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Pseudo-differential operators in the generalized weinstein setting

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Abstract
In this paper, we consider the generalized Weinstein operator $\Delta_W^{d,\alpha,n}$. For $n = 0$, we regain the classical Weinstein operator $\Delta_W^{\alpha,d}$. We introduce and study the Sobolev spaces associated with the generalized Weinstein operator and investigate their properties. Next, we introduce a class of symbols and their associated pseudo-differential operators.

Keywords Generalized Weinstein operator · Generalized Weinstein Transform · Sobolev spaces · Pseudo-differential operators

Mathematics Subject Classification 32A50 · 32B10 · 46E35 · 46F12 · 43A32

1 Introduction

In this paper, we consider the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times]0, +\infty[$, by:

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2} = \Delta_d + L_{\alpha,n} \tag{1.1}$$

where $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$, Δ_d is the Laplacian for the d first variables and $L_{\alpha,n}$ is the second-order singular differential operator on the half line given by:

$$L_{\alpha,n} = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2}. \tag{1.2}$$

For $n = 0$, we regain the classical Weinstein operator $\Delta_W^{\alpha,d}$ given by:

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$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha \tag{1.3}$$

$L_\alpha = L_{\alpha,0}$ is the Bessel operator. (see [3, 2, 4, 5, 9] and [10]).

The harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ is studied by Aboulez et al. (see [1, 6–8]).

For all $f \in L^1(\mathbb{R}_+^{d+1}, d\mu_{\alpha,d}(x))$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by:

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x)$$

where $\mu_{\alpha,d}$ is the measure defined on $\mathbb{R}_+^{d+1}$ by:

$$d\mu_{\alpha,d}(x) = x_{d+1}^{2\alpha+1}dx \tag{1.4}$$

and $\Lambda_{\alpha,d,n}$ is the generalized Weinstein kernel given by:

$$\forall x, y\in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}),$$

$x = (x', x_{d+1})$, $x' = \left(x_1, x_2, ..., x_d\right)$ and j_α is the normalized Bessel function of index α defined by:

$$\forall \xi\in \mathbb{C},\ j_\alpha(\xi) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{\xi}{2})^{2n}. \tag{1.5}$$

The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ can be written in the form

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}. \tag{1.6}$$

where $\mathscr{F}_W^{\alpha,d} = \mathscr{F}_W^{\alpha,d,0}$ is the classical Weinstein transform and $\mathscr{M}_n$ is the map defined by:

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x).$$

We designe by $\mathscr{S}_*(\mathbb{R}^{d+1})$,the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ the subspace of $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\},\ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

For all $s \in \mathbb{R}$, we define the generalized Sobolev-Weinstein space $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ as the set of all $u \in \mathscr{S}'_{n,*}$ (the strong dual of the space $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$) such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and

$$\int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi) < \infty.$$

We investigate the properties of $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$. Moreover, we introduce a class of symbols and their associated pseudo-differential operators.

The contents of the paper is as follows:

In the second section, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by the relation (1.1).

The section 3 is devoted to define and study the generalized Sobolev-Weinstein space $\mathcal{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.

In the last section, we introduce certain classes of symbols and study their associated pseudo-differential operators.

2 Preliminaires

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1}$ by the relation (1.1).

Notations. In what follows, we need the following notations:

- $\mathscr{C}_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.
- $\mathscr{M}_n$, the map defined by:

$$\forall x \in \mathbb{R}_+^{d+1}, \ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x). \tag{2.1}$$

  where $x = (x', x_{d+1})$ and $x' = \big(x_1, x_2, ..., x_d\big)$
- $L^p_{\alpha,n}(\mathbb{R}_+^{d+1}), 1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\begin{aligned} \|f\|_{\alpha,n,p} &= \left[\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1} f(x)|^p d\mu_{\alpha+2n,d}(x)\right]^{\frac{1}{p}} < +\infty, \text{ if } 1 \le p < +\infty, \\ \|f\|_{\alpha,n,\infty} &= \operatorname*{ess\ sup}_{x\in\mathbb{R}_+^{d+1}} \left|\mathscr{M}_n^{-1} f(x)\right| < +\infty, \end{aligned}$$

  where $\mu_{\alpha,d}$ is the measure given by the relation (1.4).
- $L^p_\alpha(\mathbb{R}_+^{d+1}) := L^p_{\alpha,0}(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$,and $\|f\|_{\alpha,p} := \|f\|_{\alpha,0,p}$.
- $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ repespectively stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\}, \ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

Let us begin by the following result.

Lemma 2.1 (see [1])

i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})\big(\text{resp. } \mathscr{S}_*(\mathbb{R}^{d+1})\big)$ onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ $\big(\text{resp. } \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\big)$.

ii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$L_{\alpha,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ L_{\alpha+2n}(f). \tag{2.2}$$

iii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$\Delta_W^{\alpha,d,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ \Delta_W^{\alpha+2n}(f). \tag{2.3}$$

iv) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$ and $g \in \mathscr{D}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} \Delta_W^{\alpha,d,n}(f)(x)g(x)d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} f(x)\Delta_W^{\alpha,d,n} g(x)d\mu_{\alpha,d}(x). \tag{2.4}$$

Definition 2.1 The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by:

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}), \tag{2.5}$$

where $x = (x', x_{d+1})$, $x' = \left(x_1, x_2, ..., x_d\right)$ and j_α is the normalized Bessel function of index α defined by the relation (1.5).

It is easy to see that the generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ has a unique extention to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and satisifies the following properties.

