and $\mathbb{P}^{n_1-1}$ are embeddings when restricted to Γ (this follows from Theorem 5.2): in other words,

$$S_{n_2,s_2,m_2} \cong S_{s_1+1,n_1-1,n_1+m_1-s_1-1}$$

yields another presentation for S_{n_1,s_1,m_1}, with a different embedding. We will see these phenomena in detail when dealing with two presentations of determinantal quartic K3 surfaces (abstractly but not projectively isomorphic), and of a quintic determinantal surface embedded as a codimension 2 degeneracy locus, see Sect. 2.3.

Let us now get back to $X = X_{n,s,m}$, and perform once again a *Cayley trick*. In fact, we can associate to $X_{n,s,m}$ another variety

$$Y_{n,s,m} = (\mathbb{P}^s \times \mathbb{P}^{n+m-1} \times \mathbb{P}^{n-1}, \mathcal{O}(1,1,1)), \tag{2.2}$$

defined tautologically starting from the equations of X. This will be simply given by

$$Y_{n,s,m} = V\left(\sum_{i=1}^{n} z_i(F_i \cdot \underline{y})\right).$$

Of course, the projection $\mathbb{P}^{n-1} \times \mathbb{P}^s \times \mathbb{P}^{n+m-1} \to \mathbb{P}^s \times \mathbb{P}^{n+m-1}$ restricted to Y is generically a $\mathbb{P}^{n-2}$-bundle, with special fibres the whole $\mathbb{P}^{n-1}$ over X.

Notice that $Y = Y_{n,s,m}$ is a Fano variety, simply by adjunction: on the other hand this is not the case in general for X or S: as a matter of fact, we will work only under certain (at least) non-negativity assumption for the canonical bundle of S.

In a certain sense, the main character of the whole story is precisely the Fano variety Y: we can see it as the universal variety associated to a tri-tensor $\omega \in V_{s+1}^\vee \otimes V_{n+m}^\vee \otimes V_n^\vee$ simply given by $\omega = \sum_{1 \leq i \leq n} z_i(F_i \cdot \underline{y})$.

The geometry of a tri-tensor is an old and fascinating topic, with one of the first references being [6]. See also, [26, 30] for a modern account. The degeneracy locus S, the rational variety X and all the other characters appearing in this picture can be seen to be induced by Y via the obvious projections.

Finally, we associate to $Y_{n,s,m}$ one last variety $Z_{n,s,m}$, which is far from being a Fano variety. Denote by

$$G_{n,s,m} := \mathrm{Gr}\,(2,n) \times \mathrm{Gr}\,(2,s+1) \times \mathrm{Gr}\,(n+m-2,n+m), \tag{2.3}$$

then define the vanishing locus $Z_{n,s,m} = V(\omega) \subset G_{n,s,m}$. In our notation,

$$Z_{n,s,m} = (G_{n,s,m} \quad \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee). \tag{2.4}$$

The reason for this apparently arbitrary choice is that by Borel–Bott–Weil

$$\mathrm{H}^0(\mathbb{P}^{n-1} \times \mathbb{P}^s \times \mathbb{P}^{n+m-1}, \mathcal{O}(1,1,1)) \cong \mathrm{H}^0\big(G_{n,s,m}, \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee\big).$$

Notice that this holds true for any product $\mathrm{Gr}\,(k_3,n) \times \mathrm{Gr}\,(k_1,s+1) \times \mathrm{Gr}\,(k_2,n+m)$. However, with this particular choice of ambient spaces, we have that the dimension of Z is equal to $2(s-m-1)$, i.e. $\dim Z = 2 \cdot \dim S$.

This is not a coincidence: in fact the purpose of this paper is to show that as long as the triple (n, s, m) satisfies the constraints

$$m+2 \leq s \leq 2m+3, \quad n > 2s-2m-3,$$

one has an isomorphism of schemes

$$Z_{n,s,m} \cong \mathrm{Hilb}^2(S_{n,s,m}).$$

We stress that the condition $n > 2s-2m-3$ is not necessary. In fact our proof goes via the explicit construction of a morphism to the Hilbert scheme, which exists and happens to be an isomorphism in that range. This a priori says nothing on the other cases. However, we show that for e.g. $m = 0, 1$ our bound is optimal, see Examples 7.2 and 7.5 where we explicitly compute the Hodge numbers of Z and $\mathrm{Hilb}^2(S)$ in the range $n \leq 2s-2m-3$, confirming that they are different.

2.2 A conjectural relation with the Hilbert scheme of the Fano variety *Y*

Before discussing some examples, we mention one more relation between Z and the Hilbert scheme, that we leave for future research to explore. More precisely, we conjecture that Z can be realised as a Hilbert scheme *on* Y as well. In fact, if we call $\mathbb{P}_{1,1,n-3} := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-3}$ contained fibre-wise in $\mathbb{P}_{n,s,m} := \mathbb{P}^{n-1} \times \mathbb{P}^s \times \mathbb{P}^{n+m-1}$, we can consider the incidence variety

$$F = \{(p, \mathbb{P}_{1,1,n-3}) \subset \mathbb{P}_{n,s,m} \times G_{n,s,m} \mid p \in \mathbb{P}_{1,1,n-3}\},$$

with $G_{n,s,m}$ as in (2.3).

Notice that F can be described as the zero locus

$$F = (\mathrm{Fl}\,(1,2,n) \times \mathrm{Fl}\,(1,2,s+1) \times \mathrm{Fl}\,(n+m-3, n+m-2, n+m), \mathcal{O}(1,0) \otimes \mathcal{O}(1,0) \otimes \mathcal{Q}_2)\,,$$

where the first two bundles are the pullback of the ample line bundles from $\mathbb{P}^{n-1}$ and $\mathbb{P}^s$, and the last is the pullback of the rank 2 quotient bundle in $\mathrm{Gr}\,(n+m-2, n+m)$. This implies that the projection p from F to Y is a $\mathbb{P}^{n-2} \times \mathbb{P}^{s-1} \times \mathbb{P}^{n+m-2}$-bundle, while the projection q from F to $G_{n,s,m}$ is a $\mathbb{P}^{n+m-3} \cup \mathbb{P}^{n+m-3}$ generically, degenerating to a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n+m-3}$ over Z. Our Hodge theoretic intuition suggests that one should also have an isomorphism

$$Z \cong \mathrm{Hilb}_{\mathbb{P}_{1,1,n-3}}(Y)$$

with the induced isomorphism in cohomology realised by the classical *Abel-Jacobi type* $p^* q_*$-map. However, we have not been able to prove this for the time being, and we hope to return to it in the future.

2.3 Toy case I: determinantal

As a first special sub-case, it is worth mentioning the case $m = 0$, in which case S is a determinantal hypersurface in $\mathbb{P}^s$. Also, we need $s \leq 3$, since from threefolds onwards S will in fact be singular.

With $s = 3$, the last case excluded by Theorem A, $n = 3$, is the one of a cubic surface, and we can immediately show that Z and $\mathrm{Hilb}^2(S)$ are not isomorphic: as a matter of fact, $e_{\mathrm{top}}(\mathrm{Hilb}^2(S)) = e_{\mathrm{top}}(Z) + 21$, where the discrepancy by 21 should be accounted for by the 6 exceptional lines plus the other 15 which are strict transforms of lines passing through two of the six points. This will be also discussed right after Conjecture 7.6.

If we consider $n = 4$, we have that $X \cong \Gamma \cong S$, and with three different representations. In this case the isomorphism was already known to be true from [16, Proposition 1]. In fact S is a determinantal quartic K3 surface, presented with three different models, hence Z is a hyperkähler fourfold. This construction is very classical, starting from [6], and the relations between the three models have been recently explored in [12, 24, 31].

Another interesting case which is covered by our theorem is the one of a determinantal *quintic surface*, which we will explore in detail in Sect. 3.2.

2.4 Toy case II: sub-determinantal

Another relevant case is the sub-determinantal case, i.e. for $m = 1$. In this case we can borrow some results from [19, §2.2] and [5, Proposition 3.6] to readily compute the invariants of S. We remark that our smoothness condition forces $3 \leq s \leq 5$. In fact, the k-th degeneracy locus $D_{n-k}(\varphi)$ has expected codimension $k(m+k)$ in the ambient space $\mathbb{P}^s$. Hence for $m=1$, $k=2$, it has expected codimension 6, i.e. $D_{n-2}(\varphi)=\emptyset$.

Notice how in this case the map $X\to\mathbb{P}^s$ is particularly simple, indeed it agrees with the blow up map $X=\mathrm{Bl}_S\mathbb{P}^s\to\mathbb{P}^s$.

The structure sheaf of $S=D_{n-1}(\varphi)$ admits a resolution by the so-called Eagon–Northcott complex, which has this form every time that we have a morphism of globally generated vector bundles $\mathcal{E},\mathcal{F}$ of rank $(n+1,n)$: In this case, it takes the form:

$$0\to\mathcal{F}^\vee\to\mathcal{E}^\vee\to\det(\mathcal{E}^\vee)\otimes\det(\mathcal{F})\to(\det(\mathcal{E}^\vee)\otimes\det(\mathcal{F}))|_{D_{n-1}(\varphi)}\to 0. \tag{2.5}$$

In our case it will suffices to take $\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^s}^{\oplus n+1}$ and $\mathcal{F}\cong\mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n}$.

One can use suitably twisted versions of this complex to compute some invariants of S, as shown in the next examples. Of course one could have worked directly on Γ as well, or on X, applying the blow up formula.

Proposition 2.3 *Fix $s=3$ and $n>1$. Let $\varphi:\mathcal{O}_{\mathbb{P}^3}^{\oplus n+1}\to\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus n}$ be a general morphism. Consider the smooth curve $S_n:=S_{n,3,1}=D_{n-1}(\varphi)\subset\mathbb{P}^3$. Then*

$$g(S_n)=n\binom{n}{3}-(n+1)\binom{n-1}{3},\qquad \deg(S)=\binom{n+1}{2}.$$

Proof Consider the Eagon–Northcott resolution of $\mathcal{O}_{S_n}$ from (2.5). Twisting back, we have

$$0\to\mathcal{O}_{\mathbb{P}^3}(-n-1)^{\oplus n}\to\mathcal{O}_{\mathbb{P}^3}(-n)^{\oplus n+1}\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_{S_n}\to 0.$$

We have that

$$\chi(\mathcal{O}_{S_n})=\chi(\mathcal{O}_{\mathbb{P}^3})+\chi(\mathcal{O}_{\mathbb{P}^3}(-n-1)^{\oplus n})-\chi(\mathcal{O}_{\mathbb{P}^3}(-n)^{\oplus n+1}),$$

i.e.

