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# Spectrum of periodic Sturm-Liouville problems involving additional transmission conditions

**K. Aydemir**[1] · **O. Sh. Mukhtarov**[2,3]

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**Abstract**
This paper presents some spectral properties of the Sturm-Liouville equation associated with periodic boundary conditions and additional transmission conditions at one interior singular point. We derived asymptotic formulas for the eigenvalues of the considered problem. Note that the problem under consideration has not been investigated previously. To illustrate the originality of the results we constructed a simple example of a periodic Sturm-Liouville problem with transmission conditions, whose eigenvalues may not be simple.

**Keywords** Sturm-Liouville problems · Periodic boundary conditions · Transmission conditions · Asymptotic formulas

**Mathematics Subject Classification** 34B24 · 34B27

## 1 Introduction

Nowadays, in the technological world, the scientists, bring forth at a tremendous pace new artificial specimens in which finite periodic structures are the main components. The history of periodic spectral theory starts with the investigations of Sturm and Liouville on the eigenvalues of second order differential equations of with certain boundary conditions, now referred to as Sturm-Liouville problems. Periodic Sturm-Liouville problems are used, for example, as a mathematical model of one-dimensional simplest crystals (see, for example, [28]). Historically, much of our current fundamental understanding of electronic structures of crystals was obtained through the analysis of one-dimensional crystals [21, 30]. Among the most well-known examples are the Kronig-Penney model [23], Kramer's general

✉ O. Sh. Mukhtarov
omukhtarov@yahoo.com

K. Aydemir
kadriyeaydemr@gmail.com

1 Department of Mathematics, Faculty of Art and Science, Amasya University, 05100 Amasya, Turkey

2 Department of Mathematics, Faculty of Science, Gaziosmanpaşa University, 60250 Tokat, Turkey

3 Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan

analysis of the band structure of one-dimensional crystals [22], Tamm's surface states [33], and so forth. A clear understanding of electronic states in one-dimensional finite crystals is the basis for further understandings of the electronic states in low-dimensional systems and finite crystals. For this purpose, we need to have a clear understanding of solutions of the Schrödinger equations for one-dimensional crystals. The Schrödinger differential equation for a one-dimensional crystal can be written as(see, [30])

$$y''(x) + [v(x) - \lambda]y(x) = 0 \tag{1.1}$$

where $v(x + a) = v(x)$, that is $v(x)$ the periodic potential. Eastham's book [16] provided a comprehensive and in-depth mathematical theory of a general class of periodic differential equations, where Eq. (1.1) is a specific and simple form.

Many important physical problems are described by differential equations which can be put into a form known as Sturm-Liouville equation. The general Sturm-Liouville equation can be written as

$$(-pu'(x))' + (q(x) - \lambda r(x))u(x) = 0. \tag{1.2}$$

The one-dimensional Schrödinger equation (1.1) corresponds to a specific and simple form of Sturm-Liouville equation (1.2) where $p(x) = r(x) = 1$ and $q(x) = v(x)$. Sturm-Liouville equations of the type (1.2) arise in quantum mechanics and are called one dimensional Schrödinger equations. This equation arose first in the context of the Separation of Variables method for various type partial differential equations. This and related methods continue to give rise to Sturm-Liouville problems which model phenomena such as the earth's seismic behavior, the propagation of sonar in the water stratified by varying density, and the stability and velocity of large-scale waves in the atmosphere [27].

Since there is no exact solution of Sturm-Liouville problems or it is difficult to solve, various approximate or qualitative methods need to be investigated to solve the problem. Motivated by problems of periodic motion in continuous media, such as the periodic flow of heat in a bar, Sturm and Liouville were led in 1836 to identify a class of problems in second-order differential equations that have inspired much of modern analysis and operator theory, and continue to do so. The observation that numerous vibrating physical systems admit mathematical descriptions as boundary-value problems with periodic boundary conditions, Karlin and Lee [20] undertook the study of the total positivity properties of an extensive class of periodic boundary-value problems whose differential operators are of Polya type [29]. There are several papers concerning the computation and the correction of the eigenvalues. In [14] the distribution of eigenvalues for the classical case of Sturm-Liouville problems with periodic boundary condition is derived. In [10] have been derived the computation of eigenvalues of the periodic Sturm-Liouville problem

