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On the derivations, generalized derivations and ternary derivations of degree *n*

Amin Hosseini[1] · **Mehdi Mohammadzadeh Karizaki**[2]

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Abstract

In this paper, we introduce the concepts of derivation of degree n, generalized derivation of degree n and ternary derivation of degree n, where n is a positive integer, and then we study the algebraic properties of these mappings. For instance, we study the image of derivations of degree n on algebras and in this regard we prove that, under certain conditions, every derivation of degree n on an algebra maps the algebra into its Jacobson radical. Also, we present some characterizations of these mappings on algebras. For example, under certain assumptions, we show that if f is an additive generalized derivation of degree n with an associated mapping d, then either f is a linear generalized derivation with the associated linear derivation d or f and d are identically zero. Some other related results are also established.

Keywords Derivation · Derivation of degree n · Generalized derivation of degree n · Ternary derivation of degree n · Singer-Wermer theorem

Mathematics Subject Classification Primary 47B47 · Secondary 47B48

1 Introduction and preliminaries

Let $\mathcal{R}$ be a ring and let n be a positive integer. A mapping $\Delta : \mathcal{R} \to \mathcal{R}$ is called a derivation of degree n or $\{n\}$-derivation if $\Delta(xy) = \Delta(x)y^n + x^n\Delta(y)$ holds for all $x, y \in \mathcal{R}$. Also, Δ is called a Jordan derivation of degree n or Jordan $\{n\}$-derivation if $\Delta(x^2) = \Delta(x)x^n + x^n\Delta(x)$ holds for all $x \in \mathcal{R}$. In this paper, we provide an example of a Jordan derivation of degree n which is not a derivation of degree n.

By getting the idea from cubic derivations and quadratic derivations, we define a derivation of degree n from an algebra into a module. Before stating the results of this article, let us recall some basic definitions and set the notations which we use in what follows.

✉ Mehdi Mohammadzadeh Karizaki
m.mohammadzadeh@torbath.ac.ir

Amin Hosseini
hosseini.amin82@gmail.com

[1] Kashmar Higher Education Institute, Kashmar, Iran

[2] Department of Computer Engineering, University of Torbat Heydarieh, Torbat Heydarieh, Iran

An algebra $\mathcal{A}$ is called a domain if $\mathcal{A} \neq \{0\}$, and $a = 0$ or $b = 0$, whenever $ab = 0$. A commutative domain is called an integral domain. Recall that the Jacobson radical of an algebra $\mathcal{A}$ is the intersection of all primitive ideals of $\mathcal{A}$ which is denoted by $rad(\mathcal{A})$. An algebra $\mathcal{A}$ is called semisimple if $rad(\mathcal{A}) = \{0\}$. A nonzero linear functional φ on an algebra $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. The set of all characters on $\mathcal{A}$ is denoted by $\Phi_{\mathcal{A}}$ and is called the character space of $\mathcal{A}$. We know that $\ker\varphi$ is a maximal ideal of $\mathcal{A}$ for every $\varphi \in \Phi_{\mathcal{A}}$ (see [4, Proposition 1.3.37]).

Let $\mathcal{A}$ be a complex algebra and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a derivation if it satisfies the Leibnitz's rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. In [5], Eshaghi Gordji et al. introduced the concept of a cubic derivation. A mapping $D : \mathcal{A} \to \mathcal{M}$ is called a cubic derivation if D is a cubic homogeneous mapping, that is $D(\lambda a) = \lambda^3 D(a)$ $(\lambda \in \mathbb{C},\ a \in \mathcal{A})$, and $D(ab) = D(a)b^3 + a^3 D(b)$ for all $a, b \in \mathcal{A}$. Also, a mapping $d : \mathcal{A} \to \mathcal{M}$ is called a quadratic derivation if d is a quadratic homogeneous mapping, that is $d(\lambda a) = \lambda^2 d(a)$ $(\lambda \in \mathbb{C},\ a \in \mathcal{A})$, and $d(ab) = d(a)b^2 + a^2 d(b)$ for all $a, b \in \mathcal{A}$. The most papers to date have been focused on investigating stability of cubic derivations and quadratic derivations, see, e.g. [1, 5, 6, 9, 13, 17], and references therein.

In this paper, by getting the idea from the notions of cubic derivation and quadratic derivation, we define the notion of derivation of degree n on algebras, where n is a positive integer. In what follows, let $\mathcal{A}$ be a complex algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let n be a positive integer. A mapping $\Delta : \mathcal{A} \to \mathcal{M}$ is called a derivation of degree n or $\{n\}$-derivation if it satisfies both the equations $\Delta(ab) = \Delta(a)b^n + a^n\Delta(b)$ and $\Delta(\lambda a) = \lambda^n\Delta(a)$ for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$.

Now let us to give a background about the image of derivations. The image of derivations has a fairly long history and so far, many authors have studied the image of derivations, see, e.g. [2, 3, 7, 10–12, 14–16] and references therein. As a pioneering work, Singer and Wermer [14] achieved a fundamental result which started investigation into the image of derivations on Banach algebras. The so-called Singer-Wermer theorem, which is a classical theorem of complex Banach algebra theory, states that every continuous derivation on a commutative Banach algebra maps the algebra into its Jacobson radical, and Thomas [15] proved that the Singer-Wermer theorem remains true without assuming the continuity of the derivation.

One of our aims in this research is to prove some results similar to Singer- Wermer theorem and Thomas theorem for derivations of degree n. In this regard, we first prove the following theorem which has been motivated by [7]:

Let $\mathcal{A}$ be a unital integral domain and let $\Delta : \mathcal{A} \to \mathcal{A}$ be an $\{n\}$-derivation such that its rank is at most one. Then Δ is identically zero. Using this result, it is proved that if $\mathcal{A}$ is a unital algebra and $\Delta : \mathcal{A} \to \mathcal{A}$ is an $\{n\}$-derivation such that $\Delta(a) - \Delta(b) \in ker\varphi$ whenever $a - b \in ker\varphi$ for every $a, b \in \mathcal{A}$ and every $\varphi \in \Phi_{\mathcal{A}}$, then $\Delta(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker\varphi$. If $\mathcal{A}$ is also commutative, then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. In this regard, we provide an example of an $\{n\}$ -derivation on an algebra $\mathfrak{A}$ mapping the algebra into the intersection of all characters of $\mathfrak{A}$. In addition, we prove that if $\mathcal{A}$ is a unital, commutative Banach algebra and $\Delta : \mathcal{A} \to \mathcal{A}$ is an additive $\{n\}$-derivation, then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. As another result in this regard, we prove that every $\{n\}$-derivation on finite dimensional algebras is identically zero under certain conditions. Indeed, we establish the following result. Let m be a positive integer and let $\mathcal{A}$ be an m-dimensional unital algebra with the basis $\mathfrak{B} = \{\mathfrak{b}_1, \dots, \mathfrak{b}_m\}$. Furthermore, suppose that for every integer k, $1 \leq k \leq m$, an ideal $\mathfrak{X}_k$ generated by $\mathfrak{B} - \{\mathfrak{b}_k\}$ is a proper subset of $\mathcal{A}$. If $\Delta : \mathcal{A} \to \mathcal{A}$ is an $\{n\}$-derivation such that $\Delta(a) - \Delta(b) \in \mathfrak{X}_k$ whenever $a - b \in \mathfrak{X}_k$ for every $a, b \in \mathcal{A}$ and $1 \leq k \leq m$, then Δ is identically zero.