$$1-g(S_n)=1-n\binom{n}{3}+(n+1)\binom{n-1}{3}.$$

In order to compute the degree, it suffices to check the Hilbert polynomial, which for a curve we know to be equal to $p_{S_n}(t)=dt+1-g$, where d is the degree. Since in general $p_{S_n}(t)=at+b$, we have of course $\chi(\mathcal{O}_{S_n})=p_{S_n}(0)=1-g$ and

$$\chi(\mathcal{O}_{S_n}(1))=p_{S_n}(1)=4-n\binom{n-1}{3}+(n+1)\binom{n-2}{3},$$

where we used as before the sequence (2.5). It follows that

$$a=3+n\left(\binom{n}{3}-\binom{n-1}{3}\right)-(n+1)\left(\binom{n-1}{3}-\binom{n-2}{3}\right),$$

which simplifies to $a=\binom{n+1}{2}$. The result follows. □

Proposition 2.4 *Fix $s\in\{3,4\}$ and $n>1$. Let $\varphi:\mathcal{O}_{\mathbb{P}^s}^{\oplus n+1}\to\mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n}$ be a general morphism. Then the smooth subvariety $S_{s,n}:=S_{n,s,1}=D_{n-1}(\varphi)\subset\mathbb{P}^s$, of codimension 2, has topological Euler characteristic*

$$e_{\mathrm{top}}(S_{s,n}) = \begin{cases} 4n^2 - 2n^3 + (3n-4)\binom{n}{2} - \binom{n}{3} & \text{if } s = 3 \\ n^2(10 - 10n + 3n^2) + \binom{n}{2}(-10 + 15n - 6n^2) + \binom{n}{3}(4n-5) - \binom{n}{4} & \text{if } s = 4 \end{cases}$$

Proof See Appendix 1. □

Lemma 2.5 *Fix $s = 4$. Then the smooth surface $S_{n,4,1} \subset \mathbb{P}^4$ has irregularity $q = 0$, and geometric genus*

$$p_g(S_{n,4,1}) = n\binom{n}{4} - (n+1)\binom{n-1}{4}.$$

Proof The Euler characteristic of the structure sheaf $\chi(\mathcal{O}_{S_n})$ is computed as in Proposition 2.3, using the sequence (2.5) on $\mathbb{P}^4$. We have in particular that

$$\chi(\mathcal{O}_{S_n}) = 1 + n\binom{n}{4} - (n+1)\binom{n-1}{4}.$$

Moreover, S_n is connected and $q = 0$. The first statement can be proven as in the curve case. The second follows from the isomorphism $\Gamma \cong S_n$. On the other hand, we know that $\Gamma = (\mathbb{P}^4 \times \mathbb{P}^{n-1}, \mathcal{O}(1,1)^{\oplus n+1})$. Hence, by Lefschetz hyperplane section theorem, the only weight where the cohomology of Γ has non-zero level is the middle one; therefore, $q = 0$. □

Remark 2.6 From the above lemma one immediately deduces that $p_g = q = 0$ as long as $n < 4$. Moreover the same argument tells us that for a threefold which is a degeneracy locus in $\mathbb{P}^5$, $h^1(\mathcal{O}_{S_n}) = h^2(\mathcal{O}_{S_n}) = 0$ and $h^{1,1}(S_n) = 2$.

A nice observation is that the sub-determinantal case $n = 4, s = 4, m = 1$ and the determinantal case $n = 5, s = 3, m = 0$ both give rise to a determinantal quintic, since Γ in both cases is given by

$$\Gamma = (\mathbb{P}^3 \times \mathbb{P}^4, \mathcal{O}(1,1)^{\oplus 5}),$$

albeit the role of $\mathbb{P}^3$ and $\mathbb{P}^4$ is exchanged.

3 Some examples

In this section, we collect some examples that do fall within the 'good range' prescribed by Theorem A, and that therefore realise the desired isomorphism $\mathrm{Hilb}^2(S) \cong Z$. For the sake of completeness, we write down the Hodge numbers of the varieties involved, which can be computed using the methods detailed in [9, §3.2].

3.1 The cases $n = 3, s = 3, m = 1$ and $n = 4, s = 2, m = 0$

We discuss first an example which is quite classical. Let us consider $S \subset \mathbb{P}^3$, where S is a degree 6, genus 3 space curve given by the intersection of four cubics (i.e. the maximal minors of a 4×3 matrix of linear forms).

In the notation of the previous section, according to (2.1) in the case $(n, s, m) = (3, 3, 1)$ we have $X \subset \mathbb{P}^3 \times \mathbb{P}^3$, given as the complete intersection of three divisors of bi-degree (1, 1), i.e. $X = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(1, 1)^{\oplus 3})$. This variety X is the Fano threefold **2–12** in the original Mori–Mukai notation, see [2, 9, 21].

Following the discussion of the previous section, X is identified with the blow up $\mathrm{Bl}_S\mathbb{P}^3$, see also [8, 2-12]. One can immediately compute the Hodge numbers of X, these being

$$\begin{array}{ccccccc} 0 & & 3 & & 3 & & 0 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The rational map $\eta : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ induced by this construction is the cubo-cubic Cremona transformation of $\mathbb{P}^3$ already known to Max Noether, see [22, 28] and it is the only non-trivial Cremona transformation of $\mathbb{P}^3$ that is resolved by just one blow up along a smooth curve, see [17].

The second variety in the picture is $Y = (\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2, \mathcal{O}(1, 1, 1))$. This is a Fano sevenfold, with anti-canonical class equal to $-K_Y \cong \mathcal{O}_Y(3, 3, 2)$. We can apply the Cayley trick from Y to X to determine the Hodge numbers of Y, which can be also computed using the standard Koszul resolution. These are:

$$\begin{array}{ccccccccccccccc} 0 & & 0 & & 0 & & 3 & & 3 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 9 & & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\ & & & 0 & & 0 & & 6 & & 0 & & 0 & & & \\ & & & & 0 & & 0 & & 0 & & 0 & & & & \\ & & & & & 0 & & 3 & & 0 & & & & & \\ & & & & & & 0 & & 0 & & & & & & \\ & & & & & & & 1 & & & & & & & \end{array}$$

Finally, we consider the variety $Z = (\mathrm{Gr}\,(2,4) \times \mathrm{Gr}\,(2,4) \times \mathrm{Gr}\,(2,3), \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee)$. By our theorem, $Z \cong \mathrm{Hilb}^2(S) \cong \mathrm{Sym}^2(S)$. We can check that $K_Z \cong \mathcal{O}_Z(0, 0, 1)$ and that its Hodge numbers are the expected ones, namely

$$\begin{array}{ccccc} 3 & & 10 & & 3 \\ & 3 & & 3 & \\ & & 1 & & \end{array}$$

Finally, notice that, using the notation of the previous section, the associated triple to (3, 3, 1) is (4, 2, 0). In this case $S_{4,2,0} \subset \mathbb{P}^2$ is a plane quartic curve, and Z describes its symmetric square as well.

3.2 The case $n = 5, s = 3, m = 0$ and $n = 4, s = 4, m = 1$

As before, these two cases define the same surface, in two different presentations. In fact, the first triple of invariants immediately identifies $S_{5,3,0} \subset \mathbb{P}^3$ as a quintic determinantal surface, which has Picard rank 2 in general. On the other hand $S_{4,4,1} \subset \mathbb{P}^4$ is a codimension 2 surface defined by 5 quartic equations. However, thanks to Lemma 2.2 they are both isomorphic to the same Γ, which is

$$\Gamma = (\mathbb{P}^3 \times \mathbb{P}^4, \mathcal{O}(1,1)^{\oplus 5}).$$

The Hodge numbers of S (of course regardless of the presentation) are as follows:

$$\begin{array}{ccccc} 4 & & 45 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We can consider the associated $Z = (\mathrm{Gr}(2,5) \times \mathrm{Gr}(3,5) \times \mathrm{Gr}(2,4), \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee)$, which is of course the same in both cases. The Hodge numbers of $Z \cong \mathrm{Hilb}^2(S)$ (see also Appendix 2) are:

$$\begin{array}{ccccccccc} 10 & & 184 & & 1097 & & 184 & & 10 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 4 & & 46 & & 4 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

3.3 The case $n = 6, s = 5, m = 1$

If for $m \in \{0, 1\}$ in the surface case our condition $n > 2s - 2m - 3$ corresponded essentially to a non-negative Kodaira dimension, for $(m, s) = (1, 5)$, the limit case which is not covered by Theorem A, is a threefold of general type: we are going to show in Example 7.5 that Z and $\mathrm{Hilb}^2(S)$ are *not* isomorphic. In fact, the first case with $m = 1, s = 5$ which is covered by our Theorem is for $n = 6$. In this case, the associated triple to (6, 5, 1) is again (6, 5, 1).

Our threefold $S_{6,5,1} \subset \mathbb{P}^5$ is defined by 7 minors (of degree 6): it is isomorphic to $\Gamma = (\mathbb{P}^5 \times \mathbb{P}^5, \mathcal{O}(1,1)^{\oplus 7})$.

We can compute the Hodge numbers of $S_{5,6,1}$, these being:

$$\begin{array}{ccccccc} 29 & & 520 & & 520 & & 29 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The Hodge numbers of $\mathrm{Hilb}^2(S) \cong Z \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(2,6) \times \mathrm{Gr}(5,7)$ are

$$\begin{array}{ccccccccccccc} 406 & & 15080 & & 150020 & & 271250 & & 150020 & & 15080 & & 406 \\ & 0 & & 87 & & 1560 & & 1560 & & 87 & & 0 & \\ & & 0 & & 0 & & 8 & & 0 & & 0 & & \\ & & & 29 & & 520 & & 520 & & 29 & & & \\ & & & & 0 & & 3 & & 0 & & & & \\ & & & & & 0 & & 0 & & & & & \\ & & & & & & 1 & & & & & & \end{array}$$

Notice that the Euler characteristic $e_{\mathrm{top}}(Z) = 593502$ coincides with $e_{\mathrm{top}}(\mathrm{Hilb}^2(S))$, which is computed in Appendix 2.2.

3.4 The cases $n = 4, s = 5, m = 2$ and $n = 6, s = 3, m = 0$

These two associated cases describe two different presentations for S, as a smooth determinantal sextic and as a codimension 3 surface in $\mathbb{P}^5$. Here Γ can be described as $\Gamma = (\mathbb{P}^5 \times \mathbb{P}^3, \mathcal{O}(1,1)^{\oplus 6})$. The Hodge numbers for S are:

$$\begin{array}{ccccc} 10 & & 86 & & 10 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We can compute the Hodge numbers of $\mathrm{Hilb}^2(S) \cong Z \subset \mathrm{Gr}\,(2,6) \times \mathrm{Gr}\,(2,4) \times \mathrm{Gr}\,(4,6)$, which are:

$$\begin{array}{ccccccccc} 55 & & 870 & & 3928 & & 870 & & 55 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 10 & & 87 & & 10 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

4 Key construction and preparation lemmas

In this more technical section we explain the key constructions that will allow us to prove Theorem 6.2. For the reader convenience we begin by briefly recalling the notations needed. As above, we fix integers $n \geq 3$, $m \geq 0$, and $s \in \{m+2, \dots, 2m+3\}$; we shall consider a *general* map of vector bundles

$$\varphi : \mathcal{O}_{\mathbb{P}^s}^{\oplus n+m} \to \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n} \tag{4.1}$$

along with the associated $(m+1)$-codimensional, *smooth* degeneracy locus

$$S = S_{n,s,m} = D_{n-1}(\varphi) \hookrightarrow \mathbb{P}^s.$$

Indeed, by the genericity of φ, each degeneracy locus $D_k(\varphi) = \{p \in \mathbb{P}^s \,|\, \mathrm{rank}\,(\varphi(p)) \leq k\} \subset \mathbb{P}^s$ has codimension in $\mathbb{P}^s$ equal to the expected one, namely $(n-k)(n+m-k)$. In the range $s \in \{m+2, \dots, 2m+3\}$, we have $\dim D_{n-1}(\varphi) > 0$, and the singularities are exactly in $D_{n-2}(\varphi) = \emptyset$, whence the smoothness.