$$\begin{aligned} -y''(x) + q(x)y(x) &= \mu^2 y(x), \quad x \in [0, w] \\ y(0) &= y(w), \quad y'(0) = y'(w) \end{aligned}$$

using interpolation techniques in Paley-Wiener spaces. Somali and Voger [31] investigated the computation of the eigenvalues of the same Sturm-Liouville equation defined on (0, 1) with the t-periodic boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(1)$$

where $q(x) \in C^w[0, 1]$ for some positive integer $w$, $q(x) = q(x + 1)$, $t \in (0, 2\pi)$ and $t \neq \pi$. There are also many studies about the numerical estimations of the small eigenvalues of the Sturm-Liouville operators with periodic and antiperiodic boundary conditions. Some popular methods that have been used are the finite difference method, finite element method, Prüfer transformations and the shooting method. For example, Andrew [3] considered the computations of the eigenvalues by using the finite element method. By Anderssen and Hoog [4] a simple asymptotic correction technique is shown to reduce the error in the centered finite difference estimate of the k-th eigenvalue obtained with uniform step length $h$ from $O(k^4h^2)$ to $O(kh^2)$. Vanden Berghe et al. [35] extended a modification to this classical finite difference scheme and showed that it produced a smaller local truncation error. Condon [12] proved that a simple step-dependent linear multistep method can be used to reduce the error of the eigenvalues. Ji used the shooting method in their works [17]. One of the interesting approaches was given by Dinibütün and Veliev [15]. In [24, 36] have been given a new approach for the estimations of the eigenvalues of non-self-adjoint Sturm-Liouville operators with periodic and anti periodic boundary conditions.

In recent years, periodic boundary value problems have been studied extensively in the literature (see, for example, [8, 9, 13, 18] and references therein). Motivated by the above works, we will consider the second order periodic boundary value problems with transmission conditions. We know that it is more difficult to perform a study of periodic Sturm-Liouville problem together with transmission conditions. We would like to introduce a new method for computing the eigenvalues of the Sturm-Liouville equation

$$\mathfrak{H}\varphi(t) := -\varphi''(t,\mu) + Q(t)\varphi(t,\mu) = \mu^2\varphi(t,\mu), \ \ t \in [-\pi, 0) \cup (0, \pi] \tag{1.3}$$

with periodic boundary conditions

$$\varphi(-\pi,\mu) = \varphi(\pi,\mu), \qquad \varphi'(-\pi,\mu) = \varphi'(\pi,\mu) \tag{1.4}$$

and additional transmission conditions at the interior singular point $t = 0$

$$\varphi(0^+,\mu) = \eta\varphi(0^-,\mu) \tag{1.5}$$

$$\varphi'(0^+,\mu) = \zeta\varphi(0^-,\mu) + \frac{1}{\eta}\varphi'(0^-,\mu) \tag{1.6}$$

where the potential $Q(t)$ is real-valued, continuous on $[-\pi, 0) \cup (0, \pi]$ and has a finite limits $Q(0\mp) = \lim_{t\to 0\mp} Q(t)$; $\mu$ is a complex eigenparameter, $\eta, \zeta$ are real nonzero numbers. Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the considered periodic boundary value transmission problem (1.3)–(1.6).

Transmission problems appear frequently in various fields of physics and engineering. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [26] and the references listed therein). Another completely different field is that of "hydraulic fracturing" used in order to increase the flow of oil from a reservoir into a producing oil well (see, [11]).

In recent years, Sturm-Liouville problems defined in a direct sum of Hilbert space and/or with transmission conditions have been an important research topic in mathematical physics [1, 2, 5–7, 19, 25, 32, 34]. In this study we investigate some properties of the eigenvalues and eigenfunctions of the periodic Sturm-Liouville boundary value transmission problem (1.3)–(1.6).

