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Infinitely many positive solutions for an iterative system of singular multipoint boundary value problems on time scales

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Abstract
In this paper, we consider an iterative system of singular multipoint boundary value problems on time scales. The sufficient conditions are derived for the existence of infinitely many positive solutions by applying Krasnoselskii's cone fixed point theorem in a Banach space.

Keywords Iterative system · Time scale · Singularity · Cone · Krasnoselskii's fixed point theorem · Positive solutions

Mathematics Subject Classification Primary 34N05 · Secondary 34B18

1 Introduction

Differential equations with state-dependent delays have attracted a great deal of interest to the researchers since they widely arise from application models, such as population models [4], mechanical models [19], infection disease transmission [28], the dynamics of economical systems [5], position control [9], two-body problem of classical electrodynamics [15], etc. As special type of state-dependent delay-differential equations, iterative differential equations have distinctive characteristics and have been investigated in recent years, e.g. equivariance [30], analyticity [31], convexity [27], monotonicity [16], smoothness [12]. Recently [17], Feckan, Wang and Zhao established the maximal and minimal nondecreasing bounded solutions of the following iterative functional differential equations

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$$x'(t) = g\big(t, x^{(1)}(t), x^{(2)}(t), \dots, x^{(n)}(t)\big),$$

where $x^{(i)}(t) := x(x^{(i-1)})(t)$ indicates the i-th iterate of x, where $i = 1, 2, \dots, n$, by the method of lower and upper solutions.

On the other hand, the theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. A time scale is any closed and nonempty subset of the real numbers. So, by this theory, we can extend the continuous and discrete theories to cases "in between." These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Research in this area of mathematics has exceeded by far a thousand publications, and numerous applications to literally all branches of science such as statistics, biology, economics, finance, engineering, physics, and operations research have been given. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [7, 8]. There is a great deal of research activity devoted to positive solutions of dynamic equations on time scales, see for example [14, 20, 21, 24–26] and references therein.

In [22], Liang and Zhang studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales,

$$\begin{aligned}&\big(\varphi(x^{\Delta}(t))\big)^{\nabla} + a(t)f\big(x(t)\big) = 0,\ t \in [0, a]_{\mathbb{T}}\\ &x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ x^{\Delta}(a) = 0,\end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [13], Dogan considered second order m–point boundary value problem on time scales,

$$\begin{aligned}&\big(\phi_p(x^{\Delta}(t))\big)^{\nabla} + \omega(t)f\big(t, x(t)\big) = 0,\ t \in [0, T]_{\mathbb{T}}\\ &x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \phi_p(x^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(x^{\Delta}(\xi_i)),\end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Many researchers have concentrated on studying first order iterative differential equations by different approaches such as fixed point theory, Picard's successive approximation and the technique of nonexpansive operators. But the literature related to the equations of higher order is limited since the presence of the iterates increases the difficulty of studying them. This motivates us to investigate the following second order dynamical iterative system of boundary value problems with singularities on time scales,

$$\left.\begin{aligned}&x_{\ell}^{\Delta\nabla}(t) + \lambda(t)g_{\ell}\big(x_{\ell+1}(t)\big) = 0,\ 1 \le \ell \le n,\ t \in (0, \sigma(a)]_{\mathbb{T}}\\ &x_{n+1}(t) = x_1(t),\ t \in (0, \sigma(a)]_{\mathbb{T}},\end{aligned}\right\} \tag{1}$$

$$x_{\ell}^{\Delta}(0) = 0,\ x_{\ell}(\sigma(a)) = \sum_{k=1}^{n-2} c_k x_{\ell}(\zeta_k),\ 1 \le \ell \le n, \tag{2}$$

where $n \in \mathbb{N}$, $c_k \in \mathbb{R}^+ := [0, +\infty)$ with $\sum_{k=1}^{n-2} c_k < 1$, $0 < \zeta_k < \sigma(a)/2$, $k \in \{1, 2, \dots, n-2, \}$, $\lambda(t) = \prod_{i=1}^{m} \lambda_i(t)$ and each $\lambda_i(t) \in L_\nabla^{p_i}((0, \sigma(a)]_\mathbb{T})(p_i \geq 1)$ has a singularity in the interval $(0, \sigma(a)/2]_\mathbb{T}$. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of infinitely many positive solutions for the system (1). Equation (1) in real continuous time scales describes diffusion phenomena with a source or a reaction term. For instance, in thermal conduction, it can be interpreted as the one-dimensional heat conduction equation which models the steady-states of a heated bar of length a with a controller at $\mathrm{x} = a$ that adds or removes heat according to a sensor, while the left endpoint is maintained at 0°C and g is the distributed temperature source function depending on delayed temperatures. We refer the interested reader to [10, 11] and the references therein for more details.

We assume the following conditions are true throughout the paper:

($\mathcal{H}_1$) $\mathrm{g}_\ell : [0, +\infty) \to [0, +\infty)$ is continuous.

($\mathcal{H}_2$) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \sigma(a)/2$,

$$\lim_{r\to\infty} t_r = t^* < \sigma(a)/2, \ \lim_{t\to t_r} \lambda_i(t) = +\infty, \ i = 1, 2, \dots, m.$$

Further, for each $i \in \{1, 2, \dots, m\}$, there exist $\delta_i > 0$ such that $\lambda_i(t) > \delta_i$.