Another objective of this paper is to characterize $\{n\}$-derivations, $\{n\}$-generalized derivations and $\{n\}$-ternary derivations on algebras. First, we introduce these notions. A mapping $f : \mathcal{A} \to \mathcal{M}$ is called a generalized derivation of degree n or an $\{n\}$-generalized derivation if there exists a mapping $d : \mathcal{A} \to \mathcal{M}$ such that

$$f(ab) = f(a)b^n + a^n d(b),$$
$$f(\lambda a) = \lambda^n f(a),$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. In this case, d is called an associated mapping of f.

A ternary derivation of degree n is defined as follows. A ternary derivation of degree n or an $\{n\}$-ternary derivation is a triple of mappings (d_1, d_2, d_3) from $\mathcal{A}$ into $\mathcal{M}$ such that

$$d_1(ab) = d_2(a)b^n + a^n d_3(b),$$
$$d_1(\lambda a) = \lambda^n d_1(a),$$

for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$.

For instance, we establish the result below concerning the characterization of $\{n\}$-generalized derivations. Let $\mathcal{A}$ be a unital algebra with the identity element $\mathbf{e}$, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $f : \mathcal{A} \to \mathcal{M}$ be an additive generalized $\{n\}$-derivation with an associated mapping $d : \mathcal{A} \to \mathcal{M}$ such that $d(2\mathbf{e}) = 2d(\mathbf{e})$. Then either f is a nonzero linear generalized derivation with the associated linear derivation d or f and d are identically zero.

A theorem similar to the above result is presented for the $\{n\}$-ternary derivations.

2 Definitions and examples

In this section, without further mention, $\mathbf{e}$ denotes the identity of any unital ring or algebra. We begin this section with the following definition.

Definition 1 Let $\mathcal{R}$ be a ring and let n be a positive integer. A mapping $\Delta : \mathcal{R} \to \mathcal{R}$ is called a *derivation of degree n* if

$$\Delta(xy) = \Delta(x)y^n + x^n \Delta(y)$$

holds for all $x, y \in \mathcal{R}$. Also, Δ is called a *Jordan derivation of degree n* if it satisfies

$$\Delta(x^2) = \Delta(x)x^n + x^n \Delta(x)$$

for all $x \in \mathcal{R}$.

Obviously, if Δ is a Jordan derivation of degree n on $\mathcal{R}$, then $\Delta(0) = 0$. Also, if $\mathcal{R}$ is unital with the identity element $\mathbf{e}$, then $\Delta(\mathbf{e}) = 0$. It is clear that every derivation of degree n is a Jordan derivation of degree n, but the converse is, in general, not true. In the following, we present a Jordan derivation of degree n which is not a derivation of degree n.

Example 2 Let $\mathcal{R}$ be a ring such that $x^4 = 0$ for all $x \in \mathcal{R}$, but the product of some nonzero elements of $\mathcal{R}$ is nonzero. Let

$$\mathfrak{R} = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} : x, y \in \mathcal{R} \right\}$$

Define $\Delta : \mathfrak{R} \to \mathfrak{R}$ by

$$\Delta\left(\begin{bmatrix} 0 & x & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & y^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For any $A = \begin{bmatrix} 0 & x & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{R}$, we have

$$\Delta(A^2) = \Delta\left(\begin{bmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & x^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

A straightforward verification shows that

$$\Delta(A)A^n + A^n\Delta(A) = 0,$$

for all $A \in \mathfrak{R}$ and all $n \in \mathbb{N}$. We see that Δ is a Jordan derivation of degree n for any $n \in \mathbb{N}$. Also, it is easy to see that $\Delta(A)B^n + A^n\Delta(B) = 0$ for all $A, B \in \mathfrak{R}$ and all $n \in \mathbb{N}$, but $\Delta(AB) \neq 0$ for some $A, B \in \mathfrak{R}$. It means that Δ is not a derivation of degree n for all $n \in \mathbb{N}$.

In the rest of this article, we consider derivations of degree n from algebras into modules as follows.

Definition 3 Let $\mathcal{A}$ be a complex algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let n be a positive integer. A mapping $\Delta : \mathcal{A} \to \mathcal{M}$ is called a derivation of degree n if it satisfies both of the following equations:

$$\Delta(ab) = \Delta(a)b^n + a^n\Delta(b),$$
$$\Delta(\lambda a) = \lambda^n\Delta(a),$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$.

Example 4 Let $\mathcal{A}$ an algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule, let n be a positive integer and let x_0 be an element of $\mathcal{M}$ satisfying

$$x_0\big[(ab)^n - a^nb^n\big] = \big[(ab)^n - a^nb^n\big]x_0$$

for all $a, b \in \mathcal{A}$. Define a mapping $\Delta : \mathcal{A} \to \mathcal{M}$ by $\Delta(a) = a^nx_0 - x_0a^n$ for any $a \in \mathcal{A}$. It is routine to see that $\Delta(ab) = \Delta(a)b^n + a^n\Delta(b)$ and $\Delta(\lambda a) = \lambda^n\Delta(a)$ for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. This means that Δ is an $\{n\}$-derivation. We call such mapping inner derivation of degree n or inner $\{n\}$-derivation.

