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Some applications of simultaneous continuous functional calculus

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Abstract
We define and study a continuous functional calculus for a commutative family of normal elements of a C^*-algebra. We obtain that this functional calculus is unique, continuous and satisfies the spectral mapping theorem. We also provide two applications. The first one concerns the existence of a particular orthonormal basis on a locally Hilbert space. The second for multi-dimensional continuous versions of N. Wiener and P. Lé vy theorems.

Keywords C^*-algebra · Normal element · Continuous function · Simultaneous continuous functional calculus · Weighted algebra · Wiener theorem · Lévy theorem

Mathematics subject classification 46J10 · 46H30

1 Preliminaries and introduction

Let $(A, \|.\|)$ be a C^*- algebra with unit e and involution $x \longmapsto x^*$. The *spectrum* of an element x of A will be denoted by $Sp_A x$. The *spectral radius* will be denoted by ρ_A that is, for every $a \in A$, $\rho_A(a) = \sup\left\{|z| : z \in Sp_A a\right\}$. An element h of A is called *hermitian* if $h^* = h$. The set of all hermitian elements of A will be denoted by $H(A)$. We say that the Banach algebra is *hermitian* if the spectrum of every element of $H(A)$ is real ([7]). An element x of A is called *normal* if $x^*x = xx^*$. The set of all normal elements of A will be denoted by $N(A)$. Let x be an element of A. We denote by $p_A(x)$ the square root of the spectral radius of the element x^*x, i.e., $p_A(x) = \rho_A(x^*x)^{\frac{1}{2}}$. This function $p_A$is know as Pták function. For scalars λ, we often write simply λ for the element λe of A. For a detailed account of the basic properties of C^*-algebras see [2]. Concerning basic properties of hermitian algebras see [7]. Throughout the paper, all considered algebras are complex and with unit e.

The continuous functional calculus for a single normal element of a C^*-algebra is well known ([2]). For a finite commutative family of normal bounded operators in a Hilbert space, the continuous operational calculus is given in [3]. The continuous functional

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calculus for a family of elements $(a_i)_{i\in I}$ of a C^*-algebra such that, for every $i \in I$, each a_i commutes with both $a_j$and a_j^*, for every $j \in I$, is given in [1]. Here we consider a simultaneous continuous functional calculus for a commutative family $(a_i)_{i\in I}$ of normal elements of a C^*-algebra. This functional calculus consists in giving a sense to $f(\mathbf{a})$ whenever $\mathbf{a} = (a_i)_{i\in I}$ is a commutative family of normal elements of a C^*-algebra $(A, \|.\|)$ and f is a continuous complex valued function on the simultaneous spectrum $Sp(\mathbf{a})$ of $\mathbf{a}$. We treat the fundamental properties of this functional calculus. So we show that this functional calculus is unique, continuous and satisfies the spectral mapping theorem. Once this functional calculus is defined and studied, the task is now to give some applications. We obtain a particular orthonormal basis on a locally Hilbert space. We also use weighted algebras analogues of the classical and famous theorems of N. Wiener and P. Lévy on absolutely Fourier series in order to obtain a multi-dimensional continuous versions of N. Wiener ([9]) and P. Lévy ([6]) theorems.

2 Simultaneous continuous functional calculus in C*-algebras

Let $(A, \|.\|)$ be a unital C^*-algebra and let $\mathbf{a} = (a_i)_{i\in I}$ be a commutative family of normal elements in A. By the analog of a result of Fuglede, Putnam and Rosenblum ([8], Theorem 12.16, p. 315), each a_i commutes with a_j^* as shown in the following result:

Lemma 2.1 Let $(A, \|.\|)$ *be a unital* C^**-algebra,* x *and* y *are normal elements of* A*. If* $z \in A$ *satisfies* $zx = yz$*. Then* $y^*z = zx^*$*. Hence any element that commutes with a normal element also commutes with its adjoint.*