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 2.1 [7] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, +\infty)$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$, and $\mu(t) = \sigma(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$. Other intervals can be defined similarly.

Definition 2.2 [6] Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A \backslash \Gamma_1$, then P is said to hold Δ–a.e. on A.
(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A \backslash \Gamma_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 [3, 6] Let $E \subset \mathbb{T}$ be a Δ–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be Δ–measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either

$$\int_E |f|^p(s)\Delta s < \infty \quad \text{if} \quad p \in [1, +\infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e.\ on\ E \ \text{ if } \ p = +\infty.$$

Lemma 2.4 [29] *Let $E \subset \mathbb{T}$ be a Δ–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is Δ–integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i \in I_E} \big(\sigma(t_i) - t_i\big)f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_{\mathbb{N}}(E)f(M), \ \text{if} \ \mathbb{N} \in \mathbb{T}, \\ 0, \ \text{if} \ \mathbb{N} \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Definition 2.5 [29] Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. Say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C, \quad \nabla - a.e.\ on\ E \ \text{ if } \ p = +\infty.$$

Lemma 2.6 [29] *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big)f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.7 *For any $\mathrm{y}(t) \in \mathcal{C}_{ld}((0, \sigma(a)]_{\mathbb{T}})$, the boundary value problem,*

$$\mathrm{x}_1^{\Delta\nabla}(t) + \mathrm{y}(t) = 0, \ t \in (0, \sigma(a)]_{\mathbb{T}}, \tag{3}$$

$$\mathrm{x}_1^{\Delta}(0) = 0,\ \mathrm{x}_1(\sigma(a)) = \sum_{k=1}^{n-2} c_k \mathrm{x}_1(\zeta_k) \tag{4}$$

has a unique solution

$$\mathrm{x}_1(t) = \int_0^{\sigma(a)} \aleph(t,\tau)\mathrm{y}(\tau)\nabla\tau + \frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau)\mathrm{y}(\tau)\nabla\tau, \tag{5}$$

where

$$\aleph(t,\tau) = \begin{cases} \sigma(a)-t, & \text{if } 0 \le \tau \le t \le \sigma(a), \\ \sigma(a)-\tau, & \text{if } 0 \le t \le \tau \le \sigma(a). \end{cases} \tag{6}$$

Proof Suppose x_1 is a solution of (3), then

$$\begin{aligned} \mathrm{x}_1(t) &= -\int_0^t \int_0^\tau \mathrm{y}(\tau_1)\nabla\tau_1\Delta\tau + \mathrm{A}t + \mathrm{B} \\ &= -\int_0^t (t-\tau)\mathrm{y}(\tau)\nabla\tau + \mathrm{A}t + \mathrm{B}, \end{aligned}$$

where $\mathrm{A} = \mathrm{x}_1^{\Delta}(0)$ and $\mathrm{X} = \mathrm{x}_1(0)$. Using conditions (4), we get $\mathrm{A} = 0$ and

$$\mathrm{B} = \int_0^{\sigma(a)} (\sigma(a)-\tau)\mathrm{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathrm{x}_1(\zeta_k).$$

So, we have

$$\begin{aligned} \mathrm{x}_1(t) &= -\int_0^t (t-\tau)\mathrm{y}(\tau)\nabla\tau + \int_0^{\sigma(a)} (\sigma(a)-\tau)\mathrm{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathrm{x}_1(\zeta_k) \\ &= \int_0^{\sigma(a)} \aleph(t,\tau)\mathrm{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathrm{x}_1(\zeta_k). \end{aligned} \tag{7}$$

Plugging $t = \zeta_k$ and multiplying with c_k then summing from 1 to $n-2$ in the above equation (7), we obtain

$$\mathrm{x}_1(\zeta_k) = \frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau)\mathrm{y}(\tau)\nabla\tau. \tag{8}$$

Substituting (8) into (7), we get required solution (5). This completes the proof. □

Lemma 2.8 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold. Let* $\eta \in (0, \sigma(a)/2)_{\mathbb{T}}$ *with* $\zeta_k \in [\eta, \sigma(a)-\eta]_{\mathbb{T}}$, $k \in \{1,2,\cdots,n-2\}$, *the kernel* $\aleph(t,\tau)$ *have the following properties*:

(i) $0 \le \aleph(t,\tau) \le \aleph(\tau,\tau)$ *for all* $t,\tau \in [0,\sigma(a)]_{\mathbb{T}}$,
(ii) $\frac{\eta}{\sigma(a)}\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\eta, \sigma(a)-\eta]_{\mathbb{T}}$ *and* $\tau \in [0,\sigma(a)]_{\mathbb{T}}$.

