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A note on Automatic continuity of (ψ, ϕ)-derivations

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Abstract
The main purpose of this paper is to present the conditions under which every (ψ, ϕ)-derivation is continuous on topological algebras such as normed algebras, Banach algebras and C^*-algebras.

Keywords Derivation · (ψ, ϕ)-derivation · Banach algebra · Involutive Banach algebra · Automatic continuity

Mathematics Subject Classification Primary 46H40 · Secondary 47B48

1 Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, let $\mathfrak{M}$ be a $\mathcal{B}$-bimodule and let $\psi, \phi : \mathcal{A} \to \mathcal{B}$ be two mappings. A linear mapping $d : \mathcal{A} \to \mathfrak{M}$ is called a (ψ, ϕ)-derivation if $d(ab) = d(a)\psi(b) + \phi(a)d(b)$ for all $a, b \in \mathcal{A}$. If $\mathcal{A} \subseteq \mathcal{B}$ and $\phi = I = \psi$, the identity mapping on $\mathcal{A}$, then we reach an ordinary derivation. The main objective of this study is to investigate the automatic continuity of (ψ, ϕ)-derivations on some topological algebras. Generally, the automatic continuity of a certain class of mappings, e.g. (ψ, ϕ)-derivations, is the study of (algebraic) conditions on a category, e.g. Banach algebras, which guarantee that every (ψ, ϕ)-derivation is continuous. Let us give a brief background in this regard. The theory of automatic continuity of derivations has a long history. Results on automatic continuity of linear mappings defined on Banach algebras comprise a fruitful area of research developed during the last sixty years. The reader is referred to [2, 3, 14] for a deep and extensive study on this subject. In 1958, Kaplansky [11] conjectured that every derivation on a C^*-algebra is continuous. Two years later, Sakai [15] answered to this conjecture. Indeed, he proved that every derivation on a C^*-algebra is automatically continuous and later in 1972, Ringrose [13], by using the pioneering work of Bade and Curtis [1] concerning the automatic continuity of a module homomorphism between bimodules over $C(K)$-spaces, showed that every derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is automatically continuous. Also, Johnson and Sinclair [10] investigated the continuity of derivations on semisimple Banach algebras. In [12], it is

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shown that if ψ, ϕ are continuous $*$-linear mappings, then every (ψ, ϕ)-derivation from a C^*-algebra into $B(\mathcal{H})$ is automatically continuous, and in [8] the assumption of linearity of ψ, ϕ were deleted. Moreover, Hou et al. [9] proved that if $\mathcal{X}$ is simple and ψ, ϕ are surjective and continuous mappings on $B(\mathcal{X})$, then every (ψ, ϕ)-derivation on $B(\mathcal{X})$ is continuous. For more material concerning the automatic continuity of mappings, see, e.g. [4–6, 16].

This paper consists of two sections. The main results of the paper are presented in the second section. In this section, e denotes the identity element of any unital algebra. First, we obtain a characterization of (ψ, ϕ)-derivations as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) \in Inv(\mathcal{B})$, where $Inv(\mathcal{B})$ denotes the set of all invertible elements of $\mathcal{B}$. If either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$, where $[a, b] = ab - ba$ $(a, b \in \mathcal{A})$, then $d(ab) = d(a)(d(e))^{-1}d(b)$ for all $a, b \in \mathcal{A}$. In particular, if $d(e) = e$, then d is a homomorphism. Using this result, we prove that if $\mathcal{A}$ and $\mathcal{B}$ are two topological unital algebras and $d : \mathcal{A} \to \mathcal{B}$ is a (ψ, ϕ)-derivation such that $d(e) = e$ and also if we have all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, then d, ψ and ϕ are continuous mappings. In addition, we obtain some results concerning the continuity of (ψ, ϕ)-derivations on unital involutive topological algebras. Suppose that $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ are two unital, involutive topological algebras and $d : \mathcal{A} \to \mathcal{B}$ is a (ψ, ϕ)-derivation such that d, ψ and ϕ are $(*, \star)$-mappings and $d(e) \in Inv(\mathcal{B})$. Assume that either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$. If we have all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, then d, ψ and ϕ are continuous mappings. Another result in this regard reads as follows: Let $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ be two unital involutive algebras and let $d_1, d_2 : \mathcal{A} \to \mathcal{B}$ be two $(*, \star) - (\psi, \phi)$-derivations such that $d_1(e_{\mathcal{A}})d_2(a_0) = e_{\mathcal{B}}$ or $d_1(a_0)d_2(e_{\mathcal{A}}) = e_{\mathcal{B}}$ for some $a_0 \in \mathcal{A}$. Suppose that $[\psi(a), d_2(b)] = 0 = [\phi(a), d_1(b)]$ for all $a, b \in \mathcal{A}$. Then, the mappings ψ and ϕ are linear. Moreover, suppose we have the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous. Then, d_1, d_2, ψ and ϕ are continuous linear mappings.