Proof Observe first that if a is an element of A, then $c = e^{i(a+a^*)}$ is a unitary element of A and so $\|c\| = 1$. Let x and y be normal elements of A and $z \in A$ satisfies $zx = yz$. By induction on n, one has $zx^n = y^n z$, for every $n \in \mathbb{N}^*$. Whence $ze^x = e^y z$. And therefore, $z = e^{-y} z e^x$. As the lemma hypothesis is also true for $\overline{\lambda} iy$ and $\overline{\lambda} ix$, one has $z = e^{-i\overline{\lambda} y} z e^{i\overline{\lambda} x}$. Now let $\varphi \in A'$ be arbitrary and consider the function f defined by:

$$f(\lambda) = \varphi\left[\left(e^{-i\lambda y^*} z e^{-i\lambda x^*}\right)\right], \text{ for every } \lambda \in \mathbb{C}.$$

Then

$$f(\lambda) = \varphi\left(e^{-i\left(\lambda y^* + \overline{\lambda} y\right)} z e^{i\left(\overline{\lambda} x - \lambda x^*\right)}\right).$$

Since the last exponentials are unitary elements of A, one has

$$|f(\lambda)| \leq \|\varphi\| \|z\|, \text{ for every } \lambda \in \mathbb{C}.$$

Thus f is a bounded entire function. By Liouville's theorem f is a constant, so its derivative is zero. But this derivative is $i\varphi(y^*z - zx^*)$. Since $\varphi \in A'$ was arbitrary, the Hahn-Banach theorem now gives the desired conclusion. □

Let $\mathbf{a} = (a_i)_{i\in I}$ be a commutative family of normal elements of a unital C^*-algebra $(A, \|.\|)$, i.e., for every $i \in I$, a_i is normal and $a_i a_j = a_j a_i$, for every $i,\ j \in I$. Then, by lemma 2.1, the closed subalgebra B generated by $\mathbf{a}$ and e is a unital commutative C^*-algebra. The construction of the simultaneous continuous functional calculus goes along the lines of [1]. To make the paper self-contained, we present the fundamental properties of this calculus. For $\mathbf{a} = \left(a_i\right)_{i\in I}$ a commutative family of normal elements of A, let $\widehat{\mathbf{a}}$ denote the generalized Gelfand transformation defined by:

$$\widehat{\mathbf{a}}(\chi) = \left(\chi\left(a_i\right)\right)_{i\in I} \in \mathbb{C}^I, \text{ for every } \chi \in Sp(B),$$

where $Sp(B)$ denotes the *Gelfand spectrum* of B, that is the set of non-zero characters of B. Then $\widehat{\mathbf{a}} : Sp(B) \longrightarrow \mathbb{C}^I$ is continuous and injective. The image $\widehat{\mathbf{a}}(Sp(B)) \subset \mathbb{C}^I$ is therefore a nonempty compact of $\mathbb{C}^I$, which is homeomorphic to $Sp(B)$, is called the *simultaneous spectrum* of $\mathbf{a}$ and denoted by $Sp(\mathbf{a})$. One has $Sp(\mathbf{a}) \subset \prod_{i\in I} Sp_A \mathbf{a}_i$ and in general $Sp(\mathbf{a}) \neq \prod_{i\in I} Sp_A \mathbf{a}_i$. In the case where $\mathbf{a} = a \in A$, with a a normal element of A, we have $Sp(\mathbf{a}) = Sp_A a \subset \mathbb{C}$. According to the above, $Sp(\mathbf{a})$ is homeomorphic to $Sp(B)$. This induces an isomorphism:

$$\theta : \mathcal{C}(Sp(\mathbf{a})) \longrightarrow \mathcal{C}(Sp(B)) : \theta(f) = f \circ \widehat{\mathbf{a}}$$

and Gelfand's transformation

$$\mathcal{G} : B \to \mathcal{C}(Sp(B))$$

is also a $*$-morphism of the same type. Now consider the composite isomorphism

$$\mathcal{G}^{-1} \circ \theta : \mathcal{C}(Sp(\mathbf{a})) \longrightarrow B$$

and composing this with the canonical injection of B into A, we obtain a morphism $\Phi_{\mathbf{a}} : \mathcal{C}(Sp(\mathbf{a})) \longrightarrow A$ which is defined by the equality:

$$\widehat{\Phi_{\mathbf{a}}(f)} = f \circ \widehat{\mathbf{a}}, \text{ for every } f \in \mathcal{C}(Sp(\mathbf{a})) \tag{1}$$

and more precisely by:

$$\chi\left(\Phi_{\mathbf{a}}(f)\right) = f(\chi(\mathbf{a})), \text{ for every } \chi \in Sp(B),$$