Proof (i) is evident. To prove (ii), let $t \in [\eta, \sigma(a)-\eta]_{\mathbb{T}}$ and $\tau \le t$. Then

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)}=\frac{\sigma(a)-t}{\sigma(a)-\tau}\geq\frac{\eta}{\sigma(a)}.$$

For $t\leq\tau$,

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)}=\frac{\sigma(a)-\tau}{\sigma(a)-\tau}=1\geq\frac{\eta}{\sigma(a)}.$$

This completes the proof. □

Notice that an n–tuple $(\mathrm{x}_1(t),\mathrm{x}_2(t),\mathrm{x}_3(t),\dots,\mathrm{x}_n(t))$ is a solution of the iterative boundary value problem (1)–(2) if and only if

$$\begin{aligned}\mathrm{x}_\ell(t)&=\int_0^{\sigma(a)}\aleph(t,\tau)\lambda(\tau)\mathrm{g}_\ell(\mathrm{x}_{\ell+1}(\tau))\nabla\tau\\&+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau)\lambda(\tau)\mathrm{g}_\ell(\mathrm{x}_{\ell+1}(\tau))\nabla\tau\end{aligned}$$

and

$$\mathrm{x}_{\ell+1}(t)=\mathrm{x}_1(t),\ t\in(0,a]_{\mathbb{T}},\ 1\leq\ell\leq n.$$

That is

$$\begin{aligned}\mathrm{x}_1(t)=&\int_0^{\sigma(a)}\aleph(t,\tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathrm{g}_2\left[\int_0^{\sigma(a)}\aleph(\tau_2,\tau_3)\cdots\right.\right.\\&\left.\left.\times\,\mathrm{g}_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\right]\Delta\tau_1\\&+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.\\&\left.\times\,\mathrm{g}_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1.\end{aligned}$$

Let $\mathtt{X}$ be the Banach space $C_{ld}((0,\sigma(a)]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\mathrm{x}\|=\max_{t\in(0,\sigma(a)]_{\mathbb{T}}}|\mathrm{x}(t)|$. For $\eta\in(0,\sigma(a)/2)_{\mathbb{T}}$, we define the cone $\mathtt{P}_\eta\subset\mathtt{X}$ as

$$\mathtt{P}_\eta=\left\{\mathrm{x}\in\mathtt{X}:\mathrm{x}(t)\text{ is nonnegative and }\min_{t\in[\eta,\sigma(a)-\eta]_{\mathbb{T}}}\mathrm{x}(t)\geq\frac{\eta}{\sigma(a)}\|\mathrm{x}(t)\|\right\},$$

For any $\mathrm{x}_1\in\mathtt{P}_\eta$, define an operator $\mathscr{L}:\mathtt{P}_\eta\to\mathtt{X}$ by

$$(\mathscr{L}x_1)(t) = \int_0^{\sigma(a)} \aleph(t,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\left[\int_0^{\sigma(a)} \aleph(\tau_2,\tau_3)\cdots\right.\right.$$
$$\left.\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\right]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)} \aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1.$$

Lemma 2.9 *Assume that* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *hold. Then for each* $\eta\in(0,\sigma(a)/2)_{\mathbb{T}}$, $\mathscr{L}(\mathrm{P}_\eta)\subset\mathrm{P}_\eta$ *and* $\mathscr{L}:\mathrm{P}_\eta\to\mathrm{P}_\eta$ *are completely continuous.*

Proof From Lemma 2.8, $\aleph(t,\tau)\geq 0$ for all $t,\tau\in(0,\sigma(a)]_{\mathbb{T}}$. So, $(\mathscr{L}x_1)(t)\geq 0$. Also, for $x_1\in\mathrm{P}_\eta$, we have

$$\|\mathscr{L}x_1\| = \max_{t\in(0,\sigma(a)]_{\mathbb{T}}}\int_0^{\sigma(a)} \aleph(t,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)} \aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1$$
$$\leq\int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\right]\Delta\tau_2\Big]\Delta\tau_1.$$

Again from Lemma 2.8, we get

$$\min_{t\in[\eta,a-\eta]_{\mathbb{T}}}\left\{(\mathscr{L}x_1)(t)\right\} \geq \frac{\eta}{\sigma(a)}\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.\right.$$
$$\times\, g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times\, g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1\right].$$

It follows from the above two inequalities that

$$\min_{t\in[\eta,a-\eta]_{\mathbb{T}}}\left\{(\mathscr{L}x_1)(t)\right\} \geq \frac{\eta}{\sigma(a)}\|\mathscr{L}x_1\|.$$

So, $\mathscr{L}x_1 \in P_\eta$ and thus $\mathscr{L}(P_\eta) \subset P_\eta$. Next, by standard methods and Arzela-Ascoli theorem, it can be proved easily that the operator $\mathscr{L}$ is completely continuous. The proof is complete. □

3 Infinitely many positive solutions

For the the existence of infinitely many positive solutions for iterative system of boundary value problem (1)–(2). We apply following theorems.

Theorem 3.1 **(Krasnoselskii's** [18]) *Let $\mathcal{B}$ be a cone in a Banach space $\mathcal{E}$ and Q_1, Q_2 are open sets with $0 \in Q_1, \overline{Q}_1 \subset Q_2$. Let $\mathcal{K} : \mathcal{B} \cap (\overline{Q}_2\backslash Q_1) \to \mathcal{B}$ be a completely continuous operator such that*

(a) $\|\mathcal{K}v\| \leq \|v\|,\ v \in \mathcal{B}\cap\partial Q_1$, and $\|\mathcal{K}v\| \geq \|v\|,\ v \in \mathcal{B}\cap\partial Q_2$, or
(b) $\|\mathcal{K}v\| \geq \|v\|,\ v \in \mathcal{B}\cap\partial Q_1$, and $\|\mathcal{K}v\| \leq \|v\|,\ v \in \mathcal{B}\cap\partial Q_2$.