Proposition 2.1 i) We have

$$\forall x, y \in \mathbb{R}^{d+1},\ \overline{\Lambda_{\alpha,d,n}(x,y)} = \Lambda_{\alpha,d,n}(x,-y) = \Lambda_{\alpha,d,n}(-x,y)$$

ii) For all $\beta \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}_+^{d+1}$ and $z \in \mathbb{C}^{d+1}$, we have

$$|D_z^\beta \Lambda_{\alpha,d,n}(x,z)| \leq x_{d+1}^{2n} \|x\|^{|\beta|} \exp(\|x\|\ \|\mathrm{Im} z\|), \tag{2.6}$$

where

$$D_z^\beta = \frac{\partial^\beta}{\partial z_1^{\beta_1} ... \partial z_{d+1}^{\beta_{d+1}}} \text{ and } |\beta| = \beta_1 + ... + \beta_{d+1}.$$

In particular, we have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d,n}(x,y)| \leq x_{d+1}^{2n}. \tag{2.7}$$

iii) The function $x \mapsto \Lambda_{\alpha,d,n}(x,y)$ satisifies the differential equation

$$\triangle_W^{\alpha,d,n}\left(\Lambda_{\alpha,d,n}(.,y)\right)(x) = -\|y\|^2 \Lambda_{\alpha,d,n}(x,y). \tag{2.8}$$

iv) For all x, $y \in \mathbb{C}^{d+1}$, we have

$$\Lambda_{\alpha,d,n}(x,y) = a_{\alpha+2n} e^{-i\langle x',y'\rangle} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} \cos(tx_{d+1}y_{d+1})dt \tag{2.9}$$

where a_α is the constant given by:

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}. \tag{2.10}$$

Definition 2.2 The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ is given for $f \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ by:

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x). \tag{2.11}$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by the relation (1.4).

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results.

Proposition 2.2

i) For all $f \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have

$$\|\mathscr{F}_W^{\alpha,d,n}(f)\|_{\alpha,n,\infty} \le \|f\|_{\alpha,n,1}. \tag{2.12}$$

ii) Let $m \in \mathbb{N}$ and $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$,we have

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left[\left(\triangle_W^{\alpha,d,n}\right)^m f\right](\lambda) = (-1)^m \|\lambda\|^{2m} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{2.13}$$

iii) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ and $m \in \mathbb{N}$. For all $\lambda \in \mathbb{R}_+^{d+1}$,we have

$$\left(\triangle_W^{\alpha,d,n}\right)^m \left[\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n}(f)\right](\lambda) = \mathscr{M}_n \mathscr{F}_W^{\alpha,d,n}(P_m f)(\lambda) \tag{2.14}$$

where $P_m(\lambda) = (-1)^m \|\lambda\|^{2m}$.

Proof

i) We obtain the result from the relation (2.7).

ii) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, using the relations (1.6) and (2.3) for all $\lambda \in \mathbb{R}_+^{d+1}$,we get

$$\begin{aligned}
\mathscr{F}_W^{\alpha,d,n}\left[\left(\triangle_W^{\alpha,d,n}\right)f\right](\lambda) &= \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}\left[\left(\triangle_W^{\alpha,d,n}\right)f\right](\lambda) \\
&= \mathscr{F}_W^{\alpha+2n,d}\left[\Delta_W^{\alpha+2n}\mathscr{M}_n^{-1} f\right](\lambda) \\
&= -\|\lambda\|^2 \mathscr{F}_W^{\alpha+2n,d}\left[\mathscr{M}_n^{-1} f\right](\lambda) \\
&= -\|\lambda\|^2 \mathscr{F}_W^{\alpha,d,n}(f)(\lambda)
\end{aligned}$$

which proves assertion ii).

iii) The relation (2.8) together with (2.11) give the result. □

Theorem 2.1

i) Let $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. *If* $\mathscr{F}^{\alpha,d,n}_W(f) \in L^1_{\alpha+2n}(\mathbb{R}^{d+1}_+)$, *then we have*

$$f(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(y)\Lambda_{\alpha,d,n}(-x,y)d\mu_{\alpha+2n,d}(y),\ a.e\, x \in \mathbb{R}^{d+1}_+ \tag{2.15}$$

where $C_{\alpha,d}$ is the constant given by:

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}} 2^\alpha \Gamma(\alpha+1)}. \tag{2.16}$$

ii) The Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ *is a topological isomorphism from* $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *onto* $\mathscr{S}_*(\mathbb{R}^{d+1})$ *and from* $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ *onto* $\mathscr{H}_*(\mathbb{C}^{d+1})$.

Proof

i) We obtain the result from the relation (1.6) and the fact that

$$\varphi(x) = C^2_{\alpha,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d}_W(\varphi)(y)\Lambda_{\alpha,d,0}(-x,y)d\mu_{\alpha,d}(y),\ a.e\, x \in \mathbb{R}^{d+1}_+$$

where $\varphi, \mathscr{F}^{\alpha,d}_W(\varphi) \in L^1_\alpha(\mathbb{R}^{d+1}_+)$.

ii) The transform $\mathscr{F}^{\alpha,d}_W$ is a topological isomorphism from $\mathscr{S}_*(\mathbb{R}^{d+1})$ onto itself and from $\mathscr{D}_*(\mathbb{R}^{d+1})$ onto $\mathscr{H}_*(\mathbb{C}^{d+1})$. Then using the relation (1.6) the assertion ii) is proved.

□

The following Theorem is as an immediate consequence of the relation (1.6) and the properties of the transform $\mathscr{F}^{\alpha,d}_W$ (see [6–8]).

Theorem 2.2

i) For all $f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, *we have the following Parseval formula*

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(\lambda)}d\mu_{\alpha+2n,d}(\lambda) \tag{2.17}$$

where $C_{\alpha,d}$ is the constant given by the relation (2.16).

ii) (Plancherel formula).
For all $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, *we have:*

$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2 d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda). \tag{2.18}$$

iii) (*Plancherel Theorem*):

The transform $\mathscr{F}_W^{\alpha,d,n}$ *extends uniquely to an isometric isomorphism from* $L^2(\mathbb{R}_+^{d+1}, d\mu_{\alpha,d}(x))$ *onto* $L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}d\mu_{\alpha+2n,d}(x))$.