2 Main results

In this section, without further mention, e denotes the identity element of any unital algebra. If $\mathcal{A}$ is a unital algebra, $Inv(\mathcal{A})$ denotes the set of all invertible elements of $\mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, let $\mathfrak{M}$ be a $\mathcal{B}$-bimodule and let $\psi, \phi : \mathcal{A} \to \mathcal{B}$ be two mappings. Recall that a linear mapping $d : \mathcal{A} \to \mathfrak{M}$ is called a (ψ, ϕ)-derivation if $d(ab) = d(a)\psi(b) + \phi(a)d(b)$ for all $a, b \in \mathcal{A}$. We now provide an example of this notion.

Example 2.1 Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras (finite dimensional or not). It is easy to see that $\mathfrak{A} = \mathcal{A} \times \mathcal{B}$ is an algebra by the following operations:

$$(a_1, b_1) \bullet (a_2, b_2) = (a_1 a_2, b_1 b_2);$$
$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2);$$
$$\lambda(a_1, b_1) = (\lambda a_1, \lambda b_1)$$

for all $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $\lambda \in \mathbb{C}$. Let $F, G : \mathcal{B} \to \mathcal{B}$ be two mappings. Define the mappings $d, \psi, \phi : \mathfrak{A} \to \mathfrak{A}$ by

$$d((a,b)) = (a,0),$$
$$\psi((a,b)) = (\frac{a}{2}, F(b)),$$
$$\phi((a,b)) = (\frac{a}{2}, G(b)),$$

A routine calculation shows that d is a linear (ψ, ϕ)-derivation on $\mathfrak{A}$.

We begin with the following theorem, which provides a characterization for (ψ, ϕ)-derivations.

Theorem 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) \in Inv(\mathcal{B})$. If either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$, then both ϕ and ψ are linear mappings and $d(ab) = d(a)(d(e))^{-1}d(b)$ for all $a, b \in \mathcal{A}$. In particular, if $d(e) = e$, then d is a homomorphism.*

Proof Suppose that $[\phi(e), d(e)] = 0$. Then, it is easy to see that $[\phi(e), (d(e))^{-1}] = 0$. So, $d(e) = d(e)\psi(e) + \phi(e)d(e) = d(e)(\psi(e) + \phi(e))$. This equality with the assumption that $d(e)$ is an invertible element of $\mathcal{B}$ implies that $\psi(e) + \phi(e) = e$. We have $d(a) = d(a)\psi(e) + \phi(a)d(e)$ for any $a \in \mathcal{A}$. So, $d(a)(e - \psi(e)) = \phi(a)d(e)$ and consequently,

$$\phi(a) = d(a)\phi(e)(d(e))^{-1}, \qquad (a \in \mathcal{A}). \tag{2.1}$$

Similarly, we can get that

$$\psi(a) = (d(e))^{-1}\psi(e)d(a), \qquad (a \in \mathcal{A}). \tag{2.2}$$

It follows from (2.1) and (2.2) that both ϕ and ψ are linear mappings and also we have

$$\begin{aligned} d(ab) &= d(a)\psi(b) + \phi(a)d(b) \\ &= d(a)(d(e))^{-1}\psi(e)d(b) + d(a)\phi(e)(d(e))^{-1}d(b) \\ &= d(a)(d(e))^{-1}\psi(e)d(b) + d(a)(d(e))^{-1}\phi(e)d(b) \\ &= d(a)(d(e))^{-1}(\psi(e) + \phi(e))d(b) \\ &= d(a)(d(e))^{-1}d(b), \end{aligned}$$

which means that

$$d(ab) = d(a)(d(e))^{-1}d(b), \qquad (a, b \in \mathcal{A}). \tag{2.3}$$

Clearly, if $d(e) = e$, then d is a homomorphism. Besides, we can prove that $[d(e), \psi(e)] = 0$. In view of (2.2) and (2.3), we have

$$\begin{aligned} \psi(ab) &= (d(e))^{-1}\psi(e)d(ab) \\ &= (d(e))^{-1}\psi(e)d(a)(d(e))^{-1}d(b) \\ &= \psi(a)(d(e))^{-1}d(b). \end{aligned}$$