Then $\mathcal{K}$ has a fixed point in $\mathcal{B}\cap(\overline{Q}_2\backslash Q_1)$.

Theorem 3.2 **(Hölder's Inequality** [3, 23]) *Let $f \in L^p_\nabla(I)$ with $p > 1$, $g \in L^q_\nabla(I)$ with $q > 1$, and $\frac{1}{p}+\frac{1}{q} = 1$. Then $fg \in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$. where*

$$\|f\|_{L^p_\nabla} := \begin{cases} \left[\int_I |f|^p(s)\nabla s\right]^{\frac{1}{p}}, p \in \mathbb{R}, \\ inf\left\{K \in \mathbb{R} \,/\, |f| \leq K\ \nabla - a.e.,\ on\, I\right\},\ \ p = \infty, \end{cases}$$

and $I = [a,b]_{\mathbb{T}}$. Moreover, if $f \in L^1_\nabla(I)$ and $g \in L^\infty_\nabla(I)$. Then $fg \in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^1_\nabla}\|g\|_{L^\infty_\nabla}$.

Consider the following three possible cases for $\lambda_i \in L^{p_i}_\Delta(0,\sigma(a)]_{\mathbb{T}}$:

$$\sum_{i=1}^{m}\frac{1}{p_i}<1,\ \sum_{i=1}^{m}\frac{1}{p_i}=1,\ \sum_{i=1}^{m}\frac{1}{p_i}>1.$$

Firstly, we seek infinitely many positive solutions for the case $\sum_{i=1}^{m}\frac{1}{p_i}<1.$

Theorem 3.3 *Suppose* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $t_{r+1}<\eta_r<t_r$*. Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r \text{ and } \frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N},$$

where

$$\theta=\max\left\{\left[\frac{\eta_1}{\sigma(a)}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\right.$$
$$\left.\left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\frac{\eta_1}{\sigma(a)}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\nabla\tau\right]^{-1}\right\}.$$

Assume that g_ℓ *satisfies*

(J_1) $g_\ell(x)\le\frac{\mathfrak{N}_1\Gamma_r}{2}\ \forall\, t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\le x\le\Gamma_r$, where

$$\mathfrak{N}_1<\min\left\{\left[\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1},\ \left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1}\right\},$$

(J_2) $g_\ell(x)\ge\frac{\theta\Lambda_r}{2}\ \forall\, t\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}},\ \frac{\eta_r}{\sigma(a)}\Lambda_r\le x\le\Lambda_r.$

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\{(x_1^{[r]},x_2^{[r]},\cdots,x_n^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t)\ge 0$ on $(0,\sigma(a)]_{\mathbb{T}},\ \ell=1,2,\cdots,n$ and $r\in\mathbb{N}$.

Proof Let

$$Q_{1,r}=\{x\in X:\|x\|<\Gamma_r\},\ Q_{2,r}=\{x\in X:\|x\|<\Lambda_r\}$$

be open subsets of X. Let $\{\eta_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^*<t_{r+1}<\eta_r<t_r<\frac{\sigma(a)}{2},$$

for all $r\in\mathbb{N}$. For each $r\in\mathbb{N}$, we define the cone P_{η_r} by

$$P_{\eta_r}=\left\{x\in X: x(t)\ge 0,\ \min_{t\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}}x(t)\ge\frac{\eta_r}{\sigma(a)}\|x(t)\|\right\}.$$

Let $x_1\in P_{\eta_r}\cap\partial Q_{1,r}$. Then, $x_1(\tau)\le\Gamma_r=\|x_1\|$ for all $\tau\in(0,\sigma(a)]_{\mathbb{T}}$. By (J_1) and for $\tau_{m-1}\in(0,\sigma(a)]_{\mathbb{T}}$, we have

$$\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n \le \int_0^{\sigma(a)} \aleph(\tau_n,\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_n,\tau_n)\prod_{i=1}^{m}\lambda_i(\tau_n)\nabla\tau_n.$$

There exists a $q > 1$ such that $\frac{1}{q}+\sum_{i=1}^{n}\frac{1}{p_i}=1$. So,

$$\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n \le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^q_\nabla}\left\|\prod_{i=1}^{m}\lambda_i\right\|_{L^{p_i}_\nabla}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla} \le \frac{\Gamma_r}{2} < \Gamma_r.$$

It follows in similar manner (for $\tau_{n-2}\in(0,\sigma(a)]_{\mathbb{T}}$,) that

$$\int_0^{\sigma(a)} \aleph(\tau_{n-2},\tau_{n-1})\lambda(\tau_{n-1})g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\nabla\tau_{n-1}$$
$$\le \int_0^{\sigma(a)} \aleph(\tau_{n-2},\tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1}$$
$$\le \int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_{n-1})\prod_{i=1}^{m}\lambda_i(\tau_{n-1})\nabla\tau_{n-1}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla} \le \frac{\Gamma_r}{2} < \Gamma_r.$$

Continuing with this bootstrapping argument, we get

$$\int_0^{\sigma(a)} \aleph(t,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\left[\int_0^{\sigma(a)} \aleph(\tau_2,\tau_3)\cdots\right.\right.$$
$$\left.\left.\times\, g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_3\right]\nabla\tau_2\right]\nabla\tau_1 \le \frac{\Gamma_r}{2}.$$

