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Cauchy problem of nonlinear Klein–Gordon equations with general nonlinearities

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Abstract
This paper is concerned with the Cauchy problem of nonlinear Klein–Gordon equations with general nonlinearities. We use the potential well and convexity methods to prove the global existence and finite time blow up of solution with low and critical initial energy levels. And a finite time blow up of the solution with arbitrarily positive initial energy level is proved.

Keywords Cauchy problem · Klein–Gordon equations · Global solution · Finite time blow up

Mathematics Subject Classification 35L15

1 Introduction

In this paper, we consider the following nonlinear Klein–Gordon equation with general nonlinearities

$$u_{tt} - \Delta u + u = f(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

$$u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $\mathbb{R}^n (n \geq 1)$ is a unbounded domain, Δ is the Laplacian operator on $\mathbb{R}^n$,

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$$u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n),$$
$$f(u) = a_1|u|^{p_1} + \sum_{k=2}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u, \tag{1.4}$$

and $a_k > 0, 1 \leq k \leq l, b_j > 0, 1 \leq j \leq s, a_{l+1} > 0$, p_k and q_j satisfy

$$(\mathrm{H})\begin{cases} (1)\ \text{if}\ \ n = 1,2, 1 < q_s < q_{s-1} < \cdots < q_1 = q < p = p_1 < p_2 < \cdots < p_l < \infty; \\ (2)\ \text{if}\ \ n \geq 3, 1 < q_s < q_{s-1} < \cdots < q_1 = q < p = p_1 < p_2 < \cdots < p_l \leq \frac{n+2}{n-2}. \end{cases}$$

The three-dimensional Klein–Gordon equation (1.1) was simplified as

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u + V(u)u = 0$$

to describe the quantum behavior of free particles [1]. For the one-dimensional Klein–Gordon equation

$$u_{tt} - \alpha u_{xx} + f(u) = 0, \tag{1.5}$$

when the nonlinear source $f(u) = \sin u$, $\sinh u$, e^u, $e^u + e^{-2u}$ or $e^{-u} + e^{-2u}$, the equations (1.5) are called Sine-Gordon equations, Sinh-Gordon equations, Liouville equations, Dodd–Bullough–Mikhailov equations or Tzitzeica–Dodd–Bullough equations respectively. Sine-Gordon equations and Sinh-Gordon equations are used to describe the propagation of fluxons in Josephson junction between two superconductors and the motion of a rigid pendulum attached to a stretched line, which is often seen in solid state physics and nonlinear optics [1–3]. Liouville equations are used to describe the vibration of uniformly charged plasma [4]. Dodd–Bullough–Mikhailov equations and Tzitzeica–Dodd–Bullough equations appear in various problems in fluids and quantum field theory [2, 5]. In addition, the Eq. (1.5) with nonlinear source $f(u) = -|u|^2u + |u|^4u$ are used to describe the quantum behavior of particles with spin 0 [6, 7].

In order to give a theoretical explanation from a mathematical point, we start from the well-posedness of the solution for the problem (1.1)–(1.3) to reveal the dynamic behavior of the solution. For the Cauchy problem of the Klein–Gordon equations (1.1)–(1.3) with $f(u) = |u|^{p-1}u$, Xu [9] proved the global existence and finite time blow up of the solution at low initial energy level $E(0) < d$. Li and Zhang [10] extended the result of finite time blow up and global existence to the critical initial energy level $E(0) = d$ for the Cauchy problem of the Klein–Gordon equation (1.1)–(1.3) with $f(u) = u^2 + u^3$. Wang [8] proved the finite time blow up at the arbitrarily positive initial energy level $E(0) > 0$ to the Cauchy problem of the Klein–Gordon equation (1.1)–(1.3) with $f(u) = u^{p-1}u$. Kutev et al. [11] considered Klein–Gordon equation (1.1)–(1.3) with $f(u) = \sum_{k=1}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u$ and $f(u) = a_1|u|^{p_1} + \sum_{k=2}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u$, where $a_1 > 0$, $a_k > 0$, $2 \leq k \leq l$, $b_j \geq 0$, $1 \leq j \leq s$, p_k and q_j satisfy (H). The global existence and finite time blow up of the solution at the critical initial energy level $E(0) = d$ is proved in [12]. Kutev et al. [11] considered the same Cauchy problem and proved the finite time blow up of the solution at arbitrarily positive initial energy level $E(0) > 0$.