Letting $a = e$ in the above equations, we get that

$$\psi(b) = \psi(e)(d(e))^{-1}d(b), \qquad (b \in \mathcal{A}). \tag{2.4}$$

Comparing (2.2) and (2.4), we obtain that $\psi(e)(d(e))^{-1}d(a) = (d(e))^{-1}\psi(e)d(a)$ for all $a \in \mathcal{A}$. Putting $a = e$ in the previous equation, we get that $[d(e), \psi(e)] = 0$. So, each of the equations $[d(e), \psi(e)] = 0$ or $[\phi(e), d(e)] = 0$ implies the other. □

In the following, there are some consequences of the previous theorem.

Corollary 2.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital normed algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) \in Inv(\mathcal{B})$. Suppose that either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$. Then, the continuity of d implies the continuity of both ψ and ϕ.*

Proof Using (2.1) and (2.2), we obtain the required result. □

In the following, we present some conditions that provide the continuity of (ψ, ϕ)-derivations.

Corollary 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be two topological unital algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) = e$. If we have all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, then d, ψ and ϕ are continuous mappings.*

Proof It follows from Theorem 2.2 that d is a homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Since we are assuming all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, we deduce that d is continuous. This fact along with (2.1) and (2.2) implies the continuity of ψ and ϕ. □

Remark 2.5 There are many different conditions under which a homomorphism is continuous. For instance, if $\mathcal{A}$ is a Banach $*$-algebra and $\mathcal{B}$ is a C^*-algebra, then it follows from [3, Corollary 3.2.4] that every $*$-homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ is automatically continuous. For more material about the continuity of homomorphisms and other results, see, e.g. [2, Proposition 5.1.1, Theorem 5.1.8, Theorem 5.2.4, Coroolary 5.2.5].

Let $\mathcal{A}$ be a complex algebra. Recall that an involution over $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ satisfying the following conditions for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$:

1. $(a^*)^* = a$,
2. $(ab)^* = b^*a^*$,
3. $(a + b)^* = a^* + b^*$,
4. $(\lambda a)^* = \overline{\lambda} a^*$.

An algebra $\mathcal{A}$ equipped with an involution $*$ is called an involutive algebra or $*$-algebra and is denoted, as an ordered pair, by $(\mathcal{A}, *)$. Let $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ be two involutive algebras. A mapping $T : \mathcal{A} \to \mathcal{B}$ is called a $(*, \star)$-map if $T(a) = (T(a^*))^\star$ for all $a \in \mathcal{A}$.

Theorem 2.6 *Let $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ be two unital, involutive algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that d, ψ and ϕ are $(*, \star)$-mappings and $d(e) \in Inv(\mathcal{B})$. If either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$, then $\theta = \phi + \psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and further, $d(a) = d(e)\theta(a) = \theta(a)d(e)$ for all $a \in \mathcal{A}$.*

Proof It follows from Theorem 2.2 that $\phi(e) + \psi(e) = e$ and also it follows from (2.1) that $\phi(a) = d(a)\phi(e)(d(e))^{-1}$ for all $a \in \mathcal{A}$. Since ϕ and d are $(*, \star)$-mappings, we have

$$\phi(a) = (\phi(a^*))^\star = (d(a^*)\phi(e^*)(d(e^*))^{-1})^\star = (d(e))^{-1}\phi(e)d(a) \tag{2.5}$$

Considering $\theta = \phi + \psi$ and using (2.2) and (2.5), we have

$$\theta(a) = \psi(a) + \phi(a) = (d(e))^{-1}\psi(e)d(a) + (d(e))^{-1}\phi(e)d(a) = (d(e))^{-1}d(a). \tag{2.6}$$

It is observed that

$$d(a) = d(e)\theta(a), \qquad (a \in \mathcal{A}). \tag{2.7}$$

Similarly, we get that

$$\theta(a) = d(a)(d(e))^{-1}, \qquad (a \in \mathcal{A})$$

and so, we have

$$d(a) = \theta(a)d(e), \qquad (a \in \mathcal{A}).$$

Our next task is to show that θ is a homomorphism. Using (2.3) and (2.7), we get that

$$d(a)(d(e))^{-1}d(b) = d(ab) = d(e)\theta(ab), \qquad (a, b \in \mathcal{A}). \tag{2.8}$$