where $\chi(\mathbf{a}) = \left(\chi\left(a_i\right)\right)_i$. In particular if $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp(\mathbf{a})$, we obtain $\Phi_{\mathbf{a}}\left(\mathbf{z}_i\right) = a_i$ and $\Phi_{\mathbf{a}}\left(\overline{\mathbf{z}_i}\right) = a_i^*$. Moreover, since each morphism of $\mathcal{C}(Sp(\mathbf{a}))$ into A is continuous and $\Phi_{\mathbf{a}}$ is known on all the polynomials P in z_i and $\overline{z_i}$, this implies the uniqueness of $\Phi_{\mathbf{a}}$ by the theorem of Stone-Weiertrass. Thus we obtain a continuous functional calculus, for a commutative family of normal elements of a unital C^*-algebra, which extends the one known for a finite commutative family of bounded normal operators ([3]).

Theorem 2.2 (Simultaneous continuous functional calculus). *Let* $(A, \|.\|)$ *be a unital* C^**-algebra,* $\mathbf{a} = \left(a_i\right)_{i\in I}$ *is a commutative family of normal elements of* A *and* $\mathcal{C}(Sp(\mathbf{a}))$ *is the* C^**-algebra of continuous complex-valued functions on* $Sp(\mathbf{a})$. *Then there is a unique unitary* $*$*-morphism* $\Phi_{\mathbf{a}}$ *of* $\mathcal{C}(Sp(\mathbf{a}))$ *into* A *such that*:

$$\Phi_{\mathbf{a}}\left(\mathbf{z}_i\right) = a_i, \textit{ for every } i \in I,$$

where $\mathbf{z}_i$ *denotes the function* $z \longmapsto z_i$ *on* $Sp(\mathbf{a})$. *Moreover, this* $*$*-morphism is isometric and its image* $\Phi_{\mathbf{a}}(\mathcal{C}(Sp(\mathbf{a})))$ *is the full subalgebra of* A *generated by* $\left(a_i\right)_{i\in I}$*, that is the sub-*$C^*$*-algebra of* A *generated by* e *and* $\left(a_i\right)_{i\in I}$ *and therefore consists entirely of normal elements.*

Proof It remains to show that $\Phi_{\mathbf{a}}$ is an $*$-isometry. For every $f \in \mathcal{C}(Sp(\mathbf{a}))$, one has:

$$\begin{aligned}\|f\| &= \sup\left\{\left|f(\chi\left(\left(a_i\right)\right)_{i\in I})\right| : \chi \in Sp(B)\right\}\\ &= \sup\left\{\left|f\circ\hat{\mathbf{a}}(\chi))\right| : \chi \in Sp(B)\right\}\\ &= \sup\left\{\left|\widehat{\Phi_{\mathbf{a}}(f)}(\chi))\right| : \chi \in Sp(B)\right\}\\ &= \left\|\widehat{\Phi_{\mathbf{a}}(f)}\right\|_\infty = \left\|\Phi_{\mathbf{a}}(f)\right\|.\end{aligned}$$

So

$$\left\|\Phi_{\mathbf{a}}(f)\right\| = \|f\|, \text{ for every } f \in \mathcal{C}(Sp(\mathbf{a})).$$

Now, for every polynomial P in variables z_i and $\overline{z_i}$, we have: $\Phi_{\mathbf{a}}(P) = P\left(a_i, a_i^*\right)$. So $\Phi_{\mathbf{a}}(P)^* = \Phi_{\mathbf{a}}\left(\overline{P}\right)$. The continuity of $\Phi_{\mathbf{a}}$ and the theorem of Stone-Weierstrass show that:

$$\left\|\Phi_{\mathbf{a}}(f)^*\right\| = \left\|\overline{f}\right\|, \text{ for every } f \in \mathcal{C}(Sp(\mathbf{a})).$$

□

Remark 2.3 For any polynomial P in variables z_i and $\overline{z_i}$, one has $\Phi_{\mathbf{a}}(P) = P\left(a_i, a_i^*\right)$. As in the classical case, we put $\Phi_{\mathbf{a}}(f) = f(\mathbf{a})$ which respects the multiplicative symbolism: $fg(\mathbf{a}) = f(\mathbf{a})g(\mathbf{a})$ as well as equality: $\overline{f}(\mathbf{a}) = f(\mathbf{a})^*$. So, for every $f \in \mathcal{C}(Sp(\mathbf{a}))$, we also have:

$$\|f(\mathbf{a})\| = \|f\| \text{ and } \|f(\mathbf{a})^*\| = \left\|\overline{f}\right\|.$$