As the nonlinear source terms reflect the influence of the nonlinear factors on the Cauchy problem of the Klein–Gordon equation (1.1)–(1.3), while the complex combined source terms describe the superimposition of these related factors. Obviously, the signs of the source terms greatly affect the size and direction of the combined source terms. The coefficients $a_1 > 0$, $a_k > 0$ and $b_j \geq 0$ in [11, 12], which mean that the directions of external force are positive, positive and negative respectively, and this nonlinear term is more general than the nonlinear term which appear in the damped semilinear wave equations [14], the fractional Laplacian parabolic [15], variable exponent parabolic equation [16], the parabolic systems [17], the pseudo-parabolic equation with singular potential [18], nonlinear Schröinger equation with a harmonic potential [19] and fractional p-Laplacian evolution equations [20]. In this present paper, we consider the nonlinear source $f(u) = a_1|u|^{p_1} + \sum_{k=2}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u$, where $a_1 < 0$, $a_k > 0$, $2 \leq k \leq l$, $b_j \geq 0$, $1 \leq j \leq s$, p_k and q_j satisfy (H) to reveal the negative effects of the term $a_1|u|^{p_1}$, $a_1 < 0$ and prove the well-posedness of the solution to the Cauchy problem of the Klein–Gordon equation (1.1)–(1.3).

2 Preliminaries

We commence this section by introducing the norms $\|u\|_p := \|u\|_{L^p(\mathbb{R}^n)}$, $\|u\| := \|u\|_{L^2(\mathbb{R}^n)}$ and the inner product $(u, v) := \int_{\mathbb{R}^n} uv\mathrm{d}x$. Also we introduce the norm for $H^1(\mathbb{R}^n)$

$$\|u\|_{H^1}^2 := \|u\|^2 + \|\nabla u\|^2.$$

We define two C^1 functionals on $H^1(\mathbb{R}^n) \to \mathbb{R}$, known as potential functional and Nehari function respectively as follows

$$J(u) := \frac{1}{2}\|u\|_{H^1}^2 - \sum_{k=2}^{l} \frac{a_k}{p_k+1}\|u\|_{p_k+1}^{p_k+1} - \frac{a_1}{p_1+1}\int_{\mathbb{R}^n} |u|^{p_1}u\mathrm{d}x + \sum_{j=1}^{s} \frac{b_j}{q_j+1}\|u\|_{q_k+1}^{q_k+1} \tag{2.1}$$

and

$$I(u) := \|u\|_{H^1}^2 - \sum_{k=2}^{l} a_k\|u\|_{p_k+1}^{p_k+1} - a_1\int_{\mathbb{R}^n} |u|^{p_1}u\mathrm{d}x + \sum_{j=1}^{s} b_j\|u\|_{q_j+1}^{q_j+1}. \tag{2.2}$$

We also define Nehari manifold

$$\mathcal{N} := \left\{u \in H^1(\mathbb{R}^n) |\, I(u) = 0, \|\nabla u\| \neq 0\right\}$$

and the depth of the potential well (the so-called mountain pass level in [21])

$$d := \inf_{u\in\mathcal{N}} J(u).$$

Now, we define the potential well

$$W := \left\{u \in H^1(\mathbb{R}^n) | I(u) > 0, J(u) < d\right\} \cup \{0\},$$

the outer of the potential well

$$V := \left\{u \in H^1(\mathbb{R}^n) | I(u) < 0, J(u) < d\right\}$$

and the family of potential wells

$$I_\delta(u) := \delta\|u\|_{H^1}^2 - a_1\int_{\mathbb{R}^n} |u|^{p_1}u\mathrm{d}x - \sum_{k=2}^{l} a_k\|u\|_{p_k+1}^{p_k+1} + \sum_{j=1}^{s} b_j\|u\|_{q_j+1}^{q_j+1},\ \delta > 0.$$

The corresponding Nehari manifolds and the depth of the family of potential wells are defined respectively by

$$\mathcal{N}_\delta := \left\{u \in H^1(\mathbb{R}^n)\big| I_\delta(u) = 0, \|\nabla u\| \neq 0\right\}$$

and

$$d(\delta) := \inf_{u\in\mathcal{N}_\delta} J(u). \tag{2.3}$$

Next we introduce the stable set W_δ and the unstable set V_δ defined by

$$W_\delta := \{u \in H^1(\mathbb{R}^n)\big| I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}$$

and

$$V_\delta := \{u \in H^1(\mathbb{R}^n)\big| I_\delta(u) < 0,\ J(u) < d(\delta)\}.$$

Definition 2.1 Function $u(x,\ t)$ is called a week solution to problem (1.1)–(1.3), if it satisfies

$$u \in L^\infty\left([0,T);H^1(\mathbb{R}^n)\right), u_t \in L^\infty\left([0,T);L^2(\mathbb{R}^n)\right)$$

and there holds

$$\langle u_{tt}, v\rangle + (\nabla u, \nabla v) + (u, v) = (f(u), v) \tag{2.4}$$

for any $v \in H^1(\mathbb{R}^n)$, $t \in [0,T)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\mathbb{R}^n)$ and $H_0^1(\mathbb{R}^n)$, and the following energy equality holds

$$E(t) = E(0),\ t \in (0,T), \tag{2.5}$$

where

$$E(t) := \frac{1}{2}\|u_t\|^2 + J(u). \tag{2.6}$$

Lemma 2.2 *Assume $f(u)$ satisfy (H), then we have*

$$(p+1)F(u) \le uf(u) \text{ for all } u \in \mathbb{R}, \tag{2.7}$$

where $F(u) := \int_0^u f(s)\mathrm{d}s$.

