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Maps preserving the local spectral subspace of product or Jordan triple product of operators

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Abstract

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on an infinite-dimensional Banach space X. Let λ_0 be a fixed complex scalar. Let $X_T(\{\lambda_0\})$ denote the local spectral subspace of an operator $T \in \mathcal{L}(X)$ associated with $\{\lambda_0\}$. The purpose of this paper is to characterize maps ϕ on $\mathcal{L}(X)$ that satisfy

$$X_{\phi(T)\phi(S)}(\{\lambda_0\}) = X_{TS}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

We also characterize maps ϕ on $\mathcal{L}(X)$ for which the range contains all operators of rank at most four and

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

Keywords Nonlinear preservers · Local spectral subspace · Rank one operators

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1 Introduction

Let X be an infinite-dimensional complex Banach space. We denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators on X. The local spectrum, $\sigma_T(x)$, of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$ is the complement in $\mathbb{C}$ of the union of all open subsets U of $\mathbb{C}$ for which there exists an analytic function $f : U \to X$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. For a subset F of $\mathbb{C}$, let $X_T(F)$ denote the local spectral subspace of T associated with F defined by

$$X_T(F) = \{x \in X \ : \ \sigma_T(x) \subset F\}.$$

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Note that for two subsets F and G in $\mathbb{C}$, we have $X_T(F \cap G) = X_T(F) \cap X_T(G)$. We say that an operator $T \in \mathcal{L}(X)$ has the single-valued extension property (i.e. SVEP), if for every open set $U \subset \mathbb{C}$, the only analytic solution $f : U \to X$ satisfying the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, is the zero function on U. It is well known that T has SVEP if and only if $X_T(\emptyset) = \{0\}$. We also mention that any operator of finite rank has SVEP. For more details, we refer the reader to the remarkable books [1, 10].

The problem of characterizing linear or additive maps on $\mathcal{L}(X)$ preserving local spectra was initiated in [6]. For details on this subject, we refer, for instance, to the last section of survey article [5], and the references therein.

Recently, some preserver problems concerning the local spectral subspace have been considered; see [2–4, 8, 9]. In [3] the authors studied surjective maps on $\mathcal{L}(X)$ that preserve the local spectral subspace of either products of two operators or Jordan triple products of two operators associated with non-fixed singletons. Namely, they determined the forms of all surjective maps ϕ on $\mathcal{L}(X)$ that satisfy either

$$X_{\phi(T)\phi(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}) \text{ for all } S, T \in \mathcal{L}(X) \text{ and } \lambda \in \mathbb{C},$$

or

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda\}) = X_{TST}(\{\lambda\}) \text{ for all } T, S \in \mathcal{L}(X) \text{ and } \lambda \in \mathbb{C}.$$

These results have been extended in [4] to generalized product of operators by describing surjective maps that preserve the local spectral subspace of generalized product of operators associated with non-fixed singletons. Motivated by the above results, we continue in this paper the study given in [9] for maps (not necessarily surjective) on $\mathcal{L}(X)$ that preserve the local spectral subspace of either products of two operators or Jordan triple products of two operators associated with a fixed singleton. More precisely, for a fixed complex scalar λ_0, we give a characterization of all maps ϕ on $\mathcal{L}(X)$ (not necessarily surjective) that satisfy

$$X_{\phi(T)\phi(S)}(\{\lambda_0\}) = X_{TS}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

The characterization is considered under the assumption that the range of ϕ contains all operators of rank at most two for the case when $\lambda_0 = 0$. Furthermore, we determine the forms of all maps ϕ on $\mathcal{L}(X)$ for which the range contains all operators of rank at most four and

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

2 Preliminaries

This section gathers the necessary tools that are needed afterwards. The first lemma collects some basic properties of the local spectral subspace of an operator $T \in \mathcal{L}(X)$ associated with a complex singleton $\{\lambda\}$.

Lemma 2.1 *Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold.*

(i) $X_{T-\lambda}(\{0\}) = X_T(\{\lambda\})$.
(ii) $X_{\mu T}(\{\mu\lambda\}) = X_T(\{\lambda\})$ *for every nonzero complex scalar μ.*

(iii) $\ker(T-\lambda) \subset X_T(\{\lambda\})$.

Proof See for example [1, Theorem 2.6] or [10, Proposition 1.2.16]. □

Lemma 2.2 *Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. If $X_T(\{\lambda\}) = \operatorname{span}\{x\}$ for some $x \in X$, then $Tx = \lambda x$, where $\operatorname{span}\{x\}$ denotes the subspace of X spanned by x.*

Proof See [9, Lemma 2.2]. □

For a nonzero vector x in X and a nonzero linear functional f in the topological dual X^* of X, let $x \otimes f$ stand for the operator of rank one defined by $(x \otimes f)(z) = f(z)x$ for $z \in X$. Note that $x \otimes f$ is nilpotent if and only if $f(x) = 0$, and $x \otimes f$ is idempotent if and only if $f(x) = 1$. We denote by $\mathcal{N}_1(X)$ and $\mathcal{P}_1(X)$ the set of all rank one nilpotent operators and the set of all rank one idempotent operators in $\mathcal{L}(X)$, respectively. For an integer $n \geq 1$, let $\mathcal{F}_n(X)$ denote the set of all operators of rank at most n. We can easily check the following lemma.

Lemma 2.3 *Let $x \in X$ and $f \in X^*$. Let λ_0 be a nonzero complex scalar. We have*

(i) $f(x) \neq \lambda_0$ if and only if $X_{x\otimes f}(\{\lambda_0\}) = \{0\}$.
(ii) $f(x) = \lambda_0$ if and only if $X_{x\otimes f}(\{\lambda_0\}) = \operatorname{span}\{x\}$.
(iii) $f(x) \neq 0$ if and only if $X_{x\otimes f}(\{0\}) = \ker f$.
(iv) $f(x) = 0$ if and only if $X_{x\otimes f}(\{0\}) = X$.