As an immediate consequence of (1), we have the following result:

Corollary 2.4 (Spectral mapping theorem) *Let* A, $\mathbf{a} = \left(a_i\right)_{i\in I}$ *and* $f \in \mathcal{C}(Sp(\mathbf{a}))$ *be as described in Theorem* 2.2. *Then one has*:

$$Sp(f(\mathbf{a})) = f(Sp(\mathbf{a})).$$

In the case of a family of functions $\left(f_j\right)_{j\in J}$, *with* $f_j \in \mathcal{C}(Sp(\mathbf{a}))$, $j \in J$, *where the notation* $\mathbf{b} = \left(f_j(\mathbf{a})\right)_{j\in J}$ *with* $Sp(\mathbf{b}) \subset \mathbb{C}^J$, *put* $\mathbf{f} = \left(f_j\right)_{j\in J} \in \mathcal{C}\left(Sp(\mathbf{a}), \mathbb{C}^J\right)$. *Then one has:*

Corollary 2.5 (Simultaneous spectral mapping theorem). *Let* A *and* $\mathbf{a} = \left(a_i\right)_{i\in I}$ *be as described in Theorem* 2.2 *and* $\mathbf{f} = \left(f_j\right)_{j\in J} \in C\left(Sp(\mathbf{a}), \mathbb{C}^J\right)\mathbf{f}(\mathbf{a}) = \left(f_j(\mathbf{a})\right)_{j\in J}$. *One has:*

$$Sp(\mathbf{f}(\mathbf{a})) = \mathbf{f}(Sp(\mathbf{a})).$$

Corollary 2.6 *Let* A *and* $\mathbf{a} = \left(a_i\right)_{i\in I}$ *be as described in Theorem* 2.2 *and* Put $\mathbf{f} = \left(f_j\right)_{j\in J} \in \mathcal{C}\left(Sp(\mathbf{a}), \mathbb{C}^J\right)$. $\mathbf{b} = \mathbf{f}(\mathbf{a}) = \left(f_j(\mathbf{a})\right)_{j\in J}$. *Then, for each* $g \in \mathcal{C}(Sp(\mathbf{b}))$, *one has:*

$$g(\mathbf{b}) = g(\mathbf{f}(\mathbf{a})) = (g\circ\mathbf{f})(\mathbf{a}).$$

Proof We have $Sp(\mathbf{b}) = \mathbf{f}(Sp(\mathbf{a})) \subset \mathbb{C}^J$ so $g\circ\mathbf{f}$ is well defined and $g\circ\mathbf{f} \in \mathcal{C}(Sp(\mathbf{a}))$. The map $\Psi : g \longmapsto (g\circ\mathbf{f})(\mathbf{a})$ is unitary $*$-morphism of $\mathcal{C}(Sp(\mathbf{b}))$ into A such that

$$\Psi\left(\mathbf{z}_j\right) = f_j(\mathbf{a}) = b_j \text{ , where } \mathbf{z}_j : z \longmapsto z_j$$

is the transition to the j-th coordinate of $\mathbb{C}^J$ in $\mathbb{C}$. With the uniqueness part of the Theorem 2.2, we obtain $\Psi(g) = g(\mathbf{b}) = g(\mathbf{f}(\mathbf{a}))$. The image of $\Phi_{\mathbf{a}}$ is $B \equiv \mathcal{F}(\mathbf{a})$, the commutative C^*-subalgebra of A generated by e and $(a_i)_{i\in I}$. □

For a finite commutative family of normal elements of a unital C^* -algebra, one has:

Corollary 2.7 (Simultaneous continuous functional calculus for a finite commutative family of normal elements). *Let* $(A, \|.\|)$ *be a unital* C^*-algebra, $\mathbf{a} = (a_1, ..., a_n)$ *a commutative family of normal elements of* A, *and let* B *be the* C^**-algebra with unit generated by* $a_1, ..., a_n$ *and* e.*Then there exists one and only one unitary* $*$*-morphism* $\Phi_{\mathbf{a}}$ *of* $\mathcal{C}(Sp_B(\mathbf{a}))$ *into* A *where*

$$\Phi_{\mathbf{a}}(\mathbf{z}_i) = a_i, \textit{for every } i = 1 \textit{ to } n,$$