Left multiplication of (2.8) by $(d(e))^{-1}$ and using (2.6) give

$$\theta(ab) = (d(e))^{-1}d(a)(d(e))^{-1}d(b) = \theta(a)\theta(b), \qquad (a, b \in \mathcal{A}).$$

which means that θ is a homomorphism. This proves the theorem, completely. □

An immediate corollary reads as follows:

Corollary 2.7 *Suppose that $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ are two unital, involutive topological algebras and $d : \mathcal{A} \to \mathcal{B}$ is a (ψ, ϕ)-derivation such that d, ψ and ϕ are $(*, \star)$-mappings and $d(e) \in Inv(\mathcal{B})$. Assume that either $[\psi(e), d(e)] = 0$ or $[\phi(e), d(e)] = 0$. If we have all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, then d, ψ and ϕ are continuous mappings.*

Proof It follows from Theorem 2.6 that there exists a homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ such that $d(a) = d(e)\theta(a) = \theta(a)d(e)$ for all $a \in \mathcal{A}$. Since we are assuming all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, we obtain the continuity of d. Now, Eqs. (2.1) and (2.2) imply the continuity of ϕ and ψ, respectively. □

In the following, we provide an example that shows that the conditions of Theorem 2.6 are not superfluous.

Example 2.8 Let $(\mathcal{A}, *)$ be an involutive algebra. Set $\mathcal{U} = \mathbb{C} \bigoplus \mathcal{A}$. Consider $\mathcal{U}$ as an algebra with pointwise addition, scalar multiplication and the product defined by

$$(\alpha, a) \bullet (\beta, b) = (\alpha\beta, \alpha b + \beta a), \qquad (\alpha, \beta \in \mathbb{C}, a, b \in \mathcal{A}).$$

$\mathcal{U}$ is also an involutive algebra when we define $\star : \mathcal{U} \to \mathcal{U}$ as follows:

$$(\alpha, a)^\star = (\overline{\alpha}, a^*), \qquad (\alpha \in \mathbb{C}, a \in \mathcal{A}).$$

Furthermore, $e = (1, 0)$ is the identity of $\mathcal{U}$. Let $R, S, T : \mathcal{A} \to \mathcal{A}$ be $*$-linear mappings. We define the mappings $d, \psi, \phi : \mathcal{U} \to \mathcal{U}$ by $d((\alpha, a)) = (0, T(a))$, $\psi((\alpha, a)) = (\alpha, S(a))$ and $\phi((\alpha, a)) = (\alpha, R(a))$ for all $(\alpha, a) \in \mathcal{U}$. It is clear that d, ψ, ϕ are $\star$-mappings and also d is a (ψ, ϕ)-derivation. A straightforward verification shows that $(\alpha, a)^{-1} = (\alpha^{-1}, \frac{-a}{\alpha^2})$ for all $(\alpha, a) \in \mathbb{C}\backslash\{0\} \bigoplus \mathcal{A}$. So, $Inv(\mathcal{U}) = \mathbb{C}\backslash\{0\} \bigoplus \mathcal{A}$ and obviously, $d(e) \notin Inv(\mathcal{U})$. As can be seen, $\theta = \psi + \phi$ is not a homomorphism and further $d \neq d(e)\theta$. Note that if $\mathcal{A}$ is a normed algebra, then so is $\mathcal{U}$ with the following norm:

$$\|(\alpha, a)\| = |\alpha| + \|a\|, \qquad (\alpha \in \mathbb{C}, a \in \mathcal{A}).$$

Theorem 2.9 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital algebras such that $\mathcal{B}$ is commutative. Let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) \in Inv(\mathcal{B})$. Then, $\theta = \phi + \psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and also $d(a) = d(e)\theta(a)$ for all $a \in \mathcal{A}$.*

Proof By using an argument similar to the proof of Theorem 2.6, we get the desired result. □

Corollary 2.10 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital topological algebras such that $\mathcal{B}$ is commutative. Let $d : \mathcal{A} \to \mathcal{B}$ be a (ψ, ϕ)-derivation such that $d(e) \in Inv(\mathcal{B})$. If we have all the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous, then d, ψ and ϕ are continuous mappings.*