## 2 General properties of Eigenvalues and Eigenfunctions of the problem

**Theorem 2.1** *The eigenvalues for the periodic Sturm-Liouville boundary value transmission problem* (1.3)–(1.6) *are real.*

***Proof*** Assume $\mu^2$, possibly a complex number, is an eigenvalue for (1.3)–(1.6) with eigenfunction $\chi(t,\mu)$, which may be complex-valued. That is

$$\mathfrak{H}\,\chi(t,\mu)-\mu^2\chi(t,\mu)=0 \tag{2.1}$$

and $\chi(t,\mu)$ satisfies the boundary and transmission conditions (1.4)-(1.6). Taking the complex conjugate of both sides of (2.1) and using the fact that $Q(t)$ is real-valued, we obtain

$$\overline{\mathfrak{H}\chi(t,\mu)-\mu^2\chi(t,\mu)}=\mathfrak{H}\overline{\chi}(t,\mu)-\overline{\mu^2}\,\overline{\chi}(t,\mu)=0 \tag{2.2}$$

Further, since $\eta$ and $\zeta$ are real numbers, a similar argument shows that $\overline{\chi(t,\mu)}$ also satisfies the boundary and transmission conditions (1.4)-(1.6), that is,

$$\overline{\chi}(-\pi,\mu)=\overline{\chi}(\pi,\mu)\quad \overline{\chi'}(-\pi,\mu)=\overline{\chi'}(\pi,\mu) \tag{2.3}$$

and

$$\overline{\chi}(0^+,\mu)=\eta\overline{\chi}(0^-,\mu),\quad \overline{\chi'}(0^+,\mu)=\zeta\overline{\chi'}(0^-,\mu)+\frac{1}{\eta}\overline{\chi'}(0^-,\mu). \tag{2.4}$$

Hence $\overline{\mu^2}$ is an eigenvalue with eigenfunction $\overline{\chi(t,\mu)}$. Using well-known Green's formula we find

$$\begin{aligned}&\int_{-\pi}^{0^-}\mathfrak{H}\,\chi(t,\mu)\overline{\chi}(t,\mu)dt+\int_{0^+}^{\pi}\mathfrak{H}\chi(t,\mu)\overline{\chi}(t,\mu)dt-\int_{-\pi}^{0^-}\mathfrak{H}\overline{\chi}(t,\mu)\chi(t,\mu)dt\\&\quad-\int_{0^+}^{\pi}\mathfrak{H}\overline{\chi}(t,\mu)\chi(t,\mu)dt\\&=(\chi'(t,\mu)\overline{\chi}(t,\mu)-\overline{\chi'}(t,\mu)\chi(t,\mu))|_{-\pi}^{0^-}+(\chi'(t,\mu)\overline{\chi}(t,\mu)-\overline{\chi'}(t,\mu)\chi(t,\mu))|_{0^+}^{\pi}.\end{aligned} \tag{2.5}$$

Thus, by the equalities (2.1), (2.2) and (2.5) we have

$$\begin{aligned}(\overline{\mu^2}-\mu^2)[\int_{-\pi}^{0^-}|\chi(t,\mu)|^2dt+\int_{0^+}^{\pi}|\chi(t,\mu)|^2dt]=&(\chi'(t,\mu)\overline{\chi}(t,\mu)-\overline{\chi'}(t,\mu)\chi(t,\mu))|_{-\pi}^{0^-}\\&+(\chi'(t,\mu)\overline{\chi}(t,\mu)-\overline{\chi'}(t,\mu)\chi(t,\mu))|_{0^+}^{\pi}.\end{aligned}$$

i.e.

$$
\begin{aligned}
&(\overline{\mu^2}-\mu^2)[\int_{-\pi}^{0^-}|\chi(t,\mu)|^2dt+\int_{0^+}^{\pi}|\chi(t,\mu)|^2dt] \\
&= W(\overline{\chi(t,\mu)},\chi(t,\mu);0^-)-W(\overline{\chi(t,\mu)},\chi(t,\mu);-\pi) \\
&\quad + W(\overline{\chi(t,\mu)},\chi(t,\mu);\pi)-W(\overline{\chi(t,\mu)},\chi(t,\mu);0^+).
\end{aligned}
\tag{2.6}
$$

Since $\chi(t,\mu)$ and $\overline{\chi(t,\mu)}$ satisfy the boundary transmission conditions (1.4)-(1.6), right-hand of the last equation is zero. Thus we have

$$
(\overline{\lambda}-\lambda)\left(\int_{-\pi}^{0^-}\chi(x,t)\overline{\chi(x,t)}dt+\int_{0^+}^{\pi}\chi(x,t)\overline{\chi(x,t)}dt\right)=0, \tag{2.7}
$$

where $\lambda=\mu^2$.