In the following two sections, we need the next essential lemmas.

Lemma 2.4 *Let $F \in \mathcal{L}(X)$. Let λ_0 be a nonzero complex scalar. Suppose one of the following statements holds :*

(i) $\operatorname{codim} X_{TF}(\{0\}) \leq 1$ *for all* $T \in \mathcal{F}_2(X)$.
(ii) $\operatorname{codim} X_{TFT}(\{0\}) \leq 1$ *for all* $T \in \mathcal{F}_4(X)$.
(iii) $\dim X_{TFT}(\{\lambda_0\}) \leq 1$ *for all* $T \in \mathcal{F}_4(X)$.

Then rank $F \leq 1$.

Proof Suppose that rank $F \geq 2$. Let y_1 and y_2 be two vectors linearly independent in the range of F. Let x_1 and x_2 be two vectors in X such that $y_1 = Fx_1$ and $y_2 = Fx_2$.

First, let us show that $\operatorname{codim} X_{TF}(\{0\}) \geq 2$ for some $T \in \mathcal{F}_2(X)$. Let f_1 and f_2 two linear functionals in X^* such that $f_1(y_2) = 0$, $f_1(y_1) = 1$, $f_2(y_1) = 0$ and $f_2(y_2) = 1$. Pick $T \in \mathcal{F}_2(X)$ defined by

$$T = x_1 \otimes f_1 + x_2 \otimes f_2.$$

We have $TFx_1 = x_1$ and $TFx_2 = x_2$. Then $\operatorname{span}\{x_1, x_2\} \subset \ker(TF - I)$. By Lemma 2.1 (iii), we get that $\operatorname{span}\{x_1, x_2\} \subset X_{TF}(\{1\})$. Consequently, we have

$$\begin{aligned}\operatorname{span}\{x_1,x_2\}\cap X_{TF}(\{0\}) &\subset X_{TF}(\{1\})\cap X_{TF}(\{0\})\\ &= X_{TF}(\{1\}\cap\{0\})\\ &= X_{TF}(\emptyset).\end{aligned}$$

Since, the operator TF is of finite rank, then it has SVEP, and therefore $X_{TF}(\emptyset)=\{0\}$. Hence $\operatorname{span}\{x_1,x_2\}\cap X_{TF}(\{0\})=\{0\}$. Since x_1 and x_2 are linearly independent, it follows that $\operatorname{codim} X_{TF}(\{0\})\geq 2$.

Now, let us treat the statements (ii) and (iii). Let y_3 and y_4 be two vectors in X such that y_1,y_2,y_3 and y_4 are linearly independent. Then there exist f_1,f_2,f_3 and f_4 linear functionals in X^* such that $f_i(y_j)=\delta_{ij}$ for $i,j=1,\cdots,4$. Let $T\in\mathcal{F}_4(X)$ defined by

$$T=\lambda_0 y_3\otimes f_1+\lambda_0 y_4\otimes f_2+x_1\otimes f_3+x_2\otimes f_4.$$

We have $TFTy_3=\lambda_0 y_3$ and $TFTy_4=\lambda_0 y_4$, which leads to $\operatorname{span}\{y_3,y_4\}\subset\ker(TFT-\lambda_0 I)$. Once again by Lemma 2.1 (iii), we get that $\operatorname{span}\{y_3,y_4\}\subset X_{TFT}(\{\lambda_0\})$. Therefore $\dim X_{TFT}(\{\lambda_0\})\geq 2$. On the other hand, we have

$$\begin{aligned}\operatorname{span}\{y_3,y_4\}\cap X_{TFT}(\{0\}) &\subset X_{TFT}(\{\lambda_0\})\cap X_{TFT}(\{0\})\\ &= X_{TFT}(\{\lambda_0\}\cap\{0\})\\ &= X_{TFT}(\emptyset).\end{aligned}$$

Since the operator TFT is of finite rank, then it has SVEP and so $X_{TFT}(\emptyset)=\{0\}$. Thus $\operatorname{span}\{y_3,y_4\}\cap X_{TFT}(\{0\})=\{0\}$. This implies that $\operatorname{codim} X_{TFT}(\{0\})\geq 2$. □

We end this section by the following two lemmas.

Lemma 2.5 *Let $A,B\in\mathcal{L}(X)\setminus\{0\}$ and assume that for every $P\in\mathcal{P}_1(X)$, we have $AP\in\mathcal{N}(X)$ if and only if $BP\in\mathcal{N}(X)$. Then A and B are linearly dependent. Here $\mathcal{N}(X)$ denotes the set of nilpotent operators on X.*

Proof See [11, Lemma 2.13]. □

Lemma 2.6 *Suppose that $A,B\in\mathcal{L}(X)$ are non-scalar operators. If $PAP\in\mathcal{P}(X)\setminus\{0\}$ implies $PBP\in\mathcal{P}(X)\setminus\{0\}$ for every $P\in\mathcal{P}_1(X)$, then $B=\lambda I+(1-\lambda)A$ for some $\lambda\in\mathbb{C}\setminus\{1\}$. Here $\mathcal{P}(X)$ denotes the set of idempotent operators on X.*

Proof See [12, Proposition 3]. □

3 Maps preserving the local spectral subspace of the product of operators

In this section, for a fixed complex scalar λ_0, we will characterize all maps ϕ on $\mathcal{L}(X)$ that satisfy

$$X_{\phi(T)\phi(S)}(\{\lambda_0\})=X_{TS}(\{\lambda_0\})\ \text{ for all } T,S\in\mathcal{L}(X).$$

We begin by giving the following theorem for the case where $\lambda_0=0$.