Definition 2.3 The translation operator $T_x^{\alpha,d,n}$, $x \in \mathbb{R}_+^{d+1}$, associated with the operator $\Delta_W^{\alpha,d,n}$ is defined on $\mathscr{E}_{n,*}(\mathbb{R}_+^{d+1})$ by:

$$\forall y \in \mathbb{R}_+^{d+1},\ T_x^{\alpha,d,n}f(y) = x_{d+1}^{2n} y_{d+1}^{2n} T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f(y) \tag{2.19}$$

where

$$T_x^{\alpha,d}f(y) = \frac{a_\alpha}{2}\int_0^\pi f\left(x'+y',\ \sqrt{x_{d+1}^2+y_{d+1}^2+2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}d\theta, \tag{2.20}$$

$x'+y' = \left(x_1+y_1, ..., x_d+y_d\right)$ and a_α is the constant given by (2.10).

Lemma 2.2 *Let* f_β, $\beta > 0$, *be the function defined by*:

$$\forall \xi \in \mathbb{R}_+^{d+1}, f_\beta(\xi) = \left(1+\|\xi\|^2\right)^{-\beta}.$$

Then there exists $k_\beta > 0$ *such that*

$$\forall x, y \in \mathbb{R}_+^{d+1},\ T_x^{\alpha,d}\left(f_\beta\right)(y) \le k_\beta\left(1+\|x\|^2\right)^{-\beta}\left(1+\|y\|^2\right)^{-\beta}. \tag{2.21}$$

Proof Using the relation (2.20), for all $x, y \in \mathbb{R}_+^{d+1}$,we obtain

$$\begin{aligned}
T_x^{\alpha,d}\left(f_\beta\right)(y) &= \frac{a_\alpha}{2}\int_0^\pi \left(1+\left\|x'+y'\right\|^2+x_{d+1}^2+y_{d+1}^2+2x_{d+1}y_{d+1}\cos\theta\right)^{-\beta}(\sin\theta)^{2\alpha}d\theta\\
&\le k_\beta\left(1+\|x\|^2\right)^{-\beta}\left(1+\|y\|^2\right)^{-\beta}\frac{a_\alpha}{2}\int_0^\pi(\sin\theta)^{2\alpha}d\theta\\
&\le k_\beta\left(1+\|x\|^2\right)^{-\beta}\left(1+\|y\|^2\right)^{-\beta}
\end{aligned}$$

where

$$k_\beta = \sup_{x,y\in\mathbb{R}_+^{d+1}}\left(\frac{1+\|x'+y'\|^2+\left(x_{d+1}-y_{d+1}\right)^2}{\left(1+\|x\|^2\right)\left(1+\|y\|^2\right)}\right)^{-\beta}.$$

□

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 2.3

i) *For* $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, *we have*

$$\forall x,\ y \in \mathbb{R}_+^{d+1},\ T_x^{\alpha,d,n}f(y) = T_y^{\alpha,d,n}f(x).$$

ii) *For all* $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ *and* $y \in \mathbb{R}_+^{d+1}$, *the function* $x \mapsto T_x^{\alpha,d,n}f(y)$ *belongs to* $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$.

iii) *We have*

$$\forall x \in \mathbb{R}_+^{d+1},\ \Delta_W^{\alpha,d,n} \circ T_x^{\alpha,d,n} = T_x^{\alpha,d,n} \circ \Delta_W^{\alpha,d,n}.$$

iv) *Let* $f \in L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$ *and* $x \in \mathbb{R}_+^{d+1}$. *Then* $T_x^{\alpha,d,n} f$ *belongs to* $L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\|T_x^{\alpha,d,n} f\|_{\alpha,n,p} \le x_{d+1}^{2n} \|f\|_{\alpha,n,p}. \tag{2.22}$$

v) *The function* $t \mapsto \Lambda_{\alpha,d,n}(t,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, *satisfies on* $\mathbb{R}_+^{d+1}$ *the following product formula*:

$$\forall x, y \in \mathbb{R}_+^{d+1},\ \Lambda_{\alpha,d,n}(x,\lambda)\Lambda_{\alpha,d,n}(y,\lambda) = T_x^{\alpha,d,n}\big[\Lambda_{\alpha,d,n}(.,\lambda)\big](y). \tag{2.23}$$

vi) *Let* $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $x \in \mathbb{R}_+^{d+1}$, *we have*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\big(T_x^{\alpha,d,n} f\big)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{2.24}$$

Proof The results can be obtained by a simple calculation by using the relation (2.19). □

Lemma 2.3 *Let* $f \in \mathscr{S}_{n,*}\big(\mathbb{R}^{d+1}\big)$, *for all* $x, y \in \mathbb{R}_+^{d+1}$,*we hae*

$$T_x^{\alpha,d,n}\big(\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \mathscr{M}_n f\big)(y) = \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(x,\lambda)\Lambda_{\alpha,d,n}(y,\lambda) f(\lambda)\, d\mu_{\alpha+2n,d}(\lambda). \tag{2.25}$$

Proof Let $f \in \mathscr{S}_{n,*}\big(\mathbb{R}^{d+1}\big)$. Using the the relation (2.15) and (2.24), we obtain

$$\begin{aligned}
T_x^{\alpha,d,n}\big(\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \mathscr{M}_n f\big)(y) &= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \big\{\Lambda_{\alpha,d,n}(-y,\lambda) \\
&\quad \mathscr{F}_W^{\alpha,d,n}\big(T_x^{\alpha,d,n}\big(\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \mathscr{M}_n f\big)\big)(\lambda)\big\} d\mu_{\alpha+2n,d}(\lambda) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x,\lambda)\Lambda_{\alpha,d,n}(-y,\lambda)\mathscr{F}_W^{\alpha,d,n} \\
&\quad \big(\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \mathscr{M}_n f\big)(\lambda) d\mu_{\alpha+2n,d}(\lambda) \\
&= C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x,\lambda)\Lambda_{\alpha,d,n}(-y,\lambda)\mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{F}_W^{\alpha+2n,d}(f) \\
&\quad (\lambda) d\mu_{\alpha+2n,d}(\lambda) \\
&= \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(x,-\lambda)\Lambda_{\alpha,d,n}(y,-\lambda) f(-\lambda)\, d\mu_{\alpha+2n,d}(\lambda) \\
&= \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(x,\lambda)\Lambda_{\alpha,d,n}(y,\lambda) f(\lambda)\, d\mu_{\alpha+2n,d}(\lambda).
\end{aligned}$$

