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An existence result for a singular-regular anisotropic system

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Abstract

In this paper we deal with the following singular-regular anisotropic system

$$\begin{cases} -Lu = -\sum_{i=1}^{N} \partial_i \left[\left| \partial_i u \right|^{p_i - 2} \partial_i u \right] = p \dfrac{v^q}{u^{1-p}} & in\ \Omega, \\ -Lv = -\sum_{i=1}^{N} \partial_i \left[\left| \partial_i v \right|^{p_i - 2} \partial_i v \right] = q v^{q-1} u^p & in\ \Omega, \\ u > 0 \text{ and } v > 0 & in\ \Omega, \\ u = 0 \text{ and } v = 0 & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_N$. Under some suitable conditions on the parameters p and q, we obtain existence results by using nondifferentiable variational techniques.

Keywords Anisotropic operator · Nondifferentiable variational techniques · Singular system · Approximation

Mathematic Subject Classification 35J75 · 35J67 · 35J20

1 Introduction

Problems involving a partial differential equation interacting with singular nonlinear terms, due to their importance, have been widely studied, and several works are available in the literature. In this short introduction, we will present a non-exhaustive content of the different works related to singular problems. We start by indicating to the reader the essential work [24] which contains almost all basic and advanced tools to study general singular problems. We also mention various works that consider semilinear problems with singular

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terms as [1, 6–8], and when the principal operator is a p-Laplacian like operator we mention [13, 30] and the references therein.

Problems involving the anisotropic operator

$$Lu = \sum_{i=1}^{N} \partial_i \left[\left|\partial_i u\right|^{p_i-2} \partial_i u \right]$$

encounter a great interest, we cite for example [15–17], and the references therein.

We also mention the leading works [4, 21, 22] and the very recent one [12]; where the anisotropic operator is associated to a nonlinearity, and existence, uniqueness, multiplicity and non existence results are obtained by various ways.

When a singular term is associated to the anisotropic operator L there are some few recent results, for example in [23, 28, 31], existence and regularity of solutions to equation

$$\begin{aligned} Lu &= \sum_{i=1}^{N} \partial_i \left[\left|\partial_i u\right|^{p_i-2} \partial_i u \right] \\ &= \frac{f(x)}{u^{\gamma(x)}} + \lambda g(x,u) \end{aligned}$$

where g is a regular nonlinearity in $u,$ are obtained for the case $\lambda = 0$ by approximation methods, while in [18], the authors obtained existence results for $\lambda \neq 0$ by monotonicity methods.

Systems involving operator $L,$ are less studied we cite in this direction the works [9, 18]. To the best of our knowledge, there are only two works in the literature studying problems where the anisotropic operator L is associated to nonlocal terms that are [5, 19]. We also mention [15, 29] and the references therein for results about variational techniques applied to the study of problems involving L.

Parabolic problems, with anisotropic operator $L,$ have also been studied by different authors we indicate here some of them [32, 34, 36].

Inspired by the works [7, 14, 20, 25] we consider in this paper the system

$$\begin{cases} -Lu = -\sum\limits_{i=1}^{N} \partial_i \left[\left|\partial_i u\right|^{p_i-2} \partial_i u \right] = p\dfrac{v^q}{u^{1-p}} & in\ \Omega, \\ -Lv = -\sum\limits_{i=1}^{N} \partial_i \left[\left|\partial_i v\right|^{p_i-2} \partial_i v \right] = q v^{q-1} u^p & in\ \Omega, \\ u > 0 \text{ and } v > 0 & in\ \Omega, \\ u = 0 \text{ and } v = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$ $(N \geq 2)$ and, without loss of generality, $1 \leq p_1 \leq p_2 \leq \cdots \leq p_N$. We will assume that $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$, $\overline{p} < N$, $0 < p < 1$, $q > 1$ and $p_N < p + q < \frac{N\overline{p}}{N-\overline{p}} = \overline{p}^*$. In the whole paper we will denote $\frac{\partial u}{\partial x_i} = \partial_i u$. As the considered system is singular, we will use nondifferentiable variational techniques introduced in [2, 3].

The paper is organised as follows, after this brief introduction, we present some preliminaries dealing with the functional setting associated to our problem. In the third section we study approximating regular problems, where the singular nonlinearity is replaced by a regular one. In the fourth section we prove that the sequence of solutions to the approximating problems converges to the solution of problem (1). The paper is ended with some concluding remarks.

2 Preliminaries

The natural function spaces associated to the operator L are the anisotropic Sobolev spaces

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega); \partial_i v \in L^{p_i}(\Omega) \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega)$$

endowed with the usual norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}$$

We will also use very often the following indices

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$$

and

$$\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}, \; p_\infty = \max \left\{ p_N, \overline{p}^* \right\}$$

It will be assumed throughout this paper that $p_\infty = \overline{p}^* < N$, so in this case we will have that

$$W_0^{1,(p_i)}(\Omega) \subset L^r(\Omega) \;\; \forall r \in \left[1, \overline{p}^*\right]$$

this imbedding being compact whenever $r < \overline{p}^*$. Let us recall the following Sobolev type inequalities, we refer to the early works [27, 33, 35].