Since $\chi(t,\mu)$ is a nontrivial solution to (1.3)-(1.6), then

$$
\int_{-\pi}^{0^-}|\chi(t,\mu)|^2dt+\int_{0^+}^{\pi}|\chi(t,\mu)|^2dt>0.
$$

Thus, we can divide out both sides of (2.7) by this integral to obtain $\overline{\mu^2}=\mu^2$, which means that the eigenvalue $\mu^2$ is a real number. □

***Remark 2.2*** Let $\mu^2$ be an eigenvalue. Then there is at least one real-valued eigenfunction corresponding to the eigenvalue $\mu^2$. Indeed, if $\chi(t,\mu)$ it self is not real-valued, we can obtain real-valued eigenfunctions corresponding to $\mu^2$ by taking the real or imaginary parts of $\chi(t,\mu)$. To show this, suppose $\chi(t,\mu)=\chi_1(t,\mu)+i\chi_2(t,\mu)$ be any complex valued eigenfunction corresponding to the eigenvalue $\mu^2$. This implies that not both $\chi_1(t,\mu)$, $\chi_2(t,\mu)$ are zero. It is easy to see that $\overline{\chi}(t,\mu)=\chi_1(t,\mu)-i\chi_2(t,\mu)$ is also an eigenfunction, corresponding to same eigenvalue $\mu^2$. Since $\chi(t,\mu)$ and $\overline{\chi}(t,\mu)$ are eigenfunctions of the problem (1.3)–(1.6), at least one of the real valued functions $\frac{\chi(t,\mu)+\overline{\chi}(t,\mu)}{2}=\chi_1(t,\mu)$ and $\frac{\chi(t,\mu)-\overline{\chi}(t,\mu)}{2i}=\chi_2(t,\mu)$ is not zero and therefore also is eigenfunction of the problem (1.3)–(1.6). Thus, there is at least one real valued eigenfunction, corresponding to the eigenvalue $\mu^2$.

***Remark 2.3*** Eigenvalues of the periodic Sturm-Liouville boundary value transmission problem (1.3)–(1.6) may not be simple.

To show this let us consider the following simple periodic Sturm-Liouville boundary value transmission problem consisting of the Sturm-Liouville equation

$$
-u''=\lambda u,\quad x\in[-\pi,0)\cup(0,\pi] \tag{2.8}
$$

with periodic boundary conditions

$$
u(-\pi)=u(\pi),\quad u'(-\pi)=u'(\pi) \tag{2.9}
$$

and transmission conditions

$$
u(+0)=-u(-0),\quad u'(+0)=-u'(-0) \tag{2.10}
$$

When $\lambda = 0$, we can verify easily that the periodic Sturm-Liouville boundary value transmission problem (2.8)–(2.10) has only trivial solution $u = 0$, i.e. $\lambda = 0$ is not an eigenvalue. When $\lambda < 0$, any nontrivial solution of the equation (2.8) is unbounded on $R$ and therefore no periodic nontrivial solution for this equation. Thus the periodic Sturm-Liouville boundary value transmission problem (2.8)–(2.10) does not admit negative eigenvalues. When $\lambda > 0$, the general solution is

$$w(x) = c_{1i} \cos \sqrt{\lambda} x + c_{2i} \sin \sqrt{\lambda} x$$

on the $\Omega_i$, where $\Omega_1 = [-\pi, 0), \Omega_2 = (0, \pi]$. It we impose the boundary-transmission conditions (2.9)–(2.10) on this general solution, we get the characteristic equation

$$2\lambda \cos^2(\pi\lambda) = 0$$

Consequently the periodic Sturm-Liouville boundary value transmission problem (2.8)–(2.10) have infinitely many positive eigenvalues

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, \quad n = 0, 1, ...$$

We can verify that to each of these eigenvalues $\lambda_n$ there are two linearly independent eigenfunctions

$$u_{n,1}(x) = \begin{cases} \cos\left(\frac{2n+1}{2}\right)x, \textit{ for } x \in \Omega_1 \\ -\cos\left(\frac{2n+1}{2}\right)x, \textit{ for } x \in \Omega_2 \end{cases}$$

and

$$u_{n,2}(x) = \begin{cases} \sin\left(\frac{2n+1}{2}\right)x, \textit{ for } x \in \Omega_1 \\ -\sin\left(\frac{2n+1}{2}\right)x, \textit{ for } x \in \Omega_2 \end{cases}$$

Thus each of the eigenvalues

$$\lambda_n = \frac{2n+1}{2}, \quad n = 0, 1, ...$$

has multiplicity 2, which is the maximum possible.