$\mathbf{z}_i$ *is the i-th coordinate function on* $Sp_B(\mathbf{a})$ *that is* $\mathbf{z}_i(\lambda_1, ..., \lambda_n) = \lambda_i$, *for every* $i = 1$ *to* n. *Furthermore* $\Phi_{\mathbf{a}}$ *is an isomorphism of* $\mathcal{C}(Sp_B(\mathbf{a}))$ *onto* B.

3 Some applications of simultaneous continuous functional calculus

In this section, we give some applications of the simultaneous continuous functional calculus as explored in the preceding section. Its applications concern the existence of an orthonormal basis on a locally Hilbert space H , consisting of eigenvectors of commuting normal operators acting in H. The applications also concern multi-dimensional generalization of N. Wiener ([9]) and P. Lévy ([6]) theorems.

3.1 The first application

Let Λ be a directed index set and $(H_\lambda)_{\lambda\in\Lambda}$ a family of Hilbert spaces such that

$$H_\lambda \subset H_\upsilon \text{ and } \langle\ ,\ \rangle_\lambda = \langle\ ,\ \rangle_{\upsilon|H_\lambda},\ \forall\lambda \le \upsilon, \text{ in } \Lambda,$$

and $\langle\ ,\ \rangle_\lambda$ denotes the inner product on H_λ, $\lambda \in \Lambda$.Let

$$H = \underrightarrow{\lim} H_\lambda = \bigcup_{\lambda\in\Lambda} H_\lambda$$

be the *locally convex Hilbert* space associated to $(H_\lambda)_{\lambda\in\Lambda}$ ([5], Definition 5.2). Let $\mathcal{L}(H_\lambda)$ be the C^*-algebra of all bounded linear operators on the Hilbert space H_λ, $\lambda \in \Lambda$. Then

$$\mathcal{L}(H) = \underleftarrow{\lim} \mathcal{L}(H_\lambda),\ \lambda \in \Lambda.$$

Then one has:

Theorem 3.1 *Let* $H = \underrightarrow{\lim} H_\lambda$, $\lambda \in \Lambda$, *be a locally Hilbert space whose associated Hilbert spaces* H_λ , $\lambda \in \Lambda$, *have finite dimensions and* $\mathbf{T} = (T_i)_{i\in I}$ *is a commutative family of normal operators on H. For every,* let $i \in IT_{i,\lambda} \in \mathcal{L}(H_\lambda)$, $\lambda \in \Lambda$, *such that . Then there exists* *an orthonormal basis of* $T_i = \varinjlim T_{i,\lambda}\, H$*, whose elements are eigenvectors of the operators* $T_{i,\lambda}$.

Proof Let $\mathbf{T}_\lambda = \left(T_{i,\lambda}\right)_{i\in I}$. Then $Sp(\mathbf{T}_\lambda)$ is finite, say

$$Sp(\mathbf{T}_\lambda) = \left\{\alpha_1^{(\lambda)}, ..., \alpha_{p_\lambda}^{(\lambda)}\right\},$$

with $\alpha_k^{(\lambda)} = \left(\alpha_{i,k}^{(\lambda)}\right)_{i\in I}$, for every $1 \le k \le p_\lambda$. As $1_{\left\{\alpha_k^{(\lambda)}\right\}} \in \mathcal{C}\left(Sp\left(\mathbf{T}_\lambda\right)\right)$ and since $1_{Sp(\mathbf{T}_\lambda)} = \sum_{k=1}^{p_\lambda} 1_{\left\{\alpha_k^{(\lambda)}\right\}}$, the simultaneous continuous calculus gives

$$Id_\lambda = \Phi_{\mathbf{T}_\lambda}\left(1_{Sp(\mathbf{T}_\lambda)}\right) = \sum_{k=1}^{p_\lambda} P_k^{(\lambda)},$$

where $P_k^{(\lambda)} = \Phi_{\mathbf{T}_\lambda}\left(1_{\left\{\alpha_k^{(\lambda)}\right\}}\right)$ is a hermitian projector of H_λ, for $k = 1, ..., p_\lambda$. Moreover, one has

$$T_{i,\lambda} = \mathbf{z}_i\left(\mathbf{T}_\lambda\right) = \sum_k \alpha_{i,k}^{(\lambda)} P_k^{(\lambda)}.$$