Theorem 1

$$\|v\|^{p_N}_{L^{\overline{p}^*}(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)}, \tag{2}$$

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|^{\frac{1}{N}}_{L^{p_i}(\Omega)} \quad \forall r \in \left[1, \overline{p}^*\right] \tag{3}$$

and $\forall v \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega), \overline{p} < N$

$$\left(\int_\Omega |v|^r \right)^{\frac{N}{\overline{p}} - 1} \leq C \prod_{i=1}^{N} \left(\int_\Omega |\partial_i v|^{p_i} |v|^{t_i p_i} \right)^{\frac{1}{p_i}}, \tag{4}$$

for every r and t_j chosen such a way to have

$$\begin{cases} \frac{1}{r} = \frac{\gamma_i(N-1)-1+\frac{1}{p_i}}{t_i+1} \\ \sum_{i=1}^{N} \gamma_i = 1. \end{cases}$$

We also have the following algebraic inequalities:

- There exists a $C > 0$ not depending on $\rho \in (0,1)$ such that for given $\sigma_i > 0, i = 1, 2 \dots N$ we have

$$\sum_{i=1}^{N} \sigma_i = \rho \Longrightarrow \sum_{i=1}^{N} \frac{\sigma_i^{p_i}}{p_i} \geq C\rho^{p_N} \tag{5}$$

- For $p_i \geq 2$

$$C|a-b|^{p_i} \leq \left(|a|^{p_i-2}a - |b|^{p_i-2}b\right)(a-b) \tag{6}$$

- For $1 < p_i \leq 2$

$$C\frac{|a-b|^2}{(|a|+|b|)^{2-p_i}} \leq \left(|a|^{p_i-2}a - |b|^{p_i-2}b\right)(a-b) \tag{7}$$

We need as well to recall the following truncating functions

$$T_n(s) = \begin{cases} n\frac{s}{|s|} & \textit{if} \ \ |s| > n \\ s & \textit{if} \ \ |s| \leq n \end{cases}$$

and

$$G_n(s) = s - T_n(s).$$

Definition 1 We will say that positive $u, v \in W_0^{1,(p_i)}(\Omega)$ are solutions to (1) if and only if

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega \left[\left|\partial_i u\right|^{p_i-2}\partial_i u \partial_i \varphi\right] = p \int_\Omega \frac{v^q}{u^{1-p}}\varphi, & \varphi \in C_0^1(\Omega), \\ \sum_{i=1}^{N} \int_\Omega \left[\left|\partial_i v\right|^{p_i-2}\partial_i v \partial_i \psi\right] = q \int_\Omega v^{q-1}u^p\psi, & \psi \in W_0^{1,(p_i)}(\Omega), \end{cases} \tag{8}$$

We have the following comparison principle

Proposition 2 [9] *Comparison principle*
If $u, v \in W_0^{1,(p_i)}$ are such that

$$\begin{cases} -Lu = -\sum_{i=1}^{N} \partial_i\left[\left|\partial_i u\right|^{p_i-2}\partial_i u\right] \leq -Lv = -\sum_{i=1}^{N} \partial_i\left[\left|\partial_i v\right|^{p_i-2}\partial_i v\right], \\ u \leq v \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

then $u \leq v$ a.e. in Ω.

The considered system (1) has a variational structure, so its solution can be seen as a critical point of the following functional

$$J(w,z)=\sum_{i=1}^{N}\frac{1}{p_i}\int_{\Omega}|\partial_i w|^{p_i}+\sum_{i=1}^{N}\frac{1}{p_i}\int_{\Omega}|\partial_i z|^{p_i}-\int_{\Omega}w_+^p z_+^q \tag{10}$$

which is singular when $p<1$ or/and $q<1$. Where, as usual $u_+=\max\{u,0\}$ and $u_-=\max\{-u,0\}=-\min\{u,0\}$

We will need to use the following theorem that can be find in [2, 14].

Theorem 3 (Mountain-Pass theorem for nondifferentiable functionals)
Let X, Y be two Banach spaces with $Y\subset X$, and

$$F:X\to\mathbb{R}$$

be a functional such that

(H1) *F is continuous on Y.*
(H2) *F possesses a Gateaux dérivative in X, $\langle F'(u),v\rangle$ through any direction $v\in Y$.*
(H3) *For every fixed $v\in Y$, the function $\langle F'(.),v\rangle$ is continuous in X.*
(H4) *There exists $\overline{u}\in Y$ such that*

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}F(\gamma(t))>\max\left\{F(0),F(\overline{u})\right\}$$

with

$$\Gamma=\left\{\text{Continuous }\gamma:[0,1]\to Y;\gamma(0)=0\text{ and }\gamma(1)=\overline{u}\right\}$$

(H5) *Any sequence $\left\{u_n\right\}_n$ in Y possesses a convergent subsequence in X if, for some real positive sequence $\left\{M_n\right\}_n$ and $\epsilon_n\to 0$ fulfills*

(a) *$\left\{F(u_n)\right\}_n$ is bounded*
(b) *$\left\|u_n\right\|_Y\le 2M_n$ for all n*
(c) *$\left|\langle F'(u_n),v\rangle\right|\le\epsilon_n\left[\frac{\|v\|_Y}{M_n}+\|v\|_X\right]$ for all $v\in Y$.*

Then c is a critical value of F, which means there exists $u\in Y\setminus\{0\}$ such that $F(u)=c$ and $\langle F'(u),v\rangle=0$ for all $v\in Y$.

We also introduce the following function

$$g_n(s)=\frac{s^2}{\left(s+\frac{1}{n}\right)^{2-p}} \tag{11}$$

which will play the role of a smooth approximation of s^p when $p<1$. This function has the properties

$$\frac{s^2}{(s+1)^{2-p}}\le g_n(s)\le s^p, \tag{12}$$

and

$$\frac{1}{p_N} g_n'(s)s + \left(\frac{q}{p_N} - 1\right) g_n(s) \geq \frac{p+q-p_N}{p_N} g_n(s). \tag{13}$$

and also

$$g_n'(s)s \leq 2s^p \tag{14}$$

3 The approximating problems

As $p < 1$, the functional J introduced in (10) is not differentiable (in w), so we will use the approximating functional

$$J_n(w,z) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w|^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i z|^{p_i} - \int_\Omega g_n(w_+) z_+^q$$

with $g_n(s) = \frac{s^2}{\left(s+\frac{1}{n}\right)^{2-p}}$.