Proof When $u \ge 0$, then by (i) in (H), we have

$$uf'(u) \ge pf(u)$$

and

$$\int_0^u sf'(s)\mathrm{d}s \geq p\int_0^u f(s)\mathrm{d}s = pF(u) \quad \text{for all} \quad u \geq 0,$$

which implies

$$uf(u) - \int_0^u f(s)\mathrm{d}s \geq pF(u) \quad \text{for all} \quad u \geq 0,$$

that is

$$(p+1)F(u) \leq uf(u).$$

The proof of the case $u < 0$ is similar to the case $u \geq 0$. □

Lemma 2.3 *Let $f(u)$ satisfy (H), then*

(i) $|F(u)| \leq \sum_{k=1}^{l} \frac{a_k}{p_k+1}|u|^{p_k+1} + \sum_{j=1}^{s} \frac{b_j}{q_j+1}|u|^{q_j+1}$ *for all* $u \in \mathbb{R}$;
(ii) $F(u) \geq B|u|^{p+1}$ *for* $B = F(1)$ *and all* $|u| \geq 1$.

Proof (i) From (H), it implies that

$$\begin{aligned}|F(u)| \leq &\left|\int_0^u \left(\sum_{k=1}^{l} a_k|u|^{p_k-1}u + a_1|u|^{p_1} - \sum_{j=1}^{s} b_j|u|^{q_j-1}u\right)\mathrm{d}u\right| \\ &\leq \sum_{k=1}^{l}\frac{a_k}{p_k+1}|u|^{p_k+1} + \sum_{j=1}^{s}\frac{b_j}{q_j+1}|u|^{q_j+1}.\end{aligned}$$

(ii) When $u > 0$, by (H), we have $F(u) > 0$. Then Lemma 2.2 tells

$$\frac{p+1}{u} \leq \frac{f(u)}{F(u)} = \frac{\frac{\mathrm{d}F(u)}{\mathrm{d}u}}{F(u)},$$

which gives

$$(p+1)\int_1^u \frac{1}{s}\mathrm{d}s \leq \int_1^u \frac{\mathrm{d}F(s)}{F(s)},$$

that is

$$\ln u^{p+1} \leq \ln F(u) - \ln F(1).$$

Then $F(u) \geq Bu^{p+1}$, $B = F(1)$. Similarly to the case $u \geq 1$, we obtain $F(u) \geq Bu^{p+1}$ for all $u \leq -1$. □

Lemma 2.4 (Relations between $I(u)$ *and* $\|\nabla u\|$) *Let* $\delta > 0$.

(1) *If* $0 < \|u\|_{H^1} < \gamma(\delta)$, *then* $I_\delta(u) > 0$. *In particular, if* $0 < \|u\|_{H^1} < \gamma(1)$, *then* $I(u) > 0$;
(2) *If* $I_\delta(u) < 0$, *then* $\|u\|_{H^1} > \gamma(\delta)$. *In particular, if* $I(u) < 0$, *then* $\|u\|_{H^1} > \gamma(1)$;

(3) *If* $I_\delta(u) = 0$ *and* $\|u\|_{H^1} \neq 0$, *then* $\|u\|_{H^1} \geq \gamma(\delta)$. *In particular, if* $I(u) < 0$ *and* $\|u\|_{H^1} \neq 0$, *then* $\|u\|_{H^1} \geq \gamma(1)$,

where $\gamma(\delta)$ is the unique real root of equation $\varphi(\gamma) = \delta$,

$$\varphi(\gamma) := \sum_{k=1}^{l+1} a_k C_k^{p_k+1} \gamma^{p_k-1},\ C_k := \sup_{u \in H_0^1(\mathbb{R}^n)} \frac{\|u\|_{p_k+1}}{\|u\|_{H^1}}, 1 \leq k \leq l.$$

Proof (1) From $0 < \|u\|_{H^1} \leq \gamma(\delta)$, we have $\|u\|_{q_j+1} > 0, 1 \leq j \leq s$. Hence by

$$\begin{aligned}
&\sum_{k=2}^{l} a_k \|u\|_{p_k+1}^{p_k+1} + a_1 \int_{\mathbb{R}^n} |u|^{p_1} u \mathrm{d}x - \sum_{j=1}^{s} b_j \|u\|_{q_j+1}^{q_j+1} \\
&< \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} \leq \sum_{k=1}^{l} a_k C_k^{p_k+1} \|u\|_{H^1}^{p_k+1} \\
&= \varphi(\|u\|_{H^1}) \|u\|_{H^1}^2 \leq \delta \|u\|_{H^1}^2,
\end{aligned}$$

we get $I_\delta(u) > 0$.