As already outlined in the previous sections, φ can be understood from an algebraic point of view as a matrix

$$M = \begin{pmatrix} f_1^1 & \cdots & f_{n+m}^1 \\ f_1^2 & \cdots & f_{n+m}^2 \\ \vdots & \ddots & \vdots \\ f_1^n & \cdots & f_{n+m}^n \end{pmatrix} \in \mathsf{Mat}_{n,n+m}\big(\mathrm{H}^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))\big)$$

of linear forms f_j^i depending on $s+1$ variables. We shall switch from φ to M freely in what follows.

Working in the affine setup, one is led to consider the locus

$$\widehat{S} = \left\{ v \in V_{s+1} \mid \mathrm{rank}\,(M_v) = n-1 \right\} \subset V_{s+1},$$

where $M_v \in \mathsf{Mat}_{n,n+m}(\mathbb{C})$ denotes the matrix M evaluated at the point $v \in V_{s+1}$. By the linearity of f_j^i, the subvariety $\widehat{S} \subset V_{s+1}$ descends to a subvariety $S \hookrightarrow \mathbb{P}^s = \mathbb{P}(V_{s+1})$, in such a way that $\widehat{S} \cup \{0\}$ is the affine cone over $S \hookrightarrow \mathbb{P}^s$. We shall use the notation $[v]$ to denote a point in projective space, to emphasise that we take the projective point of view.

4.1 Constructing points in the triple Grassmannian

Consider the set-theoretic map

$$\psi : S \to \mathbb{P}^{n-1}, \qquad [v] \mapsto [\alpha_v],$$

where $[\alpha_v]$ is determined by the 1-dimensional $\mathbb{C}$-vector space

$$\ker\left(\mathcal{O}_{\mathbb{P}^s}(-1)\big|_{[v]}^{\oplus n} \xrightarrow{\varphi^t_{[v]}} \mathcal{O}_{\mathbb{P}^s}\big|_{[v]}^{\oplus n+m}\right) \subset \mathcal{O}_{\mathbb{P}^s}(-1)\big|_{[v]}^{\oplus n} = V_n = \mathbb{C}^n.$$

Of course, if M is the $n \times (n+m)$ matrix of linear forms corresponding to the morphism φ, then $\alpha_v \in V_n$ is defined (up to scalar multiplication) by $M_v^t \cdot \alpha_v = 0$.

Lemma 4.1 *The association $[v] \mapsto [\alpha_v]$ defines an algebraic morphism $\psi : S \to \mathbb{P}^{n-1}$.*

Proof As already mentioned, since φ is general, one has $D_{n-2}(\varphi) = \emptyset$, and thus $\varphi_{[v]}$ has rank precisely $n-1$ for every $[v] \in S$. Therefore the sheaf $\mathcal{L} = \mathrm{coker}(\varphi)|_S$ is a locally free sheaf of rank 1, and moreover it is globally generated by n sections $\alpha^1, \dots, \alpha^n$, arising from the linear dependence relations

$$\alpha^1_{[v]} F^1_{[v]} + \cdots + \alpha^n_{[v]} F^n_{[v]} = 0, \quad [v] \in S,$$

where $F^i_{[v]}$ denotes the i-th row of the matrix associated to $\varphi_{[v]} = M_v$. The data $(\mathcal{L}, \alpha^1, \dots, \alpha^n)$ defines the sought after algebraic morphism $\psi : S \to \mathbb{P}^{n-1}$. □

Our key construction starts now. Let $[v], [w] \in S$ be distinct points and consider the space

$$\pi_{v,w} = \langle M_v^t \cdot \alpha_w, M_w^t \cdot \alpha_v \rangle \subset V_{n+m}. \tag{4.2}$$

The following lemma aims to explain the geometric role of $\pi_{v,w}$ just defined.

Lemma 4.2 *Let $[v]$, $[w]$ be two distinct points in S. Then*:

(1) $\dim \pi_{v,w} = 0$ *if and only if the line $\ell_{v,w}$ joining $[v]$, $[w]$ is entirely contained in S, and $\psi(\ell_{v,w})$ reduces to a point in $\mathbb{P}^{n-1}$.*
(2) $\dim \pi_{v,w} = 1$ *if and only if the line $\ell_{v,w}$ joining $[v]$, $[w]$ is entirely contained in S, and $\psi(\ell_{v,w})$ is a line in $\mathbb{P}^{n-1}$.*
(3) $\dim \pi_{v,w} = 2$ *if and and only if $\psi(\ell_{v,w}) \subset \mathbb{P}^{n-1}$ intersects the line between $[\alpha_v]$ and $[\alpha_w]$ in precisely two points.*

Proof We proceed case by case.

(1) $\langle \alpha_v \rangle = \langle \alpha_w \rangle$ if and only if $M_v^t \cdot \alpha_w = M_w^t \cdot \alpha_v = 0$; therefore $\dim \pi_{v,w} = 0$ if and only if $\dim \langle \alpha_v, \alpha_w \rangle = 1$ and the statement follows by

$$M^t_{\lambda v + \mu w} \cdot \alpha_v = M^t_{\lambda v} \cdot \alpha_v + M^t_{\mu w} \cdot \alpha_v = 0$$

for every $\lambda, \mu \in \mathbb{C}$.

(2) If $\dim \pi_{v,w} = 1$ then there exist $\delta_1, \delta_2 \in \mathbb{C}$ such that $\delta_1 M_w^t \cdot \alpha_v + \delta_2 M_v^t \cdot \alpha_w = 0$. Therefore $M^t_{\lambda v + \mu w}(\lambda \delta_1 \alpha_v + \mu \delta_2 \alpha_w) = \lambda\mu\big(\delta_2 M_v^t \cdot \alpha_w + \delta_1 M_w^t \cdot \alpha_v\big) = 0$, so that $[\lambda v + \mu w] \in S$ for every $\lambda, \mu \in \mathbb{C}$ and the kernels of the transpose matrices are aligned in $\mathbb{P}^{n-1}$. For the converse, first notice that $\dim \pi_{v,w} \neq 0$. Moreover, if there exists another point $[u] \in \ell_{v,w} \cap S$ with $\alpha_u = \delta_1 \alpha_v + \delta_2 \alpha_w$, and $u = \lambda v + \mu w$. Then $\lambda \delta_2 M_v^t \cdot \alpha_w + \mu \delta_1 M_w^t \cdot \alpha_v = 0$, so that $\dim \pi_{v,w} = 1$.

(3) By contradiction, suppose there exists a third point $[u] \in \ell_{v,w} \cap S$ with $\alpha_u = \delta_1 \alpha_v + \delta_2 \alpha_w$, and $u = \lambda v + \mu w$. Then $\lambda \delta_2 M_v^t \cdot \alpha_w + \mu \delta_1 M_w^t \cdot \alpha_v = M_u^t \cdot \alpha_u = 0$, so that $\dim \pi_{v,w} \leq 1$. Viceversa, if $\dim \pi_{v,w} \leq 1$ then $\ell_{v,w} \subset S$ by the above items so that $\psi(\ell_{v,w})$ intersects the line between $[\alpha_v], [\alpha_w]$ either in one point or in infinite points.

□

Definition 4.3 We shall use the shorthand notation

$$G_{n,s,m} = \mathrm{Gr}\,(2, n) \times \mathrm{Gr}\,(2, s+1) \times \mathrm{Gr}\,(n+m-2, n+m),$$

and we shall denote with the same letter $\mathcal{U}$ the tautological (sub)bundle on each Grassmannian. There is a natural section $\omega \in \mathrm{H}^0(G_{n,s,m}, \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee \boxtimes \mathcal{U}^\vee)$ associated to M, defined by

$$\omega : V_n \otimes V_{s+1} \otimes V_{n+m} \longrightarrow \mathbb{C}, \qquad (a, u, b) \mapsto a^t \cdot M_u \cdot b.$$

We denote by $Z = V(\omega) \subset G_{n,s,m}$ its zero scheme.

We note that there is an identity

$$Z = \big\{ P \in G_{n,s,m} \,:\, \omega|_P \equiv 0 \big\}$$

where, if $P = (\rho_1, \rho_2, \rho_3)$, then $\omega|_P \equiv 0$ means that $\omega(a, u, b) = 0$ for every $a \in \rho_1$, $u \in \rho_2$, $b \in \rho_3$.

Definition 4.4 To any pair of distinct points $[v], [w] \in S$ such that $\dim \pi_{v,w} = 2$ we can associate the point

$$P_{[v],[w]} = \Big(\langle \alpha_v \,,\, \alpha_w \rangle \,,\, \langle v, w \rangle \,,\, \pi^\perp_{v,w} \Big) \in G_{n,s,m},$$

where $\pi_{v,w}$ is as defined in Equation 4.2.

Remark 4.5 By Lemma 4.2, there is an immersion

$$H \to G_{n,s,m}, \quad [v] + [w] \mapsto P_{[v],[w]},$$

where $H = \{[v] + [w] \in \mathrm{Sym}^2(S) \mid [v] \neq [w] \text{ and } \dim \pi_{v,w} = 2\} \subset \mathrm{Sym}^2(S)$.

Lemma 4.6 *Let $P_{[v],[w]}$ be as in Definition 4.4, then $P_{[v],[w]} \in Z$.*

Proof We need to prove that $\omega|_{P_{[v],[w]}} \equiv 0$. Let $a = h_1\alpha_v + h_2\alpha_w$ and $u = \lambda v + \mu w$. Then

$$\begin{aligned}\omega(a,u,b) &= (h_1\alpha_v^t + h_2\alpha_w^t)\cdot M_{\lambda v+\mu w}\cdot b\\ &= h_1\mu\,(\alpha_v^t\cdot M_w)\cdot b + h_2\lambda\,(\alpha_w^t\cdot M_v)\cdot b\\ &= 0.\end{aligned}$$

□

4.2 Main technical result

In the following, given $\rho \in \mathrm{Gr}\,(2,k)$ we shall denote by $[\rho] \subset \mathbb{P}^{k-1}$ the projective line defined by ρ. Also, given two distinct points $[v]$ and $[w]$ in $\mathbb{P}^s$, we shall denote by $\ell_{v,w} \subset \mathbb{P}^s$ the line connecting them.

The following is the main technical result of the paper.

Theorem 4.7 *Let $P = (\rho_1\,,\rho_2\,,\rho_3) \in Z$. Then one of the following holds*:

a. there exist two (and only two) distinct points $[v], [w] \in S \cap [\rho_2]$ such that $P = P_{[v],[w]}$,
b. $[\rho_2] \subset S$ and $\psi([\rho_2])$ reduces to a point in $[\rho_1] \subset \mathbb{P}^{n-1}$,
c. there exists exactly one point $[v] \in S$ where $[\rho_2]$ is tangent and such that $[\alpha_v] \in [\rho_1]$.
d. $[\rho_2] \subset S$ and $[\rho_1] = \psi(\ell_{v,w}) \subset \mathbb{P}^{n-1}$.