Proposition 4 *The functional J_n fulfils all conditions of Theorem 3 for $X = W_0^{1,(p_i)}(\Omega) \times W_0^{1,(p_i)}(\Omega)$ and $Y = \left(W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)\right) \times \left(W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)\right)$.*

Proof

(H1) $J_n : Y \to \mathbb{R}$ is continuous, indeed let $\left\{\left(w_k, z_k\right)\right\}_k \subset Y$ be a strongly convergent sequence in Y, $\left(w_k, z_k\right) \to (w,z)$ so as direct consequence one have that

$$\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w_k|^{p_i} \to \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w|^{p_i},$$

$$\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w_k|^{p_i} \to \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w|^{p},$$

and as both $\{w_k\}_k$ and $\{z_k\}_k$ are bounded in $L^{\overline{p}^*}$ and $\frac{\overline{p}^*}{p+q} > 1$, so by Vitali's theorem we get

$$g_n(w_{n+}) z_{n+}^q \to g_n(w_+) z_+^q \text{ in } L^1(\Omega)$$

which means that J_n is continuous in X and consequently it is also continuous in Y.

(H2) The functional J_n is constructed in such a way to make (H2) verified.

(H3) Let $(\varphi, \psi) \in Y$ and let $\left\{\left(w_k, z_k\right)\right\}_k \subset X$ be a strongly convergent sequence in X, $\left(w_k, z_k\right) \to (w,z)$, we need to prove that

$$\left\langle J_n'\left(w_k, z_k\right), (\varphi, \psi)\right\rangle \to \left\langle J_n'(w,z), (\varphi, \psi)\right\rangle.$$

Observing that

$$\left\langle J_n'\left(w_k, z_k\right), (\varphi, \psi)\right\rangle = \int_\Omega \left(J_{n_w}\varphi + J_{n_z}\psi\right)$$
$$= \sum_{i=1}^{N}\int_\Omega \left|\partial_i w_k\right|^{p_i-2}\partial_i w_k \partial_i\varphi + \sum_{i=1}^{N}\int_\Omega \left|\partial_i z_k\right|^{p_i-2}\partial_i z_k\partial_i\psi$$
$$-\int_\Omega g_n'(w_{k+})z_{k+}^q\varphi - q\int_\Omega g_n(w_{k+})z_{k+}^{q-1}\psi.$$

The convergence of the two terms

$$\sum_{i=1}^{N}\int_\Omega \left|\partial_i w_k\right|^{p_i-2}\partial_i w_k\partial_i\varphi,$$

and

$$\sum_{i=1}^{N}\int_\Omega \left|\partial_i z_k\right|^{p_i-2}\partial_i z_k\partial_i\psi$$

is a consequence of the convergence of $\left\{\left(w_k, z_k\right)\right\}_k$. Now we deal with the third term

$$\int_\Omega g_n'(w_+)z_+^q\varphi,$$

one have that

$$g_n'(w_{k+})z_{k+}^q \to g_n'(w_+)z_+^q \text{ a.e.}$$

and as

$$\left|g_n'(w_{k+})z_{k+}^q\right| \le 2n^{1-p}z_{k+}^q$$

by the assumption $\frac{\overline{p}^*}{q} > 1$ and as z_{k+}^q is bounded in $L^{\frac{\overline{p}^*}{q}}(\Omega)$ so

$$z_{k+}^q \to z_+^q \text{ in } L^1(\Omega),$$

and

$$z_{k+}^q \to z_+^q \text{ a.e.}$$

By Vitali's theorem associated to the generalized Lebesgue theorem we arrive at the conclusion that

$$g_n'(w_{k+})z_{k+}^q \to g_n'(w_+)z_+^q \text{ in } L^1(\Omega).$$

By duality, and as $\varphi \in L^\infty(\Omega)$ we obtain the convergence of the third term. For the last term

$$q\int_\Omega g_n(w_{k+})z_{k+}^{q-1}\psi,$$

we have that

$$g_n(w_{k+})z_{k+}^{q-1} \to g_n(w_+)z_+^{q-1} \text{ a.e,}$$

and

$$\left(\left\|w_k^p\right\|_{L^{\frac{\overline{p}^*}{p}}} \le C_1 \text{ and } \left\|z_k^{q-1}\right\|_{L^{\frac{\overline{p}^*}{q-1}}} \le C_2\right) \Rightarrow \left\|g_n(w_{k+})z_{k+}^{q-1}\right\|^{\frac{\overline{p}^*}{p+q-1}} \le C_3,$$

as by hypothesis $\frac{\overline{p}^*}{p+q-1} > 1$, we obtain the equiintegraility of $g_n(w_{k+})z_{k+}^{q-1}$, Vitali's theorem allows us to conclude that

$$g_n(w_{k+})z_{k+}^{q-1} \to g_n(w_+)z_+^{q-1} \text{ in } L^1(\Omega).$$

By duality, and as $\psi \in L^\infty(\Omega)$ we obtain the convergence of the last term. In conclusion (H3) is fulfilled by J_n.