(2) The inequality $I_\delta(u) < 0$ gives

$$\begin{aligned}
\delta \|u\|_{H^1}^2 &< \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} - \sum_{j=1}^{s} b_j \|u\|_{q_j+1}^{q_j+1} \\
&< \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} \leq \varphi(\|u\|_{H^1}) \|u\|_{H^1}^2,
\end{aligned}$$

which implies $\|u\|_{H^1} > \gamma(\delta)$.

(3) If $I_\delta(u) = 0$ and $\|u\|_{H^1} \neq 0$, then by

$$\begin{aligned}
\delta \|u\|_{H^1}^2 &= \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} - \sum_{j=1}^{s} b_j \|u\|_{q_j+1}^{q_j+1} \\
&< \sum_{k=1}^{l} a_k \|u\|_{p_k+1}^{p_k+1} \leq \varphi(\|u\|_{H^1}) \|u\|_{H^1}^2,
\end{aligned}$$

we get $\|u\|_{H^1} > \gamma(\delta)$. □

Lemma 2.5 *Let* $f(u)$ *satisfy* (*H*), $u \in H^1(\mathbb{R}^n)$, $\|u\|_{H^1} \neq 0$ *and*

$$\varphi(\lambda) := \frac{1}{\lambda} \int_{\mathbb{R}^n} u f(\lambda u) \mathrm{d}x.$$

Then

(i) $\varphi(\lambda)$ *is strictly increasing on* $[0, +\infty)$;
(ii) $\lim_{\lambda \to 0} \varphi(\lambda) = 0, \ \lim_{\lambda \to +\infty} \varphi(\lambda) = +\infty$.

Proof (i) For $\lambda > 0$, the conclusion follow from

$$\frac{\mathrm{d}\varphi(\lambda)}{\mathrm{d}\lambda} = \frac{1}{\lambda}\int_{\mathbb{R}^n} u^2 f'(\lambda u)\mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}^n} uf(\lambda u)\mathrm{d}x$$
$$= \frac{1}{\lambda^3}\int_{\mathbb{R}^n} \lambda u\big(\lambda u f'(\lambda u) - f(\lambda u)\big)\mathrm{d}x > 0$$

and (i) in (H).

(ii) It follows directly from (ii) in (H) that

$$|\varphi(\lambda)| \le \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |\lambda u f(\lambda u)|\mathrm{d}x \le \frac{1}{\lambda^2}\int_{\mathbb{R}^n} \sum_{k=1}^{l} a_k |\lambda u|^{p_k+1}\mathrm{d}x = \sum_{k=1}^{l} a_k \lambda^{p_k-1}\|u\|_{p_k+1}^{p_k+1},$$

which implies that $\lim_{\lambda\to 0}\varphi(\lambda) = 0$. From Lemma 2.2 and (ii) in Lemma 2.3, we get

$$\varphi(\lambda) = \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |\lambda u f(\lambda u)|\mathrm{d}x \ge \frac{p+1}{\lambda^2}\int_{\mathbb{R}^n_\lambda} F(\lambda u)\mathrm{d}x$$
$$\ge \frac{p+1}{\lambda^2}\int_{\mathbb{R}^n_\lambda} B|\lambda u|^{p+1}\mathrm{d}x = (p+1)B\lambda^{p-1}\int_{\mathbb{R}^n_\lambda} |u|^{p+1}\mathrm{d}x,$$

where $\mathbb{R}^n_\lambda = \left\{x | x \in \mathbb{R}^n, |u| \ge \frac{1}{\lambda}\right\}$. Hence from

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^n_\lambda} |u|^{p+1}\mathrm{d}x = \|u\|_{p+1}^{p+1} > 0,$$

we have

$$\lim_{\lambda\to+\infty}\varphi(\lambda) = +\infty.$$

□

Lemma 2.6 (Properties of $J(\lambda u)$, Lemma 2.2 in [23] and Lemma 6 in [24]) *Let* $u \in H_0^1(\Omega)$ *and* $\|u\|_{H^1} \neq 0$. *Then*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.
(ii) *there exists a unique* $\lambda^* \in (0,\infty)$ *such that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)\right|_{\lambda=\lambda^*} = 0.$$

(iii) $J(\lambda u)$ *is increasing on* $0 \le \lambda \le \lambda^*$, *decreasing on* $\lambda^* \le \lambda < \infty$ *and takes the maximum at* $\lambda = \lambda^*$.
(iv) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < \infty$ *and* $I(\lambda^* u) = 0$.

