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New fixed point results for Proinov–Suzuki type contractions in metric spaces

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Abstract
We consider two new classes of contractions and obtain some new fixed point results in complete metric spaces. The mapping considered herein are not necessarily continuous on their domains. Many, well-known generalizations and extensions of the classical Banach contraction theorem have been extended and generalized. We present some illustrative examples to show the genuineness of our results. Finally, an application of our results to nonlinear integral equations is discussed.

Keywords Banach contraction · Quasi-contraction · Proinov–Suzuki contraction · Fixed point · Metric space

Mathematics Subject Classification 47H10 · 54H25

1 Introduction

In 1922, Stefan Banach obtained the following classical fixed point theorem known as *Banach contraction theorem* (BCT) which is very simple, useful, and has become a classical tool in nonlinear analysis.

Theorem 1 *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *be a contraction, that is, there exists a number* $k \in [0, 1)$ *such that for all* $x, y \in X$,

$$d(f(x), f(y)) \leq kd(x, y).$$

Then f has a unique fixed point z in X. Moreover, for an arbitrary point $x_0 \in X$ *we have* $\lim_{n\to\infty} f^n(x_0) = z$.

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The BCT has been extended and generalized by several mathematicians in various ways. Some of the earliest notable generalizations of the BCT can be found in [1, 3, 4, 8–10, 13–15, 18, 19].

In 1972, Ćirić [4] introduced the notion of quasi-contraction and obtained a very important result which generalizes the BCT and many generalizations and extensions of it.

Theorem 2 *Let (X, d) be a complete metric space and let $f : X \to X$ be a quasi-contraction, that is, there exists a number $k \in [0, 1)$ such that for all $x, y \in X$,*

$$d(f(x), f(y)) \leq k \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}. \tag{1}$$

Then f has a unique fixed point in X.

The above theorem is considered as the best generalization, amongst various type of contraction conditions compared by Rhoades [20].

Definition 1 [2] Let (X, d) be a metric space. A mapping $f : X \to X$ is said be *asymptotically regular* at some $u \in X$ if

$$\lim_{n\to\infty} d(f^n(u), f^{n+1}(u)) = 0.$$

The mapping f is said to be asymptotically regular on X if for all $x \in X$,

$$\lim_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0.$$

In 2006, Proinov [17] proved the following interesting generalization of the BCT.

Theorem 3 [17] *Suppose (X, d) is a complete metric space and $f : X \to X$ is a continuous and asymptotically regular mapping such that:*

(a) $d(f(x), f(y)) \leq \psi(\mathcal{D}(x, y))$ for all $x, y \in X$;
(b) $d(f(x), f(y)) < \mathcal{D}(x, y)$, whenever $\mathcal{D}(x, y) \neq 0$.

where and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\psi(t) \leq \varepsilon$.

Here $\mathbb{R}^+$ is the set of all non-negative real numbers, and

$$\mathcal{D}(x, y) = d(x, y) + \eta[d(x, f(x)) + d(y, f(y))], \quad \eta \geq 0.$$

Then there exists a unique fixed point $z \in X$ for f.

Further, if $\eta = 1$ and *ψ is continuous with $\psi(t) < t$ for all $t > 0$, then f need not be continuous.*

A mapping satisfying (a) and (b) is called a *Proinov contraction* [21]. The Proinov contraction is more general than the quasi-contraction:

Example 1 [21] Let $X = \{1, 2, 3\}$ be equipped with the usual metric d. Suppose $f : X \to X$ is a mapping defined as

$$f(1) = 1, \quad f(2) = 3, \quad f(3) = 1.$$

Then the mapping f does not satisfy the condition (1). However, for $\psi(t) = \frac{2t}{1+\eta}$ and $\eta > 1$ the mapping f satisfies the conditions (a) and (b).

In the BCT and most of its extensions and generalizations, the contraction condition is required to hold for all points of the underlying space. So, a natural question arises that "Can this requirement be relaxed considerably without affecting the outcome of the theorem"?