(H4) Consequently by the properties of g_n Hölder, Sobolev and Young inequalities we have

$$\begin{aligned}\int_\Omega g_n(w_+)z_+^q \le& \int_\Omega w_+^p z_+^q\\
\le&\left(\int_\Omega w_+^p\right)^{\frac{p}{\overline{p}^*}}\left(\int_\Omega z_+^{\frac{\overline{p}^* q}{\overline{p}^*-p}}\right)^{\frac{\overline{p}^*-p}{\overline{p}^*}}\\
\le& C\left(\int_\Omega w_+^p\right)^{\frac{p}{\overline{p}^*}}\left(\int_\Omega z_+^{\overline{p}^*}\right)^{\frac{q}{\overline{p}^*}}\\
\le& C\|w\|^p_{W_0^{1,(p_i)}}\|z\|^q_{W_0^{1,(p_i)}}\\
\le& \varepsilon\|w\|^{p_N}_{W_0^{1,(p_i)}(\Omega)} + C(\varepsilon)\|z\|^{\frac{qp_N}{qp_N-p}}_{W_0^{1,(p_i)}(\Omega)}.\end{aligned}$$

Now using (5) consequently for $\sigma_i = \left\|\partial_i w\right\|_{L^{p_i}(\Omega)}$ and $\sigma_i = \left\|\partial_i z\right\|_{L^{p_i}(\Omega)}$, associated to the latter inequation we obtain

$$\begin{aligned}J_n(w,z) \ge& C_1\|w\|^{p_N} + C_2\|z\|^{p_N} - \varepsilon\|w\|^{p_N} - C(\varepsilon)\|z\|^{\frac{qp_N}{qp_N-p}}\\
\ge& C(\|w\|^{p_N} + \|z\|^{p_N}) - \varepsilon\|w\|^{p_N} - \varepsilon\|z\|^{p_N} + \varepsilon\|z\|^{p_N} - C(\varepsilon)\|z\|^{\frac{qp_N}{qp_N-p}}\\
\ge& (C-\varepsilon)(\|w\|^{p_N} + \|z\|^{p_N}) + \varepsilon\|z\|^{p_N} - C(\varepsilon)\|z\|^{\frac{qp_N}{qp_N-p}}\\
\ge& (C-\varepsilon)(\|w\|^{p_N} + \|z\|^{p_N}) + f(\|z\|)\end{aligned}$$

where

$$f(t) = \varepsilon t^{p_N} - C(\varepsilon)t^{\frac{qp_N}{qp_N-p}},$$

so it is always possible to have for $t > T$

$$f(t) > 0,$$

choosing

$$\|w\|^{p_N} + \|z\|^{p_N} = R,$$

we obtain

$$J_n(w, z) > \alpha \text{ for } \|w\|^{p_N} + \|z\|^{p_N} = R.$$

$\psi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega), \|\psi\|_{L^\infty(\Omega)} = 1$ that is $0 \leq \psi \leq 1$, and for $t \geq 1$

$$g_n(t\psi) \geq \frac{(t\psi)^2}{(1 + t\psi)^{2-p}}$$

and as $p + q > p_N > p_1$

$$\begin{aligned} J_n(t\psi, t\psi) =& 2\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\partial_i t\psi\right|^{p_i} - \int_\Omega g_n(t\psi)(t\psi)^q \\ \leq& \frac{2Ct^{p_N}}{p_1} \|\psi\|^{p_N}_{W_0^{1,(p_i)}(\Omega)} - \int_\Omega \frac{(t\psi)^{2+q}}{(1 + t\psi)^{2-p}} \\ \leq& C_1 t^{p_N} - C_2 \frac{(t)^{2+q}}{(1 + t)^{2-p}} \end{aligned}$$

since $p + q > p_N$ we have that

$$\lim_{t\to+\infty} \left(C_1 t^{p_N} - C_2 \frac{(t)^{2+q}}{(1 + t)^{2-p}} \right) = -\infty$$

and thus for large $\bar{t} >> 1$ we have

$$J_n(\bar{t}\psi, \bar{t}\psi) < 0,$$

so if we choose

$$\left(\overline{w}, \overline{z}\right) = (\bar{t}\psi, \bar{t}\psi)$$

and for $t \in [0, 1]$

$$\overline{\gamma}(t) := \left(\overline{\gamma_1}(t), \overline{\gamma_2}(t)\right) = \left(t\overline{w}, t\overline{z}\right)$$

then

$$\overline{\gamma} \in \Gamma$$

in view of the previous computation, we obtain for $J_n(w, z) > \alpha$ and $\|w\|^{p_N} + \|z\|^{p_N} = R$

$$\max_{t\in[0,1]} J_n\left(\gamma_1(t), \gamma_2(t)\right) \geq \alpha \text{ for } \gamma(.) = \left(\gamma_1(.), \gamma_2(.)\right) \in \Gamma$$

we finally reach that

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_n\big(\gamma_1(t),\gamma_2(t)\big) \geq \alpha > 0 = \max\left\{J_n(0,0), J_n\big(\overline{w},\overline{z}\big)\right\}.$$

(H5) We assume that condition (a), (b) and (c) in (H5) Theorem 3 are fulfilled by $J_n = F$ and the sequence $\left\{\big(w_k, z_k\big)\right\}_k \subset Y$, we claim that $\left\{\big(w_k, z_k\big)\right\}_k$ is bounded in X. As a direct consequence of hypothesis (a), one have that

$$\sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega \left|\partial_i w\right|^{p_i} + \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega \left|\partial_i z\right|^{p_i} - \int_\Omega g_n(w_{k+})z_{k+}^q \leq C$$

by (b) and (c), we have that

$$\begin{aligned}-\left\langle J_n'\big(w_k,z_k\big),\big(w_k,z_k\big)\right\rangle &= -\sum_{i=1}^{N}\int_\Omega \left|\partial_i w_k\right|^{p_i} - \sum_{i=1}^{N}\int_\Omega \left|\partial_i z_k\right|^{p_i}\\ &\quad + \int_\Omega g_n'(w_{k+})z_{k+}^q w_{k+} + q\int_\Omega g_n(w_{k+})z_{k+}^q\\ &\leq \varepsilon_k\left[2 + \left\|\big(w_k,z_k\big)\right\|_X\right].\end{aligned}$$