Theorem 3.1 *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map such that $\mathcal{F}_2(X) \subset \phi(\mathcal{L}(X))$. Then ϕ satisfies*

$$X_{\phi(T)\phi(S)}(\{0\}) = X_{TS}(\{0\}) \text{ for all } T, S \in \mathcal{L}(X), \tag{3.1}$$

if and only if given any $T \in \mathcal{L}(X)$ there exists a nonzero scalar $c_T \in \mathbb{C}$ such that $\phi(T) = c_T T$.

Proof Assume that (3.1) holds. Let us divide the proof into two claims.

Claim 1 Given any nonzero operator $F \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(X)$, there exists a nonzero scalar $c_F \in \mathbb{C}$ such that $\phi(F) = c_F F$.

Let x be a nonzero vector in X and f be a nonzero linear functional in X^* such that $f(x) \neq 0$. By Lemma 2.3 (iii), (iv), we have

$$\ker f \subseteq X_{Tx\otimes f}(\{0\}) = X_{\phi(T)\phi(x\otimes f)}(\{0\})$$

for all $T \in \mathcal{L}(X)$. Then codim $X_{\phi(T)\phi(x\otimes f)}(\{0\}) \leq 1$ for all $T \in \mathcal{L}(X)$. Since $\mathcal{F}_2(X) \subset \phi(\mathcal{L}(X))$, it follows from Lemma 2.4 (i) that rank $\phi(x \otimes f) \leq 1$. Write $\phi(x \otimes f) = y \otimes g$ for some $y \in X$ and $g \in X^*$. By Lemma 2.3 (iii), we have

$$\ker f = X_{f(x)x\otimes f}(\{0\}) = X_{\phi(x\otimes f)^2}(\{0\}) = X_{(y\otimes g)^2}(\{0\}) = X_{g(y)y\otimes g}(\{0\}).$$

This implies that $g(y) \neq 0$, and so $\ker f = \ker g$. From this, we get that g is nonzero and $g \in \text{span}\{f\}$. Moreover, we obtain that y is nonzero and $f(y) \neq 0$, which implies, since f is arbitrary and $f(x) \neq 0$, that $y \in \text{span}\{x\}$. Finally, $\phi(x \otimes f) = c_{x,f} x \otimes f$ for some nonzero scalar $c_{x,f} \in \mathbb{C}$.

Claim 2 Given any $T \in \mathcal{L}(X)$, there is a nonzero scalar $c_T \in \mathbb{C}$ such that $\phi(T) = c_T T$.

Let T be a nonzero operator in $\mathcal{L}(X)$. Then there is a nonzero vector $x \in X$ such that $Tx \neq 0$. Pick $f \in X^*$ such that $f(x) \neq 0$ and $f(Tx) \neq 0$. Taking into account Claim 1, we have

$$\ker f = X_{Tx\otimes f}(\{0\}) = X_{\phi(T)\phi(x\otimes f)}(\{0\}) = X_{c_{x,f}\phi(T)x\otimes f}(\{0\} = X_{\phi(T)x\otimes f}(\{0\}).$$

This gives that $f(\phi(T)x) \neq 0$, and so $\phi(T) \neq 0$. Now, let us check that for all $P \in \mathcal{P}_1(X)$, we have $TP \in \mathcal{N}(X)$ if and only if $\phi(T)P \in \mathcal{N}(X)$. Let $P = z \otimes h$ where $z \in X$ and $h \in X^*$ such that $h(z) = 1$. As just above, we have $X_{Tz\otimes h}(\{0\}) = X_{\phi(T)z\otimes h}(\{0\})$. Therefore

$$\begin{aligned} TP \in \mathcal{N}(X) &\Longleftrightarrow h(Tz) = 0 \\ &\Longleftrightarrow X_{Tz\otimes h}(\{0\}) = X \\ &\Longleftrightarrow X_{\phi(T)z\otimes h}(\{0\}) = X \\ &\Longleftrightarrow h(\phi(T)z) = 0 \\ &\Longleftrightarrow \phi(T)P \in \mathcal{N}(X). \end{aligned}$$

By Lemma 2.5, it follows that $\phi(T)$ and T are linearly dependent, and hence there is a nonzero scalar $c_T \in \mathbb{C}$ such that $\phi(T) = c_T T$. It remains to show that $\phi(0) = 0$. Suppose, for a contradiction, that $\phi(0) \neq 0$. Then there is a nonzero vector $x \in X$ such that $\phi(0)x \neq 0$. Pick a nonzero linear functional $f \in X^*$ such that $f(\phi(0)x) \neq 0$. For $T = x \otimes f$, we have

$$X = X_{0T}(\{0\}) = X_{\phi(0)\phi(T)}(\{0\}) = X_{c_T\phi(0)T}(\{0\}) = X_{\phi(0)x\otimes f}(\{0\}) = \ker f,$$

a contradiction.

The converse implication is trivial. ☐

Now, let us treat the case when $\lambda_0 \neq 0$. Before that we state the following elementary lemma.

Lemma 3.2 *Let $A, B \in \mathcal{L}(X)$. The following statements are equivalent.*

(i) $A = B$.
(ii) $X_{AT}(\{\lambda\}) = X_{BT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{L}(X)$.
(iii) $X_{AT}(\{\lambda\}) = X_{BT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{F}_n(X)$.
(iv) $X_{AT}(\{\lambda\}) = X_{BT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{F}_1(X) \setminus \{0\}$.
(v) $X_{AT}(\{\lambda_0\}) = X_{BT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{L}(X)$.
(vi) $X_{AT}(\{\lambda_0\}) = X_{BT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{F}_n(X)$.
(vii) $X_{AT}(\{\lambda_0\}) = X_{BT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{F}_1(X) \setminus \{0\}$.