In 2008, Suzuki [24] made a significant beginning in this direction. He introduced a new type of contraction and obtained the following simple and important generalization of the BCT:

Theorem 4 *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\phi(k)d(x,f(x)) \leq d(x,y) \text{ implies } d(f(x),f(y)) \leq kd(x,y), \tag{2}$$

where $\phi : [0,1) \to (\frac{1}{2}, 1]$ is a nonincreasing function defined by

$$\phi(k) = \begin{cases} 1, & \text{if } 0 \leq k \leq \frac{(\sqrt{5}-1)}{2} \\ (1-k)k^{-2}, & \text{if } \frac{(\sqrt{5}-1)}{2} \leq k \leq 2^{-\frac{1}{2}} \\ (1+k)^{-1}, & \text{if } 2^{-\frac{1}{2}} \leq k < 1. \end{cases}$$

Then there exists a unique fixed point $z \in X$ for f.

A mapping f satisfying (2) is called as *Suzuki contraction* [22]. The following example shows the generality of Theorem 4 over Theorem 1.

Example 2 [22]. Let $X = \{(1,1),(4,1),(1,4),(4,5),(5,4)\}$ with the metric d defined as follows

$$d((x^{(1)},x^{(2)}),(y^{(1)},y^{(2)})) = \left|x^{(1)} - y^{(1)}\right| + \left|x^{(2)} - y^{(2)}\right|.$$

Define a mapping $f : X \to X$ by

$$f(x^{(1)},x^{(2)}) = \begin{cases} (x^{(1)},1), & \text{if } x^{(1)} \leq x^{(2)} \\ (1,x^{(2)}), & \text{if } x^{(1)} > x^{(2)}. \end{cases}$$

Then f satisfies all the hypotheses of Theorem 4 and (1, 1) is the unique fixed point of f. However, for $x = (4,5)$ and $y = (5,4)$

$$d(f(x),f(y)) = 6 > 2 = d(x,y).$$

Thus f does not satisfy the assumptions in Theorem 1 for any $k \in [0,1)$.

Remark 1 We note that

1. A mapping satisfying (2) need not be continuous.
2. A metric space X is complete if and only if every Suzuki contraction mapping on X has a fixed point.

Some of the recent extensions and generalizations of the Banach, Proinov and Suzuki contractions can be found in [5, 6, 11, 12, 16, 23].

In the present paper, motivated by the results of Proinov [17], Suzuki [24] and others, we consider two new classes of contractions and present some existence results in complete metric spaces. Many well-known classical results can be directly obtained from our theorems. Some useful examples are discussed to illustrate facts. We also discuss an application of our results to nonlinear integral equations.

2 Proinov–Suzuki type contractions

Now, we consider the notion of *Proinov–Suzuki contraction* as follows:

Definition 2 Let (X, d) be a metric space. A mappings $f : X \to X$ will be called a *Proinov–Suzuki contraction* if for all $x, y \in X$,

$$\frac{1}{2}d(x,f(x)) \leq d(x,y) \text{ implies } d(f(x),f(y)) \leq \psi(\mathcal{D}(x,y)), \tag{PS}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

Now we present our first main theorem.

Theorem 5 *Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous and asymptotically regular Proinov–Suzuki contraction mapping. Then f has a unique fixed point.*

Further, if $\eta = 1$ then f need not be continuous.

Proof Pick $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = f^n(x_0) = f(x_{n-1})$ for all $n \in \mathbb{N}$. Since f is asymptotically regular, i.e., $\lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists $k \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \geq k$,

$$d(x_n, x_{n+1}) \leq \varepsilon. \tag{3}$$

We show that the sequence $\{x_n\}$ is Cauchy. Suppose that $\{x_n\}$ is not Cauchy. Then for any $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{4}$$

We may assume that

$$d(x_{m_k-1}, x_{n_k}) < \varepsilon,$$

by choosing m_k to be the smallest number exceeding n_k for which (4) holds. Using the triangle inequality, we get

$$\begin{aligned}\varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon.\end{aligned}$$