□

Definition 2.4 Let $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$.The generalized Weinstein convolution product of f and g is given by:

$$\forall x \in \mathbb{R}_+^{d+1}, f *_{\alpha,n} g(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d,n} f(-y) g(y)\, d\mu_{\alpha,d}(y). \tag{2.26}$$

Proposition 2.4 *For all* $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$,*we have* $f *_{\alpha,n} g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$*and*

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g). \tag{2.27}$$

Proof Let $f, g \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. We have

$$\|f *_{\alpha,n} g\|_{\alpha,n,1} \leq \|f\|_{\alpha,n,1}\|g\|_{\alpha,n,1}.$$

Now using Fubini's theorem and the relation (2.24), we obtain

$$\begin{aligned}
\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g)(\lambda) &= \int_{\mathbb{R}^{d+1}_+}\left(\int_{\mathbb{R}^{d+1}_+} T_x^{\alpha,d,n}f(-y)g(y)d\mu_{\alpha,d}(y)\right)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x)\\
&= \int_{\mathbb{R}^{d+1}_+} g(y)\left(\int_{\mathbb{R}^{d+1}_+} T_{-y}^{\alpha,d,n}f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x)\right)d\mu_{\alpha,d}(y)\\
&= \int_{\mathbb{R}^{d+1}_+} g(y)\mathscr{F}_W^{\alpha,d,n}\left(T_{-y}^{\alpha,d,n}f\right)(\lambda)d\mu_{\alpha,d}(y)\\
&= \mathscr{F}_W^{\alpha,d}(f)(\lambda)\int_{\mathbb{R}^{d+1}_+} g(y)\Lambda_{\alpha,d,n}(y,\lambda)d\mu_{\alpha,d}(y)\\
&= \mathscr{F}_W^{\alpha,d}(f)(\lambda)\mathscr{F}_W^{\alpha,d}(g)(\lambda).
\end{aligned}$$

□

Remark 2.1 From the relation (2.27), we deduce that

$$f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}) \Rightarrow f *_{\alpha,n} g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}).$$

Notations. We denoted by:

· $\mathscr{S}'_*$, the strong dual of the space $\mathscr{S}_*(\mathbb{R}^{d+1})$.

· $\mathscr{S}'_{n,*}$, the strong dual of the space $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$.

Definition 2.5 The generalized Fourier-Weinstein transform of a distribution $u \in \mathscr{S}'_{n,*}$ is defined by:

$$\forall \phi \in \mathscr{S}_*(\mathbb{R}^{d+1}),\ \langle \mathscr{F}_W^{\alpha,d,n}(u),\ \phi\rangle = \langle u,\ \left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}(\phi)\rangle. \tag{2.28}$$

The following proposition is as an immediate consequence of Theorem 2.1.

Proposition 2.5 *The transform $\mathscr{F}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathscr{S}'_{n,*}$ onto $\mathscr{S}'_*$.*

Lemma 2.4 *Let $m \in \mathbb{N}$ and $u \in \mathscr{S}'_{n,*}$,we have*

$$\left(\mathscr{F}_W^{\alpha,d,n}\right)\left[(\Delta_W^{\alpha,d,n})^m u\right] = (-1)^m\|x\|^{2m}\left(\mathscr{F}_W^{\alpha,d,n}\right)(u) \tag{2.29}$$

where

$$\forall \phi \in \mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right),\ \langle \Delta_W^{\alpha,d,n}u,\ \phi\rangle = \langle u,\ \Delta_W^{\alpha,d,n}\phi\rangle. \tag{2.30}$$

Proof Let $m \in \mathbb{N}$ and $u \in \mathscr{S}'_{n,*}$,by invoking (2.13), (2.28) and (2.30), for all $\phi \in \mathscr{S}_*\left(\mathbb{R}^{d+1}\right)$,we can write

$$
\begin{aligned}
\langle \left(\mathscr{F}_W^{\alpha,d,n}\right)\left[(\Delta_W^{\alpha,d,n})^m u\right],\ \phi\rangle &= \langle (\Delta_W^{\alpha,d,n})^m u,\ \left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}(\phi)\rangle \\
&= \langle u,\ (\Delta_W^{\alpha,d,n})^m \left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}(\phi)\rangle \\
&= \langle u,\ \left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}((-1)^m \|x\|^{2m}\phi)\rangle \\
&= \langle \mathscr{F}_W^{\alpha,d,n}(u),\ (-1)^m \|x\|^{2m}\phi)\rangle \\
&= \langle (-1)^m \|x\|^{2m} \mathscr{F}_W^{\alpha,d,n}(u), \phi)\rangle.
\end{aligned}
$$

Which completes the proof. □

3 Sobolev spaces associated with the generalized Weinstein operator

The goal of this section is to introduce and study the Sobolev spaces associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.

Definition 3.1 For $s \in \mathbb{R}$, we define the generalized Sobolev-Weinstein space of order s, that will be denoted $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$, as the set of all $u \in \mathscr{S}'_{n,*}$ such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and

$$
\int_{\mathbb{R}_+^{d+1}} (1 + \|\lambda\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) < +\infty. \tag{3.1}
$$

We provide $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ with the inner product

$$
\langle u, v\rangle_{s,\alpha,n} = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^s \mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)} d\mu_{\alpha+2n,d}(\xi) \tag{3.2}
$$

and the norm

$$
\|u\|_{\mathscr{H}^{s,\alpha,n}} = \left[C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\right]^{\frac{1}{2}}. \tag{3.3}
$$

The following properties of the spaces $\mathscr{H}^{s,\alpha,n}$ can easily be established.