Example 5 Let $\mathcal{A}$ be a commutative algebra, let n be an arbitrary positive integer and let

$$\mathfrak{A} = \left\{\begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix} : a, b, c, e \in \mathcal{A}\right\}$$

It is clear that $\mathfrak{A}$ is a non-commutative algebra. Define $\Delta : \mathfrak{A} \to \mathfrak{A}$ by

$$\Delta\left(\begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & b^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that for any $A = \begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix} \in \mathfrak{A}$ and any $k \in \mathbb{N}$, we have

$$A^k = \begin{bmatrix} 0 & ac^{k-1} & be^k \\ 0 & c^k & 0 \\ 0 & 0 & e^k \end{bmatrix}$$

One can easily get that $\Delta(AB) = \Delta(A)B^n + A^n\Delta(B)$ and $\Delta(\lambda A) = \lambda^n\Delta(A)$ for all $A, B \in \mathfrak{A}$ and all $\lambda \in \mathbb{C}$, which means that Δ is a derivation of degree n on $\mathfrak{A}$.

Example 6 Let $\mathcal{A}$ be an algebra, let n be an arbitrary positive integer and let

$$\mathfrak{A} = \left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathcal{A}\right\}$$

Define $\Delta : \mathfrak{A} \to \mathfrak{A}$ by

$$\Delta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c^n \\ 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to see that $\Delta(AB) = \Delta(A)B^n + A^n\Delta(B)$ and $\Delta(\lambda A) = \lambda^n\Delta(A)$ for all $A, B \in \mathfrak{A}$ and all $\lambda \in \mathbb{C}$, which means that Δ is a derivation of degree n on $\mathfrak{A}$.

Definition 7 Let $\mathcal{A}$ be an algebra and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A mapping $f : \mathcal{A} \to \mathcal{M}$ is called a generalized derivation of degree n or an $\{n\}$-generalized derivation if there exists a mapping $d : \mathcal{A} \to \mathcal{M}$ such that

$$f(ab) = f(a)b^n + a^n d(b),$$
$$f(\lambda a) = \lambda^n f(a),$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. In this case, d is called an associated map of f.

Example 8 Let $\mathcal{A}$ an algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule, let n be a positive integer and let x_0 and y_0 be two elements of $\mathcal{M}$ satisfying

$$y_0\left[(ab)^n - a^n b^n\right] = \left[(ab)^n - a^n b^n\right]x_0,$$

for all $a, b \in \mathcal{A}$. Define the mappings $f, d : \mathcal{A} \to \mathcal{M}$ by $f(a) = a^n x_0 - y_0 a^n$ and $d(a) = a^n x_0 - x_0 a^n$ for any $a \in \mathcal{A}$. It is routine to see that $f(ab) = f(a)b^n + a^n d(b)$ and $f(\lambda a) = \lambda^n f(a)$ for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. This means that f is an $\{n\}$-generalized derivation with the associated mapping d. We call such mapping inner generalized derivation of degree n or inner $\{n\}$-generalized derivation.

In the following, we define a ternary derivation of degree $\{n\}$.

Definition 9 Let $\mathcal{A}$ be an algebra and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A ternary derivation of degree n or an $\{n\}$-ternary derivation is a triple of mappings (d_1, d_2, d_3) from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\begin{aligned} d_1(ab) &= d_2(a)b^n + a^n d_3(b), \\ d_1(\lambda a) &= \lambda^n d_1(a), \end{aligned}$$

for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$.

3 Results and proofs

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a field $\mathbb{F}$. Throughout this section, a mapping $D : \mathcal{A} \to \mathcal{B}$ is called a *rank-one mapping* if there exist a nonzero element $\mathfrak{b}$ of $\mathcal{B}$ and a functional $\mu : \mathcal{A} \to \mathbb{F}$ such that $D(a) = \mu(a)\mathfrak{b}$ for all $a \in \mathcal{A}$.

We begin our results with the following theorem.

Theorem 10 *Let $\mathcal{A}$ be a unital integral domain and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a derivation of degree n such that its rank is at most one. Then Δ is identically zero.*

Proof Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a derivation of degree n such that its rank is at most one. We are going to show that $\Delta(\mathcal{A}) = \{0\}$. Suppose that Δ is a rank-one mapping. So there exist a nonzero element $\mathfrak{c}$ of $\mathcal{A}$ and a functional $\mu : \mathcal{A} \to \mathbb{C}$ such that $\Delta(a) = \mu(a)\mathfrak{c}$ for all $a \in \mathcal{A}$. To obtain a contradiction, suppose there exists a nonzero element $\mathfrak{a} \in \mathcal{A}$ such that $\Delta(\mathfrak{a}) \neq 0$. It is clear that $\mu(\mathfrak{a}) \neq 0$. We observe two cases for $\Delta(\mathfrak{c})$.

Case 1. $\Delta(\mathfrak{c}) = 0$. In this case, we have $\mu(\mathfrak{c})\mathfrak{c} = 0$ and it implies that $\mu(\mathfrak{c}) = 0$. We have the following expressions:

$$\begin{aligned} \mu(\mathfrak{a}^2)\mathfrak{c} &= \Delta(\mathfrak{a}^2) \\ &= \Delta(\mathfrak{a})\mathfrak{a}^n + \mathfrak{a}^n\Delta(\mathfrak{a}) \\ &= 2\mathfrak{a}^n\Delta(\mathfrak{a}) \\ &= 2\mathfrak{a}^n\mu(\mathfrak{a})\mathfrak{c} \\ &= 2\mu(\mathfrak{a})\mathfrak{a}^n\mathfrak{c}. \end{aligned}$$

Since we are assuming that $\Delta(\mathfrak{c}) = 0$, we have

$$\begin{aligned} 0 &= (\mu(\mathfrak{a}^2))^n\Delta(\mathfrak{c}) = \Delta(\mu(\mathfrak{a}^2)\mathfrak{c}) = \Delta(2\mu(\mathfrak{a})\mathfrak{a}^n\mathfrak{c}) \\ &= 2^n(\mu(\mathfrak{a}))^n[\Delta(\mathfrak{a}^n)\mathfrak{c}^n + \mathfrak{a}^{n^2}\Delta(\mathfrak{c})] \\ &= 2^n(\mu(\mathfrak{a}))^n\Delta(\mathfrak{a}^n)\mathfrak{c}^n \end{aligned}$$