Hence $d(x_{m_k}, x_{n_k}) \to \varepsilon$, as $k \to \infty$. Now, by (3) and (4), we have $\frac{1}{2}d(x_{n_k}, x_{n_k+1}) \le d(x_{m_k}, x_{n_k})$. Since f is Proinov–Suzuki contraction, (PS) implies

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &= d(f(x_{m_k}), f(x_{n_k})) \\ &\le \psi(d(x_{m_k}, x_{n_k}) + \eta[d(x_{m_k}, x_{m_k+1}) + d(x_{n_k}, x_{n_k+1})]). \end{aligned}$$

Letting $n \to \infty$, gives

$$\varepsilon \le \psi(\varepsilon) < \varepsilon,$$

a contradiction unless $\varepsilon = 0$. Thus the sequence $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges to a point $z \in X$. If f is continuous then z is obviously a fixed point of f.

Now suppose that $\eta = 1$ and f is not continuous. We show that for any $n \in \mathbb{N}$ either

$$\frac{1}{2}d(x_n, x_{n+1}) \le d(x_n, z) \quad \text{or} \quad \frac{1}{2}d(x_{n+1}, x_{n+2}) \le d(x_{n+1}, z). \tag{5}$$

Assume the contrary, that is, we suppose that for some $n > k$

$$d(x_n, z) < \frac{1}{2}d(x_n, x_{n+1}) \quad \text{and} \quad d(x_{n+1}, z) < \frac{1}{2}d(x_{n+1}, x_{n+2}).$$

Then by the triangle inequality, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, z) + d(x_{n+1}, z) \\ &< \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}) \\ &= d(x_n, x_{n+1}) \end{aligned}$$

a contradiction and (5) holds. In the case $\frac{1}{2}d(x_n, x_{n+1}) \le d(x_n, z)$ by (PS), we have

$$\begin{aligned} d(x_{n+1}, z) = d(f(x_n), f(z)) &\le \psi(\mathcal{D}(x_n, z)) \\ &= \psi(d(x_n, z) + d(x_n, f(x_n)) + d(z, f(z))). \end{aligned}$$

Letting $n \to \infty$, gives $d(z, f(z)) \le \psi(d(z, f(z))) < d(z, f(z))$, a contradiction unless $f(z) = z$ is fixed point of f. We get the same conclusion in the other case. The uniqueness of fixed point follows easily. □

If we take $\mathcal{D}(x, y) = d(x, y)$ in Theorem 5, we obtain following generalization of Boyd and Wong [3, Th. 1].

Theorem 6 *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping such that*

$$\frac{1}{2}d(x, f(x)) \le d(x, y) \text{ implies } d(f(x), f(y)) \le \psi(d(x, y)) \tag{6}$$

Then there exists a unique fixed point $z \in X$ for f.

Proof Pick $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = f^n(x_0) = f(x_{n-1})$ for all $n \in \mathbb{N}$. Since $\frac{1}{2}d(x_{n-1}, x_n) \le d(x_{n-1}, x_n)$ by (6), we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\le \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \end{aligned}$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below, so it has a limit c. Suppose that $c > 0$. Then by the above inequality, we get $d(x_n, x_{n+1}) \le \psi(d(x_{n-1}, x_n))$. So that

$$c = \limsup_{t \to c+} \psi(t) \le \psi(c),$$

which a contradiction. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and f is asymptotically regular. Rest of the proof may be completed following the proof of Theorem 5. □

Now, we consider another class of mappings:

Definition 3 Let (X, d) be a metric space and let $f : X \to X$ be a mapping such that for all $x, y \in X$,

$$d(x, f(y)) \le Md(x, y) + \mu[d(x, f(x)) + d(f^j(x), f^{j+1}(x))] \tag{7}$$

where $j \in \mathbb{N}$, $M \in (0, 1)$ and $\mu \in [0, \infty)$.