Proposition 3.1 (i) For all $s \in \mathbb{R}$, we have

$$
\mathscr{S}_{n,*}(\mathbb{R}^{d+1}) \subset \mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1}).
$$

(ii) We have

$$
\mathscr{H}^{0,\alpha,n}(\mathbb{R}_+^{d+1}) = L_{\alpha,n}^2(\mathbb{R}_+^{d+1}).
$$

(iii) For all $s,\ t \in \mathbb{R},\ t > s$, the space $\mathscr{H}^{t,\alpha,n}(\mathbb{R}_+^{d+1})$ is continuously contained in $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$..

Proposition 3.2 *The space $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ provided with the norm $\|.\|_{\mathscr{H}^{s,\alpha,n}}$ is a Banach space.*

Proof Let $\left(f_m\right)_{m\in\mathbb{N}}$ be a Cauchy sequence of $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$. From the definition of the norm $\|.\|_{\mathscr{H}^{s,\alpha,n}}$, it is clear that $\left(\mathscr{F}_W^{\alpha,d,n}\left(f_m\right)\right)_{m\in\mathbb{N}}$ is a Cauchy sequence of $L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))$.

Since $L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))$ is complete, there exists a function $f\in L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))$ such that

$$\lim_{m\to+\infty}\|\mathscr{F}_W^{\alpha,d,n}\left(f_m\right)-f\|_{L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))}=0. \tag{3.4}$$

Then $f\in\mathscr{S}_*$ and $h=\left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}(f)\in\mathscr{S}_{n,*}$.

So, $\mathscr{F}_W^{\alpha,d,n}(h)=f\in L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))$, which proves that $h\in\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\|f_m-h\|_{\mathscr{H}^{s,\alpha,n}}=\|\mathscr{F}_W^{\alpha,d,n}\left(f_m\right)-f\|_{L^2(\mathbb{R}_+^{d+1}, C^2_{\alpha+2n,d}(1+\|x\|^2)^s d\mu_{\alpha+2n,d}(x))}\underset{m\to+\infty}{\to}0.$$

Hence, $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ is complete. □

Proposition 3.3 *Let $s,t\in\mathbb{R}$. The operator $\mathscr{O}_t$ defined by*:

$$\forall x\in\mathbb{R}_+^{d+1},\ \mathscr{O}_t u(x)=C^2_{\alpha+2n,d}\int_{\mathbb{R}_+^{d+1}}(1+\|\xi\|)^t\Lambda_{\alpha,d,n}(-x,\xi)\mathscr{F}_W^{\alpha,d,n}(u)(\xi)d\mu_{\alpha+2n,d}(\xi)$$

is a isomorphism from $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ onto $\mathscr{H}^{s-t,\alpha,n}(\mathbb{R}_+^{d+1})$.

Proof Let $s,t\in\mathbb{R}$ and $u\in\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$. The function:

$\xi\mapsto(1+\|\xi\|)^t(1+\|\xi\|^2)^{\frac{s-t}{2}}\mathscr{F}_W^{\alpha,d,n}(u)(\xi)$ belongs to $L^2_{\alpha+2n}(\mathbb{R}_+^{d+1})$ and have

$$\forall\xi\in\mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(\mathscr{O}_t u\right)(\xi)=(1+\|\xi\|)^t\mathscr{F}_W^{\alpha,d,n}(u)(\xi).$$

Thus

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}}(1+\|\xi\|^2)^{s-t}\left|\mathscr{F}_W^{\alpha,d,n}(\mathscr{O}_t u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\\ &\le\kappa_t\int_{\mathbb{R}_+^{d+1}}(1+\|\xi\|^2)^{s}\left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\end{aligned}$$

where

$$\kappa_t=\sup_{x\in\mathbb{R}_+^{d+1}}\left[\frac{(1+\|x\|)^{2t}}{(1+\|x\|^2)^t}\right]\le 2^{|t|}.$$

Then, $\mathscr{O}_t u\in\mathscr{H}^{s-t,\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\left\|\mathscr{O}_t u\right\|_{\mathscr{H}^{s-t,\alpha,n}}\le 2^{\frac{|t|}{2}}\|u\|_{\mathscr{H}^{s,\alpha,n}}.$$

Now, let $v\in\mathscr{H}^{s-t,\alpha,n}(\mathbb{R}_+^{d+1})$ and put

$$u=\left(\mathscr{F}_W^{\alpha,d,n}\right)^{-1}\left((1+\|\xi\|)^{-t}\mathscr{F}_W^{\alpha,d,n}(v)\right).$$

From the definition of the operator $\mathscr{O}_t$, we have $\mathscr{O}_t u = v$ and we get

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^s \left|\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\\
&\quad = \int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^s (1+\|\xi\|)^{-2t} \left|\mathscr{F}_W^{\alpha,d,n}(v)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi)\\
&\quad \leq 2^{|t|} \int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^{s-t} \left|\mathscr{F}_W^{\alpha,d,n}(v)(\xi)\right|^2 d\mu_{\alpha+2n,d}(\xi).
\end{aligned}$$

Hence, $u \in \mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and we obtain

$$\|u\|_{\mathscr{H}^{s,\alpha,n}} \leq 2^{\frac{|t|}{2}} \left\|\mathscr{O}_t u\right\|_{\mathscr{H}^{s-t,\alpha,n}}.$$

Which completes the proof. □

Remark 3.1 The dual of $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$can be identified with $\mathscr{H}^{-s,\alpha,n}(\mathbb{R}_+^{d+1})$. The relation of the identification is as follows:

$$\langle u, v\rangle_{0,\alpha,n} = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)} d\mu_{\alpha+2n,d}(\xi), \tag{3.5}$$

with $u \in \mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ and $v \in \mathscr{H}^{-s,\alpha,n}(\mathbb{R}_+^{d+1})$.