Proof Consider the linear subspace

$$W_{(\rho_1,\rho_2)} = \left\{M_u^t \cdot a \,|\, a \in \rho_1,\ u \in \rho_2\right\} \subset V_{n+m}.$$

Now, since $(\rho_1,\rho_2,\rho_3) \in Z$, we have $W_{(\rho_1,\rho_2)} \subset \rho_3^{\perp}$ so that $\dim W_{(\rho_1,\rho_2)} \leq 2$. Let us proceed case by case.

- Suppose $\dim W_{(\rho_1,\rho_2)} = 0$ first. This means that $M_u^t \cdot a = 0$ for every $u \in \rho_2$ and for every $a \in \rho_1$. This is impossible since it would imply $\dim\ker(M_u^t) \geq 2$, i.e. $\mathrm{rank}\,(M_u) < n-1$. But this is in contradiction with the generality assumption on M.
- Next, let us suppose $\dim W_{(\rho_1,\rho_2)} = 1$. This means that we can find a basis $\{M_{u_1}^t \cdot a_1\}$ for $W_{(\rho_1,\rho_2)}$. We can complete to bases $\{a_1, a_2\} \subset \rho_1$ and $\{u_1, u_2\} \subset \rho_2$, in such a way that

$$M_{u_1}^t \cdot a_2 = M_{u_2}^t \cdot a_1 = 0\,.$$

  In fact, if $\{a_1, a_2'\}$ is any basis for ρ_1, then $M_{u_1}^t \cdot (ha_1 + a_2') = 0$ for some $h \in \mathbb{C}$. Hence it is sufficient to choose $a_2 = ha_1 + a_2'$. A similar argument provides the required choice of $u_2 \in \rho_2$. In particular, $[u_1], [u_2] \in S$ and by assumption $M_{u_1}^t \cdot a_1 + M_{u_2}^t \cdot a_2 = 0$ (up to a possible rescale of a_2). Therefore

$$M_{\lambda u_1+\mu u_2}^t \cdot (\mu a_1 + \lambda a_2) = \lambda\mu(M_{u_1}^t \cdot a_1 + M_{u_2}^t \cdot a_2) = 0\,,$$

  so that $[\rho_2] \subset S$ and $[\rho_1] = \psi(\ell_{v,w}) \subset \mathbb{P}^{n-1}$. This is the case ***d*** in the statement.

- Finally, let us suppose $\dim W_{(\rho_1,\rho_2)} = 2$, which means $W_{(\rho_1,\rho_2)} = \rho_3^{\perp}$. Choose bases $\{a_1, a_2\} \subset \rho_1$ and $\{u_1, u_2\} \subset \rho_2$, define

$$v_{i,j} = M^t_{u_i} \cdot a_j, \qquad i,j \in \{1,2\} \ .$$

Now, if $\dim\langle v_{1,1}\ ,\ v_{2,1}\rangle = 1$ then there exist $\delta_1, \delta_2 \in \mathbb{C}$ such that $\delta_1 v_{1,1} + \delta_2 v_{2,1} = 0$. It follows that $M^t_{\delta_1 u_1 + \delta_2 u_2} \cdot a_1 = 0$ so that $[\delta_1 u_1 + \delta_2 u_2] \in S$. Now if $\delta_1 \neq 0$ we define $u_1' = \delta_1 u_1 + \delta_2 u_2$ and we replace the basis $\{u_1, u_2\}$ with $\{u_1', u_2\}$. Since $\dim W_{(\rho_1,\rho_2)} = 2$ there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ such that

$$\gamma_1 M^t_{u_2} \cdot a_1 + \gamma_2 M^t_{u_1'} \cdot u_2 + \gamma_3 M^t_{u_2} \cdot a_2 = 0 \qquad \text{so that} \qquad M^t_{\gamma_2 u_1' + \gamma_3 u_2} \cdot (\gamma_1 a_1 + \gamma_3 a_2) = 0 \ .$$

Therefore, if $\gamma_3 \neq 0$ we have two distinct points in S (u_1' and $\gamma_2 u_1' + \gamma_3 u_2$) so that the statement follows by Lemma 4.2 (being one of the cases ***a***, ***b***, ***d***). On the other hand, if $\gamma_3 = 0$ we may assume

$$v_{1,1} = 0, \quad v_{1,2} = v_{2,1}, \quad W_{(\rho_1,\rho_2)} = \langle v_{1,2}\ ,\ v_{2,2}\rangle \ ,$$

which gives item ***c*** of the statement. One can argue in a similar way for the case $\dim\langle v_{1,2}\ ,\ v_{2,2}\rangle = 1$ simply reordering the basis $\{a_1, a_2\}$. We are only left with the case

$$W_{(\rho_1,\rho_2)} = \langle v_{1,1}, v_{2,1}\rangle = \langle v_{1,2}, v_{2,2}\rangle \ .$$

Hence there exists a matrix $\Phi \in \mathsf{Mat}_{2,2}(\mathbb{C})$ realising a coordinate change

$$\begin{pmatrix} v_{1,2} & v_{2,2} \end{pmatrix} = -\begin{pmatrix} v_{1,1} & v_{2,1} \end{pmatrix} \cdot \Phi,$$

where we adopted the notation $\begin{pmatrix} v_{1,j} & v_{2,j} \end{pmatrix}$ to denote the $(n+m) \times 2$ matrix whose columns are $v_{1,j}$ and $v_{2,j}$. Our aim is now to study vectors $v \in \rho_2$ corresponding to points $[v] \in S$ with the additional property that $[\alpha_v] \in [\rho_1]$. Such a point is given by the choice of a nonzero vector

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in \mathbb{C}^2$$

together with scalars $\delta_1, \delta_2 \in \mathbb{C}$, not both vanishing, such that

$$M^t_{\lambda u_1 + \mu u_2} \cdot (\delta_1 a_1 + \delta_2 a_2) = 0.$$

In particular, it is not restrictive to assume $\delta_2 \neq 0$ since $\dim\langle v_{1,1}\ ,\ v_{2,1}\rangle = 2$. Rename $\delta = \delta_1 \delta_2^{-1} \in \mathbb{C}$ and consider the following equalities:

$$
\begin{aligned}
M^t_{\lambda u_1+\mu u_2}\cdot(\delta a_1+a_2) &= \lambda\delta M^t_{u_1}\cdot a_1+\mu\delta M^t_{u_2}\cdot a_1+\lambda M^t_{u_1}\cdot a_2+\mu M^t_{u_2}\cdot a_2\\
&= \lambda\delta v_{1,1}+\mu\delta v_{2,1}+\lambda v_{1,2}+\mu v_{2,2}\\
&= \begin{pmatrix} v_{1,1} & v_{2,1}\end{pmatrix}\cdot\begin{pmatrix}\delta\lambda\\ \delta\mu\end{pmatrix}+\begin{pmatrix} v_{1,2} & v_{2,2}\end{pmatrix}\cdot\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\\
&= \begin{pmatrix} v_{1,1} & v_{2,1}\end{pmatrix}\cdot\begin{pmatrix}\delta\lambda\\ \delta\mu\end{pmatrix}-\begin{pmatrix} v_{1,1} & v_{2,1}\end{pmatrix}\cdot\Phi\cdot\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\\
&= \begin{pmatrix} v_{1,1} & v_{2,1}\end{pmatrix}\cdot\{\delta\cdot\mathrm{id}-\Phi\}\cdot\begin{pmatrix}\lambda\\ \mu\end{pmatrix}.
\end{aligned}
$$

Now, since $\dim\langle v_{1,1}, v_{2,1}\rangle = 2$ the last line vanishes if and only if δ is an eigenvalue of Φ and $\begin{pmatrix}\lambda & \mu\end{pmatrix}^t$ is an eigenvector relative to δ. Since $\mathbb{C}$ is algebraically closed, we conclude that the line $[\rho_2]\subset\mathbb{P}^s$ always intersects S in (at least) one point $[v]=[\lambda u_1+\mu u_2]$ satisfying $[\alpha_v]\in[\rho_1]$. More precisely we have the following three possibilities.

a. *The matrix* Φ *admits two different eigenvalues* δ *and* θ. In this case we have two (independent) eigenvectors

$$
\begin{pmatrix}\lambda_\delta\\ \mu_\delta\end{pmatrix},\quad \begin{pmatrix}\lambda_\theta\\ \mu_\theta\end{pmatrix}
$$

and the above discussion provides two distinct points

$$
\begin{aligned}
[v] &= [\lambda_\delta u_1+\mu_\delta u_2]\in S\cap[\rho_2],\\
[w] &= [\lambda_\theta u_1+\mu_\theta u_2]\in S\cap[\rho_2].
\end{aligned}
$$

Notice that by Lemma 4.2 either we are in case ***d*** of the statement or the points $[v],[w]\in S\cap[\rho_2]$ are the only ones satisfying the additional property $[\alpha_v],[\alpha_w]\in[\rho_1]$. Clearly, in this last case $\rho_1=\langle\alpha_v,\alpha_w\rangle$, $\rho_2=\langle v,w\rangle$, and $W_{(\rho_1,\rho_2)}=\langle M^t_v\alpha_w, M^t_w\alpha_v\rangle=\pi_{v,w}$; therefore $(\rho_1,\rho_2,\rho_3)=P_{[v],[w]}$. This is item ***a*** in the statement.

b. *The matrix* Φ *admits one eigenvalue* δ *whose eigenspace is* 2*-dimensional.* In this case every non-trivial $\begin{pmatrix}\lambda & \mu\end{pmatrix}^t\in\mathbb{C}^2$ is an eigenvector so that the line defined by $[\rho_2]$ in $\mathbb{P}^s$ is entirely contained in S. On the other hand the matrix M^t_v admits the same kernel $\delta a_1+a_2\in\rho_1$ for every $v\in\rho_2$. This is item ***b*** in the statement.

c. *The matrix* Φ *admits only one eigenvalue* δ *whose eigenspace is* 1*-dimensional.* In this case any eigenvector $\begin{pmatrix}\lambda & \mu\end{pmatrix}^t$ corresponds to the same point $[v]=[\lambda u_1+\mu u_2]\in S$. Hence $[v]$ is the only point in the intersection $[\rho_2]\cap S$ such that $[\alpha_v]\in[\rho_1]$. Moreover, in this case the *algebraic* multiplicity of δ is 2; i.e. the multiplicity of the intersection $[\rho_2]\cap S$ is 2 at $[v]$. This is item ***c*** in the statement.

The proof is now complete. □

5 Existence of special lines

As in the previous section, we fix integers $n\geq 3$, $m\geq 0$, $s\in\{m+2,\dots,2m+3\}$ and a general map of vector bundles φ as in (4.1). Moreover, we shall use the following terminology: a line $\ell\subset S=D_{n-1}(\varphi)\subset\mathbb{P}^s$ is said to be *of type* ***b*** (resp. *of type* ***d***) if it arises from a point $P\in Z$ satisfying condition ***b*** (resp. condition ***d***) in Theorem 4.7.

5.1 Excluding lines of type *b*

The first aim of this section is to understand the fibres of the map ψ, and we will be particularly interested in the existence of points $[\alpha] \in \mathbb{P}^{n-1}$ whose fibre $\psi^{-1}([\alpha])$ is a line in S.