Note that if $\mathcal{A}$ is an $*$-algebra, then a straightforward verification shows that $\mathcal{A} \times \mathcal{A}$ is also an $*$-algebra by regarding the following structure:

1. $(a, b) + (c, d) = (a + c, b + d)$;
2. $\lambda(a, b) = (\lambda a, \lambda b)$:
3. $(a, b).(c, d) = (ac, bd)$;
4. $(a, b)^* = (a^*, b^*)$;

for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Similar to the $(*, \star)$-mappings, a bi-mapping $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is a $(*, \star)$-mapping if $\Omega(a, b) = (\Omega(a^*, b^*))^\star$ for all $a, b \in \mathcal{A}$. Let $\psi, \phi : \mathcal{A} \to \mathcal{B}$ be two mappings. A bi-linear mapping (i.e., linear in both arguments) $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is called a left two variable (ψ, ϕ)-derivation if $\Omega(ab, c) = \Omega(a, c)\psi(b) + \phi(a)\Omega(b, c)$ for all $a, b, c \in \mathcal{A}$. A right two variable (ψ, ϕ)-derivation is defined, similarly. A bi-linear mapping $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is said to be a two variable (ψ, ϕ)-derivation if it is both a left-and a right two variable (ψ, ϕ)-derivation. A $(*, \star)$-left two variable (ψ, ϕ)-derivation means a left two variable (ψ, ϕ)-derivation $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$, whenever Ω, ψ and ϕ are $(*, \star)$-mappings.

Theorem 2.11 *Let $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ be two unital involutive algebras and let $d_1, d_2 : \mathcal{A} \to \mathcal{B}$ be two $(*, \star) - (\psi, \phi)$-derivations such that $d_1(e)d_2(a_0) = e$ or $d_1(a_0)d_2(e) = e$ for some $a_0 \in \mathcal{A}$. Suppose that $[\psi(a), d_2(b)] = 0 = [\phi(a), d_1(b)]$ for all $a, b \in \mathcal{A}$. Then, the mappings ψ and ϕ are linear. Moreover, suppose we have the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous. Then, d_1, d_2, ψ and ϕ are continuous linear mappings.*

Proof Define $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ by $\Omega(a, b) = d_1(a)d_2(b)$. It is easy to see that Ω is a $(*, \star)-$ two variable (ψ, ϕ)-derivation. So, we have

$$\begin{aligned}\Omega(ab,c) &= \left(\Omega(b^*a^*,c^*)\right)^\star \\ &= \left(\Omega(b^*,c^*)\psi(a^*)+\phi(b^*)\Omega(a^*,c^*)\right)^\star \\ &= \psi(a)\Omega(b,c)+\Omega(a,c)\phi(b)\end{aligned}$$

Moreover, we know that $\Omega(ab,c)=\Omega(a,c)\psi(b)+\phi(a)\Omega(b,c)$ for all $a,b,c\in\mathcal{A}$. Hence, we have the following expressions:

$$\begin{aligned}\Omega(ab,c) &= \frac{1}{2}\Omega(ab,c)+\frac{1}{2}\Omega(ab,c) \\ &= \frac{\Omega(a,c)\psi(b)+\phi(a)\Omega(b,c)}{2}+\frac{\psi(a)\Omega(b,c)+\Omega(a,c)\phi(b)}{2}\end{aligned}$$

So,

$$\Omega(ab,c)=\Omega(a,c)\left(\frac{\psi(b)+\phi(b)}{2}\right)+\left(\frac{\psi(a)+\phi(a)}{2}\right)\Omega(b,c),$$

for all $a,b,c\in\mathcal{A}$. Considering $\frac{\psi+\phi}{2}=\Sigma$, we see that

$$\Omega(ab,c)=\Omega(a,c)\Sigma(b)+\Sigma(a)\Omega(b,c),$$

for all $a,b,c\in\mathcal{A}$. Let a_0 be an element of $\mathcal{A}$ such that $d_1(e)d_2(a_0)=e$. So, it is observed that $\Omega(e,a_0)=e$. It follows from [7, Theorem 2.16] that there exists a unital homomorphism $\Theta:\mathcal{A}\to\mathcal{B}$ defined by $\Theta(a)=\Omega(a,a_0)$ such that $\Omega(a,b)=\Theta(ab)(\Theta(a_0))^{-1}$ for all $a,b\in\mathcal{A}$ and also $\frac{\psi(a)+\phi(a)}{2}=\Sigma(a)=\frac{\Omega(a,a_0)}{2}=\frac{\Theta(a)}{2}$ for all $a\in\mathcal{A}$. Consequently, $\Theta=\phi+\psi$. Since $d_1(e)d_2(a_0)=e$ and $[\psi(a),d_2(b)]=0$ for all $a,b\in\mathcal{A}$, we have