Lemma 2.7 (Properties of $d(\delta)$, Lemma 3 in [25]) $d(\delta)$ *possesses the following properties*

(i) $d(\delta) > a(\delta)\gamma^2(\delta)$ *for* $0 < \delta < \frac{p+1}{2}$, *where* $a(\delta) := \frac{1}{2} - \frac{\delta}{p+1}$.
(ii) $\lim_{\delta\to 0} d(\delta) = d(0)$ *and there exists a unique* $b > \frac{p+1}{2}$ *such that* $d(\delta_0) = 0$ *and* $d(\delta) > 0$ *for* $0 \le \delta < b$.

(iii) *$d(\delta)$ is increasing on $0 \le \delta \le 1$, decreasing on $1 \le \delta \le \delta_0$ and takes the maximum d=$d(1)$ at $\delta = 1$.*

Lemma 2.8 (Invariance sets for $E(0) < d$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1(x) \in L^2(\mathbb{R}^n)$. Suppose that $0 \le e < d$, δ_1 and δ_2 are the two roots of equation $d(\delta) = e$. Then,*

(i) *the solution of problem (1.1)–(1.3) with $0 < E(0) \le e$ belongs to W_δ for $\delta_1 < \delta < \delta_2$, provided that $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$;*
(ii) *the solution of problem (1.1)–(1.3) with $0 < E(0) \le e$ belongs to V_δ for $\delta_1 < \delta < \delta_2$, provided that $I(u_0) < 0$.*

Proof Assume $u = u(t)$ is the solution to problem (1.1)–(1.3) with $0 < E(0) \le e$, $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$ and T is the maximum existence time of $u(t)$. If $\|u_0\|_{H^1} = 0$, then $u_0 \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. If $I(u_0) > 0$, then by (2.3) and

$$\frac{1}{2}\|u_1\| + J(u_0) = E(0) = d(\delta_1) = d(\delta_2) < d(\delta),\ \delta \in (\delta_1, \delta_2),$$

it follows that $I_\delta(u_0) > 0$ and $J(u_0) < d(\delta)$, i.e., $u_0(x) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. Next we prove $u(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in (0, T)$. Arguing by contradiction, we suppose that there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_\delta$, i.e.,

$$I_\delta(u_0) = 0,\ \|u(t_0)\|_{H^1} \ne 0 \quad \text{and} \quad J(u(t_0)) = d(\delta),\ \delta \in (\delta_1, \delta_2).$$

From (2.5), we obtain

$$\frac{1}{2}\|u_t\|^2 + J(u) = E(0) < d(\delta),\ \delta \in (\delta_1, \delta_2),\ t \in (0, T), \tag{2.8}$$

which implies that $J(u(t_0)) = d(\delta)$ is impossible. If $I_\delta(u(t_0)) = 0$ and $\|u(t_0)\|_{H^1} \ne 0$, by (2.3), it follows that $J(u(t_0)) \ge d(\delta)$, which contradicts (2.8). Similarly, we can achieve the second statement. □

3 Global solution and finite time blow up for $E(0) < d$

Theorem 3.1 (Global existence for $E(0) < d$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$ and $u_1(x) \in L^2(\mathbb{R}^n)$. Suppose that $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$. Then problem (1.1)–(1.3) admits a global weak solution*

$$u(t) \in L^\infty\big((0,\infty); H^1(\mathbb{R}^n)\big)$$

with

$$u_t(t) \in L^\infty\big((0,\infty); L^2(\mathbb{R}^n)\big)$$

and $u(t) \in W$ for $t \in (0, +\infty)$.

Proof Let $\{\omega_j(x)\}$ be a system of base functions in $H_0^1(\mathbb{R}^n)$. Construct approximate solutions of problem (1.1)–(1.3) as

$$u_m(x,t)=\sum_{j=1}^{m} g_{jm}(t)\omega_j(x), m=1,2,\cdots$$

satisfying

$$\langle u_{mtt},\omega_s\rangle+(\nabla u_m,\nabla\omega_s)+(u_m,\omega_s)=(f(u_m),\omega_s),\quad s=1,2,\dots,m,$$
$$u_m(x,0)=\sum_{j=1}^{m} g_{jm}(0)\omega_j(x)\to u_0(x)\in H_0^1(\mathbb{R}^n)$$

and

$$u_{mt}(x,0)=\sum_{j=1}^{m} g'_{jm}(0)\omega_j(x)\to u_1(x)\in L^2(\mathbb{R}^n).$$