It follows that the restriction of $T_{i,\lambda}$ to $H_k^{(\lambda)} = P_k^{(\lambda)}\left(H_\lambda\right)$ is a homothety with ratio $\alpha_{i,k}^{(\lambda)}$. For $k = 1, ..., p_\lambda$, let $B_k^{(\lambda)}$ be a basis of $H_k^{(\lambda)}$. Then $B^{(\lambda)} = B_1^{(\lambda)} \cup B_1^{(\lambda)} \cup ... \cup B_{p_\lambda}^{(\lambda)}$ is an orthonormal basis of H_λ, whose elements are eigenvectors of the operators $T_{i,\lambda}$. Finally

$$B = \varinjlim B^{(\lambda)} = \bigcup_{\lambda\in\Lambda} B^{(\lambda)}$$

is the desired orthonormal basis of H. □

3.2 Multi-dimensional generalization of P. Lévy theorem

Let $p \in]1, +\infty[$, and $\omega : \mathbb{Z}^k \longrightarrow [1, +\infty[$, $k \in \mathbb{N}^*$ be a map on $\mathbb{Z}^k$. We say that ω is a weight on $\mathbb{Z}^k$ if:

$$c(\omega) = \sum_{n\in\mathbb{Z}^k} \omega(n)^{\frac{1}{1-p}} < +\infty. \tag{2}$$

We write $l_\omega^p\left(\mathbb{Z}^k\right)$ for the space of all sequences $\left(a_n\right)_{n\in\mathbb{Z}}$ with $a_n \in \mathbb{C}$ and

$$\sum_{n\in\mathbb{Z}^k} \left|a_n\right|^p \omega(n) < +\infty.$$

A weight ω on $\mathbb{Z}^k$ is said to be m-convolutive if there exists $\gamma = \gamma(\omega) > 0$ such that:

$$\omega^{\frac{1}{1-p}} * \omega^{\frac{1}{1-p}} \le \gamma\omega^{\frac{1}{1-p}}, \tag{3}$$

where $*$ denotes the convolution product, and it is said to be regular if:

$$\lim_{|m|\longrightarrow+\infty}\omega(m)^{\frac{1}{m_j}}=1,\ \text{for every}\ j=1,\dots,k \tag{4}$$

where $|m|=|m_1|+\cdots+|m_k|$ denotes the length of $m=(m_1,\dots,m_k)\in\mathbb{Z}^k$. For $m=(m_1,\dots,m_k)\in\mathbb{Z}^k$ and $t=(t_1,\dots,t_k)\in\mathbb{R}^k$, we will use the notation $(m,t)=m_1t_1+\cdots+m_kt_k$. We consider the following weighted space:

$$\mathcal{A}_k^p(\omega)=\left\{f:\mathbb{R}^k\longrightarrow\mathbb{C}:f(t)=\sum_{n\in\mathbb{Z}^k}a_ne^{i(n,t)}:\ (a_n)_{n\in\mathbb{Z}^k}\in l_\omega^p(\mathbb{Z}^k)\right\}.$$

The space $\mathcal{A}_k^p(\omega)$ endowed with the norm $\|.\|_{p,\omega}$ defined by:

$$\|f\|_{p,\omega}=\left(\sum_{n\in\mathbb{Z}}|a_n|^p\omega(n)\right)^{\frac{1}{p}},\ \text{for every}\ f\in\mathcal{A}_k^p(\omega),$$

and with the classical pointwise multiplication is a commutative semi-simple Banach algebra ([4]) with unit element $\hat{e}$ given by $\hat{e}(t)=1,(t\in\mathbb{R})$. We consider, in the algebra $\mathcal{A}_k^p(\omega)$, the involution $f\longmapsto f^*$ defined by:

$$f^*(t)=\sum_{n\in\mathbb{Z}}\overline{a_{-n}}e^{int},\ \text{for every}\ f\in\mathcal{A}_k^p(\omega).$$