Fix $[\alpha] \in \mathbb{P}^{n-1}$ and observe that

$$\psi^{-1}([\alpha]) = \left\{[v] \in S \mid M_v^t \cdot \alpha = 0\right\} \subset S \tag{5.1}$$

is nothing but the solutions set of a linear system of $n+m$ equations in $s+1$ variables, namely an intersection of $n+m$ hyperplanes in $\mathbb{P}^s$. Therefore the fibre (5.1) is always linear. Moreover, it can be described by means of a matrix $A_\alpha \in \mathsf{Mat}_{n+m,s+1}(\mathbb{C})$, and by the linearity with respect to α we get an immersion

$$f : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P} = \mathbb{P}\big(\mathsf{Mat}_{n+m,s+1}(\mathbb{C})\big), \quad [\alpha] \mapsto [A_\alpha]. \tag{5.2}$$

Remark 5.1 Notice that an additional condition $s \le n+m$ is essential in order to obtain 0-dimensional fibres of ψ, and similarly $s \le n+m+1$ is necessary in order to obtain 1-dimensional fibres, as well as $s \le n+m+2$ for 2-dimensional fibres.

Let us denote by $N_k \subset \mathbb{P}$ the subvariety of matrices of rank at most k. We can easily compute the codimension of N_k in $\mathbb{P}$ as

$$\operatorname{codim}(N_k) = (n+m-k)(s+1-k),$$

so that in particular assuming $s \le n+m$ one finds

$$\begin{aligned}\operatorname{codim}(N_s) &= n+m-s \ge 0\\ \operatorname{codim}(N_{s-1}) &= 2(n+m-s+1) \ge 2\\ \operatorname{codim}(N_{s-2}) &= 3(n+m-s+2) \ge 6.\end{aligned}$$

Theorem 5.2 *Let $\psi : S \to \mathbb{P}^{n-1}$ and $f : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}$ be the maps defined by Lemma 4.1 and (5.2) respectively. Fix integers $n \ge 3$, $m \ge 0$, $s \in \{m+2, \dots, 2m+3\}$.*

(i) Assume $s = n+m$. Then

(1) ψ is surjective and its generic fibre is a point.

(2) $f \circ \psi$ admits 1-dimensional fibres precisely over $\mathrm{Im}(f) \cap N_{s-1}$.

(ii) Assume $s < n+m$. Then

(1) ψ is a closed immersion if and only if $n > 2s - 2m - 3$, in which case the image of the composition $f \circ \psi$ is $\mathrm{Im}(f) \cap N_s \subset \mathbb{P}$,

(2) $f \circ \psi$ admits 1-dimensional fibres if and only if $n \leq 2s - 2m - 3$, and such fibres arise precisely over $\mathrm{Im}(f) \cap N_{s-1}$,

(3) $f \circ \psi$ admits 2-dimensional fibres if and only if $n \leq \frac{1}{2}(3s - 3m - 7)$, and such fibres arise precisely over $\mathrm{Im}(f) \cap N_{s-2}$,

Proof Let us proceed by steps.

(i) First suppose that $s = n + m$. As already observed the fibre $\psi^{-1}([\alpha])$ is cut by $n + m$ hyperplanes in $\mathbb{P}^s$, hence the generic fibre reduces to a point. Moreover, the fibre is 1-dimensional at those $[\alpha]$ such that $f([\alpha]) \in \mathrm{Im}(f) \cap N_{s-1} \subset \mathbb{P}$, which has dimension $(n-1) - 2 = n - 3 \geq 0$.

(ii) Now assume $s < n + m$. Then the fibre $\psi^{-1}([\alpha])$ is a point (respectively a line) precisely at those $[\alpha]$ such that $f([\alpha]) \in \mathrm{Im}(f) \cap N_k \subset \mathbb{P}$ with $k = s < n + m$ (respectively $k = s - 1 < n + m$). Therefore the image of ψ describes a subvariety of $\mathbb{P}^{n-1}$ of dimension

$$\dim \psi(S) = (n-1) - \operatorname{codim}(N_s) = (n-1) - (n + m - s) = s - m - 1 = \dim(S) ,$$

while the 1-dimensional fibres of ψ (if they exist) are mapped onto a locus of dimension

$$(n-1) - \operatorname{codim}(N_{s-1}) = (n-1) - 2(n + m - s + 1) = 2s - n - 2m - 3 .$$

The condition $n > 2s - 2m - 3$ is the same as $\operatorname{codim}(N_{s-1}) = 2(n + m - s + 1) > n - 1$, which in turn is equivalent to require that N_{s-1} is empty; here we are using the genericity of the original matrix M (hence of the form ω) from which it follows the genericity of the immersion of $\mathbb{P}^{n-1}$ in $\mathbb{P}$ through f. Hence the fibres of the map consist of at most one point if and only if $n > 2s - 2m - 3$, in which case ψ is a closed immersion, as wanted. Finally, the fibres of dimension at least 2 arise over $\mathrm{Im}(f) \cap N_{s-2}$, for which the expected dimension is

$$(n-1) - \operatorname{codim}(N_{s-2}) = (n-1) - 3(n + m - s + 2) = 3s - 2n - 3m - 7.$$

This number is non-negative if and only if $n \leq \frac{1}{2}(3s - 3m - 7)$, as required.

□

5.2 Excluding lines of type *d*

The next aim of this section is to show that the lines described by item ***d*** of Theorem 4.7 do not occur whenever $n > 2s - 3m - 2$. Recall that these are the lines $\ell \subset S$ such that the image $\ell' = \psi(\ell)$ remains a line in $\mathbb{P}^{n-1}$.

Theorem 5.3 *Let* $n \geq 3$, $m \geq 0$ *and* $s \in \{m+2, \ldots, 2m+3\}$.

- If $n > 2s - 3m - 1$ then the composition

$$\pi_Z : Z \xhookrightarrow{\iota} G_{n,s,m} \xrightarrow{\mathrm{pr}_{12}} \mathrm{Gr}(2,n) \times \mathrm{Gr}(2,s+1)$$

is injective, where ι is the natural inclusion and pr_{12} is the natural projection.

- A line $\ell \subset S \subset \mathbb{P}^s$ such that $\ell' = \psi(\ell)$ remains a line in $\mathbb{P}^{n-1}$ exists if and only if the map π_Z admits an $(n+m-2)$-dimensional linear fibre.

Proof Let us proceed by steps.

- The fibre of π_Z over $(\rho_1, \rho_2) \in \mathrm{Gr}\,(2,n) \times \mathrm{Gr}\,(2,s+1)$ can be easily described as

$$\pi_Z^{-1}(\rho_1,\rho_2) = \left\{ (\rho_1,\rho_2,\rho_3) \in G_{n,s,m} \,\middle|\, \rho_3 \subset W^{\perp}_{(\rho_1,\rho_2)} \right\}$$

where $W_{(\rho_1,\rho_2)} = \langle M_u^t \cdot a \,|\, a \in \rho_1,\ u \in \rho_2 \rangle \subset V_{n+m}$. Notice that

$$W_{(\rho_1,\rho_2)} = \langle M_{u_i}^t \cdot a_j \,|\, 1 \le i,j \le 2 \rangle$$

where $\{a_1,a_2\}$ and $\{u_1,u_2\}$ are arbitrary bases for ρ_1 and ρ_2 respectively. In particular, $\dim W_{(\rho_1,\rho_2)} \le 4$. Since $\rho_3 \in \mathrm{Gr}\,(n+m-2, n+m)$, we deduce

$$\pi_Z^{-1}(\rho_1,\rho_2) = \begin{cases} \emptyset & \text{if } \dim W_{(\rho_1,\rho_2)} \ge 3 \\ \left(\rho_1,\rho_2,W^{\perp}_{(\rho_1,\rho_2)}\right) & \text{if } \dim W_{(\rho_1,\rho_2)} = 2 \\ \mathrm{Gr}\,\left(n+m-2, W^{\perp}_{(\rho_1,\rho_2)}\right) & \text{if } \dim W_{(\rho_1,\rho_2)} = 1 \end{cases}$$

Notice that $\dim W_{(\rho_1,\rho_2)} \ne 0$, otherwise we would have points $u \in V_{s+1}$ satisfying $\mathrm{rank}\, M_u = n-2$ and this is excluded since $D_{n-2}(\varphi) = \emptyset$. In particular, for $\dim W_{(\rho_1,\rho_2)} = 1$ we have

$$\mathrm{Gr}\,\left(n+m-2, W^{\perp}_{(\rho_1,\rho_2)}\right) \cong \mathbb{P}^{n+m-2}.$$

On the other hand Z cannot contain an $(n+m-2)$-dimensional subspace whenever $\dim(Z) = 2(s-m-1) < n+m-2$, i.e. when $n > 2s-3m$. Moreover, in the case $n = 2s-3m$, the irreducibility of Z together with the non injectivity of the map π_Z would imply $Z \cong \mathbb{P}^{n+m-2}$, which is impossible because otherwise the map π_Z would be constant so that S would reduce to a line $S = [\rho_2] \cong \mathbb{P}^1 \subset \mathbb{P}^s$. Of course this is false being $n \ge 3$.

- We claim that the existence of a line $\ell \subset S$ such that $\ell' = \psi(\ell)$ remains a line in $\mathbb{P}^{n-1}$ is equivalent to the existence of a $(n+m-2)$-dimensional fibre of the map π_Z. In fact, by Lemma 4.2 the existence of such a line ℓ is equivalent to a point

$$(\rho_1,\rho_2) \in \mathrm{Gr}\,(2,n) \times \mathrm{Gr}\,(2,s+1)$$

with $[\rho_1] = \ell'$ and $[\rho_2] = \ell$, that moreover satisfies $\dim W_{(\rho_1,\rho_2)} = 1$. As shown in the first item this is equivalent to the condition $\dim \pi_Z^{-1}(\rho_1,\rho_2) = n+m-2$.

□

In Corollary 5.5 we will be able to give a better bound than the one in Theorem 5.3 in the cases $m = 0$ and $m = 1$.

Remark 5.4 Notice that for large values of m, the bound $n > 2s - 2m - 3$ obtained in Theorem 5.2 is stronger than the one obtained in Theorem 5.3. More precisely,

$$n > 2s - 2m - 3 \qquad \Longrightarrow \qquad n > 2s - 3m - 1$$

as soon as $m \geq 2$.

5.3 Conclusions

We now summarise the main results of this section in the following corollary.

Corollary 5.5 *Let $n \geq 3$, $m \geq 0$ and $s \in \{m+2, \dots, 2m+3\}$. Assume $n > 2s - 2m - 3$ and let $(\rho_1, \rho_2, \rho_3) \in Z$.*

Then one of the following holds:

(1) *there exist two (and only two) distinct points $[v], [w] \in S \cap [\rho_2]$ such that $P = P_{[v],[w]}$,*
(2) *there exists exactly one point $[v] \in S$ where $[\rho_2]$ is tangent and such that $[\alpha_v] \in [\rho_1]$.*

Proof If $m \geq 2$, then by Remark 5.4 the statement is an immediate consequence of Theorem 4.7, Theorem 5.2, Theorem 5.3.

For $m = 1$ our assumption becomes $n > 2s - 5$, so that the hypothesis of Theorem 5.2 are satisfied while Theorem 5.3 works as soon as $n > 2s - 4$. Let us prove by hand that choosing $m = 1$ and $n = 2s - 4$ the map

$$\pi_Z : Z \overset{\iota}{\hookrightarrow} G_{n,s,m} \xrightarrow{\mathrm{pr}_{12}} \mathrm{Gr}(2, n) \times \mathrm{Gr}(2, s+1)$$

is still injective. Here ι is the natural inclusion and π is the natural projection. The idea is to exclude high dimensional fibres following the proof of Theorem 5.3.