Multiplying by $\frac{1}{p_N}$ both sides of the last inequality

$$\begin{aligned}&-\int_\Omega g_n(w_{k+})z_{k+}^q + \frac{1}{p_N}\int_\Omega g_n'(w_{k+})z_{k+}^q w_{k+} + \frac{q}{p_N}\int_\Omega g_n(w_{k+})z_{k+}^q\\ &\leq C + \frac{\varepsilon_k}{p_N}\left[2 + \left\|\big(w_k,z_k\big)\right\|_X\right]\end{aligned}$$

we then arrive at

$$\frac{1}{p_N}\int_\Omega \left[\frac{1}{p_N}g_n'(w_{k+})w_{k+} + \left(\frac{q}{p_N} - 1\right)g_n(w_{k+})\right]z_{k+}^q \leq C + \frac{\varepsilon_k}{p_N}\left[2 + \left\|\big(w_k,z_k\big)\right\|_X\right]$$

by the properties of g_n

$$\frac{p+q+p_N}{p_N}\int_\Omega g_n(w_{k+})z_{k+}^q \leq C + \frac{\varepsilon_k}{p_N}\left[2 + \left\|\big(w_k,z_k\big)\right\|_X\right]$$

as $p + q > p_N$

$$\frac{1}{p_N}\left\|\big(w_k,z_k\big)\right\|_X \leq C + \frac{p+q+p_N}{p_N}\left(C + \frac{\varepsilon_k}{p_N}\left[2 + \left\|\big(w_k,z_k\big)\right\|_X\right]\right)$$

as $\varepsilon_k \to 0$ the claim follows, that is $\left\{\left\|\big(w_k,z_k\big)\right\|_X\right\}_k$ is bounded, and by the sequel $\left\{\big(w_k,z_k\big)\right\}_k$ weakly converges in X. We will now prove strong convergence of the sequence $\left\{\big(w_k,z_k\big)\right\}_k$. First observe that if M_k in (b) of (H5) is such that $M_k \to 0$ then immediately $\left\{\big(w_k,z_k\big)\right\}_k$ strongly converges. So we will assume without loss of generality that $M_k \to l > 0$ (or more precisely that $\liminf M_k > 0$) and we have

$$\left\langle J_n'\left(w_k, z_k\right), \left(w_k - T_h\left(w_k\right), 0\right)\right\rangle$$
$$= \sum_{i=1}^{N} \int_\Omega \left|\partial_i w_k\right|^{p_i-2} \partial_i w_k \partial_i\left(w_k - T_h\left(w_k\right)\right) - \int_\Omega g_n'(w_{k+}) z_{k+}^q \left(w_k - T_h\left(w_k\right)\right)$$

Using hypotheses (b), (c) and the fact that $\left\{w_k\right\}_k$ is bounded in $W_0^{1,(p_i)}(\Omega)$ we came to

$$\begin{aligned}
&\left\langle J_n'\left(w_k, z_k\right), \left(w_k - T_h\left(w_k\right), 0\right)\right\rangle \\
&\le \varepsilon_k \left[\frac{\|w_k - T_h(w_k)\|_{W_0^{1,(p_i)}(\Omega)\cap L^\infty(\Omega)}}{M_k} + \|w_k - T_h(w_k)\|_{W_0^{1,(p_i)}(\Omega)}\right] \\
&\le \varepsilon_k \left[\frac{2M_k + h + C_1}{M_k} + C_2\right] \\
&\le \varepsilon_k \left[\frac{h + C_1}{M_k} + C_3\right] \\
&\le \widetilde{\varepsilon_k} + h\widetilde{\varepsilon_k}
\end{aligned}$$

Now observe that as $\left\{w_k\right\}_k$ is bounded, we must have that $w_{k+}^p z_{k+}^q$ is bounded in $L^{\frac{\overline{p}^*}{p+q}}(\Omega)$and as $\frac{\overline{p}^*}{p+q} > 1$, Vitali's theorem and generelized Lebesgue theorem give

$$\int_\Omega g_n'(w_{k+}) z_{k+}^q w_{k+} = \int_\Omega g_n'(w) z_{k+}^q + \widetilde{\varepsilon_k}$$

Now using the fact

$$g_n'(w_{k+}) z_{k+}^q T_h(w) \le 2n^{1-q} z_{k+}^q h$$

and as $2n^{1-q} z_{k+}^q h$ is bounded in $L^{\frac{\overline{p}^*}{q}}(\Omega)$ for $\frac{\overline{p}^*}{q} > 1$, newly Vitali's theorem and generelized Lebesgue theorem give that

$$\int_\Omega g_n'(w_{k+}) z_{k+}^q T_h(w) = \int_\Omega g_n'(w_+) z_+^q T_h(w) + \widetilde{\varepsilon_k}$$

The previous computations added to hypotheses (a), (b) and (c) lead us to

$$\left|\sum_{i=1}^{N} \int_\Omega \left|\partial_i w_k\right|^{p_i-2} \partial_i w_k \partial_i\left(w_k - T_h(w)\right) - \int_\Omega g_n'(w_{k+}) z_{k+}^q \left(w_k - T_h(w)\right)\right| \le o(1) + ho(1). \tag{15}$$