**Theorem 2.4** *The eigenfunctions $\upsilon_1$ and $\upsilon_2$ that corresponding to distinct eigenvalues of the periodic Sturm-Liouville boundary value transmission problem* (1.3)–(1.6) *are orthogonal, i.e.*

$$< \upsilon_1, \upsilon_2 >= \int_{-\pi}^{0-} \overline{\upsilon_1} \upsilon_2 dx + \int_{0+}^{\pi} \upsilon_1 \overline{\upsilon_2} dx = 0$$

***Proof*** Let $\mu^2$ and $\nu^2$ be distinct eigenvalues with corresponding eigenfunctions $\chi(t, \mu)$ and $\psi(t, \nu)$. We can assume, by Theorem 2.1, $\chi(t, \mu)$ and $\psi(t, \nu)$ are real-valued functions. By applying the Green's formula we have

$$\int_{-\pi}^{0^-} \mathfrak{H}\, \chi(t,\mu)\psi(t,\nu)dt + \int_{0^+}^{\pi} \mathfrak{H}\chi(t,\mu)\psi(t,\nu)dt - \int_{-\pi}^{0^-} \mathfrak{H}\psi(t,\nu)\chi(t,\mu)dt - \int_{0^+}^{\pi} \mathfrak{H}\psi(t,\nu)\chi(t,\mu)dt$$
$$= (\chi'(t,\mu)\psi(t,\nu) - \psi'(t,\nu)\chi(t,\mu))|_{-\pi}^{0^-} + (\chi'(t,\mu)\psi(t,\nu) - \psi'(t,\nu)\chi(t,\mu))|_{0^+}^{\pi}. \tag{2.11}$$

The last equality yields

$$(\nu^2-\mu^2)\left[\int_{-\pi}^{0^-} \chi(t,\mu)\psi(t,\nu)dt + \int_{0^+}^{\pi} \chi(t,\mu)\psi(t,\nu)dt\right] = (\chi'(t,\mu)\psi(t,\nu) - \psi(t,\nu)\chi(t,\mu))|_{-\pi}^{0^-}$$
$$+ (\chi'(t,\mu)\psi(t,\nu) - \psi(t,\nu)\chi(t,\mu))|_{0^+}^{\pi}.$$

Since $\chi(t,\mu)$ and $\psi(t,\nu)$ satisfy the boundary transmission conditions (1.4)-(1.6), right-hand of the last equation is zero. Thus we have

$$(\nu^2-\mu^2)\left[\int_{-\pi}^{0^-} \chi(t,\mu)\psi(t,\nu)dt + \int_{0^+}^{\pi} \chi(t,\mu)\psi(t,\nu)dt = 0\right]. \tag{2.12}$$

Since $\nu^2 \neq \mu^2$, we see that

$$\int_{-\pi}^{0^-} \chi(t,\mu)\psi(t,\nu)dt + \int_{0^+}^{\pi} \chi(t,\mu)\psi(t,\nu)dt = 0. \tag{2.13}$$

Therefore the eigenfunctions $\chi(t,\mu)$ and $\psi(t,\nu)$ are orthogonal. □

## 3 Asymptotic formulas for the eigenvalues of the considered problem

Let $\chi_1(t,\mu)$, $\chi_2(t,\mu)$, $\psi_1(x,\mu)$ and $\psi_2(t,\mu)$ be solutions of the equation (1.3) under the initial conditions

$$\chi_1(-\pi,\mu) = 1, \quad \chi_1'(-\pi,\mu) = 0, \tag{3.1}$$

$$\chi_2(\pi,\mu) = 1, \quad \chi_2'(\pi,\mu) = 0, \tag{3.2}$$

$$\psi_1(-\pi,\mu) = 0, \quad \psi_1'(-\pi,\mu) = 1, \tag{3.3}$$

and

$$\psi_2(\pi,\mu) = 0, \quad \psi_2'(\pi,\mu) = 1, \tag{3.4}$$

respectively. We can prove that each of these solutions are entire functions of parameter $\mu \in \mathbb{C}$ for each fixed $t \in [-\pi, 0) \cup (0, \pi]$. It is obvious that the solutions $\chi_i(t,\mu)$, $\psi_i(t,\mu)$ becomes linear independent on $\Omega_i$.