Since $\mathcal{A}$ is a domain and $\mu(\mathfrak{a})$ and $\mathfrak{c}$ are nonzero, we get that $\Delta(\mathfrak{a}^n) = 0$. Using induction, for any $m \in \mathbb{N}$, one can easily prove that

$$\Delta(a^m) = \Sigma_{k=1}^{m} a^{(k-1)n}\Delta(a)a^{(m-k)n}$$

in which $a^0 = \mathbf{e}$. So we have

$$\begin{aligned} 0 &= \Delta(\mathfrak{a}^n) = \Delta(\mathfrak{a}^{n-1}\mathfrak{a}) \\ &= \Delta(\mathfrak{a}^{n-1})\mathfrak{a}^n + \mathfrak{a}^{n(n-1)}\Delta(\mathfrak{a}) \\ &= \left[\Sigma_{k=1}^{n-1}\mathfrak{a}^{(k-1)n}\Delta(\mathfrak{a})\mathfrak{a}^{(n-1-k)n}\right]\mathfrak{a}^n + \mathfrak{a}^{n(n-1)}\Delta(\mathfrak{a}) \\ &= \Sigma_{k=1}^{n-1}\left[\Delta(\mathfrak{a})\mathfrak{a}^{n^2-n}\right] + \mathfrak{a}^{n^2-n}\Delta(\mathfrak{a}) \\ &= n\Delta(\mathfrak{a})\mathfrak{a}^{n^2-n}, \end{aligned}$$

which implies that $\Delta(\mathfrak{a}) = 0$, a contradiction.

Case 2. $\Delta(\mathfrak{c}) \neq 0$. In this case, we have $\mu(\mathfrak{c}) \neq 0$. Now look at the following statements:

$$\mu(\mathfrak{c}^2)\mathfrak{c} = \Delta(\mathfrak{c}^2) = \Delta(\mathfrak{c})\mathfrak{c}^n + \mathfrak{c}^n\Delta(\mathfrak{c}) = 2\mathfrak{c}^n\Delta(\mathfrak{c}) = 2\mu(\mathfrak{c})\mathfrak{c}^{n+1} \quad (1)$$

If $\mu(\mathfrak{c}^2) = 0$, then it follows from (1) that either $\mu(\mathfrak{c}) = 0$ or $\mathfrak{c} = 0$, and we know that both of them are nonzero. So $\mu(\mathfrak{c}^2) \neq 0$. Putting $\frac{\mu(\mathfrak{c}^2)}{2\mu(\mathfrak{c})} = \alpha$ in (1), we have $\mathfrak{c}(\mathfrak{c}^n - \alpha\mathbf{e}) = \mathfrak{c}^{n+1} - \alpha\mathfrak{c} = 0$. In view of this assumption that $\mathcal{A}$ is a domain, we infer that $\mathfrak{c} = 0$, a contradiction, or $\mathfrak{c}^n = \alpha\mathbf{e}$. So we have

$$\begin{aligned} 0 &= \alpha^n\Delta(\mathbf{e}) = \Delta(\alpha\mathbf{e}) = \Delta(\mathfrak{c}^n) = \Delta(\mathfrak{c}^{n-1}\mathfrak{c}) \\ &= \Delta(\mathfrak{c}^{n-1})\mathfrak{c}^n + \mathfrak{c}^{n(n-1)}\Delta(\mathfrak{c}) \\ &= \left[\Sigma_{k=1}^{n-1}\mathfrak{c}^{(k-1)n}\Delta(\mathfrak{c})\mathfrak{c}^{(n-1-k)n}\right]\mathfrak{c}^n + \mathfrak{c}^{n(n-1)}\Delta(\mathfrak{c}) \\ &= \Sigma_{k=1}^{n-1}\left[\Delta(\mathfrak{c})\mathfrak{c}^{n^2-n}\right] + \mathfrak{c}^{n^2-n}\Delta(\mathfrak{c}) \\ &= n\Delta(\mathfrak{c})\mathfrak{c}^{n^2-n}. \end{aligned}$$

Reusing the assumption that $\mathcal{A}$ is a domain, we get that $\mathfrak{c} = 0$ or $\Delta(\mathfrak{c}) = 0$, which these are contradictions. It is observed that both Cases 1 and 2 lead to contradictions. Therefore, there is no element $\mathfrak{a}$ of $\mathcal{A}$ such that $\Delta(\mathfrak{a}) \neq 0$, and consequently, Δ must be zero. □

In the following, we provide some examples that show that the conditions of Theorem 10 are not superfluous.

Example 11

(i) Let n be a positive number. Define $\Delta : \mathbb{R} \to \mathbb{R}$ by

$$\Delta(a) = \begin{cases} a^n \ln(\mid a \mid) & a \neq 0, \\ 0 & a = 0. \end{cases}$$

One can easily check that $\Delta(ab) = \Delta(a)b^n + a^n\Delta(b)$ for all $a, b \in \mathbb{R}$ and also it is clear that the rank of Δ is at most one, but we observe that $\Delta(\alpha a) \neq \alpha^n\Delta(a)$ for some $\alpha, a \in \mathbb{R}$. We see that Δ is a nonzero mapping.

(ii) In Example 5, considering $\mathcal{A} = \mathbb{C}$, we see that $\Delta : \mathfrak{A} \to \mathfrak{A}$ defined by

$$\Delta\left(\begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & b^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = b^n \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a nonzero, rank one derivation of degree n. Note that $\mathfrak{A}$ is not an integral domain.

In the following theorem, we present some conditions under which every derivation of degree n on an algebra maps the algebra into its Jacobson radical.

Theorem 12 *Let $\mathcal{A}$ be a unital algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a derivation of degree n such that $\Delta(a) - \Delta(b) \in ker\varphi$ whenever $a - b \in ker\varphi$ for every $a, b \in \mathcal{A}$ and every $\varphi \in \Phi_{\mathcal{A}}$. In this case, $\Delta(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$. If $\mathcal{A}$ is also commutative, then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Proof Let φ be an arbitrary character on $\mathcal{A}$. We define a mapping $\Omega : \frac{\mathcal{A}}{\ker\varphi} \to \frac{\mathcal{A}}{\ker\varphi}$ by $\Omega(a + \ker\varphi) = \Delta(a) + \ker\varphi$ for every $a \in \mathcal{A}$. Ω is a derivation of degree n on the algebra $\frac{\mathcal{A}}{\ker\varphi}$. It is clear that the algebra $\frac{\mathcal{A}}{\ker\varphi}$ is a unital, integral domain and it follows from [4, Proposition 1.3.37] that $\dim(\frac{\mathcal{A}}{\ker\varphi}) = 1$. So the rank of Ω is at most one. Now, Theorem 10 yields that Ω is identically zero, and it means that $\Delta(\mathcal{A}) \subseteq \ker\varphi$. Since we are assuming φ is an arbitrary element of $\Phi_{\mathcal{A}}$, $\Delta(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$. It is obvious that if $\mathcal{A}$ is commutative, then $\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi = rad(\mathcal{A})$ (see [4]). Hence, we deduce that $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. □