Proof Obviously (i)⇒(ii)⇒(iii)⇒(iv) and (i)⇒(v)⇒(vi)⇒(vii) hold. Since $X_{\lambda T}(\{\lambda_0\}) = X_T\left(\left\{\frac{\lambda_0}{\lambda}\right\}\right)$ for all nonzero $\lambda \in \mathbb{C}$ and all operators $T \in \mathcal{L}(X)$, we see that (ii)⟺(v), and (iii)⟺(vi), and (iv)⟺(vii). Lastly, it suffices to prove that (iv)⇒(i). Indeed, assume that (iv) holds, and suppose for the sake of contradiction that $Ax \neq Bx$ for some nonzero $x \in X$. Note that either $Ax \neq 0$ or $Bx \neq 0$, and let $f \in X^*$ such that $f(Ax) = 1$ and $f(Bx) \neq 1$. We have

$$\{0\} = X_{B(x\otimes f)}(\{1\}) = X_{A(x\otimes f)}(\{1\}) = \mathrm{span}\{Ax\},$$

and this contradiction shows that $A = B$. □

The following theorem gives a characterization of maps ϕ on $\mathcal{L}(X)$ that satisfy the equation (3.2) below. Observe that we have no surjectivity assumption on the map ϕ in this case.

Theorem 3.3 *Let λ_0 be a nonzero fixed scalar in $\mathbb{C}$. A map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ satisfies*

$$X_{\phi(T)\phi(S)}(\{\lambda_0\}) = X_{TS}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X), \tag{3.2}$$

if and only if there exists $\epsilon \in \{\pm 1\}$ such that $\phi(T) = \epsilon T$ for all $T \in \mathcal{L}(X)$.

Proof Assume that (3.2) holds. Let $T \in \mathcal{L}(X)$ be a nonzero operator. It is trivial that $\phi(T)x$ and Tx are linearly dependent for $x \in X$ such that $Tx = 0$. Now, let $x \in X$ such that $Tx \neq 0$. Pick $f \in X^*$ such that $f(Tx) = \lambda_0$. By Lemma 2.3 (ii), we have

$$\mathrm{span}\{x\} = X_{x\otimes T^*f}(\{\lambda_0\}) = X_{(x\otimes f)T}(\{\lambda_0\}) = X_{\phi(x\otimes f)\phi(T)}(\{\lambda_0\}).$$

From Lemma 2.2, it follows that $\phi(x \otimes f)\phi(T)x = \lambda_0 x$, and so $\phi(T)\phi(x \otimes f)\phi(T)x = \lambda_0\phi(T)x$. Hence $\phi(T)x \in \ker(\phi(T)\phi(x \otimes f) - \lambda_0 I)$. Therefore, by Lemma 2.1 (iii), we obtain that

$$\ker(\phi(T)\phi(x \otimes f) - \lambda_0 I) \subset X_{\phi(T)\phi(x\otimes f)}(\{\lambda_0\}) = X_{Tx\otimes f}(\{\lambda_0\}) = \mathrm{span}\{Tx\}.$$

Consequently, $\phi(T)x \in \mathrm{span}\{Tx\}$. Thus, for every $x \in X$ the vectors $\phi(T)x$ and Tx are linearly dependent. By [7, Theorem 2.3], either

i) the operators $\phi(T)$ and T are linearly dependent, that is, $\phi(T) = c_T T$ for some scalar $c_T \in \mathbb{C}$, or

ii) there exists a nonzero vector $y_T \in X$ such that $T = y_T \otimes g_T$ and $\phi(T) = y_T \otimes h_T$ for some linearly independent functionals g_T and h_T in X^*.

Suppose that ii) holds. Then there is $z \in X$ such that $g_T(z) = 1$ and $h_T(z) = 0$. We can find a functional $k \in X^*$ such that $k(y_T) = \lambda_0$ and $k(z) \neq 0$. Set $P = z \otimes k$. According to the cases i) and ii) above, one can see that $\phi(P) = c_P P$ or $\phi(P) = z \otimes l$ where l is a functional linearly independent of k. Since $TP = y_T \otimes k$ and $\phi(T)\phi(P) = 0$, it follows that

$$\operatorname{span}\{y_T\} = X_{y_T \otimes k}(\{\lambda_0\}) = X_{TP}(\{\lambda_0\}) = X_{\phi(T)\phi(P)}(\{\lambda_0\}) = X_0(\{\lambda_0\}) = \{0\},$$

a contradiction. Thus the second case can not occur. Consequently, we have $\phi(T) = c_T T$ for all nonzero operators $T \in \mathcal{L}(X)$. Let $\mu_0 \in \mathbb{C}$ such that $\mu_0^2 = \lambda_0$. We will prove that there exists $\epsilon \in \{\pm 1\}$ such that $\phi(\mu_0 F) = \epsilon \mu_0 F$ for all $F \in \mathcal{F}_1(X) \setminus \{0\}$. Let us start by checking first that $\phi(\mu_0 I) = \epsilon \mu_0 I$. Indeed, since

$$X = X_{\mu_0^2 I}(\{\lambda_0\}) = X_{\phi(\mu_0 I)^2}(\{\lambda_0\}) = X_{c_{\mu_0 I}^2 \mu_0^2 I}(\{\lambda_0\}),$$