$$\begin{aligned}e=d_1(e)d_2(a_0) &= d_1(e)\psi(e)d_2(a_0)+\phi(e)d_1(e)d_2(a_0) \\ &= d_1(e)d_2(a_0)\psi(e)+\phi(e)d_1(e)d_2(a_0) \\ &= \psi(e)+\phi(e).\end{aligned}$$

So, we have the following statements:

$$\begin{aligned}\Theta(a)=\Omega(a,a_0) &= d_1(a)d_2(a_0) \\ &= d_1(a)\psi(e)d_2(a_0)+\phi(a)d_1(e)d_2(a_0) \\ &= d_1(a)d_2(a_0)\psi(e)+\phi(a) \\ &= \Theta(a)\psi(e)+\phi(a),\end{aligned}$$

which means that

$$\phi(a)=\Theta(a)\phi(e), \qquad (a\in\mathcal{A}). \tag{2.9}$$

Reasoning like above, one can easily get that

$$\psi(a)=\psi(e)\Theta(a), \qquad (a\in\mathcal{A}). \tag{2.10}$$

Putting $\Theta(a)=\psi(a)+\phi(a)$ $(a\in\mathcal{A})$ in Eq. (2.9) and using $\psi(e)+\phi(e)=e$, we obtain that

$$\psi(a)=\Theta(a)\psi(e), \qquad (a\in\mathcal{A}). \tag{2.11}$$

Similarly, we get that

$$\phi(a) = \phi(e)\Theta(a), \qquad (a \in \mathcal{A}). \tag{2.12}$$

Since Θ is a linear mapping, the above discussion implies that both ψ and ϕ are linear mappings and further, if $\psi(e) = \phi(e)$, then $\phi = \psi$. Now, we are going to prove that d_1, d_2, ψ and ϕ are continuous linear mappings. Note that Θ is continuous, since we are assuming the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous. This fact with Eqs. (2.9) and (2.10) (or (2.11) and (2.12)) imply that both ψ and ϕ are continuous linear mappings. Now, our task is to prove that d_1 and d_2 are continuous. We know that $\Sigma(a) = \frac{\Omega(a,a_0)}{2} = \frac{\Theta(a)}{2}$ for all $a \in \mathcal{A}$. So, Σ is continuous and also $\Sigma(e) = \frac{e}{2}$. Using an argument similar to the one given above, it can be shown that

$$d_1(ab) = d_1(a)\Sigma(b) + \Sigma(a)d_1(b)$$
$$d_2(ab) = d_2(a)\Sigma(b) + \Sigma(a)d_2(b)$$

for all $a, b \in \mathcal{A}$ Thus, we have $d_1(a) = d_1(a)\Sigma(e) + \Sigma(a)d_1(e) = \frac{d_1(a)}{2} + \Sigma(a)d_1(e)$ for all $a \in \mathcal{A}$. Hence,

$$d_1(a) = 2\Sigma(a)d_1(e), \qquad (a \in \mathcal{A}) \tag{2.13}$$

and similarly, we get that

$$d_2(a) = 2\Sigma(a)d_2(e), \qquad (a \in \mathcal{A}) \tag{2.14}$$

Equations (2.13) and (2.14) with the continuity of Σ give that both d_1 and d_2 are continuous. Thereby, our proof is complete. □

An immediate consequence of Theorem 2.11 reads as follows:

Corollary 2.12 *Let $(\mathcal{A}, *)$ and $(\mathcal{B}, \star)$ be two unital involutive algebras and let $d : \mathcal{A} \to \mathcal{B}$ be a $(*, \star) - (\psi, \phi)$-derivation such that $d(e)d(a_0) = e$ or $d(a_0)d(e) = e$ for some $a_0 \in \mathcal{A}$. Suppose that $[\psi(a), d(b)] = 0 = [\phi(a), d(b)]$ for all $a, b \in \mathcal{A}$. Then, the mappings ψ and ϕ are linear. Moreover, suppose we have the conditions under which every homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is continuous. Then, d, ψ and ϕ are continuous linear mappings.*

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