An immediate corollary of the previous theorem is as follows:

Corollary 13 *Let $\mathcal{A}$ be a unital algebra such that $\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi = \{0\}$ and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a derivation of degree n such that $\Delta(a) - \Delta(b) \in ker\varphi$ whenever $a - b \in ker\varphi$ for every $a, b \in \mathcal{A}$ and every $\varphi \in \Phi_{\mathcal{A}}$. Then Δ is identically zero.*

Proof According to [8, Proposition 2.10], the algebra $\mathcal{A}$ is commutative and semisimple. Now the previous theorem gives the result. □

Remark 14 In this remark, we show that the image of derivation of degree n presented in Example 5 is contained in $\bigcap_{\varphi\in\Phi_{\mathfrak{A}}} \ker\varphi$. Let $\mathcal{A}$ be a unital commutative Banach algebra and let

$$\mathfrak{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix} : a, b, c, e \in \mathcal{A} \right\}$$

Note that $\mathfrak{A}$ is a non-commutative algebra. Since $\mathcal{A}$ is a unital commutative Banach algebra, it follows from [4, Theorem 2.3.1] that its character space is a non-empty set, i.e. $\Phi_{\mathcal{A}} \neq \phi$. Let φ be a character of $\mathcal{A}$. We define $\theta_\varphi : \mathfrak{A} \to \mathbb{C}$ by $\theta_\varphi\left(\begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix}\right) = \varphi(c)$. It is clear that θ_φ is a character on $\mathfrak{A}$ and it is easy to see that

$$ker(\theta_\varphi) = \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} \\ 0 & ker(\varphi) & 0 \\ 0 & 0 & \mathcal{A} \end{bmatrix} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & x & 0 \\ 0 & 0 & e \end{bmatrix} \ : \ a, b, e \in \mathcal{A}, x \in \ker(\varphi) \right\}.$$

Also, if we define $\theta_\varphi \ : \ \mathfrak{A} \to \mathbb{C}$ by $\theta_\varphi\left(\begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & e \end{bmatrix}\right) = \varphi(e)$, then we deduce that θ_φ is a character on $\mathfrak{A}$. It is easy to see that

$$ker(\theta_\varphi) = \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} & 0 \\ 0 & 0 & ker(\varphi) \end{bmatrix} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & z \end{bmatrix} \ : \ a, b, c \in \mathcal{A}, z \in \ker(\varphi) \right\}.$$

Therefore, $\Phi_{\mathfrak{A}} = \{\theta_\varphi \ : \ \varphi \in \Phi_{\mathcal{A}}\}$. It is observed that $\Delta(\mathfrak{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathfrak{A}}} \ker \theta_\varphi$.

In the next theorem, we prove that every derivation of degree n on a unital finite-dimensional algebra is identically zero under certain conditions. Let m be a positive integer and let $\mathcal{A}$ be an m-dimensional unital algebra with the basis $\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_m\}$.

Theorem 15 *Suppose that for every integer k, $1 \le k \le m$, an ideal $\mathfrak{X}_k$ generated by $\mathfrak{B} - \{\mathfrak{b}_k\}$ is a proper subset of $\mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a derivation of degree n such that $\Delta(a) - \Delta(b) \in \mathfrak{X}_k$ whenever $a - b \in \mathfrak{X}_k$ for every $a, b \in \mathcal{A}$ and every $1 \le k \le m$. Then Δ is identically zero.*

Proof It is clear that $\dim(\frac{\mathcal{A}}{\mathfrak{X}_k}) = 1$ for every $k \in \{1, ..., m\}$. We show that $\mathfrak{X}_k$ is a maximal ideal of $\mathcal{A}$ for each $k \in \{1, ..., m\}$. If $\mathfrak{X}_k$ is not a maximal ideal of $\mathcal{A}$ for some k, $1 \le k \le m$, then there exists a maximal ideal $\mathfrak{M}_k$ of $\mathcal{A}$ such that $\mathfrak{X}_k \subset \mathfrak{M}_k \subset \mathcal{A}$, and so $m - 1 = \dim(\mathfrak{X}_k) < \dim(\mathfrak{M}_k) < m$, a contradiction. Hence, every $\mathfrak{X}_k$ is a maximal ideal of $\mathcal{A}$. Moreover, it follows from Proposition 1.3.37 and Corollary 1.4.38 of [4] that for every maximal ideal $\mathfrak{X}_k$ $(1 \le k \le m)$ there exists a character $\varphi_k \in \Phi_{\mathcal{A}}$ such $\mathfrak{X}_k = ker\varphi_k$. So the algebra $\frac{\mathcal{A}}{\mathfrak{X}_k}$ is an integral domain. Now Theorem 10 yields that $\Omega : \frac{\mathcal{A}}{\mathfrak{X}_k} \to \frac{\mathcal{A}}{\mathfrak{X}_k}$ defined by $\Omega(a + \mathfrak{X}_k) = \Delta(a) + \mathfrak{X}_k$, which is a derivation of degree n, is identically zero. This means that $\Delta(\mathcal{A}) \subseteq \mathfrak{X}_k$, for every $k \in \{1, ..., m\}$, and so $\Delta(\mathcal{A}) \subseteq \bigcap_{k=1}^n \mathfrak{X}_k$. Now suppose that there is an element $\mathfrak{a}$ of $\mathcal{A}$ such that $\Delta(\mathfrak{a}) \neq 0$. Since $\mathfrak{B} = \{\mathfrak{b}_1, ..., \mathfrak{b}_m\}$ is a basis for $\mathcal{A}$, there exist the complex numbers μ_{i_j}, and the elements $\mathfrak{b}_{i_j}$ of $\mathfrak{B}$ such that

$$\Delta(\mathfrak{a}) = \sum_{j=1}^r \mu_{i_j}\mathfrak{b}_{i_j} = \mu_{i_1}\mathfrak{b}_{i_1} + \mu_{i_2}\mathfrak{b}_{i_2} + ... + \mu_{i_r}\mathfrak{b}_{i_r}, \quad (r \le m).$$

We know that $\Delta(\mathcal{A}) \subseteq \mathfrak{X}_k$ for every $k \in \{1, ..., m\}$. So we can assume that $\Delta(\mathcal{A}) \subseteq \mathfrak{X}_{i_1} = \mathfrak{B} - \{\mathfrak{b}_{i_1}\}$. Thus, we have

$$\Delta(\mathfrak{a}) = \mu_{i_1}\mathfrak{b}_{i_1} + \mu_{i_2}\mathfrak{b}_{i_2} + ... + \mu_{i_r}\mathfrak{b}_{i_r} \in \mathfrak{X}_{i_1}.$$

The previous equation asserts that $\mathfrak{b}_{i_1} \in \mathfrak{X}_{i_1}$, which is a contradiction. This contradiction proves our claim. □

In the following, we are going to characterize $\{n\}$-derivations, $\{n\}$-generalized derivations and $\{n\}$-ternary derivations on algebras under certain conditions.