This involution is continuous for the algebra is semi-simple. For a regular weight function ω on $\mathbb{Z}^k$ satisfying

$$\omega(n+m)\le\omega(n)\omega(m),\ \text{for every}\ n,m\in\mathbb{Z}^k \tag{5}$$

every character of $\mathcal{A}_k^p(\omega)$ is an evaluation at some point $t_0\in\mathbb{R}$ ([4]). Moreover $(\mathcal{A}_k^p(\omega),\|.\|_{p,\omega})$ is a hermitian algebra. As $\mathcal{A}_k^p(\omega)$ is commutative, the Ptàk function $p_{\mathcal{A}_k^p(\omega)}$, of the algebra $\mathcal{A}_k^p(\omega)$, given by:

$$p_{\mathcal{A}_k^p(\omega)}(f)=\rho_{\mathcal{A}_k^p(\omega)}(f^*f)^{1/2},\ \text{for every}\ f\in\mathcal{A}_k^p(\omega),$$

is an algebra norm, on $\mathcal{A}_k^p(\omega)$, such that

$$\rho_{\mathcal{A}_k^p(\omega)}(f^*f)=p_{\mathcal{A}_k^p(\omega)}(f)^2.$$

So, for every $f\in\mathcal{A}_k^p(\omega)$, one has:

$$p_{\mathcal{A}_k^p(\omega)}(f)=\rho_{\mathcal{A}_k^p(\omega)}(f)=\sup_{t\in\mathbb{R}^k}|f(t)|.$$

Now consider the completion $\widehat{\mathcal{A}}_k^p(\omega)$ of the algebra $\left(\mathcal{A}_k^p(\omega),p_{\mathcal{A}_k^p(\omega)}\right)$. With the norm $\|.\|_\infty$ defined by:

$$\|f\|_\infty=\sup_{t\in\mathbb{R}^k}|f(t)|,f\in\widehat{\mathcal{A}}_k^p(\omega),$$

the algebra $\left(\widehat{\mathcal{A}}_k^p(\omega),\|.\|_\infty\right)$ is commutative, semi-simple and with unit element $\hat{e}$ given by $\hat{e}(t)=1\ (t\in\mathbb{R})$. Moreover, it is not difficult to show that, every character of the algebra $\widehat{\mathcal{A}}_k^p(\omega)$ is an evaluation at some $t^0\in\mathbb{R}^k$, where $t^0=(t_1^0,...,t_k^0)$ with $0\le t_j^0<2\pi$, for every $j=1,...,k$, and so, for every $f\in\widehat{\mathcal{A}}_k^p(\omega)$, one has:

$$Sp(f) = \{f(t) \,:\, t \in [0, 2\pi[^k\}.$$

It follows that, for every family $\mathbf{f} = (f_j)_{j\in I} : \mathbb{R}^k \longrightarrow \mathbb{C}^I$ of elements of $\widehat{\mathcal{A}}_k^p(\omega)$, one has:

$$\mathrm{Im}\mathbf{f} = \prod_{j\in I} \{f_j(t) \,:\, t \in [0, 2\pi[^k\} = \prod_{j\in I} Sp_{\widehat{\mathcal{A}}_k^p(\omega)}(f_j) \supset Sp(\mathbf{f}).$$

Now let $f \in \widehat{\mathcal{A}}_k^p(\omega)$ and $(f_m)_m$ be a sequence of elements of $\mathcal{A}_k^p(\omega)$, which uniformly converges to f. Then, for every m, there exists $(a_{m,n})_{n\in\mathbb{Z}} \in l_\omega^p(\mathbb{Z}^k)$ such that

$$f_m(t) = \sum_{n\in\mathbb{Z}} a_{m,n} e^{i(n,t)}.$$

We consider, in the algebra $\widehat{\mathcal{A}}_k^p(\omega)$, the algebra involution $f \longmapsto f^*$ defined by:

$$f^*(t) = \lim_{m\longrightarrow+\infty} \sum_{n\in\mathbb{Z}} \overline{a_{m,-n}} e^{i(n,t)}, \text{ for every } f \in \widehat{\mathcal{A}}_k^p(\omega).$$

With this involution, the algebra $\left(\widehat{\mathcal{A}}_k^p(\omega), \|.\|_\infty\right)$ becomes a C^*-algebra.