Also, we note that

$$\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_2\right]\nabla\tau_1$$
$$\leq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1(\Gamma_r)\nabla\tau_1$$
$$\leq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\leq\frac{\Gamma_r}{2}.$$

Thus, $(\mathscr{L}x_1)(t)\leq\frac{\Gamma_r}{2}+\frac{\Gamma_r}{2}=\Gamma_r$. Since $\Gamma_r=\|x_1\|$ for $x_1\in P_{\eta_r}\cap\partial Q_{1,r}$, we get

$$\|\mathscr{L}x_1\|\leq\|x_1\|. \tag{9}$$

Next, let $t\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}$. Then,

$$\Lambda_r=\|x_1\|\geq x_1(t)\geq\min_{t\in[\eta_r,a-\eta_r]_{\mathbb{T}}}x_1(t)\geq\frac{\eta_r}{\sigma(a)}\|x_1\|\geq\frac{\eta_r}{\sigma(a)}\Lambda_r.$$

By (J_2) and for $\tau_{n-1}\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}$, we have

$$\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))]\nabla\tau_n$$
$$\geq\int_{\eta_r}^{\sigma(a)-\eta_r}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n$$
$$\geq\frac{\eta_r}{\sigma(a)}\frac{\theta\Lambda_r}{2}\int_{\eta_r}^{\sigma(a)-\eta_r}\aleph(\tau_n,\tau_n)\lambda(\tau_n))\nabla\tau_n$$
$$\geq\frac{\eta_r}{\sigma(a)}\frac{\theta\Lambda_r}{2}\int_{\eta_r}^{\sigma(a)-\eta_r}\aleph(\tau_n,\tau_n)\prod_{i=1}^{m}\lambda_i(\tau_n))\nabla\tau_n$$
$$\geq\frac{\eta_1}{\sigma(a)}\frac{\theta\Lambda_r}{2}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau_n,\tau_n)\nabla\tau_n$$
$$\geq\frac{\Lambda_r}{2}.$$

and

$$\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times g_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_2\right]\nabla\tau_1$$
$$\geq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\eta_1}{\sigma(a)}\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1(\Gamma_r)\nabla\tau_1$$
$$\geq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\eta_1}{\sigma(a)}\frac{\theta\Lambda_r}{2}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau_1,\tau_1)\nabla\tau_1$$

Continuing with bootstrapping argument, we get $(\mathscr{L}x_1)(t)\geq\frac{\Lambda_r}{2}+\frac{\Lambda_r}{2}=\Lambda_r$. Thus, if $x_1\in P_{\eta_r}\cap\partial P_{2,r}$, then

$$\|\mathscr{L}x_1\|\geq\|x_1\|. \tag{10}$$

It is evident that $0\in Q_{2,k}\subset\overline{Q}_{2,k}\subset Q_{1,k}$. From (9),(10), it follows from Theorem 3.1 that the operator $\mathscr{L}$ has a fixed point $x_1^{[r]}\in P_{\eta_r}\cap\left(\overline{Q}_{1,r}\backslash Q_{2,r}\right)$ such that $x_1^{[r]}(t)\geq 0$ on $(0,a]_{\mathbb{T}}$, and $r\in\mathbb{N}$. Next setting $x_{m+1}=x_1$, we obtain infinitely many positive solutions $\{(x_1^{[r]},x_2^{[r]},\ldots,x_m^{[r]})\}_{r=1}^{\infty}$ of (1)–(2) given iteratively by

$$x_\ell(t)=\int_0^{\sigma(a)}\aleph(t,\tau)\lambda(\tau)g_\ell(x_{\ell+1}(\tau))\nabla\tau,\ t\in(0,\sigma(a)]_{\mathbb{T}},\ \ell=n,n-1,\ldots,1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{m}\frac{1}{p_i}=1$, we have the following theorem.

Theorem 3.4 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $t_{r+1}<\eta_r<t_r$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r\quad\text{and}\quad\frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N}.$$

Assume that g_ℓ *satisfies* (J_2) *and*

(J_3) $\quad g_\ell(x)\leq\dfrac{\mathfrak{N}_2\Gamma_r}{2}\,\forall\, t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\leq x\leq\Gamma_r$, where

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L_\nabla^\infty}\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}}\right]^{-1},\ \left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L_\nabla^\infty}\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}}\right]^{-1}\right\}.$$

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\left\{\left(x_1^{[r]},x_2^{[r]},\ldots,x_n^{[r]}\right)\right\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t)\geq 0$ on $(0,\sigma(a)]_{\mathbb{T}}$, $\ell=1,2,\ldots,n$ and $r\in\mathbb{N}$.