(A) Set $s = 5, n = 6, m = 1$. We have to exclude the existence of a $\mathbb{P}^5 \subset Z$. However, in this case Z is a sixfold with $h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 3$, which in this case is equal to the Picard rank. In fact, Pic(Z) is generated by the restrictions of the three Plücker line bundles from the ambient Grassmannians. Hence, by degree reasons, since Z is smooth and of degree > 1, it cannot contain a $\mathbb{P}^5$.
(B) Set $s = 4, n = 4, m = 1$. We have to exclude the existence of a $\mathbb{P}^3 \subset Z$. This time, we know that Pic(S) is only generically of rank 2, and the same holds for Z (in fact $h^{2,0}(Z) = 4$). For the same reasons above, we can therefore exclude the existence of a $\mathbb{P}^3$ for a general Z. But this is enough, since we started by hypothesis from a general matrix, and S - which is a isomorphic to a determinantal quintic hypersurface also described as a complete intersection in $\mathbb{P}^4 \times \mathbb{P}^3$ - in this case the Picard group will be $\mathbb{Z}^2$ and generated by the two hyperplane classes of $\mathbb{P}^4$ and $\mathbb{P}^3$, and will not even contain lines by a Noether–Lefschetz type argument, see [20] and also [7, Theorem 1.2]. Similarly Z won't contain a copy of $\mathbb{P}^3$.

Hence the statement is proven for every $m \geq 1$. We are only left with the case $m = 0$. In this case our assumption becomes $n > 2s - 3$, so that the hypothesis of Theorem 5.2 are satisfied while Theorem 5.3 works as soon as $n > 2s - 1$. Hence we only need to check the following cases, where $m = 0$ and either $n = 2s - 1$ or $n = 2s - 2$.

(C) Set $s = 2, n = 3, m = 0$. Just observe that in this case S is a curve of genus $g \neq 0$, so that in particular it does not contain lines and the conclusion follows by the second item in Theorem 5.3.

(D) Set $s = 3, n = 5, m = 0$. We have to exclude the existence of $\mathbb{P}^3 \subset Z$. To this aim, it is sufficient to run exactly the same argument as in case (B).

(E) Set $s = 3, n = 4, m = 0$. In this case S and $\psi(S)$ in $\mathbb{P}^3$ are precisely the K3 surfaces studied by Oguiso in [23, 24]. Notice that a generic determinantal K3 surface does not contain lines, since the Picard lattice is

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$$

and the square of every other element is divisible by 4. Therefore we do not have (-2)-curves in general. Hence the second item in Theorem 5.3 implies the injectivity of the map $\pi_Z : Z \to \mathrm{Gr}\,(2,4) \times \mathrm{Gr}\,(2,4)$ as required.

□

6 Hilbert squares of degeneracy loci

In this section we finally prove our main theorem, namely Theorem A.

We denote by $\mathrm{Fl}\,(1,2,n)$ and by $\mathrm{Fl}\,(1,2,s+1)$ the appropriate flag varieties. Moreover, we denote by

$$\Gamma_\psi \subset \mathbb{P}^{n-1} \times S \subset \mathbb{P}^{n-1} \times \mathbb{P}^s$$

the graph of the morphism ψ of Lemma 4.1.

In the category of $\mathbb{C}$-schemes, we consider the limit $\mathcal{V}$ of the following diagram of solid arrows

$$\begin{tikzcd}
 & & \mathcal{V} \arrow[dll, dotted] \arrow[d, dotted] \arrow[drr, dotted] & & \\
\Gamma_\psi \arrow[dr] & & \mathrm{Fl}(1,2,n)\times\mathrm{Fl}(1,2,s+1) \arrow[dl] \arrow[dr] & & Z \arrow[dl] \\
 & \mathbb{P}^{n-1}\times\mathbb{P}^s & & \mathrm{Gr}(2,n)\times\mathrm{Gr}(2,s+1) &
\end{tikzcd}$$

Notice that, set-theoretically, $\mathcal{V}$ can be described as

$$\mathcal{V} = \left\{ \big([v], \psi([v]), (\rho_1, \rho_2, \rho_3)\big) \in \Gamma_\psi \times Z \mid [v] \in [\rho_2], \psi([v]) \in [\rho_1] \right\} \hookrightarrow \Gamma_\psi \times Z$$

and via the natural isomorphism $\Gamma_\psi \xrightarrow{\sim} S$ we make the identification

$$\mathcal{V} = \left\{ \left([v], (\rho_1, \rho_2, \rho_3) \right) \mid [v] \in [\rho_2], \psi([v]) \in [\rho_1] \right\} \hookrightarrow S \times Z.$$

Composing with the projection $S \times Z \to Z$, we obtain a morphism

$$\pi : \mathcal{V} \hookrightarrow S \times Z \to Z.$$

We now show that this morphism defines a modular map $Z \to \operatorname{Hilb}^2(S)$.

Lemma 6.1 *Let $n \geq 3$, $m \geq 0$, $s \in \{m+2, \ldots, 2m+3\}$. Assume $n > 2s - 2m - 3$. Then the natural morphism*

$$\pi : \mathcal{V} \hookrightarrow S \times Z \to Z$$

is a flat family of length 2 subschemes of S.

Proof Since Z is smooth, in particular reduced, it is enough to prove that the fibre over any closed point is a finite subscheme of length 2. Flatness is then automatic.

In fact, the fibre $\pi^{-1}(\rho_1, \rho_2, \rho_3)$ over a point $P = (\rho_1, \rho_2, \rho_3) \in Z$ is of the form

$$\pi^{-1}(P) = \{([v], P) | [v] \in S \cap [\rho_2], \ [\alpha_v] \in [\rho_1]\}.$$

By Corollary 5.5, this is a length 2 subscheme of $S \times \{P\}$ if $n > 2s - 2m - 3$, which we are assuming. □

In particular, if $n > 2s - 2m - 3$, the morphism π gives rise, via the universal property of the Hilbert scheme, to a morphism

$$\vartheta : Z \to \operatorname{Hilb}^2(S).$$

Theorem 6.2 *Let $n \geq 3$, $m \geq 0$, $s \in \{m+2, \ldots, 2m+3\}$. Assume $n > 2s - 2m - 3$. Then the morphism $\vartheta : Z \to \operatorname{Hilb}^2(S)$ is an isomorphism.*

Proof To prove ϑ is an isomorphism, by Zariski's Main Theorem it is enough to prove it is bijective, since both source and target are smooth $\mathbb{C}$-varieties of the same dimension $2(s - m - 1)$.

On $\mathbb{C}$-valued points, the morphism ϑ is defined by

$$\vartheta(P) = [\pi^{-1}(P)] \ \in \ \operatorname{Hilb}^2(S).$$

By the uniqueness conditions spelled out in Corollary 5.5, the map ϑ is injective. By the same argument, one can see that $\vartheta(B)$ is an injective map of sets for every $\mathbb{C}$-scheme B. Thus ϑ is a proper monomorphism, i.e. a closed immersion.

Since source and target are smooth of the same dimension, ϑ is an lci morphism of codimension 0, hence the tangent map $T_Z \to \vartheta^* T_{\operatorname{Hilb}^2(S)}$ is an isomorphism, in particular ϑ is étale. Thus it is an open and closed map to a connected scheme, hence it is surjective.

□

7 Geometric examples

Our aim is to list some interesting examples of varieties arising as degeneracy loci that can be described by Theorem 5.2.

Example 7.1 Let us study in more detail the case $m = 1$. Recall that we are only interested in applications with $n \geq 3$ and $s \in \{3, 4, 5\}$. Theorem 5.2 above proves that the map ψ does not contract lines inside S precisely when one of the following conditions is satisfied:

- $n \geq 3$ for curves in $\mathbb{P}^3$ (we excluded the case of the twisted cubic obtained for $n = 2$),
- $n \geq 4$ for surfaces in $\mathbb{P}^4$,
- $n \geq 6$ for threefolds in $\mathbb{P}^5$.

Moreover, again by Theorem 5.2, under the assumption $n \geq 3$ the map ψ does not admit 2-dimensional fibres.

Example 7.2 (White surfaces) Fix $m \geq 0$ and choose $s = m + 3$ and $n = s - m = 3$. Now, the degeneracy locus S_m is a surface in $\mathbb{P}^{m+3}$. Moreover, by Theorem 5.2 the map $\psi : \mathsf{S}_m \to \mathbb{P}^2$ is surjective and generically injective. The exceptional divisor (i.e. the union of the 1-dimensional fibres) arises over a 0-dimensional locus so that S_m is the blow up of $\mathbb{P}^2$ at c points. Again by Theorem 5.2, c can be easily computed as the degree of N_{s-1} in $\mathbb{P}(\mathsf{Mat}_{s,s+1}(\mathbb{C}))$, namely

$$c = \frac{(s+1)!}{(s-1)!2!} = \binom{m+4}{2}.$$

We also observe the following:

- For $m = 0$ we obtain the determinantal cubic surface $\mathsf{S}_0 \subset \mathbb{P}^3$ realised as the blow up of $\mathbb{P}^2$ in 6 points.
- For $m = 1$ we recover the classical construction of the Bordiga surface $\mathsf{S}_1 \subset \mathbb{P}^4$ realised as the blow up of $\mathbb{P}^2$ in 10 points, see e.g. [26]. In this case $e_{\mathrm{top}}(Z) = 94$, with $h^{1,1} = 12$, $h^{2,2} = 68$ and the other relevant Hodge numbers being 0. On the other hand, $\mathrm{Hilb}^2(\mathsf{S}_1)$ has topological Euler characteristic 104, with $h^{2,2} = 78$.

In the general case $\mathsf{S}_m \subset \mathbb{P}^{m+3}$ is nothing but the $(m + 3)$-th White surface named after F. Puryer White, see [32].

Example 7.3 (Generalised Bordiga scrolls over $\mathbb{P}^2$) Fix $m \geq 1$ and choose $s = m + 4$ and $n = s - m - 1 = 3$. Notice that the condition $m \geq 1$ ensures that $s \in \{m + 2, \dots, 2m + 3\}$. In this case the degeneracy locus B_m is a threefold in $\mathbb{P}^{m+4}$. Since the fibre of the map $\psi : \mathsf{B}_m \to \mathbb{P}^2$ is cut by $n + m = s - 1$ equations, the generic fibre of ψ is 1-dimensional. On the other hand, following the same argument of the proof of Theorem 5.2 it is immediate to see that 2-dimensional fibres of $f \circ \psi$ may only arise over $\mathrm{Im}(f) \cap N_{s-3} = \emptyset$, being $\mathrm{codim}\,(N_{s-3}) = 8$. Hence the map ψ is surjective and realises $\mathsf{B}_m \subset \mathbb{P}^{m+4}$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, so that B_m is the projectivisation of a rank 2 vector bundle over $\mathbb{P}^2$.

In particular, for $m = 1$ we recover the classical construction of the Bordiga scroll $\mathsf{B}_1 \subset \mathbb{P}^5$, i.e. the (rational, non Fano) variety described by $\mathbb{P}_{\mathbb{P}^2}(E)$, with E a rank 2 stable bundle with $c_1(E) = 4$, $c_2(E) = 10$, see e.g. [25].