Let us construct the fundamental solutions

$$\chi(t,\mu)=\begin{cases}\chi_1(t,\mu) \text{ for } t\in[-\pi,0)\\ \chi_2(t,\mu) \text{ for } t\in(0,\pi]\end{cases},\ \psi(t,\mu)=\begin{cases}\psi_1(t,\mu) \text{ for } t\in[-\pi,0)\\ \psi_2(t,\mu) \text{ for } t\in(0,\pi]\end{cases}$$

Now, let $\varphi_0(t,\mu_0)$ be any eigenfunction corresponding to the eigenvalue $\mu_0$. Since for any $\mu\in\mathbb{C}$ the solutions $\chi_i(t,\mu)$ and $\psi_i(t,\mu)$ become linear independent solutions of the equation (1.3) defined on $\Omega_i(i=1,2)$, the solution $\varphi_0(t,\mu_0)$ of the same equation defined on whole $\Omega_1\cup\Omega_2$ can be written as

$$\varphi_0(t,\mu_0)=\begin{cases}\kappa_1\psi_1(t,\mu)+\kappa_2\chi_1(t,\mu) \text{ for } t\in[-\pi,0)\\ \kappa_3\psi_2(t,\mu)+\kappa_4\chi_2(t,\mu) \text{ for } t\in(0,\pi]\end{cases} \tag{3.5}$$

where at least one of the constants $\kappa_1,\kappa_2,\kappa_3,\kappa_4$ is not zero. Since $\varphi_0(t,\mu_0)$ is an eigenfunction of the problem (1.3)–(1.6), the boundary transmission conditions

$$\varphi_0(-\pi,\mu_0)=\varphi_0(\pi,\mu_0) \text{ and } \varphi_0'(-\pi,\mu_0)=\varphi'(\pi,\mu_0) \tag{3.6}$$

and

$$\varphi_0(0^+,\mu_0)=\eta\varphi_0(0^-,\mu_0) \quad \varphi_0'(0^+,\mu_0)=\zeta\varphi_0(0^-,\mu_0)+\frac{1}{\eta}\varphi_0'(0^-,\mu_0) \tag{3.7}$$

are satisfied. By using the equalities (3.6)–(3.7) in the (3.5) we obtain a homogeneous system of linear algebraic equations for the determination of the unknown constants $\kappa_1,\kappa_2,\kappa_3$ and $\kappa_4$ whose determinants is

$$\begin{vmatrix}\chi_2'(0^+,\mu_0)-\eta\chi_1(0^-,\mu_0) & \psi_2'(0^+,\mu_0)-\eta\psi_1(0^-,\mu_0)\\ \chi_2'(0^+,\mu_0)-\zeta\chi_1(0^-,\mu_0)-\frac{1}{\eta}\chi_1'(0^-,\mu_0) & \psi_2'(0^+,\mu_0)-\zeta\psi_1(0^-,\mu_0)-\frac{1}{\eta}\psi_1'(0^-,\mu_0)\end{vmatrix} \tag{3.8}$$

Since the function $\varphi_0(t,\mu_0)$ is an eigenfunction, this determinant is zero. So we can define the characteristic function $\nabla(\mu)$ as

$$\begin{aligned}\nabla(\mu):=&[\chi'(0^+,\mu)-\eta\chi(0^-,\mu)]\,[\psi'(0^+,\mu)-\zeta\psi(0^-,\mu)-\frac{1}{\eta}\psi'(0^-,\mu)]\\ &-[\chi'(0^+,\mu)-\zeta\chi(0^-,\mu)-\frac{1}{\eta}\chi'(0^-,\mu)][\psi'(0^+,\mu)-\eta\psi(0^-,\mu)]\end{aligned} \tag{3.9}$$

Thus we have proved the following theorem.