Proof For a fixed r, let $Q_{1,r}$ be as in the proof of Theorem 3.3 and let $x_1\in P_{\eta_r}\cap\partial Q_{2,r}$. Again

$$x_1(\tau) \le \Gamma_r = \|x_1\|,$$

for all $\tau \in (0, \sigma(a)]_{\mathbb{T}}$. By (J_3) and for $\tau_{\ell-1} \in (0, \sigma(a)]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n &\le \int_0^{\sigma(a)} \aleph(\tau_n, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n \\ &\le \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_n, \tau_n)\prod_{i=1}^{m}\lambda_i(\tau_n)\nabla\tau_n \\ &\le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L_\nabla^\infty}\left\|\prod_{i=1}^{m}\lambda_i\right\|_{L_\nabla^{p_i}} \\ &\le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L_\nabla^\infty}\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}} \le \frac{\Gamma_r}{2} < \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{n-2} \in (0, \sigma(a)]_{\mathbb{T}}$,) that

$$\begin{aligned}&\int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\nabla\tau_{n-1} \\ &\quad\le \int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1} \\ &\quad\le \int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1} \\ &\quad\le \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1})\prod_{i=1}^{m}\lambda_i(\tau_{n-1})\nabla\tau_{n-1} \\ &\quad\le \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L_\nabla^\infty}\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}} \le \frac{\Gamma_r}{2} < \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}&\int_0^{\sigma(a)} \aleph(t, \tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1, \tau_2)\lambda(\tau_2)g_2\left[\int_0^{\sigma(a)} \aleph(\tau_2, \tau_3)\cdots\right.\right. \\ &\quad\left.\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_3\right]\nabla\tau_2\right]\nabla\tau_1 \le \frac{\Gamma_r}{2}.\end{aligned}$$

Also, we note that

$$\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times\,\mathrm{g}_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_2\right]\nabla\tau_1$$
$$\le\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathrm{g}_1(\Gamma_r)\nabla\tau_1$$
$$\le\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\le\frac{\Gamma_r}{2}.$$

Thus, $(\mathscr{L}\mathrm{x}_1)(t)\le\frac{\Gamma_r}{2}+\frac{\Gamma_r}{2}=\Gamma_r$. Since $\Gamma_r=\|\mathrm{x}_1\|$ for $\mathrm{x}_1\in\mathtt{P}_{\eta_r}\cap\partial\mathtt{Q}_{1,r}$, we get

$$\|\mathscr{L}\mathrm{x}_1\|\le\|\mathrm{x}_1\|. \tag{11}$$

Now define $\mathtt{Q}_{2,r}=\{\mathrm{x}_1\in\mathtt{X}:\|\mathrm{x}_1\|<\Lambda_r\}$. Let $\mathrm{x}_1\in\mathtt{P}_{\eta_r}\cap\partial\mathtt{Q}_{2,r}$ and let $\tau\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}$. Then, the argument leading to (11) can be done to the present case. Hence, the theorem. □

Lastly, the case $\sum_{i=1}^{m}\frac{1}{p_i}>1$.

Theorem 3.5 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^\infty$ *be a sequence with* $t_{r+1}<\eta_r<t_r$. *Let* $\{\Gamma_r\}_{r=1}^\infty$ *and* $\{\Lambda_r\}_{r=1}^\infty$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r\quad\text{and}\quad\frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N}.$$

Assume that g_ℓ *satisfies* (J_2) *and*

(J_4) $\mathrm{g}_\ell(\mathrm{x})\le\frac{\mathfrak{N}_2\Gamma_r}{2}\ \forall\, t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\le\mathrm{x}\le\Gamma_r$, where

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^1_\nabla}\right]^{-1},\left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^1_\nabla}\right]^{-1}\right\}.$$

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\{(\mathrm{x}_1^{[r]},\mathrm{x}_2^{[r]},\ldots,\mathrm{x}_n^{[r]})\}_{r=1}^\infty$ such that $\mathrm{x}_\ell^{[r]}(t)\ge0$ on $(0,\sigma(a)]_{\mathbb{T}}$, $\ell=1,2,\ldots,n$ and $r\in\mathbb{N}$.

Proof The proof is similar to the proof of Theorem 3.1. So, we omit the details here. □

4 Example

In this section, we provide two examples to check validity of our main results.

Example 4.1 Consider the following boundary value problem on $\mathbb{T}=[0,1]$.

$$\left.\begin{aligned}&x_\ell''(t)+\lambda(t)g_\ell(x_{\ell+1}(t))=0, t\in(0,\sigma(1)]_{\mathbb{T}},\ \ell=1,2,3,4,\\&x_5(t)=x_1(t), t\in(0,\sigma(1)]_{\mathbb{T}},\end{aligned}\right\}\tag{12}$$

$$x_\ell'(0)=0,\ \ x_\ell(1)=\frac{1}{2}x_\ell\left(\frac{1}{3}\right)+\frac{1}{3}x_\ell\left(\frac{1}{4}\right),\tag{13}$$

where we take $n=4, m=2,\ c_1=\frac{1}{2},\ c_2=\frac{1}{3},\ \zeta_1=\frac{1}{3},\ \zeta_2=\frac{1}{4}$ and $\lambda(t)=\lambda_1(t)\lambda_2(t)$ in which

$$\lambda_1(t)=\frac{1}{|t-\frac{1}{4}|^{\frac{1}{2}}}\quad\text{and}\quad\lambda_2(t)=\frac{1}{|t-\frac{3}{4}|^{\frac{1}{2}}}.$$