We were not able to find a precise reference for the threefolds described in Example 7.3, so that we decided to call these threefolds *generalised Bordiga scrolls*, in analogy with the classical Bordiga scroll, see e.g. [25].

Example 7.4 (White varieties) Choose $m \geq 0$, and $3 \leq n \leq m+3$. Fix $s = n+m \in \{m+3, \dots, 2m+3\}$. Denote by $\mathsf{W}_{m,n} = D_{n-1}(\varphi)$ the usual degeneracy locus. Then by Theorem 5.2 the map $\psi : \mathsf{W}_{m,n} \to \mathbb{P}^{n-1}$ is surjective and generically injective. In particular, $\dim \mathsf{W}_{m,n} = n-1$. Moreover, 1-dimensional fibres arise over an $(n-3)$-dimensional locus.

We also observe the following:

- For any $m \geq 0$, $\mathsf{W}_{m,3}$ is nothing but the $(m+3)$-th White surface denoted by S_m in Example 7.2.
- For any $m \geq 1$, $\mathsf{W}_{m,4} \subset \mathbb{P}^{m+4}$ is a threefold that contains a $\mathbb{P}^1$-scroll over a curve $\mathsf{W}'_{m,4} \subset \mathbb{P}^3$. We can also compute the degree and the genus of $\mathsf{W}'_{m,4}$ as
$$\deg(\mathsf{W}'_{m,4}) = \deg(N_{m+3}) = \binom{m+5}{2}$$
$$g(\mathsf{W}'_{m,4}) = (m+4)\binom{m+4}{3} - (m+5)\binom{m+3}{3},$$
as proved in Proposition 2.3.

We were not able to find a precise reference for the construction spelled out in Example 7.4, so we decided to call these $(n-1)$-folds *White varieties*, in analogy with the usual White surfaces described in Example 7.2.

Apart from the limit case of White varieties $(s = n+m)$ we provide examples for which $s \leq n+m$ but Z need not to be isomorphic to $\mathrm{Hilb}^2(S)$. More precisely, it may be interesting to investigate the limit case when $n = 2s-2m-3$. Notice that, given $m \geq 0$, assuming $s \in \{m+2, \dots, 2m+3\}$ the system
$$\begin{cases} s \leq n+m \\ n = 2s-2m-3 \geq 3 \end{cases}$$
implies $s \geq m+3$ and $n \leq 2m+3$.

Example 7.5 $(n = 2s-2m-3)$ Fix $m \geq 0$, $s \in \{m+3, \dots, 2m+3\}$, and choose $n = 2s-2m-3$. By Theorem 5.2 the map ψ maps the degeneracy locus $\mathsf{M}_{m,s} \subset \mathbb{P}^s$ onto a certain variety $\mathsf{M}'_{m,s} \subset \mathbb{P}^{n-1}$ of dimension $s-m-1$, having a finite number c of special points over which the fibres are 1-dimensional. Notice that it is easy to compute c, since it equals the degree of N_{s-1} inside $\mathbb{P}(\mathsf{Mat}_{n+m,s+1}(\mathbb{C}))$, which is given by the formula [26]
$$\begin{aligned} c = \deg(N_{s-1}) &= \prod_{i=0}^{1} \frac{(n+i+m)!\, i!}{(s+i-1)!\,(n+m-s+i+1)!} \\ &= \frac{1}{s}\binom{2s-m-2}{s-1}\binom{2s-m-3}{s-1}. \end{aligned}$$

Remarkably, the same proof of Theorem 5.3 excludes lines of type $\boldsymbol{d}$ as soon as $m \geq 3$. Notice that the choice $s = m + 3$ implies $n = 3$, so that in particular we recover the White surfaces described in Example 7.2, i.e. $\mathsf{M}_{m,m+3} = \mathsf{S}_m$.

$m = 0$ In this case we only have the White surface $\mathsf{M}_{0,3} = \mathsf{S}_0 \subset \mathbb{P}^3$.

$m = 1$ In this case, apart from the Bordiga surface $\mathsf{M}_{1,4} = \mathsf{S}_1$ already discussed in Example 7.2, we may only choose $s = n = 5$. Then $\mathsf{M}_{1,5}$ is a threefold in $\mathbb{P}^5$. By the above formula there are $c = 105$ fibres of dimension 1 and the image of $\mathsf{M}_{1,5}$ inside $\mathbb{P}^4$ is a determinantal threefold $\mathsf{M}'_{1,5} = f^{-1}(\mathrm{Im}(f) \cap N_5)$ of degree 6, whose singular locus consists exactly of these 105 points. In fact, $\mathsf{M}_{1,5}$ is a small resolution of $\mathsf{M}'_{1,5}$. Notice how in this case $e_{\mathrm{top}}(Z) = 46158$ and $e_{\mathrm{top}}(\mathrm{Hilb}^2(S_{1,5})) = 46053$ by Appendix 2.2. Their difference is exactly 105 so that in particular $Z \ncong \mathrm{Hilb}^2(\mathsf{M}_{1,5})$.

$m = 2$ In this case, apart from the White surface $\mathsf{M}_{2,5} = \mathsf{S}_2$, we may choose $s = 6$ and $n = 5$ or $s = n = 7$. Now, by Theorem 5.2, $\mathsf{M}_{2,6} \subset \mathbb{P}^6$ is a threefold, and the map ψ contracts $c = \frac{1}{6}\binom{8}{5}\binom{7}{5} = 196$ lines. On the other hand $\mathsf{M}_{2,7} \subset \mathbb{P}^6$ is a fourfold, and the map ψ contracts $c = \frac{1}{7}\binom{10}{6}\binom{9}{6} = 2520$ lines.

Example 7.5 leads us to formulate the following conjecture.

Conjecture 7.6 *Fix* $m \geq 1$, $s \in \{m+3, \ldots, 2m+3\}$, *and choose* $n = 2s - 2m - 3$. *Then*

$$e_{\mathrm{top}}(\mathrm{Hilb}^2(\mathsf{M}_{m,s})) - e_{\mathrm{top}}\left(Z_{n,s,m}\right) = (-1)^{\dim(\mathsf{M}_{m,s})}\frac{1}{s}\binom{2s-m-2}{s-1}\binom{2s-m-3}{s-1}.$$

Notice that in Examples 7.2 and 7.5 we have shown that the above conjecture holds true for $m = 1$. We also did the computation for White surfaces taking higher values of m confirming the prediction of Conjecture 7.6.

On the other hand we excluded the case $m = 0$, for which the conjecture is easily seen to fail. However, this can be justified by noticing that $\mathsf{M}_{0,3} = \mathsf{S}_0$ contains 15 lines of type $\boldsymbol{d}$ (arising as birational transforms of lines in $\mathbb{P}^2$ passing through 2 out of the 6 points of $\mathbb{P}^2$), and indeed we compute the difference to be

$$e_{\mathrm{top}}(\mathrm{Hilb}^2(\mathsf{M}_{0,3})) - e_{\mathrm{top}}(Z_{3,3,0}) = 6 + 15\ .$$

As already remarked in Example 7.5 it is immediate to see that for $m \geq 3$ the varieties $\mathsf{M}_{m,s}$ do not admit lines of type $\boldsymbol{d}$, and actually we do not expect this to happen even in the cases $m = 1$ and $m = 2$.

We are particularly interested in Conjecture 7.6 since it would imply for instance that the bound provided by Theorem 6.2 is optimal.

Appendix 1: Euler characteristic of Hilbert squares

The goal of this appendix is to give a detailed proof of Proposition 2.4. We shall exploit a nontrivial Chern class calculation on (smooth) degeneracy loci following Pragacz [27].

Fix $m = 1$ throughout this section. Let $s \in \{3, 4\}$, and consider, as ever, a general map $\varphi : \mathcal{F} \to \mathcal{E}$ between vector bundles $\mathcal{F} = \mathcal{O}_{\mathbb{P}^s}^{\oplus n+1}$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus n}$. The k-th degeneracy locus of φ is the closed subscheme $D_k(\varphi) \subset \mathbb{P}^s$ defined by the condition $\operatorname{rank}(\varphi) \leq k$, which is (locally) equivalent to the vanishing of the $(k+1)$-minors of φ. We are interested in the case $k = n - 1$, which leads to $D_{n-2}(\varphi)$ of expected codimension 6, and $D_{n-1}(\varphi)$ of expected codimension 2. Since φ is general, we have $D_{n-2}(\varphi) = \emptyset$, so that $D_{n-1}(\varphi) \subset \mathbb{P}^s$ is a smooth subvariety of codimension 2. In the case $s = 4$, we shall denote it by $S_n \subset \mathbb{P}^4$, whereas in the case $s = 3$ we shall denote it by $C_n \subset \mathbb{P}^3$.

We start assuming $s = 4$, the case $s = 3$ being essentially a truncation of the case $s = 4$. Let $H \in A^1(\mathbb{P}^4)$ denote the first Chern class of $\mathcal{O}_{\mathbb{P}^4}(1)$. The ordinary Segre class of $\mathcal{E}$ is the class

$$\widetilde{s}(\mathcal{E}) = \sum_{0 \leq i \leq 4} \widetilde{s}_i(\mathcal{E}) = (1 + H)^{-n},$$

with $\widetilde{s}_i(\mathcal{E}) \in A^i(\mathbb{P}^4) = \mathbb{Z}[H^i]$ sitting in codimension i. Inverting the Chern class

$$c(\mathcal{E}) = 1 + nH + \binom{n}{2} H^2 + \binom{n}{3} H^3 + \binom{n}{4} H^4$$

we find

$$\begin{aligned}
\widetilde{s}_1(\mathcal{E}) &= -c_1(\mathcal{E}) = -nH \\
\widetilde{s}_2(\mathcal{E}) &= s_1(\mathcal{E})^2 - c_2(\mathcal{E}) = \left[n^2 - \binom{n}{2} \right] H^2 \\
\widetilde{s}_3(\mathcal{E}) &= -s_1(\mathcal{E})c_2(\mathcal{E}) - s_2(\mathcal{E})c_1(\mathcal{E}) - c_3(\mathcal{E}) = \left[-n^3 - \binom{n}{3} + 2n\binom{n}{2} \right] H^3 \\
\widetilde{s}_4(\mathcal{E}) &= -s_1(\mathcal{E})c_3(\mathcal{E}) - s_2(\mathcal{E})c_2(\mathcal{E}) - s_3(\mathcal{E})c_1(\mathcal{E}) - c_4(\mathcal{E}) \\
&= \left[n^4 + 2n\binom{n}{3} - 3n^2\binom{n}{2} + \binom{n}{2}^2 - \binom{n}{4} \right] H^4.
\end{aligned}$$