**Theorem 3.1** *The eigenvalues of the periodic Sturm-Liouville boundary value transmission problem* (1.3)–(1.6) *are the roots of the characteristic functions* $\nabla(\mu)$.

**Theorem 3.2** *Let* $\mu=\sigma+i\tau$. *Then the asymptotic estimates*

$$\frac{d^k\chi(t,\mu)}{dt^k}=\begin{cases}\frac{d^k}{dt^k}\cos(\mu(t+\pi))+O(|\mu|^{k-2}e^{|\tau|(t+\pi)}) \text{ for } t\in[-\pi,0)\\ \\ \frac{d^k}{dt^k}\cos(\mu(\pi-t))+O(|\mu|^{k-2}e^{|\tau|(\pi-t)}) \text{ for } t\in(0,\pi]\end{cases} \tag{3.10}$$

*and*

$$\frac{d^k\psi(t,\mu)}{dt^k} = \begin{cases} \mu^{k-1}\frac{d^k}{dt^k}\sin(\mu(t+\pi)) + O(|\mu|^{k-2}e^{|\tau|(t+\pi)}) \text{ for } t\in[-\pi,0) \\ \mu^{k-1}\frac{d^k}{dt^k}\sin(\mu(t-\pi)) + O(|\mu|^{k-2}e^{|\tau|(\pi-t)}) \text{ for } t\in(0,\pi] \end{cases} \tag{3.11}$$

*are valid as* $|\mu|\to\infty$ *(*$k=0,1$*). These asymptotic estimates are holds uniformly with respect to the variable* $t$*.*

***Proof*** Substituting $\mu^2\chi_i(t,\mu) + \frac{d^2}{dx^2}\chi_i(t,\mu)$ and $\mu^2\psi_i(t,\mu) + \frac{d^2}{dx^2}\psi_i(t,\mu)$ instead of $Q(t)\chi_i(t,\mu)$ and $Q(t)\psi_i(t,\mu)$ in the corresponding integral terms respectively, and then integrating by parts twice, we obtain the following integral and integro-differential equations

$$\frac{d^k\chi(t,\mu)}{dt^k} = \begin{cases} \frac{d^k}{dt^k}\cos(\mu(t+\pi)) + \frac{1}{\mu}\int\limits_{-\pi}^{t}\frac{d^k}{dt^k}\sin(\mu(t-\xi))Q(\xi)\chi_1(\xi,\mu)d\xi \text{ for } t\in[-\pi,0) \\ \frac{d^k}{dt^k}\cos(\mu(\pi-t)) + \frac{1}{\mu}\int\limits_{t}^{\pi}\frac{d^k}{dx^k}\sin(\mu(\xi-t))Q(\xi)\chi_2(\xi,\mu)d\xi \text{ for } t\in(0,\pi] \end{cases} \tag{3.12}$$

and

$$\frac{d^k\psi(t,\mu)}{dt^k} = \begin{cases} \frac{1}{\mu}\frac{d^k}{dt^k}\sin(\mu(t+\pi)) + \frac{1}{\mu}\int\limits_{-\pi}^{t}\frac{d^k}{dx^k}\sin(\mu(t-\xi))Q(\xi)\psi_1(\xi,\mu)d\xi \text{ for } t\in[-\pi,0) \\ \frac{1}{\mu}\frac{d^k}{dt^k}\sin(\mu(t-\pi)) + +\frac{1}{\mu}\int\limits_{t}^{\pi}\frac{d^k}{dx^k}\sin(\mu(\xi-t))Q(\xi)\psi_2(\xi,\mu)d\xi \text{ for } t\in(0,\pi] \end{cases} \tag{3.13}$$

Denote $\Omega(t,\mu) = e^{-|\tau|(t+\pi)}\chi_1(t,\mu)$ and let

$$\omega(\mu) = \sup_{t\in[-\pi,0)\cup(0,\pi]}|\Omega(t,\mu)| \tag{3.14}$$

Using the integral equation (3.12) we have

$$|\,\Omega(t,\mu)\,| \le e^{|\tau|(t+\pi)}e^{-|\tau|(t+\pi)} + \frac{1}{|\mu|}\int\limits_{-\pi}^{t}|\sin(\mu(t-\xi))||Q(\xi)||\chi_1(\xi,\mu)|e^{-|\tau|(t+\pi)}d\xi \tag{3.15}$$