Then $\sum_{k=1}^{n-2}c_k=\frac{5}{6}<1$ and $\delta_1=\delta_2=(4/3)^{1/2}$. For $\ell=1,2,3,4$, let

$$g_\ell(x)=\begin{cases}0.05\times10^{-4}, x\in(10^{-4},+\infty),\\ \frac{62\times10^{-(4r+3)}-0.05\times10^{-4r}}{10^{-(4r+3)}-10^{-4r}}(x-10^{-4r})+0.05\times10^{-8r},\\ \ x\in\left[10^{-(4r+3)},10^{-4r}\right],\\ 62\times10^{-(4r+3)}, x\in\left(\frac{1}{5}\times10^{-(4r+3)},10^{-(4r+3)}\right),\\ \frac{62\times10^{-(4r+3)}-0.05\times10^{-8r}}{0.05\times10^{-(4r+3)}-10^{-(4r+4)}}(x-10^{-(4r+4)})+0.05\times10^{-8r},\\ \ x\in\left(10^{-(4r+4)},\frac{1}{5}\times10^{-(4r+3)}\right],\\ 0,\ x=0,\end{cases}$$

for all $r\in\mathbb{N}$. Let

$$t_r=\frac{31}{64}-\sum_{k=1}^{r}\frac{1}{4(k+1)^4}\ \text{ and }\ \eta_r=\frac{1}{2}(t_r+t_{r+1}),\ r\in\mathbb{N},$$

then

$$\eta_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$$

and

$$t_{r+1}<\eta_r<t_r,\ \eta_r>\frac{1}{5}.$$

Therefore,

$$\frac{\eta_r}{a}=\frac{\eta_r}{1}>\frac{1}{5},\ r\in\mathbb{N}.$$

It is clear that

$$t_1=\frac{15}{32}<\frac{1}{2},\ t_r-t_{r+1}=\frac{1}{4(r+2)^4},\ r\in\mathbb{N}.$$

Since $\sum_{j=1}^{\infty}\frac{1}{j^4}=\frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}$, it follows that

$$t^*=\lim_{r\to\infty}t_r=\frac{31}{64}-\sum_{k=1}^{\infty}\frac{1}{4(r+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}=0.46.$$

Also, we have

$$\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\Delta\tau=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}(1-\tau)d\tau=0.03.$$

Thus, we get

$$\theta=\max\left\{\frac{1}{0.0163},\frac{1}{5\times 0.0163}\right\}=61.35.$$

Next, let $0<\mathfrak{a}<1$ be fixed. Then $\lambda_1,\lambda_2\in L^{1+\mathfrak{a}}[0,1]$. A simple calculations shows that

$$\int_0^{\sigma(1)}\lambda_1(t)\lambda_2(t)dt=\pi-\ln(7-4\sqrt{3}).$$

So, let $p_i=1$ for $i=1,2$. Then

$$\prod_{i=1}^{m}\|\lambda_i\|_{L_{\nabla}^{p_i}}=\pi-\ln(7-4\sqrt{3})\approx 5.78,$$

and also $\|\aleph\|_{L_{\nabla}^{\infty}}=1$. Therefore,

$$\mathfrak{N}_1<\left[\|\aleph\|_{\infty}\prod_{i=1}^{m}\|\lambda_i\|_{L_{\nabla}^{p_i}}\right]^{-1}\approx 0.173.$$

Taking $\mathfrak{N}_1=\frac{1}{10}$. In addition if we take

$$\Gamma_r=10^{-4r},\ \Lambda_r=10^{-(4r+3)},$$

then

$$\begin{aligned}\Gamma_{r+1}=10^{-(4r+4)}&<\frac{1}{5}\times 10^{-(4r+3)}<\frac{\eta_r}{a}\Lambda_r\\&<\Lambda_r=10^{-(4r+3)}<\Gamma_r=10^{-4r},\end{aligned}$$

$\theta\Lambda_r=61.35\times 10^{-(4r+3)}<\frac{1}{10}\times 10^{-4r}=\mathfrak{N}_1\Gamma_r$, $r\in\mathbb{N}$ and $g_\ell(\ell=1,2,3,4)$ satisfies the following growth conditions:

$$\begin{aligned}g_\ell(x)&\le\mathfrak{N}_1\Gamma_r=\frac{1}{10}\times 10^{-4r},\quad x\in\left[0,10^{-4r}\right],\\g_\ell(x)&\ge\theta\Lambda_r=61.35\times 10^{-(4r+3)},\quad x\in\left[\frac{1}{5}\times 10^{-(4r+3)},10^{-(4r+3)}\right],\end{aligned}$$

for $r \in \mathbb{N}$. Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the iterative boundary value problem (1) has infinitely many solutions $\{(x_1^{[r]}, x_2^{[r]}, x_3^{[r]}, x_4^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t) \geq 0$ on $[0, 1]$, $\ell = 1, 2, 3, 4$ and $r \in \mathbb{N}$.