it follows that $c_{\mu_0 I}^2 \mu_0^2 = \lambda_0$, and so $c_{\mu_0 I}^2 = 1$. Therefore there exists $\epsilon \in \{\pm 1\}$ such that $c_{\mu_0 I} = \epsilon$. Now, let $F \in \mathcal{F}_1(X) \setminus \{0\}$. Let x be a non-zero vector in X such that Fx and x are linearly independent. Then $Fx - x$ and x are linearly independent too. Thus there exists a linear functional f in X^* such that $f(Fx - x) = 0$ and $f(x) = 1$. Since

$$\begin{aligned}\operatorname{span}\{x\} = X_{\lambda_0 x \otimes f}(\{\lambda_0\}) &= X_{\phi(\mu_0 I)\phi(\mu_0 x \otimes f)}(\{\lambda_0\}) \\ &= X_{(\epsilon \mu_0 I)(c_{\mu_0 x \otimes f} \mu_0 x \otimes f)}(\{\lambda_0\}) \\ &= X_{\epsilon \lambda_0 c_{\mu_0 x \otimes f} x \otimes f}(\{\lambda_0\}),\end{aligned}$$

we get that $f(\epsilon \lambda_0 c_{\mu_0 x \otimes f} x) = \lambda_0$, which implies that $c_{\mu_0 x \otimes f} = \epsilon$. Moreover, we have

$$\begin{aligned}\operatorname{span}\{Fx\} = X_{\lambda_0 Fx \otimes f}(\{\lambda_0\}) &= X_{\phi(\mu_0 F)\phi(\mu_0 x \otimes f)}(\{\lambda_0\}) \\ &= X_{(c_{\mu_0 F} \mu_0 F)(c_{\mu_0 x \otimes f} \mu_0 x \otimes f)}(\{\lambda_0\}) \\ &= X_{(\epsilon \lambda_0 c_{\mu_0 F} Fx \otimes f)}(\{\lambda_0\}).\end{aligned}$$

This yields that $f(\epsilon \lambda_0 c_{\mu_0 F} Fx) = \lambda_0$, and so $c_{\mu_0 F} = \epsilon$. Therefore $\phi(\mu_0 F) = \epsilon \mu_0 F$. Consequently, we obtain that $\phi(F) = \epsilon F$ for all $F \in \mathcal{F}_1(X) \setminus \{0\}$. Lastly, Let $T \in \mathcal{L}(X)$. We have

$$X_{TF}(\{\lambda_0\}) = X_{\phi(T)\phi(F)}(\{\lambda_0\}) = X_{\phi(T)(\epsilon F)}(\{\lambda_0\}) = X_{\epsilon \phi(T) F}(\{\lambda_0\}),$$

for all $F \in \mathcal{F}_1(X) \setminus \{0\}$. By Lemma 3.2, it follows that $\phi(T) = \epsilon T$.

The converse implication is obvious. □

4 Maps preserving the local spectral subspace of the Jordan triple product of operators

In this section, we will characterize for a fixed complex scalar λ_0 all maps ϕ on $\mathcal{L}(X)$ with a slight surjectivity assumption on them and satisfy

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

As in the previous section we deal first with the case when $\lambda_0 = 0$.

Theorem 4.1 *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{L}(X))$. Then ϕ satisfies*

$$X_{\phi(T)\phi(S)\phi(T)}(\{0\}) = X_{TST}(\{0\}) \text{ for all } T, S \in \mathcal{L}(X), \tag{4.1}$$

if and only if given any $T \in \mathcal{L}(X)$, there exists a nonzero scalar $d_T \in \mathbb{C}$ such that $\phi(T) = d_T T$.

Proof Assume that (4.1) holds. We will prove the following claims.

Claim 1 For all nonzero operator $F \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(X)$, there is a nonzero scalar $d_F \in \mathbb{C}$ such that $\phi(F) = d_F F$.

Let $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. See that

$$\ker T^* f \subseteq X_{Tx\otimes T^*f}(\{0\}) = X_{\phi(T)\phi(x\otimes f)\phi(T)}(\{0\})$$

for all $T \in \mathcal{L}(X)$. Then $\operatorname{codim} X_{\phi(T)\phi(x\otimes f)\phi(T)}(\{0\}) \leq 1$ for all $T \in \mathcal{L}(X)$. Since $\mathcal{F}_4(X) \subset \phi(\mathcal{L}(X))$, it follows from Lemma 2.4 (ii) that $\operatorname{rank} \phi(x \otimes f) \leq 1$. Write $\phi(x \otimes f) = y \otimes g$ for some vector y in X and some linear functional g in X^*. We have

$$\ker f = X_{f(x)^2 x\otimes f}(\{0\}) = X_{\phi(x\otimes f)^3}(\{0\}) = X_{g(y)^2 y\otimes g}(\{0\}).$$

This implies that $g(y) \neq 0$, and so $\ker f = \ker g$. As in the proof of Claim 1 related to Theorem 3.1, we get that $0 \neq g \in \operatorname{span}\{f\}$ and $0 \neq y \in \operatorname{span}\{x\}$. Therefore there is a nonzero scalar $d_{x,f} \in \mathbb{C}$ such that $\phi(x \otimes f) = d_{x,f} x \otimes f$.

Claim 2 For all $T \in \mathcal{L}(X)$ there is a nonzero scalar $d_T \in \mathbb{C}$ such that $\phi(T) = d_T T$.