From (3.15) we get $\omega(\mu) = O(1)$ as $|\mu|\to\infty$. Consequently

$$\chi_1(t,\mu) = O(e^{|\tau|(t+\pi)}) \tag{3.16}$$

as $|\mu|\to\infty$. The estimate (3.12) for the case $k=0$ is obtained by substituting (3.16) in the integral term on the right-hand side of (3.12). The case $k=1$ of the (3.15) follows at once on differentiating (3.12) and making the same procedure as in the case $k=0$. The proof of (3.13) is similar. □

**Theorem 3.3** *Let* $\mu = \sigma + i\tau$*. Then the characteristic function* $\nabla(t)$ *has the following asymptotic formula*

$$\nabla(\mu) = 2 - \mu\cos(2\mu\pi) + O\left(\frac{1}{|\mu|}e^{2\pi|\tau|}\right) \tag{3.17}$$

***Proof*** The proof is immediate by substituting (3.12)–(3.13) into (3.9). □

Now we are ready to derive the needed asymptotic formulas for the eigenvalues.

**Theorem 3.1** *The periodic Sturm-Liouville boundary value transmission problem* (1.3)–(1.6) *has a precisely countable many real eigenvalues* $\mu_1, \mu_2, ...$ *satisfying the following asymptotic equality as* $n \to \infty$

$$\mu_n = \frac{n}{2} + \frac{1}{4} + O\left(\frac{1}{n}\right). \tag{3.18}$$

***Proof*** Denoting $\nabla_0(\mu) = 2 - \mu\cos(2\mu\pi)$ and $\nabla_1(\mu) := \nabla(\mu) - \nabla_0(\mu)$, we write $\nabla(\mu) = \nabla_0(\mu) + \nabla_1(\mu)$. By applying the well-known Rouche Theorem which asserts that if $f(\mu)$ and $g(\mu)$ are analytic inside and on a closed contour $\mathfrak{C}$, and $|g(\mu)| < |f(\mu)|$ on $\mathfrak{C}$ then $f(\mu)$ and$f(\mu) + g(\mu)$ have the same number zeros inside $\mathfrak{C}$ provided that each zero is counted according to their multiplicity. It is easy to show that there is $\epsilon > 0$ small enough, such that $|\nabla_0(\mu)| > |\nabla_1(\mu)|$ on the contours

$$\mathfrak{C}_n := \left\{\mu \in \mathbb{C} : |\mu| = \frac{n}{2} + \frac{1}{4} + \epsilon\right\}$$

for sufficiently large $n$. Let $\mu_0^2 \le \mu_1^2 \le ...$ be zeros of $\nabla(\mu)$ . Since inside the contour $\mathfrak{C}_n$, $\nabla_0(\mu)$ has zeros at points $\widetilde{\mu}_n = \frac{n}{2} + \frac{1}{4}$, $n \in \mathbb{Z}$ the number of zeros $\nabla_0(\mu)$ inside the contour $\mathfrak{C}_n^{\pm}$ is equal to $2n$. Then by applying the well-known Rouche Theorem we find that inside the contour $\mathfrak{C}_n$ the function $\nabla(\mu)$ has exactly $2n$ zeros for sufficiently large $n$. Again, by the Rouche Theorem inside the contour $\gamma_n := \{\rho \in \mathbb{C} : |\rho - \widetilde{\mu}_n| < \epsilon\}$ there is exactly one zero $\mu_n$ of $\nabla(\mu)$ for sufficiently small $\epsilon > 0$ and sufficiently large $n$. Thus we have

$$\mu_n = \frac{n}{2} + \frac{1}{4} + \epsilon_n \tag{3.19}$$

where $|\epsilon_n| < \epsilon$ for all $n$. Substituting (3.19) into (3.17) we have $\epsilon_n = O(\frac{1}{n})$, so we have proved that

$$\mu_n = \frac{n}{2} + \frac{1}{4} + O\left(\frac{1}{n}\right)$$

as $n \to \infty$, which completes the proof. □

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