Then by the same arguments used in the proof of Theorem 3.2 in [13], for sufficiently large m and $t\in(0,+\infty)$ we obtain

$$\frac{1}{2}\|u_{mt}\|^2+J(u_m)=E_m(0)<d \tag{3.1}$$

and $u_m(t)\in W$. From (3.1) and

$$\begin{aligned} J(u_m)&\ge\frac{1}{2}\|u_m\|_{H^1}^2-\frac{1}{p+1}\left(\sum_{k=2}^{l}a_k\|u\|_{p_k+1}^{p_k+1}+\int_{\mathbb{R}^n}|u|^{p_1}u\mathrm{d}x-\sum_{j=1}^{s}b_j\|u\|_{q_k+1}^{q_k+1}\right)\\ &=\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_m\|_{H^1}^2+\frac{1}{p+1}I(u_m)\ge\frac{p-1}{2(p+1)}\|u_m\|_{H^1}^2, \end{aligned} \tag{3.2}$$

for sufficiently large m it follows that

$$\frac{1}{2}\|u_{mt}\|^2+\frac{p-1}{2(p+1)}\|u_m\|_{H^1}^2<d,\ t\in(0,+\infty), \tag{3.3}$$

for sufficiently large m which implies that

$$\|u_m\|_{H^1}^2<\frac{2(p+1)}{p-1}d,\ t\in(0,+\infty) \tag{3.4}$$

and

$$\|u_{mt}\|^2<2d,\ t\in(0,+\infty). \tag{3.5}$$

From the definition of C_k and (3.3), for sufficiently large m we have

$$\|u_m\|_{p_k+1}^2\le C_k^2\|u_m\|_{H^1}^2<C_k^2\frac{2(p+1)}{p-1}d,\ 1\le k\le l,\ t\in(0,+\infty). \tag{3.6}$$

From (3.4)–(3.6) and compactness method it follows that problem (1.1)–(1.3) admits a global weak solution $u(t) \in L^\infty\big([0,\infty);H_0^1(\mathbb{R}^n)\big)$ with $u_t(t) \in L^\infty\big([0,\infty);L^2(\mathbb{R}^n)\big)$. Finally by Lemma 2.8, for $t \in (0,+\infty)$ we have $u(t) \in W$. □

Theorem 3.2 (Finite time blow up for $E(0) < d$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1(x) \in L^2(\mathbb{R}^n)$. Assume that $E(0) < d$ and $I(u_0) < 0$, then the solution to problem (1.1)–(1.3) blows up in finite time.*

Proof Arguing by contradiction, we assume the maximum existence time $T = +\infty$. First, for any $T > 0$ we define

$$M(t) := \|u(t)\|^2, \tag{3.7}$$

then

$$M'(t) = 2(u_t, u) \tag{3.8}$$

and

$$M''(t) = 2\|u_t\|^2 - 2I(u) \tag{3.9}$$

due to (2.4). From (3.1) and (3.2), we have

$$2I(u) \le 2(p+1)E(0) - (p+1)\|u_t\|^2 - (p-1)\|u\|_{H^1}^2. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain

$$M''(t) \ge (p-1)M(t) + (p+3)\|u_t\|^2 - 2(p+1)E(0). \tag{3.11}$$

Now, we consider the following two cases respectively.

(i) If $0 < E(0) < d$, then from Lemma 2.8, it follows that $u(t) \in V_\delta$ for $1 < \delta < \delta_2$ and $t > 0$, where δ_2 is the same as that in Theorem 2.8. Thus $I_\delta(u) < 0$ and $\|u\|_{H^1} > \gamma(\delta)$ for $1 < \delta < \delta_2$ and $t > 0$. Therefore, we obtain $I_{\delta_2}(u) \le 0$ and $\|u\|_{H^1} \ge \gamma(\delta_2)$ for $t > 0$ and by (3.9), for $t \in [0,T)$ we have

$$\begin{aligned} M''(t) \ge -2I(u) =& 2(\delta_2 - 1)\|u\|_{H^1}^2 - 2I_{\delta_2}(u) \\ \ge& 2(\delta_2 - 1)\gamma^2(\delta_2) > 0, \end{aligned}$$

then

$$M'(t) \ge 2(\delta_2 - 1)\gamma^2(\delta_2)t + M'(0),$$

which shows that there exists a $t_0 \ge 0$ such that

$$M'(t) > M'(t_0) > 0$$

and

$$M(t) \ge M'(t_0)(t - t_0) + M(t_0),\ t \ge t_0.$$

Hence for a sufficiently large t, we get

$$(p-1)M(t) > 2(p+1)E(0)$$

and

$$M''(t) > (p+3)\|u_t\|^2. \tag{3.12}$$

Finally, Schwarz inequality tells

$$M(t)M''(t) - \frac{p+3}{4}(M'(t))^2 \geq M(t)\big(M''(t) - (p+3)\|u_t\|^2\big) > 0,$$

then

$$(M^{-\alpha}(t))'' = \frac{-\alpha}{M(t)^{\alpha+2}}\big(M(t)M''(t) - (\alpha+1)(M'(t))^2\big) < 0, \ \alpha = \frac{p-1}{4}$$

and

$$\lim_{t\to T^*} M(t) = +\infty$$

for some $T^* > 0$, which contradicts $T = +\infty$.