Due to weak convergence of $\left\{w_k\right\}_k$ to w in $W_0^{1,(p_i)}(\Omega)$ we have,

$$\sum_{i=1}^{N} \int_\Omega \left|\partial_i w\right|^{p_i-2} \partial_i w \partial_i\left(w_k - T_h(w)\right) = o(1)$$

so from (15) and the latter observation

$$\sum_{i=1}^{N}\int_{\Omega}\Big(\left|\partial_i w_k\right|^{p_i-2}\partial_i w_k-\left|\partial_i T_h(w)\right|^{p_i-2}\partial_i T_h(w)\Big)\partial_i\big(w_k-T_h(w)\big)$$
$$-\int_{\Omega} g_n'(w_+)z_+^q\big(w-T_h(w)\big)$$
$$\le o(1)+ho(1),$$

using Lebesgue theorem we have

$$\lim_{n\to\infty}\int_{\Omega} g_n'(w_+)z_+^q\big(w-T_h(w)\big)=0$$

we thus obtain

$$\sum_{i=1}^{N}\int_{\Omega}\Big(\left|\partial_i w_k\right|^{p_i-2}\partial_i w_k-\left|\partial_i T_h(w)\right|^{p_i-2}\partial_i T_h(w)\Big)\partial_i\big(w_k-T_h(w)\big)\le o(1)+ho(1).$$

By (6) and (7) we conclude that

$$\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i\big(w_k-T_h(w)\big)\right|^{p_i}\le o(1)+ho(1). \tag{16}$$

we also observe that

$$\left|\partial_i\big(w_k-w\big)\right|^{p_i}\le 2^{p_i-1}\Big(\left|\partial_i\big(w_k-T_h(w)\big)\right|^{p_i}+\left|\partial_i G_h(w)\right|^{p_i}\Big) \tag{17}$$

so for $h>\widetilde{h}$

$$\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i G_h(w)\right|^{p_i}\le o(1)$$

and by (16) and (17)

$$\limsup_{k\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i\big(w_k-w\big)\right|^{p_i}=ho(1)$$

and in conclusion

$$w_k\to w \text{ strongly in } W_0^{1,(p_i)}(\Omega).$$

The same reasonning applied to $\{z_k\}_k$ instead of $\{w_k\}_k$ leads us to the conclusion

$$z_k\to z \text{ strongly in } W_0^{1,(p_i)}(\Omega).$$

This concludes the proof of the theorem. □

Proposition 5 *For each $n\in\mathbb{N}$ there exists a nonnegative couple $(w_n,z_n)\in Y$ such that*

$$\begin{cases} \sum_{i=1}^{N} \int_{\Omega} \left[|\partial_i w_n|^{p_i-2} \partial_i w_n \partial_i \varphi \right] = \int_{\Omega} g_n'(w_n) z_n^q \varphi & \varphi \in W_0^{1,(p_i)}(\Omega), \\ \sum_{i=1}^{N} \int_{\Omega} \left[|\partial_i z_n|^{p_i-2} \partial_i z_n \partial_i \psi \right] = q \int_{\Omega} g_n(w_n) z_n^{q-1} \psi & \psi \in W_0^{1,(p_i)}(\Omega), \end{cases} \tag{18}$$

Proof Let

$$\Gamma = \left\{ \text{Continuous } \gamma : [0,1] \to Y; \gamma(0) = (0,0) \text{ and } \gamma(1) = (\overline{w}, \overline{z}) \right\}$$

as before

$$J_n(w_n, z_n) = \sum_{i=1}^{N} \frac{1}{p_i} \int_{\Omega} |\partial_i w_n|^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i} \int_{\Omega} |\partial_i z_n|^{p_i} - \int_{\Omega} g_n(w_{n+}) z_{n+}^q$$

and

$$c_n = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_n(\gamma(t))$$

considering

$$(\overline{w}, \overline{z}) = (\bar{t}\chi, \bar{t}\chi)$$

a similar reasonning as the one made in the proof of the previous theorem and using Theorem 3 lead us to that c_n is a critical point of J_n so there exists a (w_n, z_n) such that for $(\varphi, \psi) \in Y$

$$J_n(w_n, z_n) = c_n \text{ and } \left\langle J_n'\left(w_n, z_n\right), (\varphi, \psi) \right\rangle = 0$$

choosing firstly $\varphi = 0$ and secondly $\psi = 0$ we obtain (18).

Now as $c_n > 0$ and by the reasonning made in the proof of the previous theorem we must have that at least one of w_n and z_n must be non identically null. We claim that both w_n and z_n are not identically null, indeed let us assume by contradiction that $z_n \equiv 0$ if we put $\varphi = w_n$ in (18) we came to the conclusion that also $w_n \equiv 0$ which is not possible, so necessarily both w_n and z_n are not identically equal to zero

To prove the nonnegativity of w_n and z_n, we only have to take $\varphi = w_n^-$ and $\psi = z_n^-$ in (18). □

4 Passage to the limit

Now we will pass to the limit, as n goes to $+\infty$ in $\left\{ \left(w_n, z_n\right) \right\}_n$.

Proposition 6 *Let* $\left\{ \left(w_n, z_n\right) \right\}_n$ *be the sequence introduced in the previous proposition, then*

$$w_n \rightharpoonup w \text{ weakly in } W_0^{1,(p_i)}(\Omega),$$

and

$$z_n \rightharpoonup z \text{ weakly in } W_0^{1,(p_i)}(\Omega).$$

Proof Let

$$\Gamma = \left\{ \text{Continuous } \gamma : [0,1] \to Y; \gamma(0) = (0,0) \text{ and } \gamma(1) = (\overline{w}, \overline{z}) \right\}$$

as before

$$J_n(w_n, z_n) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w_n|^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i z_n|^{p_i} - \int_\Omega g_n(w_{n+}) z_{n+}^q$$

and

$$c_n = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_n(\gamma(t))$$

so

$$J_n(w_n, z_n) \le \max_{t \in [0,1]} (t\overline{w}, t\overline{z})$$

since

$$\gamma : t \mapsto (t\overline{w}, t\overline{z}) \in \Gamma.$$

Repeating the same reasonning as the one done in the proof of Theorem 3 we arrive at

$$J_n(t\overline{w}, t\overline{z}) \le Ct^{p_N} - C_2 \frac{t^{q+2}}{(1+t)^{2-p}}$$

and so

$$J_n(w_n, z_n) \le \max_{t \in [0,1]} \left[Ct^{p_N} - C_2 \frac{t^{q+2}}{(1+t)^{2-p}} \right] = T$$

we deduce that

$$\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i w_n|^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i z_n|^{p_i} - \int_\Omega g_n(w_{n+}) z_n^q \le T. \tag{19}$$