Theorem 16 *Let $\mathcal{A}$ be a unital algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $\Delta : \mathcal{A} \to \mathcal{M}$ be an additive $\{n\}$-derivation. Then either Δ is a nonzero linear derivation or Δ is identically zero.*

Proof Since Δ is an additive mapping, $\Delta(a(b+c)) = \Delta(ab) + \Delta(ac)$ for all $a, b, c \in \mathcal{A}$. We have

$$\Delta(a(b+c)) = \Delta(a)(b+c)^n + a^n\Delta(b) + a^n\Delta(c). \tag{2}$$

Also, we have

$$\Delta(ab) + \Delta(ac) = \Delta(a)b^n + a^n\Delta(b) + \Delta(a)c^n + a^n\Delta(c). \tag{3}$$

Comparing (2) and (3), we get that

$$\Delta(a)\left[(b+c)^n - b^n - c^n\right] = 0, \qquad (a, b, c \in \mathcal{A}). \tag{4}$$

Putting $b = c = \mathbf{e}$ in (4), we arrive at

$$(2^n - 2)\Delta(a) = 0, \qquad (a \in \mathcal{A}).$$

It follows from the previous equation that either $n = 1$, which means that Δ is a nonzero linear derivation from $\mathcal{A}$ into $\mathcal{M}$ or Δ is identically zero. By the way, in both cases Δ is a derivation on $\mathcal{A}$. □

Corollary 17 *Let $\mathcal{A}$ be a unital, commutative Banach algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be an additive $\{n\}$-derivation for some $n \in \mathbb{N}$. Then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Proof It follows from the previous theorem that Δ is a derivation and now [15, Theorem 4.4] yields the required result. □

Theorem 18 *Let $\mathcal{A}$ be a unital algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $f : \mathcal{A} \to \mathcal{M}$ be a generalized $\{n\}$-derivation with an associated mapping $d : \mathcal{A} \to \mathcal{M}$. Then d is an $\{n\}$-derivation if and only if $f(\mathbf{e})\left[(bc)^n - b^n c^n\right] = 0$ for all $b, c \in \mathcal{A}$.*

Proof For every $a, b, c \in \mathcal{A}$, we have

$$f(abc) = f(a)(bc)^n + a^n d(bc).$$

On the other hand, we have

$$f(abc) = f(ab)c^n + (ab)^n d(c) = f(a)b^n c^c + a^n d(b)c^n + (ab)^n d(c).$$

Comparing the last two equations, we get that

$$f(a)\left[(bc)^n - b^n c^n\right] = a^n\left[d(b)c^n - d(bc)\right] + (ab)^n d(c). \tag{5}$$

Putting $a = \mathbf{e}$ in (5), we have

$$f(\mathbf{e})\big[(bc)^n - b^n c^n\big] = d(b)c^n - d(bc) + b^n d(c).$$

If follows from the previous equation that $f(\mathbf{e})\big[(bc)^n - b^n c^n\big] = 0$ if and only if $d(bc) = d(b)c^n + b^n d(c)$ for all $b, c \in \mathcal{A}$. We know that $f(\lambda a) = \lambda^n f(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Hence, for any $a, b \in \mathcal{A}$ and any $\lambda \in \mathbb{C}$, we have the following statements:

$$f(a)(\lambda b)^n + a^n d(\lambda b) = f(a\lambda b) = \lambda^n f(a)b^n + \lambda^n a^n d(b),$$

which implies that $a^n d(\lambda b) = \lambda^n a^n d(b)$. Putting $a = \mathbf{e}$ in the previous equation, we get that $d(\lambda b) = \lambda^n d(b)$ for all $b \in \mathcal{A}$. This means that d is an $\{n\}$-derivation. □

Theorem 19 *Let $\mathcal{A}$ be a unital algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $f : \mathcal{A} \to \mathcal{M}$ be an additive generalized $\{n\}$-derivation with an associated mapping $d : \mathcal{A} \to \mathcal{M}$ such that $d(2\mathbf{e}) = 2d(\mathbf{e})$. Then either f is a nonzero linear generalized derivation with the associated linear derivation d or f and d are identically zero.*

Proof Since f is an additive mapping, $f(a(b + c)) = f(ab) + f(ac)$ for all $a, b, c \in \mathcal{A}$. We have

$$f(a(b + c)) = f(a)(b + c)^n + a^n d(b + c). \tag{6}$$

Also, we have

$$f(ab) + f(ac) = f(a)b^n + a^n d(b) + f(a)c^n + a^n d(c). \tag{7}$$

Comparing (6) and (7), we get that

$$f(a)\big[(b + c)^n - b^n - c^n\big] = a^n\big[d(b) + d(c) - d(b + c)\big], \ \ (a, b, c \in \mathcal{A}). \tag{8}$$

Setting $b = c = \mathbf{e}$ in (8) and using the assumption that $d(2\mathbf{e}) = 2d(\mathbf{e})$, we arrive at

$$(2^n - 2)f(a) = 0, \qquad (a \in \mathcal{A}). \tag{9}$$

We consider the following two cases:

Case 1. $2^n - 2 = 0$. Then $n = 1$ and this means that f is a linear generalized derivation with an associated mapping $d : \mathcal{A} \to \mathcal{M}$. Now we show that d is a linear derivation. Since $n = 1$, it follows from (8) that

$$0 = a\big[d(b) + d(c) - d(b + c)\big], \ \ (a, b, c \in \mathcal{A}). \tag{10}$$

Putting $a = \mathbf{e}$ in (10), we see that d is an additive mapping. Also, note that $f(\lambda a) = \lambda^n f(a) = \lambda f(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Similar to the proof of Theorem 18, one can easily show that $d(\lambda a) = \lambda d(a)$ for all $a \in \mathcal{A}$ and we leave it to the interested reader. So d is a linear derivation.