Let T be a nonzero operator in $\mathcal{L}(X)$. Then there is a nonzero vector $x \in X$ such that $Tx \neq 0$. Pick $f \in X^*$ such that $f(x) \neq 0$ and $f(Tx) \neq 0$. Taking into account the previous claim, we obtain

$$\begin{aligned}\ker f = X_{f(Tx)x\otimes f}(\{0\}) &= X_{\phi(x\otimes f)\phi(T)\phi(x\otimes f)}(\{0\})\\ &= X_{d_{x,f}^2 f(\phi(T)x)x\otimes f}(\{0\})\\ &= X_{f(\phi(T)x)x\otimes f}(\{0\}).\end{aligned}$$

This gives that $f(f(\phi(T)x)x) \neq 0$, and so $\phi(T) \neq 0$. Now, let us check that $TP \in \mathcal{N}(X)$ if and only if $\phi(T)P \in \mathcal{N}(X)$ for all $P \in \mathcal{P}_1(X)$. Let $z \in X$ and $h \in X^*$ such that $h(z) = 1$, and set $P = z \otimes h$. Since

$$X_{h(Tz)z\otimes h}(\{0\}) = X_{\phi(z\otimes h)\phi(T)\phi(z\otimes h)}(\{0\}) = X_{d_{z,h}^2 h(\phi(T)z)z\otimes h}(\{0\}) = X_{h(\phi(T)z)z\otimes h}(\{0\}),$$

we have

$$\begin{aligned}TP \in \mathcal{N}(X) &\Longleftrightarrow h(Tz) = 0\\ &\Longleftrightarrow X_{h(Tz)z\otimes h}(\{0\}) = X\\ &\Longleftrightarrow X_{h(\phi(T)z)z\otimes h}(\{0\}) = X\\ &\Longleftrightarrow h(\phi(T)z) = 0\\ &\Longleftrightarrow \phi(T)P \in \mathcal{N}(X).\end{aligned}$$

By Lemma 2.5, it follows that $\phi(T)$ and T are linearly dependent, and hence there is a nonzero scalar $d_T \in \mathbb{C}$ such that $\phi(T) = d_T T$. It remains to show that $\phi(0) = 0$. Suppose, for a contradiction, that $\phi(0) \neq 0$. Then there is a nonzero vector x in X such that $\phi(0)x \neq 0$. Pick a nonzero linear functional $f \in X^*$ such that $f(x) \neq 0$ and $f(\phi(0)x) \neq 0$. For $T = x \otimes f$, we have

$$X = X_{T0T}\{0\}) = X_{\phi(T)\phi(0)\phi(T)}(\{0\}) = X_{d_T^2 T\phi(0)T}(\{0\}) = X_{f(\phi(0)x)x\otimes f}(\{0\}) = \ker f,$$

a contradiction.

The converse is trivial. □

We give the following two lemmas which will be used to treat the case where $\lambda_0 \neq 0$.

Lemma 4.2 *Let λ_0 be a nonzero fixed scalar in $\mathbb{C}$, and let $\mu_0 \in \mathbb{C}$ such that $\mu_0^3 = \lambda_0$. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map satisfying*

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X). \tag{4.2}$$

Then there exists $\alpha \in \mathbb{C}$ satisfying $\alpha^3 = 1$ such that $\phi(\mu_0 I) = \alpha\mu_0 I$.

Proof Assume that (4.2) holds. Let x be a nonzero vector in X. Pick $f \in X^*$ such that $f(x) = \mu_0$. We have

$$X_{\phi(x\otimes f)^3}(\{\lambda_0\}) = X_{(x\otimes f)^3}(\{\lambda_0\}) = X_{f(x)^2 x\otimes f}(\{\lambda_0\}) = \text{span}\{x\}.$$

Then $\phi(x \otimes f)^3 x = \lambda_0 x$, which implies that $\phi(x \otimes f)x \in \ker(\phi(x \otimes f)^3 - \lambda_0 I)$. From Lemma 2.1 (iii), it follows that $\phi(x \otimes f)x \in \text{span}\{x\}$, and so $\phi(x \otimes f)x = \mu x$ for some complex scalar μ depending a priori on x and f. Therefore

$$\phi(x \otimes f)\phi(\mu_0 I)\phi(x \otimes f)x = \mu\phi(x \otimes f)\phi(\mu_0 I)x.$$

Since $\phi(x \otimes f)\phi(\mu_0 I)\phi(x \otimes f)x = \lambda_0 x$, which comes from

$$X_{\phi(x\otimes f)\phi(\mu_0 I)\phi(x\otimes f)}(\{\lambda_0\}) = X_{(x\otimes f)(\mu_0 I)(x\otimes f)}(\{\lambda_0\}) = X_{x\otimes f}(\{\mu_0\}) = \text{span}\{x\},$$

we get that $\lambda_0 x = \mu\phi(x \otimes f)\phi(\mu_0 I)x$. Then

$$\lambda_0\phi(\mu_0 I)x = \mu\phi(\mu_0 I)\phi(x \otimes f)\phi(\mu_0 I)x.$$

Since, from the fact that

$$X_{\phi(\mu_0 I)\phi(x\otimes f)\phi(\mu_0 I)}(\{\lambda_0\}) = X_{(\mu_0 I)(x\otimes f)(\mu_0 I)}(\{\lambda_0\}) = X_{x\otimes f}(\{\mu_0\}) = \text{span}\{x\},$$

we have $\phi(\mu_0 I)\phi(x \otimes f)\phi(\mu_0 I)x = \lambda_0 x$, it follows that $\phi(\mu_0 I)x = \mu x$. This implies that $\phi(\mu_0 I) = \mu I$ with μ is constant. Since

$$X_{\mu^3 I}(\{\lambda_0\}) = X_{\phi(\mu_0 I)^3}(\{\lambda_0\}) = X_{\mu_0^3 I}(\{\lambda_0\}) = X,$$

then $\mu^3 = \lambda_0$, that is $\left(\frac{\mu}{\mu_0}\right)^3 = 1$. Set $\alpha = \frac{\mu}{\mu_0}$. Thus $\phi(\mu_0 I) = \alpha\mu_0 I$, as desired. □

Lemma 4.3 *Let λ_0 be a nonzero fixed scalar in $\mathbb{C}$. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{L}(X))$ and*

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

Then

(i) rank $\phi(F) = 1$ *for all* $F \in \mathcal{F}_1(X) \setminus \{0\}$.
(ii) *There exists* $\alpha \in \mathbb{C}$ *satisfying* $\alpha^3 = 1$ *such that* $\phi(F) = \alpha F$ *for all* $F \in \mathcal{F}_1(X) \setminus \{0\}$.