We set $s_i = (-1)^i \widetilde{s}_i(\mathcal{E})$ for $0 \leq i \leq 4$. Then, unraveling [27, Example 5.8 (ii)], we have, for the smooth surface $S_n \subset \mathbb{P}^4$, an identity

$$e_{\mathrm{top}}(S_n) = s_2 c_2(\mathbb{P}^4) - \left[s_{(2,1)} + 2s_3 \right] c_1(\mathbb{P}^4) + s_{(2,1,1)} + 3s_{(3,1)} + 3s_4, \tag{7.1}$$

given the Schur polynomials

$$\begin{aligned}
s_{(2,1)} &= \begin{vmatrix} s_2 & s_3 \\ s_0 & s_1 \end{vmatrix} = \begin{vmatrix} s_2 & s_3 \\ 1 & s_1 \end{vmatrix} = s_2 s_1 - s_3 \\
s_{(3,1)} &= \begin{vmatrix} s_3 & s_4 \\ s_0 & s_1 \end{vmatrix} = \begin{vmatrix} s_3 & s_4 \\ 1 & s_1 \end{vmatrix} = s_3 s_1 - s_4 \\
s_{(2,1,1)} &= \begin{vmatrix} s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 \\ 0 & s_0 & s_1 \end{vmatrix} = \begin{vmatrix} s_2 & s_3 & s_4 \\ 1 & s_1 & s_2 \\ 0 & 1 & s_1 \end{vmatrix} = s_2(s_1^2 - s_2) - (s_1 s_3 - s_4).
\end{aligned}$$

Expanding, we obtain

$$
\begin{aligned}
s_2c_2(\mathbb{P}^4) &= 10n^2 - 10\binom{n}{2} \\
\left[s_{(2,1)} + 2s_3\right]c_1(\mathbb{P}^4) &= 5(s_2s_1 + s_3)H = 10n^3 - 15n\binom{n}{2} + 5\binom{n}{3} \\
s_{(2,1,1)} &= n\binom{n}{3} - \binom{n}{4} \\
3s_{(3,1)} &= \binom{n}{2}\left[3n^2 - 3\binom{n}{2}\right] - 3n\binom{n}{3} + 3\binom{n}{4} \\
3s_4 &= 3n^4 + 6n\binom{n}{3} - 9n^2\binom{n}{2} + 3\binom{n}{2}^2 - 3\binom{n}{4}.
\end{aligned}
$$

Formula (7.1) then yields

$$
e_{\mathrm{top}}(S_n) = n^2(10 - 10n + 3n^2) + \binom{n}{2}(-10 + 15n - 6n^2) + \binom{n}{3}(4n - 5) - \binom{n}{4}.
$$

In the case of a smooth determinantal *curve* $C_n \subset \mathbb{P}^3$, i.e. when we set $s = 3$, we only need to use

$$
s_0 = 1, \quad s_1 = nH, \quad s_2 = \left[n^2 - \binom{n}{2}\right]H^2, \quad s_3 = \left[n^3 + \binom{n}{3} - 2n\binom{n}{2}\right]H^3.
$$

In this case, [27, Example 5.8 (i)] gives

$$
\begin{aligned}
e_{\mathrm{top}}(C_n) &= s_2c_1(\mathbb{P}^3) - s_{(2,1)} - 2s_3 = 4Hs_2 - (s_2s_1 - s_3) - 2s_3 = 4Hs_2 - s_2s_1 - s_3 \\
&= 4n^2 - 4\binom{n}{2} - n^3 + n\binom{n}{2} - n^3 - \binom{n}{3} + 2n\binom{n}{2} \\
&= 4n^2 - 2n^3 + (3n - 4)\binom{n}{2} - \binom{n}{3}.
\end{aligned}
$$

The formulas for $e_{\mathrm{top}}(S_n)$ and $e_{\mathrm{top}}(C_n)$ prove Proposition 2.4.

Appendix 2: Hodge–Deligne polynomial of Hilbert squares

We again set $m = 1$ throughout this section. We shall consider once more smooth (sub-determinantal) degeneracy loci $S = D_{n-1}(\varphi) \subset \mathbb{P}^s$ (of dimension 2 or 3), and we shall compute the Hodge–Deligne polynomial

$$
E(\mathrm{Hilb}^2(S); u, v) = \sum_{p,q \geq 0} h^{p,q}(\mathrm{Hilb}^2(S))(-u)^p(-v)^q \in \mathbb{Z}[u, v]
$$

via standard motivic techniques, exploiting the power structure on the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$ [14], as well as our knowledge of the Hodge numbers of S (cf. Sect. 3).

2.1. Surface case: $(s, n, m) = (4, 4, 1)$

Let us consider the smooth determinantal surface $S_4 = D_3(\varphi) \subset \mathbb{P}^4$. By Göttsche's formula [13] for the motive of the Hilbert scheme of points on a surface, combined with the main result of [14], there is an identity

$$\sum_{n \geq 0} \left[\operatorname{Hilb}^n(S_4)\right] q^n = \prod_{n>0} \left(1 - \mathbb{L}^{n-1} q^n\right)^{-[S_4]}$$

in $K_0(\operatorname{Var}_{\mathbb{C}})[\![q]\!]$, where exponentiation is to be thought of in the language of power structures. The Hodge–Deligne polynomial of a smooth projective $\mathbb{C}$-variety Y is the polynomial

$$E(Y; u, v) = \sum_{p,q \geq 0} h^{p,q}(Y)(-u)^p(-v)^q \in \mathbb{Z}[u, v].$$

We have, on $\mathbb{Z}[u, v]$, the power structure defined by the identity

$$(1 - q)^{-f(u,v)} = \prod_{i,j} \left(1 - u^i v^j q\right)^{-p_{ij}}$$

if $f(u, v) = \sum_{i,j} p_{ij} u^i v^j$. Looking at the Hodge diamond depicted in Sect. 3.2, we deduce

$$E(S_4; u, v) = 1 + 4u^2 + 45uv + 4v^2 + u^2 v^2,$$

and since $E(-)$ defines a morphism $K_0(\operatorname{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$ of *rings with power structure* sending $\mathbb{L} \mapsto uv$, we have an identity

$$\begin{aligned}
\sum_{n \geq 0} E(\operatorname{Hilb}^n(S_4); u, v) q^n &= \prod_{n>0} \left(1 - u^{n-1} v^{n-1} q^n\right)^{-E(S_4; u,v)} \\
&= \prod_{n>0} (1 - q)^{-E(S_4; u,v)} \Big|_{q \mapsto u^{n-1} v^{n-1} q^n} \\
&= \prod_{n>0} \left(1 - u^{n-1} v^{n-1} q^n\right)^{-1} \left(1 - u^{n+1} v^{n-1} q^n\right)^{-4} \cdot \\
&\qquad \cdot (1 - u^n v^n q^n)^{-45} \left(1 - u^{n-1} v^{n+1} q^n\right)^{-4} \left(1 - u^{n+1} v^{n+1} q^n\right)^{-1}
\end{aligned}$$

where the substitution $q \mapsto u^{n-1} v^{n-1} q^n$ is possible thanks to the properties of a power structure.

Expanding and isolating the coefficient of q^2 gives

$$\begin{aligned}
E(\operatorname{Hilb}^2(S_4); u, v) = &1 + 46uv + 4(u^2 + v^2) + 1097u^2 v^2 + 184(uv^3 + u^3 v) \\
&+ 10(u^4 + v^4) + 46u^3 v^3 + 4(u^4 v^2 + u^2 v^4) + u^4 v^4,
\end{aligned}$$

in full agreement with the Hodge diamond depicted in Sect. 3.2.

2.2. Threefold case: $(s, n, m) = (5, 5, 1)$

In the case $(s, n, m) = (5, 5, 1)$, we obtain a smooth threefold $S_{5,5,1} \subset \mathbb{P}^5$ outside the 'good range' of Theorem A, cf. Example 7.5. There is an identity [14, 29]

$$\mathsf{Z}_{S_{5,5,1}}(q)=\sum_{n\geq 0}\left[\operatorname{Hilb}^n(S_{5,5,1})\right]q^n=\left(\sum_{n\geq 0}\left[\operatorname{Hilb}^n(\mathbb{A}^3)_0\right]q^n\right)^{[S_{5,5,1}]}$$

in $K_0(\operatorname{Var}_{\mathbb{C}})[\![q]\!]$, where $\operatorname{Hilb}^n(\mathbb{A}^3)_0$ denotes the punctual Hilbert scheme, namely the subscheme of $\operatorname{Hilb}^n(\mathbb{A}^3)$ parametrising subschemes entirely supported at the origin $0\in\mathbb{A}^3$. Let us define classes $\Omega_n\in K_0(\operatorname{Var}_{\mathbb{C}})$ via the relation

$$\sum_{n\geq 0}\left[\operatorname{Hilb}^n(\mathbb{A}^3)_0\right]q^n=\operatorname{Exp}\left(\sum_{n>0}\Omega_n q^n\right)=\prod_{n>0}(1-q^n)^{-\Omega_n}.$$

Since $\operatorname{Hilb}^1(\mathbb{A}^3)_0=\operatorname{Spec}\mathbb{C}$ and $\operatorname{Hilb}^2(\mathbb{A}^3)_0=\mathbb{P}^2$, one can easily compute $\Omega_1=1$ and $\Omega_2=\mathbb{L}+\mathbb{L}^2$. Therefore

$$\mathsf{Z}_{S_{5,5,1}}(q)=\prod_{n>0}(1-q^n)^{-\Omega_n[S_{5,5,1}]},$$

which implies

$$\sum_{n\geq 0}E(\operatorname{Hilb}^n(S_{5,5,1});u,v)q^n=\prod_{n>0}(1-q^n)^{-E(\Omega_n;u,v)E(S_{5,5,1};u,v)}. \tag{B.1}$$

One can compute the Hodge–Deligne polynomial of $S_{5,5,1}$ to be

$$E(S_{5,5,1};u,v)=1+2uv+2u^2v^2+u^3v^3-(5u^3+151u^2v+151uv^2+5v^3),$$

so that extracting the coefficient of q^2 from (B.1), one obtains

$$\begin{aligned}E(\operatorname{Hilb}^2(S_{5,5,1});u,v)=&\left[\frac{(1-u^3q)^5(1-v^3q)^5(1-uv^2q)^{151}(1-u^2vq)^{151}}{(1-q)(1-uvq)^2(1-u^2v^2q)^2(1-u^3v^3q)}\right]_{q^2}\\&+(uv+u^2v^2)E(S_{5,5,1};u,v).\end{aligned}$$

In particular, the topological Euler characteristic is

$$e_{\mathrm{top}}(\operatorname{Hilb}^2(S_{5,5,1}))=E(\operatorname{Hilb}^2(S_{5,5,1});1,1)=46053=e_{\mathrm{top}}(Z_{5,5,1})-105.$$

B.3. Threefold case: $(s,n,m)=(5,6,1)$

In the case $(s,n,m)=(5,6,1)$, we get a smooth threefold $S_{5,6,1}\subset\mathbb{P}^5$. Using the Hodge diamond depicted in Sect. 3.3, one has

$$E(S_{5,6,1};u,v)=1+2uv+2u^2v^2+u^3v^3-(29u^3+520u^2v+520uv^2+29v^3).$$

Formula (B.1) applied to this case yields

$$\begin{aligned}E(\operatorname{Hilb}^2(S_{5,6,1});u,v)=&\left[\frac{(1-u^3q)^{29}(1-v^3q)^{29}(1-uv^2q)^{520}(1-u^2vq)^{520}}{(1-q)(1-uvq)^2(1-u^2v^2q)^2(1-u^3v^3q)}\right]_{q^2}\\&+(uv+u^2v^2)E(S_{5,6,1};u,v).\end{aligned}$$

In particular,

$$e_{\mathrm{top}}(\mathrm{Hilb}^2(S_{5,6,1})) = E(\mathrm{Hilb}^2(S_{5,6,1});1,1) = 593502,$$

in complete agreement with what one gets out of the Hodge diamond for Z depicted in Sect. 3.3.

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