Case 2. $2^n - 2 \neq 0$. It follows from (9) that f is identically zero. This fact with $f(ab) = f(a)b^n + a^n d(b)$ imply that $a^n d(b) = 0$ for all $a, b \in \mathcal{A}$. Putting $a = \mathbf{e}$ in the previous equation, we infer that d is identically zero. By the way, in both above-mentioned cases f is a generalized derivation with an associated derivation d on $\mathcal{A}$. $\square$

In the following, we present a characterization of $\{n\}$-ternary derivations on algebras.

Theorem 20 *Let $\mathcal{A}$ be a unital algebra, let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $(d_1, d_2, d_3) : \mathcal{A} \to \mathcal{M}$ be an $\{n\}$-ternary derivation. Let $d_3(2\mathbf{e}) = 2d_3(\mathbf{e})$ or $d_2(2\mathbf{e}) = 2d_2(\mathbf{e})$. If d_1 is an additive mapping, then either all the mappings d_1, d_2 and d_3 are linear and (d_1, d_2, d_3) is a ternary derivation on $\mathcal{A}$ or $d_1 = d_2 = d_3 = 0$.*

Proof Suppose that $d_3(2\mathbf{e}) = 2d_3(\mathbf{e})$. Let a, b, c be arbitrary elements of $\mathcal{A}$. We have the following expressions:

$$d_1(a(b+c)) = d_2(a)(b+c)^n + a^n d_3(b+c). \tag{11}$$

On the other hand, we have

$$\begin{aligned} d_1(a(b+c)) &= d_1(ab) + d_1(ac) \\ &= d_2(a)b^n + a^n d_3(b) + d_2(a)c^n + a^n d_3(c) \\ &= d_2(a)(b^n + c^n) + a^n(d_3(b) + d_3(c)), \end{aligned}$$

which means that

$$d_1(a(b+c)) = d_2(a)(b^n + c^n) + a^n(d_3(b) + d_3(c)). \tag{12}$$

Comparing (11) and (12), we get that

$$d_2(a)\big[(b+c)^n - b^n - c^n\big] = a^n\big[d_3(b) + d_3(c) - d_3(b+c)\big]. \tag{13}$$

Putting $b = c = \mathbf{e}$ in (13) and using the assumption that $d_3(2\mathbf{e}) = 2d_3(\mathbf{e})$, we get that

$$(2^n - 2)d_2(a) = 0 \ \textit{for all } a \in \mathcal{A}. \tag{14}$$

We have two cases concerning $2^n - 2$ as follows:

Case 1. $2^n - 2 = 0$. So $n = 1$ and it follows from (13) that

$$0 = a\big[d_3(b) + d_3(c) - d_3(b+c)\big]. \tag{15}$$

Setting $a = \mathbf{e}$ in (15), we see that d_3 is an additive mapping. We know that $d_1(\lambda a) = \lambda^n d_1(a) = \lambda d_1(a)$ for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Hence, for any $a, b \in \mathcal{A}$ and any $\lambda \in \mathbb{C}$, we have the following statements:

$$d_2(a)(\lambda b) + a d_3(\lambda b) = d_1(a\lambda b) = \lambda d_2(a)b + \lambda a d_3(b),$$

which implies that $ad_3(\lambda b) = \lambda a d_3(b)$. Putting $a = \mathbf{e}$ in the previous equation, we get that $d_3(\lambda b) = \lambda d_3(b)$ for all $b \in \mathcal{A}$. This means that d_3 is a linear mapping. Similarly, we can show that d_2 is a linear mapping. Hence, (d_1, d_2, d_3) is a ternary derivation on $\mathcal{A}$.

Case 2. $2^n - 2 \neq 0$. Then equation (14) yields that d_2 must be zero. Considering this case and using $d_1(ab) = d_2(a)b^n + a^n d_3(b)$ for all $a, b \in \mathcal{A}$, we get that

$$d_1(ab) = a^n d_3(b) \quad \textit{for all } a, b \in \mathcal{A}. \tag{16}$$

We know that d_1 is an additive mapping. So we have $d_1((b+c)a) = d_1(ba) + d_1(ca)$ for all $a, b, c \in \mathcal{A}$. This equation along with (16) imply that

$$\left[(b+c)^n - b^n - c^n\right] d_3(a) = 0, \quad \textit{for all } a, b, c \in \mathcal{A}. \tag{17}$$

Putting $b = c = \mathbf{e}$ in (17) and considering the assumption that $2^n - 2 \neq 0$, we infer that $d_3 = 0$ and it follows from (16) that so is d_1. Therefore, d_1, d_2 and d_3 are zero. Reasoning like above, we obtain the required result if we assume that $d_2(2\mathbf{e}) = 2d_2(\mathbf{e})$. Note, however, that in both above-mentioned cases, (d_1, d_2, d_3) is a ternary derivation. □

In the next theorem, we present a characterization of $\{n\}$-generalized derivations using some functional equations.

Theorem 21 *Let $\mathcal{A}$ be a unital algebra, let n be a positive integer, and let $d_1, d_2, d_3 : \mathcal{A} \to \mathcal{A}$ be mappings satisfying*

$$d_1(ab) = d_2(a)b^n + a^n d_3(b) = d_3(a)b^n + a^n d_2(b) \tag{18}$$

$$d_1(\lambda a) = \lambda^n d_1(a) \tag{19}$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. Furthermore, assume that $d_i(\mathbf{e})\left[a^n b^n - (ab)^n\right] = 0$ for all $a, b \in \mathcal{A}$ and $i \in \{2, 3\}$. Then there exists an $\{n\}$-derivation $\Delta : \mathcal{A} \to \mathcal{A}$ such that d_1, d_2 and d_3 are $\{n\}$-generalized derivations with the associated $\{n\}$-derivation Δ.