Proof (i) Let x be a nonzero vector in X and f be a nonzero linear functional in X^*. We have

$$X_{\phi(T)\phi(x\otimes f)\phi(T)}(\{\lambda_0\}) = X_{T(x\otimes f)T}(\{\lambda_0\}) = X_{Tx\otimes T^*f}(\{\lambda_0\}) \subseteq \text{span}\{Tx\}$$

for all $T \in \mathcal{L}(X)$. Then $\dim X_{\phi(T)\phi(x\otimes f)\phi(T)}(\{\lambda_0\}) \leq 1$ for all $T \in \mathcal{L}(X)$. Since $\mathcal{F}_4(X) \subset \phi(\mathcal{L}(X))$, it follows from Lemma 2.4 (iii) that rank $\phi(x \otimes f) \leq 1$. Assume, for a contradiction that $\phi(x \otimes f) = 0$. Let $z \in X$ such that z and $(x \otimes f)z$ are linearly independent. Pick $h \in X^*$ such that $h((x \otimes f)z) = h(z) = \eta$ where $\eta^2 = \lambda_0$. We have

$$\text{span}\{z\} = X_{h((x\otimes f)z)z\otimes h}(\{\lambda_0\}) = X_{\phi(z\otimes h)\phi(x\otimes f)\phi(z\otimes h)}(\{\lambda_0\}) = X_0(\{\lambda_0\}) = \{0\},$$

a contradiction. Therefore rank $\phi(x \otimes f) = 1$.

(ii) Let $\mu_0 \in \mathbb{C}$ such that $\mu_0^3 = \lambda_0$. Let us first check that there exists $\alpha \in \mathbb{C}$ satisfying $\alpha^3 = 1$ such that $\phi(\mu_0 P) = \alpha\mu_0 P$ for all $P \in \mathcal{P}_1(X)$. Let $P \in \mathcal{P}_1(X)$, and write $P = x \otimes f$ where $x \in X$ and $f \in X^*$ such that $f(x) = 1$. By (i), we have $\phi(\mu_0 x \otimes f) = y \otimes g$ for some vector y in X and some linear functional g in X^*. Using Lemma 2.1 (ii) and Lemma 4.2, we get that

$$\text{span}\{x\} = X_{(\mu_0 I)(\mu_0 x\otimes f)(\mu_0 I)}(\{\lambda_0\}) = X_{\phi(\mu_0 I)\phi(\mu_0 x\otimes f)\phi(\mu_0 I)}(\{\lambda_0\}) = X_{y\otimes g}(\{\alpha\mu_0\}).$$

By Lemma 2.3 (i), (ii), it follows that $g(y) = \alpha\mu_0$ and so span$\{x\}$ = span$\{y\}$. Then y is nonzero and $y = \beta x$ for some nonzero complex scalar β. Therefore, we obtain that $g(x) = g(\frac{1}{\beta}y) = \frac{\alpha\mu_0}{\beta} = \frac{\alpha\mu_0}{\beta}f(x)$, which implies that $g = \frac{\alpha\mu_0}{\beta}f$. Thus $\phi(\mu_0 x \otimes f) = \alpha\mu_0 x \otimes f$. Now, Let $F \in \mathcal{F}_1(X) \setminus \{0\}$. We have

$$\begin{aligned} X_{PFP}(\{1\}) = X_{(\mu_0 P)(\mu_0 F)(\mu_0 P)}(\{\lambda_0\}) &= X_{\phi(\mu_0 P)\phi(\mu_0 F)\phi(\mu_0 P)}(\{\lambda_0\}) \\ &= X_{(\alpha\mu_0 P)\phi(\mu_0 F)(\alpha\mu_0 P)}(\{\lambda_0\}) \\ &= X_{P\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)\right)P}(\{1\}). \end{aligned}$$

Since $PFP = f(Fx)x \otimes f$ and $P\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)\right)P = f\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)x\right)x \otimes f$, we get that

$$\begin{aligned} PFP \in \mathcal{P}(X) \setminus \{0\} &\Longrightarrow f(f(Fx)x) = 1 \\ &\Longrightarrow X_{PFP}(\{1\}) = \text{span}\{x\} \\ &\Longrightarrow X_{P\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)\right)P}(\{1\}) = \text{span}\{x\} \\ &\Longrightarrow f\left(f\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)x\right)x\right) = 1 \\ &\Longrightarrow P\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)\right)P \in \mathcal{P}(X) \setminus \{0\}. \end{aligned}$$

Therefore

$$PFP \in \mathcal{P}(X) \setminus \{0\} \Longrightarrow P\left(\frac{1}{\alpha\mu_0}\phi(\mu_0 F)\right)P \in \mathcal{P}(X) \setminus \{0\},$$

for all $P \in \mathcal{P}_1(X)$. Note that F is non-scalar, and by Lemma 4.3 (i), $\phi(\mu_0 F)$ is non-scalar too. Then by Lemma 2.6, there exists $\lambda_F \in \mathbb{C} \setminus \{1\}$ such that $\frac{1}{\alpha\mu_0}\phi(\mu_0 F) = \lambda_F I + (1-\lambda_F)F$, that is $\alpha\mu_0\lambda_F I = \phi(\mu_0 F) - \alpha\mu_0(1-\lambda_F)F$. Since, by Lemma 4.3 (i), the rank of $\phi(\mu_0 F) - \alpha\mu_0(1-\lambda_F)F$ is at most two, it follows that the rank of $\alpha\mu_0\lambda_F I$ is also at most two. This forces that $\lambda_F = 0$. Finally, we conclude that $\phi(F) = \alpha F$ for all $F \in \mathcal{F}_1(X) \setminus \{0\}$. □

Now, we give the following elementary lemma.