(ii) When $E(0) \leq 0$, by (3.11), we obtain (3.12). The remaining proof is similar to the case (i). □

4 Global existence and finite time blow up for $E(0) = d$

Theorem 4.1 (Global existence for $E(0) = d$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1(x) \in L^2(\mathbb{R}^n)$. Suppose that $E(0) = d$ and $u_0 \in W$, then problem (1.1)–(1.3) admits a global weak solution $u(t) \in L^\infty\big([0,T);H^1(\mathbb{R}^n)\big)$ with $u_t(t) \in L^\infty\big([0,T);L^2(\mathbb{R}^n)\big)$.*

Proof We prove this theorem considering two cases (i) and (ii).

(i) In the case $\|u_0\|_{H^1} \neq 0$, let $\lambda_m = 1 - \frac{1}{m}$ and $u_{0m} = \lambda_m u_0$, $m = 2, 3, \cdots$. Consider the initial conditions

$$u(x,0) = u_{0m}(x), \quad u_t(x,0) = u_1(x) \tag{4.1}$$

and the corresponding problem (1.1), (4.1). From $I(u_0) > 0$ and (iii), (iv) in Lemma 2.6, we have

$$\lambda^* = \lambda^*(u_0) \geq 1 \ \text{ and } \ J(u_{0m}) = J(\lambda_m u_0) < J(u_0). \tag{4.2}$$

From (2.1), we replace $\|u_{0m}\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}$ by $I(u_{0m})$ to have

$$\begin{aligned} J(u_{0m}) \geq& \frac{1}{2}\|\nabla u_{0m}\|^2 + \frac{1}{2}\|u_{0m}\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} u_{0m} f(u_{0m})\mathrm{d}x \\ =& \frac{p-1}{2(p+1)}\|u_{0m}\|^2_{H^1} + \frac{1}{p+1}I(u_{0m}) > 0. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$0 < E_m(0) \equiv J(u_{0m}) + \frac{1}{2}\|u_1\|^2 < \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d.$$

Similar to the proof of Theorem 3.1, we finish this proof.

(ii) We discuss the case $\|u_m\|_{H^1}^2 = 0$, which implies $J(u_0) = 0$ and $\frac{1}{2}\|u_1\|^2 = E(0) = d$. Let $\lambda_m = 1 - \frac{1}{m}$, $u_{1m}(x) = \lambda_m u_1(x)$, $m = 2, 3, \dots$. We take initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_{1m}(x) \tag{4.4}$$

and consider the corresponding problem (1.1) and (4.4). From $J(u_0) = 0$ and (2.6), we have

$$0 < E_m(0) = \frac{1}{2}\|u_{1m}\|^2 + J(u_0) = \frac{1}{2}\|\lambda_m u_1\|^2 < \frac{1}{2}\|u_1\|^2 = E(0) = d.$$

The remainder proof is similar to part (i) of this Theorem. □

Lemma 4.2 (Invariance of V' for $E(0) = d$, Lemma 2.7 in [22]) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1(x) \in L^2(\mathbb{R}^n)$. Suppose that $E(0) = d$ and $(u_0, u_1) \geq 0$, then the set*

$$V' = \left\{u \in H^1(\mathbb{R}^n) \middle|\, I(u) < 0\right\}$$

is invariant under the flow of problem (1.1)–(1.3).

Theorem 4.3 (Finite time blow up for $E(0) = d$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1(x) \in L^2(\mathbb{R}^n)$. Assume that $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$. Then the solution to problem* (1.1)–(1.3) *blows up in finite time.*

Proof Let $u(t)$ be any weak solution to problem (1.1)–(1.3) with $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$ and T be the maximum existence time of $u(t)$. We prove $T < +\infty$. Arguing by contradiction, we suppose $T = \infty$. Recalling auxiliary function $M(t)$ as (3.7) shows and from Lemma 4.2, we have

$$M''(t) = 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle = 2\|u_t\|^2 - 2I(u) > 0,\ t \in (0, +\infty), \tag{4.5}$$

which implies that $M'(t)$ is strictly increasing on $(0, \infty)$. Hence for any $t_0 > 0$, we get

$$M'(t) > M'(t_0) > M'(0) \geq 0,\ t \in (0, +\infty),$$

then

$$M(t) \geq M'(t_0)(t - t_0) + M(t_0) \geq M'(t_0)(t - t_0),\ t \in (0, +\infty).$$

Similarly arguments to Theorem 3.2, we derive the conclusion. □

5 Finite time blow up for $E(0) > 0$

Lemma 5.1 *Let $u_0(x) \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ and $(u_0, u_1) \geq 0$. Suppose that u is a solution of the problem* (1.1)–(1.3), *then the map $\{t \mapsto \|u\|^2\}$ is strictly increasing as long as $u(t) \in V'$.*