Proposition 3.4 *Let $s_1, s, s_2 \in \mathbb{R}$, satisfying $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a nonnegative constant C_ε such that for all $u \in \mathscr{H}^{s_2,\alpha,n}(\mathbb{R}_+^{d+1})$, we have*

$$\|u\|_{\mathscr{H}^{s,\alpha,n}} \leq C_\varepsilon \|u\|_{\mathscr{H}^{s_1,\alpha,n}} + \varepsilon \|u\|_{\mathscr{H}^{s_2,\alpha,n}}. \tag{3.6}$$

Proof Let $s_1, s_2 \in \mathbb{R}$ and $s = (1-t)s_1 + ts_2, t \in]0,\ 1[$. Let $u \in \mathscr{H}^{s_2,\alpha,n}(\mathbb{R}_+^{d+1})$. We put $t = \dfrac{1}{p}$ and $1-t = \dfrac{1}{q}$, applying the Hölder's inequality, we get

$$\begin{aligned}
\|u\|_{\mathscr{H}^{s,\alpha,n}} &\leq \|u\|_{\mathscr{H}^{s_1,\alpha,n}}^{1-t} \times \|u\|_{\mathscr{H}^{s_2,\alpha,n}}^t\\
&\leq \left(\varepsilon^{\frac{-t}{1-t}} \|u\|_{\mathscr{H}^{s_1,\alpha,n}}\right)^{1-t} \times \left(\varepsilon \|u\|_{\mathscr{H}^{s_2,\alpha,n}}\right)^t\\
&\leq \varepsilon^{\frac{s-s_1}{s-s_2}} \|u\|_{\mathscr{H}^{s_1,\alpha,n}} + \varepsilon \|u\|_{\mathscr{H}^{s_2,\alpha,n}}.
\end{aligned}$$

Then the relation (3.6) is proved. □

Proposition 3.5 *Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$. Then for all $\varepsilon > 2m$, the operator $\left(\Delta_W^{\alpha,d,n}\right)^m$ is continuous from $\mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$ into $\mathscr{H}^{s-\varepsilon,\alpha,n}(\mathbb{R}_+^{d+1})$.*

Proof Let $m \in \mathbb{N}$, $\varepsilon > 2m$, $s \in \mathbb{R}$ and $u \in \mathscr{H}^{s,\alpha,n}(\mathbb{R}_+^{d+1})$.

Using the relation (2.29), we can see that $\left(\Delta_W^{\alpha,d}\right)^m u \in \mathscr{H}^{s-\varepsilon,\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\|\left(\Delta_W^{\alpha,d}\right)^m u\|_{\mathscr{H}^{s-\varepsilon,\alpha,n}} \leq \|u\|_{\mathscr{H}^{s,\alpha,n}}.$$

Thus the proof is finished. □

4 Pseudo-differential operators

*Notations*We need the following notations

- For $r \geq 0$, we designate by $\mathcal{S}^r$, the space of C^∞−function a on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ such that for each compact set $K \subset \mathbb{R}^{d+1}$ and each $\beta, \gamma \in \mathbb{N}$, there exists a constant $C = C(K, \beta, \gamma)$ satisfying:

$$\forall (x,\ \xi) \in K \times \mathbb{R}^{d+1},\ \left| D_\xi^\beta D_x^\gamma a(x,\ \xi) \right| \leq C(1 + \|\xi\|^2)^{\frac{r}{2}}. \tag{4.1}$$

- For $r, l \in \mathbb{R}$ with $l > 0$, we denote by $\mathcal{S}^{r,l}$, the space consits of all C^∞−function a on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ such that for each $\beta, \gamma \in \mathbb{N}$, there exist a positive constant $C = C(r, l, \beta, \gamma)$ satisfying the relation:

$$\forall (x,\ \xi) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1},\ \left| D_\xi^\beta D_x^\gamma a(x,\ \xi) \right| \leq C(1 + \|\xi\|^2)^{\frac{r}{2}} (1 + \|x\|^2)^{-\frac{l}{2}}. \tag{4.2}$$

Definition 4.1 The pseudo-differential operator $A\big(a, \Delta_W^{\alpha,d,n}\big)$ associated with $a(x, \xi) \in \mathcal{S}^r$ is defined for $u \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ by:

$$\left[A\big(a, \Delta_W^{\alpha,d,n}\big) u\right](x) = \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x, y) a(x, y) \mathscr{F}_W^{\alpha,d,n}(u)(y) d\mu_{\alpha+2n,d}(y). \tag{4.3}$$

Theorem 4.1 *If $a(x,\ \xi) \in \mathcal{S}^r$, then its associated pseudo-differential operator $A\big(a,\ \Delta_W^{\alpha,d}\big)$ is a well-defined mapping from $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ into $C^\infty\big(\mathbb{R}^{d+1}\big)$.*

Proof Let $a(x,\ \xi) \in \mathcal{S}^r$ and $s > r + \dfrac{d}{2} + \alpha + 2n + 1$. From the relation (4.1), we have for any compact set $K \subset \mathbb{R}^{d+1}$ and any $\gamma \in \mathbb{N}$,

$$\forall (x,\ \xi) \in K \times \mathbb{R}^{d+1},\ \left| D_x^\gamma a(x,\ \xi) \right| \leq C(1 + \|\xi\|^2)^{\frac{r}{2}}. \tag{4.4}$$

Let $u \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ and $x \in K$, using the relations (4.4), (2.7) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \left| a(x,\ \xi) \Lambda_{\alpha,d,n}(-x, \xi) \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \right| d\mu_{\alpha+2n,d}(\xi) \\
&\leq C x_{d+1}^{2n} \int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^{\frac{r}{2}} \left| \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \right| d\mu_{\alpha+2n,d}(\xi) \\
&\leq \frac{C}{C_{\alpha+2n,d}} x_{d+1}^{2n} \left(\int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^{r-s} d\mu_{\alpha+2n,d}(\xi) \right)^{\frac{1}{2}} \|u\|_{\mathscr{H}^{s,\alpha,n}}.
\end{aligned}$$

This relation proves that $A\big(a,\ \Delta_W^{\alpha,d}\big)(u)$ is well-defined and continuous on $\mathbb{R}_+^{d+1}$.