Lemma 4.4 *Let $A, B \in \mathcal{L}(X)$. The following statements are equivalent.*

(i) $A = B$.
(ii) $X_{TAT}(\{\lambda\}) = X_{TBT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{L}(X)$.
(iii) $X_{TAT}(\{\lambda\}) = X_{TBT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{F}_n(X)$.
(iv) $X_{TAT}(\{\lambda\}) = X_{TBT}(\{\lambda\})$ *for all nonzero* $\lambda \in \mathbb{C}$, *and all* $T \in \mathcal{F}_1(X) \setminus \{0\}$.
(v) $X_{TAT}(\{\lambda_0\}) = X_{TBT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{L}(X)$.
(vi) $X_{TAT}(\{\lambda_0\}) = X_{TBT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{F}_n(X)$.
(vii) $X_{TAT}(\{\lambda_0\}) = X_{TBT}(\{\lambda_0\})$ *for some nonzero* $\lambda_0 \in \mathbb{C}$, *and all* $T \in \mathcal{F}_1(X) \setminus \{0\}$.

Proof As in the proof of Lemma 3.2, it suffices to show that (iv)⇒(i). Assume that (iv) holds. Let x be a nonzero vector in X, and let us prove that $f(Ax) = f(Bx)$ for all $f \in X^*$. Let $f \in X^*$ such that $f(x) \neq 0$, and set $F = x \otimes f$. Note that $FAFx = f(Ax)f(x)x$ and $FBFx = f(Bx)f(x)x$, so $x \in \ker(FAF - f(Ax)f(x)I) \cap \ker(FBF - f(Bx)f(x)I)$. By Lemma 2.1 (iii), we get that

$$x \in X_{FAF}(\{f(Ax)f(x)\}) \cap X_{FBF}(\{f(Bx)f(x)\}).$$

Suppose that $f(Ax) \neq 0$. We have $X_{FAF}(\{f(Ax)f(x)\}) = X_{FBF}(\{f(Ax)f(x)\})$, it follows then that

$$x \in X_{FBF}(\{f(Ax)f(x)\}) \cap X_{FBF}(\{f(Bx)f(x)\}) = X_{FBF}(\{f(Ax)f(x)\} \cap \{f(Bx)f(x)\}).$$

Then $X_{FBF}(\{f(Ax)f(x)\} \cap \{f(Bx)f(x)\}) \neq \{0\}$. Since $X_{FBF}(\emptyset) = \{0\}$ because FBF has SVEP, we obtain that $f(Ax) = f(Bx)$. We proceed similarly if we suppose that $f(Bx) \neq 0$. Consequently, we get that $f(Ax) = f(Bx)$ for all $f \in X^*$ such that $f(x) \neq 0$. Now, let $f \in X^*$ such that $f(x) = 0$. Pick $g \in X^*$ such that $g(x) \neq 0$. We have $g(Ax) = g(Bx)$, and since $(f+g)(x) \neq 0$, we also have $(f+g)(Ax) = (f+g)(Bx)$. This implies that $f(Ax) = f(Bx)$. Finally, we get that $Ax = Bx$ for every $x \in X$, and consequently $A = B$. □

Finally, we establish the following theorem concerning the case where $\lambda_0 \neq 0$.

Theorem 4.5 *Let λ_0 be a nonzero fixed scalar in $\mathbb{C}$. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{L}(X))$. Then ϕ satisfies*

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X),$$

if and only if there exists $\alpha \in \mathbb{C}$ *satisfying* $\alpha^3 = 1$ *such that* $\phi(T) = \alpha T$ *for all* $T \in \mathcal{L}(X)$.

Proof Assume that 4.2 holds. Let $T \in \mathcal{L}(X)$. According to Lemma 4.3 (ii), we have

$$X_{FTF}(\{\lambda_0\}) = X_{\phi(F)\phi(T)\phi(F)}(\{\lambda_0\}) = X_{(\alpha F)\phi(T)(\alpha F)}(\{\lambda_0\}) = X_{F\left(\frac{1}{\alpha}\phi(T)\right)F}(\{\lambda_0\}),$$

for all $F \in \mathcal{F}_1(X) \setminus \{0\}$. By Lemma 4.4, we get that $\phi(T) = \alpha T$.

The converse is clear. □

5 Concluding remarks

We finally state the following remarks.

(i) Note that our results remain valid in finite-dimensional case. Namely, our proofs require only that the dimension should be no less than 2 for the usual product, and no less than 4 for the Jordan triple product.

(ii) Let λ_0 be a fixed complex scalar. We would like to point out that one may ask about the characterization of all pairs of surjective maps ϕ_1 and ϕ_2 on $\mathcal{L}(X)$ that satisfy

$$X_{\phi_1(T)\phi_2(S)}(\{\lambda_0\}) = X_{TS}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

It seems that the description of such maps could not be immediately deduced from the proofs and techniques used above in Sect. 3. Similar question can be asked when replacing the usual product by Jordan triple product. In other words, this is to describe all surjective maps ϕ_1 and ϕ_2 on $\mathcal{L}(X)$ that satisfy

$$X_{\phi_1(T)\phi_2(S)\phi_1(T)}(\{\lambda_0\}) = X_{TST}(\{\lambda_0\}) \text{ for all } T, S \in \mathcal{L}(X).$$

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