Proof Putting $b = \mathbf{e}$ in (18), we obtain

$$d_1(a) = d_2(a) + a^n d_3(\mathbf{e}) = d_3(a) + a^n d_2(\mathbf{e}), \tag{20}$$

and taking $a = \mathbf{e}$ in (18), we see that

$$d_1(b) = d_2(\mathbf{e})b^n + d_3(b) = d_3(\mathbf{e})b^n + d_2(b). \tag{21}$$

Comparing (20) and (21), we get that

$$d_i(\mathbf{e})a^n = a^n d_i(\mathbf{e}) \tag{22}$$

for all $a \in \mathcal{A}$ and $i \in \{1, 2, 3\}$. It follows from (20) and (22) that

$$d_3(a) = d_2(a) + \big(d_3(\mathbf{e}) - d_2(\mathbf{e})\big)a^n = d_2(a) + a^n\big(d_3(\mathbf{e}) - d_2(\mathbf{e})\big),$$

for all $a \in \mathcal{A}$. Using (20), we have

$$d_2(a)b^n + a^n d_3(b) = d_1(ab) = d_2(ab) + d_3(\mathbf{e})(ab)^n$$

and so

$$\begin{aligned} d_2(ab) &= d_2(a)b^n + a^n d_3(b) - d_3(\mathbf{e})(ab)^n \\ &= d_2(a)b^n + a^n\left[d_2(b) + (d_3(\mathbf{e}) - d_2(\mathbf{e}))b^n\right] - d_3(\mathbf{e})(ab)^n \\ &= d_2(a)b^n + a^n d_2(b) - d_2(\mathbf{e})a^n b^n \end{aligned}$$

We define $\Delta : \mathcal{A} \to \mathcal{A}$ by $\Delta(a) = d_2(a) - d_2(\mathbf{e})a^n$. So by (22) and the assumption that $d_i(\mathbf{e})\left[a^n b^n - (ab)^n\right] = 0$ for all $a, b \in \mathcal{A}$ and $i \in \{2, 3\}$, we have the following expressions:

$$\begin{aligned}\Delta(ab) &= d_2(ab) - d_2(\mathbf{e})(ab)^n \\ &= d_2(a)b^n + a^n d_2(b) - d_2(\mathbf{e})a^n b^n - d_2(\mathbf{e})(ab)^n \\ &= \left[d_2(a) - d_2(\mathbf{e})a^n\right]b^n + a^n\left[d_2(b) - d_2(\mathbf{e})b^n\right] \\ &= \Delta(a)b^n + a^n\Delta(b),\end{aligned}$$

which means that

$$\Delta(ab) = \Delta(a)b^n + a^n\Delta(b), \quad \textit{for all } a, b \in \mathcal{A}.$$

Our next task is to show that $\Delta(\lambda a) = \lambda^n\Delta(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Before that, we prove that $d_2(\lambda a) = \lambda^n d_2(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We know that $d_1(\lambda a) = \lambda^n d_1(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. So we have

$$d_1(\lambda ab) = \lambda^n d_1(ab) = \lambda^n d_2(a)b^n + \lambda^n a^n d_3(b)$$

and on the other hand

$$d_1(\lambda ab) = d_2(\lambda a)b^n + \lambda^n a^n d_3(b)$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. By comparing these two equations related to $d_1(\lambda ab)$, we deduce that $\lambda^n d_2(a)b^n = d_2(\lambda a)b^n$. Putting $b = \mathbf{e}$ in the previous equation, we get that $d_2(\lambda a) = \lambda^n d_2(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Consequently, $\Delta(\lambda a) = \lambda^n\Delta(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. So Δ is an $\{n\}$-derivation. Using this fact, we have

$$\begin{aligned}d_2(ab) &= \Delta(ab) + d_2(\mathbf{e})(ab)^n \\ &= \Delta(a)b^n + a^n\Delta(b) + d_2(\mathbf{e})a^n b^n \\ &= (\Delta(a) + d_2(\mathbf{e})a^n)b^n + a^n\Delta(b) \\ &= d_2(a)b^n + a^n\Delta(b) \\ &= \Delta(a)b^n + a^n d_2(b),\end{aligned}$$

which means that

$$d_2(ab) = d_2(a)b^n + a^n\Delta(b) = \Delta(a)b^n + a^n d_2(b), \quad \textit{for all } a, b \in \mathcal{A}.$$

So d_2 is an $\{n\}$-generalized derivation with the associated $\{n\}$derivation Δ. Using a similar argument, one can easily show that

$$d_3(ab) = d_3(a)b^n + a^n d_3(b) - d_3(\mathbf{e})(ab)^n, \quad \textit{for all } a, b \in \mathcal{A}.$$

By defining $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(a) = d_3(a) - d_3(\mathbf{e})a^n$ and by reasoning like the mapping d_2, it is observed that d_3 is an $\{n\}$-generalized derivation with the associated $\{n\}$-derivation δ. In the following, we show that $\delta = \Delta$. We know that $\Delta(a) = d_2(a) - d_2(\mathbf{e})a^n$ and it follows from (21) that $d_2(a) = d_3(a) + a^n d_2(\mathbf{e}) - d_3(\mathbf{e})a^n$ for all $a \in \mathcal{A}$. So we have

$$\begin{aligned}\Delta(a) &= d_2(a) - d_2(\mathbf{e})a^n \\ &= d_3(a) + a^n d_2(\mathbf{e}) - d_3(\mathbf{e})a^n - d_2(\mathbf{e})a^n \\ &= d_3(a) - d_3(\mathbf{e})a^n \\ &= \delta(a)\end{aligned}$$

for all $a \in \mathcal{A}$. Hence, both d_2 and d_3 are $\{n\}$-generalized derivations with the associated $\{n\}$-derivation Δ. We are now ready to show that d_1 is also an $\{n\}$-generalized derivation with the associated $\{n\}$-derivation Δ. We know that $d_1(a) = d_2(a) + d_3(\mathbf{e})a^n$ and $d_2(a) = \Delta(a) + d_2(\mathbf{e})a^n$ for all $a \in \mathcal{A}$. Hence, we have

$$d_1(a) = \Delta(a) + d_2(\mathbf{e})a^n + d_3(\mathbf{e})a^n = \Delta(a) + d_1(\mathbf{e})a^n,$$

which means that d_1 is an $\{n\}$-generalized derivation with the associated $\{n\}$-derivation Δ, as required. □

We conclude this paper with the following questions.

Question 22 Let $\mathcal{A}$ be an algebra or ring, let $n > 1$ be a positive integer, and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a mapping such that $\Delta(a^2) = \Delta(a)a^n + a^n\Delta(a)$ holds for all $a \in \mathcal{A}$. Under what conditions we have $\Delta(ab) = \Delta(a)b^n + a^n\Delta(b)$ for all $a, b \in \mathcal{A}$?

Question 23 Let $\mathcal{A}$ be a unital algebra or ring, let $n > 1$ be a positive integer, and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying

$$\Delta(a^m) = \Sigma_{k=1}^{m} a^{(k-1)n}\Delta(a)a^{(m-k)n}$$

in which $a^0 = \mathbf{e}$, for all $a \in \mathcal{A}$ and for some positive integer m. Under what conditions we have $\Delta(ab) = \Delta(a)b^n + a^n\Delta(b)$ for all $a, b \in \mathcal{A}$?

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