By the same argument, we can prove

$$\int_{\mathbb{R}_+^{d+1}} \left| D_x^\gamma a(x, \xi) \Lambda_{\alpha,d,n}(-x, \xi) \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \right| d\mu_{\alpha,d}(\xi) \leq C' \|u\|_{\mathscr{H}^{s,\alpha,n}}$$

where C' is a positive constant.

Consequently, in vertue of Leibniz formula, we obtain the result. □

The next lemma plays an important role in this section.

Lemma 4.1 *Let $t \geq 0$ and $l > 2\alpha + 4n + d + 2$. Then, for all $a(x, \ \xi) \in \mathcal{S}^{r,l}$, we have:*

$$\left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(\xi)\right| \leq C(1+\|y\|^2)^{\frac{r}{2}}(1+\|\xi\|^2)^{-\frac{t}{2}}, \tag{4.5}$$

where C is a constant depending on r, t, α, d, n and l.

Proof Let $k \in \mathbb{N}$. By invoking (2.7), (2.13) and (4.2), we obtain

$$\begin{aligned}
\|\xi\|^{2k}\left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(\xi)\right| &= \left|\mathscr{F}_W^{\alpha,d,n}\left[\left(\triangle_W^{\alpha,d,n}\right)^k\left(\mathscr{M}_n a(.,y)\right)(\xi)\right]\right| \\
&\leq \int_{\mathbb{R}_+^{d+1}} \left|\left(\triangle_{W,x}^{\alpha,d,n}\right)^k\left(x_{d+1}^{2n} a(x,y)\right)\right| \left|\Lambda_{\alpha,d,n}(x,\xi)\right| d\mu_{\alpha,d}(x) \\
&\leq \int_{\mathbb{R}_+^{d+1}} \left|\left(\triangle_{W,x}^{\alpha,d,n}\right)^k\left(x_{d+1}^{2n} a(x,y)\right)\right| x_{d+1}^{2n} d\mu_{\alpha,d}(x) \\
&\leq C_1(1+\|y\|^2)^{\frac{r}{2}} \int_{\mathbb{R}_+^{d+1}} (1+\|x\|^2)^{-\frac{l}{2}}\left(1+x_{d+1}^{4n}\right) d\mu_{\alpha,d}(x) \\
&\leq C_2(1+\|y\|^2)^{\frac{r}{2}}
\end{aligned}$$

where $l > 2\alpha + 4n + d + 2$ and

$$C_2 = C_1 \int_{\mathbb{R}_+^{d+1}} (1+\|x\|^2)^{-\frac{l}{2}}\left(1+x_{d+1}^{4n}\right) d\mu_{\alpha,d}(x) = C_2(r,k,\alpha,d,n).$$

We put $m = \left[\frac{t}{2}\right] + 1, \ t \geq 0$, where $\left[\frac{t}{2}\right]$ is the integer part of $\frac{t}{2}$. We get

$$\begin{aligned}
(1+\|\xi\|^2)^m \left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(\xi)\right| &= \sum_{k=0}^{m} C_m^k \|\xi\|^{2k} \left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(\xi)\right| \\
&\leq \sum_{k=0}^{m} C_m^k C_2(r,k,\alpha,d,n)(1+\|y\|^2)^{\frac{r}{2}} \\
&\leq C(1+\|y\|^2)^{\frac{r}{2}}
\end{aligned}$$

where C is a constant depending on r, t, α, d, n and l.

Hence, we obtain

$$\begin{aligned}
\left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(\xi)\right| &\leq C(1+\|y\|^2)^{\frac{r}{2}}(1+\|\xi\|^2)^{-m} \\
&\leq C(1+\|y\|^2)^{\frac{r}{2}}(1+\|\xi\|^2)^{-\frac{t}{2}}.
\end{aligned}$$

□

The following theorem gives an alternative form of $A\left(a, \ \Delta_W^{\alpha,d}\right)$ which will be useful in the sequel.

Theorem 4.2 *Let* $l > 2\alpha + 4n + d + 2$, $a(x,\ \xi) \in \mathcal{S}^{r,l}$ *and* $u \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$. *Then, the pseudo-differential operator* $A\big(a,\ \Delta_W^{\alpha,d}\big)$ *admits the following representation*:

$$\big[A\big(a, \Delta_W^{\alpha,d}\big)u\big](x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\Lambda_{\alpha,d,n}(-x,\xi)\times \tag{4.6}$$

$$\left[\int_{\mathbb{R}^{d+1}_+}\mathscr{M}^{-1}_{n,\xi}\mathscr{M}^{-1}_{n,y}T^{\alpha,d,n}_{-y}\mathscr{M}_n\mathscr{F}^{\alpha,d,n}_W\big(\mathscr{M}_n a(.,y)\big)(\xi)\mathscr{F}^{\alpha,d,n}_W(u)(y)d\mu_{\alpha+2n,d}(y)\right]d\mu_{\alpha+2n,d}(\xi)$$

where all involved integrals are absolutely convergent.

Proof We put

$$g_x(y,\xi) = \Lambda_{\alpha,d,n}(-x,\xi)\mathscr{M}^{-1}_{n,\xi}\mathscr{M}^{-1}_{n,y}T^{\alpha,d,n}_{-y}\mathscr{M}_n\mathscr{F}^{\alpha,d,n}_W\big(\mathscr{M}_n a(.,y)\big)(\xi)\mathscr{F}^{\alpha,d,n}_W(u)(y).$$

We shall prove that g_x belongs to $L^1\big(\mathbb{R}^{d+1}_+\times\mathbb{R}^{d+1}_+,\ d\mu_{\alpha+2n,d}(y)d\mu_{\alpha+2n,d}(\xi)\big)$.