Example 4.2 Let $\mathbb{T} = \{0\} \cup [1/2, 1] \cup \left\{ \frac{1}{2^{k+1}} : k \in \mathbb{N} \right\}$. Consider the boundary value problem

$$\left.\begin{aligned} &x_\ell^{\Delta\nabla}(t) + \lambda(t) g_\ell(x_{\ell+1}(t)) = 0,\ t \in (0, \sigma(1)]_{\mathbb{T}},\ \ell = 1, 2, 3, \\ &x_4(t) = x_1(t),\ t \in (0, \sigma(1)]_{\mathbb{T}}, \end{aligned}\right\} \tag{14}$$

$$x_\ell'(0) = 0, \quad x_\ell(1) = \frac{1}{5} x_\ell\left(\frac{1}{4}\right), \tag{15}$$

where we take $n = 3, m = 2, c_1 = \frac{1}{5}, \zeta_1 = \frac{1}{4}$ and $\lambda(t) = \lambda_1(t)\lambda_2(t)$ in which

$$\lambda_1(t) = \frac{1}{|t - \frac{2}{5}|^{1/4}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{1/4}}.$$

Then $\sum_{k=1}^{n-2} c_k = \frac{1}{5} < 1$ and $\delta_1 = \delta_2 = (4/3)^{1/4}$. For $\ell = 1, 2, 3$, let

$$g_\ell(x) = \begin{cases} \frac{1}{5} \times 10^{-9}, x \in (10^{-9}, +\infty), \\ \frac{62 \times 10^{-(8r+3)} - \frac{1}{5} \times 10^{-(8r+1)}}{10^{-(8r+3)} - 10^{-(8r+1)}} (x - 10^{-(8r+1)}) + \frac{1}{5} \times 10^{-(8r+1)}, \\ x \in \left[10^{-(8r+3)}, 10^{-(8r+1)}\right], \\ 62 \times 10^{-(8r+3)}, x \in \left(\frac{1}{5} \times 10^{-(8r+3)}, 10^{-(8r+3)}\right), \\ \frac{62 \times 10^{-(8r+3)} - \frac{1}{5} \times 10^{-(8r+4)}}{\frac{1}{5} \times 10^{-(8r+3)} - 10^{-(8r+4)}} (x - 10^{-(8r+4)}) + \frac{1}{5} \times 10^{-(8r+4)}, \\ x \in \left(10^{-(8r+4)}, \frac{1}{5} \times 10^{-(8r+3)}\right], \\ 0,\ x = 0, \end{cases}$$

for all $r \in \mathbb{N}$.

Let t_r, η_r be the same as in example 4.1. Then $\eta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}, t_{r+1} < \eta_r < t_r$, $\eta_r > \frac{1}{5}$ and $t_1 = \frac{15}{32} < \frac{1}{2}$, $t_r - t_{r+1} = \frac{1}{4(r+2)^4}$, $r \in \mathbb{N}$. Also, $t^* = \lim_{r \to \infty} t_r = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.46$. Also, we have

$$\int_{\eta_1}^{\sigma(a) - \eta_1} \aleph(\tau, \tau) \Delta\tau = \int_{\frac{15}{32} - \frac{1}{648}}^{1 - \frac{15}{32} + \frac{1}{648}} (1 - \tau) d\tau = 0.03.$$

Thus, we get

$$\theta = \max\left\{ \frac{1}{0.0161845}, \frac{1}{4 \times 0.0161845} \right\} = 61.79.$$

By Lemma 2.4, we obtain

$$\int_0^{\sigma(1)} \lambda_1(t)\lambda_2(t)dt = \int_{\frac{1}{2}}^{1} \lambda_1(t)\lambda_2(t)dt + \sum_{k=1}^{\infty} \left[\sigma\left(\frac{1}{2^k}\right) - \frac{1}{2^k}\right]\lambda_1\left(\frac{1}{2^k}\right)\lambda_2\left(\frac{1}{2^k}\right)$$
$$\approx 2.311909422$$

So, let $p_i = 1$ for $i = 1, 2$. Then

$$\prod_{i=1}^{m} \|\lambda_i\|_{L_\nabla^{p_i}} \approx 2.311909422,$$

and also $\|\aleph\|_{L_\nabla^\infty} = 1$. Therefore,

$$\mathfrak{N}_1 < \left[\|\aleph\|_\infty \prod_{i=1}^{m} \|\lambda_i\|_{L_\nabla^{p_i}}\right]^{-1} \approx 0.4325428974.$$

Taking $\mathfrak{N}_1 = \frac{1}{3}$. In addition, if we take

$$\Gamma_r = 10^{-8r} \text{ and } \Lambda_r = 10^{-(8r+3)},$$

then

$$\Gamma_{r+1} = 10^{-(8r+8)} < \frac{1}{5} \times 10^{-(8r+3)} < \frac{\eta_r}{a}\Lambda_r < \Lambda_r = 10^{-(8r+3)} < \Gamma_r = 10^{-8r},$$
$$\theta\Lambda_r = 61.79 \times 10^{-(8r+3)} < \frac{1}{3} \times 10^{-8r} = \mathfrak{N}_1\Gamma_r,\ r \in \mathbb{N}$$

and $g_\ell(\ell = 1, 2, 3)$ satisfies the following growth conditions:

$$g_\ell(x) \le \mathfrak{N}_1\Gamma_r = \frac{1}{3} \times 10^{-8r}, \quad x \in \left[0, 10^{-8r}\right],$$
$$g_\ell(x) \ge \theta\Lambda_r = 61.79 \times 10^{-(8r+3)}, \quad x \in \left[\frac{1}{5} \times 10^{-(8r+3)}, 10^{-(8r+3)}\right],$$

for $r \in \mathbb{N}$. Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the iterative boundary value problem (1) has infinitely many solutions $\{(x_1^{[r]}, x_2^{[r]}, x_3^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t) \ge 0$ on [0, 1], $\ell = 1, 2, 3$ and $r \in \mathbb{N}$.

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