Theorem 3.2 (Multi-dimensional continuous version of P. Lévy theorem) *Let* $p \in]1, +\infty[$ *and* ω *be an m-convolutive and regular weight on* $\mathbb{Z}^k$ *satisfying* (5). *Let* $f(t) = (f_j(t))_{j\in I} : \mathbb{R}^k \longrightarrow \mathbb{C}^I$ *be a family of functions. Suppose that, for* $j \in I$, *there exists a sequence of periodic functions*

$$F_{m,j}(t) = \sum_{n\in\mathbb{Z}^k} a_{m,n,j} e^{i(n,t)}$$

such that:

(i) *For every* m,

$$\left(\sum_{n\in\mathbb{Z}^k} \left|a_{m,n,j}\right|^p \omega(n)\right)^{\frac{1}{p}} < +\infty, \textit{for every } j \in I.$$

(ii) *For every* $j \in I$,

$$\sup_{t\in\mathbb{R}^k} \left|F_{m,j}(t) - f_j(t)\right| \underset{m\longrightarrow+\infty}{\longrightarrow} 0.$$

Let Φ *be a continuous function on the image of the function f. Then, there exists a sequence of periodic functions*

$$G_m(t) = \sum_{n\in\mathbb{Z}^k} b_{m,n} e^{i(n,t)}$$

such that:

$$\left(\sum_{n\in\mathbb{Z}} |b_{m,n}|^p \omega(n)\right)^{\frac{1}{p}} < +\infty$$

and

$$\sup_{t\in\mathbb{R}^k}\left|G_m(t)-\Phi\circ f(t)\right| \underset{m\longrightarrow+\infty}{\longrightarrow} 0.$$

Proof Observe first that by (i) and (ii), $f_j \in \widehat{\mathcal{A}}_k^p(\omega)$, for every $j \in I$. Moreover $\Phi \in \mathcal{C}(Sp(f))$. Then, by Theorem 2.2, $\Phi(f) \in \widehat{\mathcal{A}}_k^p(\omega)$. So, by (1),

$$\widehat{\Phi(f)} = \Phi\circ\hat{f}.$$

It follows that, for every $t \in [0, 2\pi[^k$, one has:

$$\Phi(f)(t) = \chi_t\left[\Phi(f)\right] = \Phi\circ\hat{f}\left(\chi_t\right) = \Phi\circ f(t),$$

where χ_t is the evaluation at t. Whence $\Phi(f) = \Phi\circ f$, and the theorem follows. □

As a consequence, we have:

Theorem 3.3 (Multi-dimensional generalization of N. Wiener theorem). *Let* $p \in]1, +\infty[$ *and* ω *be an m-convolutive and regular weight on* $\mathbb{Z}^k$ *satisfying* (5). *Let be a family of functions. Suppose that, for* $f(t) = \left(f_j(t)\right)_{j\in I} : \mathbb{R}^k \longrightarrow \mathbb{C}^I$ $j \in I$, *there exists a sequence of periodic functions*

$$F_{m,j}(t) = \sum_{n\in\mathbb{Z}^k} a_{m,n,j}e^{i(n,t)}$$

such that:

(i) *For every m,*

$$\left(\sum_{n\in\mathbb{Z}^k}\left|a_{m,n,j}\right|^p\omega(n)\right)^{\frac{1}{p}} < +\infty, \textit{for every } j \in I.$$

(ii) *For every* $j \in I$,

$$\sup_{t\in\mathbb{R}^k}\left|F_{m,j}(t)-f_j(t)\right| \underset{m\longrightarrow+\infty}{\longrightarrow} 0.$$

If $f(t) \neq 0$, *for every* $t \in \mathbb{R}^k$, *then there exists a sequence of periodic functions*

$$G_m(t) = \sum_{n\in\mathbb{Z}^k} b_{m,n}e^{i(n,t)}$$

such that:

$$\left(\sum_{n\in\mathbb{Z}}\left|b_{m,n}\right|^p\omega(n)\right)^{\frac{1}{p}} < +\infty$$

and

$$\sup_{t\in\mathbb{R}^k}\left|G_m(t)-f^{-1}(t)\right| \underset{m\longrightarrow+\infty}{\longrightarrow} 0.$$

Declarations

Conflict of interest On behalf of all authors, there is no conflict of interest.

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