Let $t > l$ and $\gamma > \frac{r}{2} - \frac{t}{2} + \frac{d}{2} + \alpha + 2n + 1$. Using the relations (2.19), (2.7) and (4.5), we obtain

$$\begin{aligned}&\Big|\mathscr{M}^{-1}_{n,\xi}\mathscr{M}^{-1}_{n,y}T^{\alpha,d,n}_{-y}\mathscr{M}_n\mathscr{F}^{\alpha,d,n}_W\big(\mathscr{M}_n a(.,\ y)\big)(\xi)\Big|\\ &\quad\le C_1(1+\|y\|^2)^{\frac{r}{2}}\Big|T^{\alpha+2n,d}_{-y}\Big[(1+\|x\|^2)^{-\frac{t}{2}}\Big](\xi)\Big|.\end{aligned}$$

Hence from (2.21), we get

$$\Big|\mathscr{M}^{-1}_{n,\xi}\mathscr{M}^{-1}_{n,y}T^{\alpha,d,n}_{-y}\mathscr{M}_n\mathscr{F}^{\alpha,d,n}_W\big(\mathscr{M}_n a(.,\ y)\big)(\xi)\Big| \le C_2(1+\|y\|^2)^{\frac{r-t}{2}}(1+\|\xi\|^2)^{-\frac{t}{2}} \tag{4.7}$$

where C_2 is a constant depending on r, t, α, d, n and l. On the other hand since $u \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, then there exist $C_3 > 0$ such that

$$\forall y \in \mathbb{R}^{d+1}_+,\ \Big|\mathscr{F}^{\alpha,d,n}_W(u)(y)\Big| \le C_3(1+\|y\|^2)^{-\gamma}.$$

Hence, we get

$$\big|g_x(y,\xi)\big| \le Cx^{2n}_{d+1}(1+\|y\|^2)^{\frac{r}{2}-\frac{t}{2}-\gamma}(1+\|\xi\|^2)^{-\frac{t}{2}}.$$

Since $t > l > 2\alpha + 4n + d + 2$ and $\gamma > \frac{r}{2} - \frac{t}{2} + \frac{d}{2} + \alpha + 2n + 1$, the function g_x belongs to $L^1\big(\mathbb{R}^{d+1}_+\times\mathbb{R}^{d+1}_+, d\mu_{\alpha+2n,d}(y)d\mu_{\alpha+2n,d}(\xi)\big)$. So, the result follows by applying the inverse theorem and using the relation (2.25). □

Now, we are in a situation to establish the fundamental result of this section given by the following result.

Theorem 4.3 *Let* s, $\frac{l}{2} > \alpha + 2n + \frac{d}{2} + 1$, $a(x,\ \xi) \in \mathcal{S}^{r,l}$ *and* $A\big(x,\ \Delta_W^{\alpha,d,n}\big)$ *be the associated pseudo-differential operator. Then* $A\big(a,\ \Delta_W^{\alpha,d,n}\big)$ *maps continuously from* $\mathscr{H}^{s+r,\alpha,n}(\mathbb{R}^{d+1}_+)$ *to* $\mathscr{H}^{s,\alpha,n}(\mathbb{R}^{d+1}_+)$. *Moreover, we have*

$$\forall u \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}), \left\| A\left(a,\ \Delta_W^{\alpha,d,n}\right)u \right\|_{\mathscr{H}^{s,\alpha,n}} \leq k\|u\|_{\mathscr{H}^{s+r,\alpha,n}} \tag{4.8}$$

where k is a constant depending on s, r, α, d, n and l.

Proof Let $s, \frac{l}{2} > \alpha + 2n + \frac{d}{2} + 1$. We put

$$\varphi_s(\xi) = \int_{\mathbb{R}_+^{d+1}} \mathscr{M}_{n,\xi}^{-1}\mathscr{M}_{n,y}^{-1}T_y^{\alpha,d,n}\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_n a(.,y)\right)(-\xi)\mathscr{F}_W^{\alpha,d,n}(u)(y)d\mu_{\alpha+2n,d}(y)$$

From the relations (4.7), we have

$$\left|\varphi_s(\xi)\right| \leq C_2(1+\|\xi\|^2)^{-\frac{t}{2}} \int_{\mathbb{R}_+^{d+1}} (1+\|y\|^2)^{\frac{r-t}{2}} \left|\mathscr{F}_W^{\alpha,d,n}(u)(y)\right| d\mu_{\alpha+2n,d}(y).$$

Hence using the Cauchy-Schwartz inequality, we obtain

$$C_{\alpha+2n,d}(1+\|\xi\|^2)^{\frac{s}{2}}\left|\varphi_s(\xi)\right| \leq C_3(1+\|\xi\|^2)^{\frac{s}{2}-\frac{t}{2}}\|u\|_{\mathscr{H}^{s+r,\alpha,n}}$$

where

$$C_3 = C_2\left(\int_{\mathbb{R}_+^{d+1}} (1+\|y\|^2)^{-s-t} d\mu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}}.$$

Then

$$\left\| A\left(a,\ \Delta_W^{\alpha,d,n}\right)u \right\|_{\mathscr{H}^{s,\alpha,n}} = C_{\alpha+2n,d}\left\|(1+\|\xi\|^2)^{\frac{s}{2}}\varphi_s\right\|_{\alpha+2n,2} \leq k\|u\|_{\mathscr{H}^{s+r,\alpha,n}}$$

where $t > |s| + \alpha + 2n + \dfrac{d}{2} + 1$ and

$$k = C_3\left(\int_{\mathbb{R}_+^{d+1}} (1+\|\xi\|^2)^{s-t} d\mu_{\alpha+2n,d}(\xi)\right)^{\frac{1}{2}}.$$

□

Declaration

Conflict of interest The author "Hassen Ben Mohamed" declares that he has no conict of interest. The author "Youssef Bettaibi" declares that he has no conict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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