Proof Recalling the function $M(t)$ as (3.7) shows and by (3.9) and $u \in V'$, we have

$$M''(t) = 2\langle u, u_{tt}\rangle + 2\|u_t\|^2 = -2I(u) + 2\|u_t\|^2 > 0,\ t \in (0, +\infty).$$

Similarly arguments to Theorem 4.3, we know that $M(t)$ is strictly increasing on $[0, +\infty)$. □

Lemma 5.2 (Invariance of the unstable set V' for $E(0) > 0$) *Let $u_0(x) \in H^1(\mathbb{R}^n)$ and $u_1(x) \in L^2(\mathbb{R}^n)$. Assume that (H), $(u_0, u_1) \geq 0$, $u_0 \in V'$ and*

$$\|u_0\|^2 > \frac{2(p+1)}{p-1}E(0) > 0 \tag{5.1}$$

hold, then $u \in V'$ for all $t \in [0, T)$.

Proof We prove $u(t) \in V'$ for all $t \in [0, T)$. By contradiction, suppose that there is a $t_0 \in (0, T)$ such that $u \in \mathcal{N}$ and $I(u(t)) < 0$ for all $t \in [0, t_0)$. The Lemma 5.1 tells that $M(t)$ is strictly increasing on the interval $[0, t_0)$, which implies that

$$M(t) > \|u_0\|^2 > \frac{2(p+1)}{p-1}E(0),\ t \in [0, t_0).$$

Thus the continuity of $u(t)$ in time tells

$$M(t_0) = \|u(t_0)\|^2 > \frac{2(p+1)}{p-1}E(0). \tag{5.2}$$

Then by (2.2) and (2.5), for $t \in [0, t_0]$ we obtain

$$\begin{aligned} E(0) = E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_{H^1}^2 - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\|u\|_{p_k+1}^{p_k+1} + \sum_{j=1}^{s} \frac{b_j}{q_j+1}\|u\|_{q_k+1}^{q_k+1} \\ \geq& \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|u\|^2 + \frac{1}{p+1}I(u). \end{aligned} \tag{5.3}$$

We substitute $t = t_0$ into (5.3) and by the fact that $I(u(t_0)) = 0$ to obtain

$$\|u(t_0)\|^2 \leq \frac{2(p+1)}{p-1}E(0),$$

which contradicts (5.2). So we complete the proof. □

Theorem 5.3 (Finite time blow up for $E(0) > 0$) *Let $f(u)$ satisfy (H), $u_0(x) \in H^1(\mathbb{R}^n)$ and $u_1(x) \in L^2(\mathbb{R}^n)$. Suppose that $E(0) > 0$, $I(u_0) < 0$, $(u_0, u_1) \geq 0$ and (5.1) hold, then the corresponding solution $u(x, t)$ of problem (1.1)–(1.3) blows up in finite time.*

Proof By contradiction, we suppose that $u(t)$ is global in time. For any $T > 0$, from (3.7), the Schwarz inequality and (4.5), we obtain

$$
\begin{aligned}
&M''(t)M(t) - \frac{p+3}{4}(M'(t))^2 \\
&\geq M(t)\big(M''(t) - (p+3)\|u_t\|^2\big) \\
&= M(t)\big(-2I(u) - (p+1)\|u_t\|^2\big) \\
&= M(t)\xi(t), \ t \in [0, T),
\end{aligned}
\tag{5.4}
$$

where

$$\xi(t) := -2I(u) - (p+1)\|u_t\|^2, \ t \in [0, T). \tag{5.5}$$

By (5.3), we have

$$(p+1)\|u_t\|^2 \leq -(p-1)\|u\|_{H^1}^2 + 2(p+1)E(0) - 2I(u), \ t \in [0, T). \tag{5.6}$$

We substitute (5.6) into (5.5), by Lemma 5.1 and (5.1) to obtain

$$\xi(t) \geq (p-1)\|u_0\|^2 - 2(p+1)E(0) > 0, \ t \in [0, T),$$

then

$$\xi(t) > \delta, \ t \in [0, T) \tag{5.7}$$

for a constant $\delta > 0$. On the other hand, the Lemma 5.2 tells that $I(u(t)) < 0$ for all $t \in [0, T)$. Similar arguments in Lemma 5.1, we know that $M(t)$ is strictly increasing on $[0, T)$. The continuity of $u(t)$ in t tells

$$M(t) \geq \rho, \ t \in [0, T)$$

for a constant $\rho > 0$. Hence from (5.4) and (5.7), we have

$$M''(t)M(t) - \frac{p+3}{4}(M'(t))^2 > \rho\delta, \ t \in [0, T).$$

Then similar arguments in the proof of Theorem 3.2, we achieve the conclusion. □

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