Using $\varphi = w_n$ as test function in its equation in (18) we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i w_n|^{p_i} = \int_\Omega g_n'(w_n) z_n^q w_n$$

and using $\psi = z_n$ as test function its equation in () we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i z_n|^{p_i} = q \int_\Omega g_n(w_n) z_n^q$$

by (19) we arrive at

$$\int_\Omega \left[\frac{1}{p_N} \int_\Omega g_n'(w_n)w_n + \left(\frac{q}{p_N} - 1 \right) g_n(w_n) \right] z_n^q \le T$$

by the properties of g_n we have that

$$\frac{p+q-p_N}{p_N} \int_\Omega g_n(w_n) z_n^q \le T$$

in conclusion $\int_\Omega g_n(w_n)z_n^q$ is bounded independently of $n \in \mathbb{N}$, and consequently in view of (19) both $\{w_n\}_n$ and $\{z_n\}_n$ are bounded in $W_0^{1,(p_i)}(\Omega)$ which implies that

$$w_n \rightharpoonup w \text{ weakly in } W_0^{1,(p_i)}(\Omega),$$

and

$$z_n \rightharpoonup z \text{ weakly in } W_0^{1,(p_i)}(\Omega).$$

□

Proposition 7 *Neither w nor z can be identically equal to zero.*

Proof Let us assume by contradiction that $z = 0$, so $z_n \rightharpoonup 0$ weakly in $W_0^{1,(p_i)}(\Omega)$ and by the sequel $\{z_n\}_n$ converges strongly to 0 in any $L^r(\Omega)$ for every $r < \overline{p}^*$, since $p + q < \overline{p}^*$ we can always choose r in such a way to have $q < \frac{\overline{p}^* q}{\overline{p}^* - q} < r < \overline{p}^*$ and by the properties of g_n associated to Hölder inequality we have

$$\begin{aligned} \int_\Omega g_n(w_n) z_n^q &\le \int_\Omega w_n^p z_n^q \\ &\le \left(\int_\Omega w_n^{\frac{pr}{r-q}} \right)^{\frac{1-p}{r}} \left(\int_\Omega z_n^r \right)^{\frac{q}{r}} \end{aligned}$$

since $p < \overline{p}^*$ and by the choice of r we have $\frac{pr}{r-q} < \overline{p}^*$, and thus

$$\int_\Omega g_n(w_n) z_n^q \le C \left(\int_\Omega z_n^r \right)^{\frac{q}{r}}$$

and as $\{z_n\}_n$ is assumed to converge to 0

$$\int_\Omega g_n(w_n) z_n^q \to 0.$$

Choosing $\psi = z_n$ in the second equation of (18), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i z_n|^{p_i} \to 0$$

and so $z_n \rightharpoonup 0$ strongly in $W_0^{1,(p_i)}(\Omega)$. Now choosing $\varphi = w_n$ in the first equation of (18), we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |\partial_i w_n|^{p_i} &= \int_{\Omega} g_n'(w_n) w_n z_n^q \\ &\leq 2 \int_{\Omega} w_n^p z_n^q\end{aligned}$$

and thus also $w_n \rightharpoonup 0$ strongly in $W_0^{1,(p_i)}(\Omega)$. Finally we must have in that case

$$\lim_{n \to +\infty} J_n(w_n, z_n) = 0$$

which is a contradiction. With a similar reasonning we came to the same conclusion if we assume by contradiction that $w = 0$. □

Theorem 8 *Under the extra condition $p_1 > 2$, w and z solve problem* (1) *in the sense of* (8).

Proof To prove this theorem we need to pass to the limit when n goes to $+\infty$ in (18).

We begin by the second equation in (18), which represents the regular part of the system. As $\{w_n\}_n$ and $\{z_n\}_n$ converge strongly in $L^r(\Omega)$ for every $r < \overline{p}^*$, so by the properties of g_n and in view of $p + q < \overline{p}^*$, we obtain

$$g_n(w_n) z_n^{q-1} \to q w^p z^{q-1} \text{ strongly in } \left(W_0^{1,(p_i)}(\Omega)\right)',$$

so we conclude that z verify the second equation of (8), and as $p_1 > 2$ the anisotropic operator verify a strong maximum principle, see for instance [15], and thus $z > 0$ in Ω.

Now we turn our attention to the first equation which represents the singular part of (1), observe that the right hand side of this equation converges almost everywhere to $p \frac{z^q}{w^{1-p}}$ which is singular on the set where w vanishes. Since w is not identically equal to zero, one can always find two real numbers $a,\ b$ such that $0 < a < b$,

$$|\{x \in \Omega,\ a < w(x) < b\}| > 0$$

and

$$|\{x \in \Omega,\ w(x) = a\}| \cup |\{x \in \Omega,\ w(x) = b\}| = 0$$

where $|A|$ stands for the Lebesgue measure of the set A.

From the properties of g_n we have

$$g\prime_n(s) \geq M \text{ for } s \in [a, b] \text{ and } b > \frac{1}{n}$$

thus

$$\begin{aligned} g\prime_n(w_n)z_n^q \geq & M\chi_{\{x\in\Omega,\ a<w_n(x)<b\}} z_n^q \\ \geq & M\chi_{\{x\in\Omega,\ a<w_n(x)<b\}} T_1\left(z_n^q\right). \end{aligned}$$

where χ_A denotes the characteristic function of the set A.

Let y_n be solution to the auxiliary problem

$$\begin{cases} -Ly_n = -\sum_{i=1}^{N} \partial_i \left[\left| \partial_i y_n \right|^{p_i-2} \partial_i y_n \right] = M\chi_{\{x\in\Omega,\ a<w_n(x)<b\}} T_1\left(z_n^q\right) & \text{in } \Omega, \\ y_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that y_n always exists by classical theory as the right hand side belongs to $L^\infty(\Omega)$. The comparison principle leads us to the fact that

$$w_n \geq y_n.$$

From one hand by strong maximum principle we have that

$$y_n \geq C_K \text{ on every } K \subset\subset \Omega$$

On the other hand, a simple modification of De Giorgi theorem [26], allows us to obtain the uniform convergence of $\left\{y_n\right\}_n$ to $y \in W_0^{1,(p_i)}(\Omega)$ on every $K \subset\subset \Omega$; that is in particular

$$y_n(x) \geq y(x) - \varepsilon \geq C_K - \varepsilon \text{ for all } x \in K \text{ and } n \geq n_\varepsilon,$$

which leads to

$$w_n \geq \widetilde{C}_K \text{ in } K \text{ for } n \geq n_\varepsilon.$$

Now we are allowed to pass to the limit in the right hand side of the first equation of (18) as

$$g_n'(s) \leq 2s^{p-1}$$

and using $\varphi \in C_0^1(\Omega)$ as test function, considering $K = \{x \in \Omega,\ \varphi(x) > 0\}$ we obtain

$$\left| g_n'(w_n) z_n^q \varphi \right| \leq \frac{p}{\left(\widetilde{C}_K\right)^{1-p}} z_n^q \|\varphi\|_{L^\infty}$$

and as $q < \overline{p}^*$, by generelized Lebesgue theorem and Vitali's theorem we get

$$\lim_{n\to\infty} \int_\Omega g_n'(w_n) z_n^q \varphi = p \int_\Omega \frac{z^q}{w^{1-p}},$$

this associated to the weak convergence of $\left\{w_n\right\}_n$ to w lead us to the conlusion that w verify the first equation of (8), which ends the proof. □

5 Concluding remarks

We give here some remarks and observations.

1. We can generalize all the results obtained here to the more general problem

$$\begin{cases} -L_{(p_i)}u = -\sum_{i=1}^{N} \partial_i\left[\left|\partial_i u\right|^{p_i-2}\partial_i u\right] = p\dfrac{v^q}{u^{1-p}} & in\ \Omega, \\ -L_{(q_i)}v = -\sum_{i=1}^{N} \partial_i\left[\left|\partial_i v\right|^{q_i-2}\partial_i v\right] = qv^{q-1}u^p & in\ \Omega, \\ u > 0 \text{ and } v > 0 & in\ \Omega, \\ u = 0 \text{ and } v = 0 & on\ \partial\Omega, \end{cases}$$

   but this will generate a huge number of indices which can be tedious for the reader.
2. By some minor modifications, we can obtain similar results for anisotropic-isotropic problem of the form

$$\begin{cases} -L_{(p_i)}u = -\sum_{i=1}^{N} \partial_i\left[\left|\partial_i u\right|^{p_i-2}\partial_i u\right] = p\dfrac{v^q}{u^{1-p}} & in\ \Omega, \\ -\Delta_m v = -div\left(|\nabla v|^{m-2}\nabla v\right) = qv^{q-1}u^p & in\ \Omega, \\ u > 0 \text{ and } v > 0 & in\ \Omega, \\ u = 0 \text{ and } v = 0 & on\ \partial\Omega, \end{cases}$$

3. The regular-regular case corresponding to $p > 1$ and $q > 1$, can also be studied, by the use of classical variational methods and similar existence results can be obtained.
4. The hypothesis $p_1 > 2$ was introduced in the last theorem, as—to the best of our knowledge—strong maximum principle for anisotropic operator L is established only in that case.
5. The singular-singular case corresponding to $p < 1$ and $q < 1$, can be studied in a similar way by introducing the following definition of solution

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega \left[\left|\partial_i u\right|^{p_i-2}\partial_i u \partial_i \varphi\right] = p\int_\Omega \dfrac{v^q}{u^{1-p}}\varphi & \varphi \in C_0^1(\Omega), \\ \sum_{i=1}^{N} \int_\Omega \left[\left|\partial_i v\right|^{p_i-2}\partial_i v \partial_i \psi\right] = q\int_\Omega v^{q-1}u^p\psi & \psi \in C_0^1(\Omega), \end{cases}$$

   and an approximation energy functional of the form

$$J_n(w,z) = \sum_{i=1}^{N} \frac{1}{p_i}\int_\Omega |\partial_i w|^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i}\int_\Omega |\partial_i z|^{p_i} - \int_\Omega g_n(w_+)h_n\left(z_+\right)$$

   with

$$g_n(s) = \frac{s^2}{\left(s+\frac{1}{n}\right)^{2-p}} \text{ and } h_n(s) = \frac{s^2}{\left(s+\frac{1}{n}\right)^{2-q}}$$

and all the steps can be reproduced, but not for the last theorem as we need to have $p_1 > 2$ so the condition $p_N < p + q$ cannot be fulfilled as $p < 1, q < 1$ and $p_N > p_1 > 2$. Thus one have to use some other techniques to prove existence of solution.

6. In the particular case, $p_1 = p_2 = \cdots = p_N = P$, all the results obtained here are still vaild, but we underline that in this case we deal with an isotropic operator that absolutely doesn't coincide with the usual P-Laplace operator. Instead we obtain a closely related operator called the orthotropic P-Laplace operator or the pseudo P-Laplace operator who has an importance on its own. We invite the reader interested in this operator to see the very recent works [10, 11] and the references therein.

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