The above class of mappings contains many important classes of mappings. A number of contractions listed in [20] are particular cases of the following mapping:

Definition 4 Let (X, d) be a metric space and let $f : X \to X$ be a mapping such that for all $x, y \in X$,

$$\begin{aligned} d(f(x), f(y)) \le k \max \Big\{ & d(x, y), \frac{d(x, f(x)) + d(y, f(y))}{2}, d(x, f(y)), d(y, f(x)), \\ & d(f^2(x), x), d(f^2(x), f(x)), d(f^2(x), y), d(f^2(x), f(y)) \Big\}, \end{aligned} \tag{8}$$

where $k \in [0, 1)$ is fixed.

Now, we show that a mapping satisfying (8) also satisfies (7).

Proposition 1 *Let (X, d) be a metric space and let $f : X \to X$ satisfies (8). Then f also satisfies (7) but the converse need not be true.*

Proof We consider the following cases:

Case (i) $d(f(x), f(y)) \le kd(x, y)$. Using the triangle inequality, we get

$$\begin{aligned} d(x, f(y)) &\le d(x, f(x)) + d(f(x), f(y)) \\ &\le kd(x, y) + d(x, f(x)). \end{aligned}$$

Case (ii) $d(f(x), f(y)) \le k\frac{d(x, f(x)) + d(y, f(y))}{2}$. Then

$$\begin{aligned} d(x,f(y)) &\le d(x,f(x)) + d(f(x),f(y)) \\ &\le d(x,f(x)) + k\frac{d(x,f(x)) + d(y,f(y))}{2} \\ &\le \frac{3}{2}d(x,f(x)) + \frac{1}{2}kd(x,y) + \frac{1}{2}d(x,f(y)). \end{aligned}$$

It implies that

$$d(x,f(y)) \le kd(x,y) + 3d(x,f(x)).$$

Case (iii) $d(f(x),f(y)) \le kd(x,f(y))$. Then

$$d(x,f(y)) \le d(x,f(x)) + kd(x,f(y))$$

and

$$d(x,f(y)) \le kd(x,y) + \frac{1}{(1-k)}d(x,f(x)).$$

Case (iv) $d(f(x),f(y)) \le kd(y,f(x))$. Then

$$\begin{aligned} d(x,f(y)) &\le d(x,f(x)) + kd(y,f(x)) \\ &\le d(x,f(x)) + kd(x,y) + kd(x,f(x)) \\ &\le kd(x,y) + (1+k)d(x,f(x)). \end{aligned}$$

Case (v) $d(f(x),f(y)) \le kd(f^2(x),x)$. Then

$$\begin{aligned} d(x,f(y)) &\le d(x,f(x)) + kd(f^2(x),x) \\ &\le d(x,f(x)) + kd(f^2(x),f(x)) + kd(x,f(x)) \\ &\le (1+k)d(x,f(x)) + kd(f^2(x),f(x)). \end{aligned}$$

Case (vi) $d(f(x),f(y)) \le kd(f^2(x),f(x))$. Then

$$d(x,f(y)) \le d(x,f(x)) + kd(f^2(x),f(x)).$$

Case (vii) $d(f(x),f(y)) \le kd(f^2(x),y)$. Then

$$\begin{aligned} d(x,f(y)) &\le d(x,f(x)) + kd(f^2(x),y) \\ &\le d(x,f(x)) + kd(f^2(x),f(x)) + kd(f(x),y) \\ &\le kd(x,y) + (1+k)[d(x,f(x)) + d(f^2(x),f(x))]. \end{aligned}$$

Case (viii) $d(f(x),f(y)) \le kd(f^2(x),f(y))$. Then

$$\begin{aligned} d(x,f(y)) &\le d(x,f(x)) + kd(f^2(x),f(y)) \\ &\le d(x,f(x)) + kd(f^2(x),f(x)) + kd(f(x),x) + kd(x,f(y)) \end{aligned}$$

and

$$d(x,f(y)) \le kd(x,y) + \frac{2}{(1-k)}[d(x,f(x)) + d(f^2(x),f(x))].$$

Thus f satisfies (7) with $M = k$, $\mu = \max\left\{3, \frac{2}{1-k}\right\}$ and $j = 2$. □

The following example shows that the converse of above proposition need not be true.

Example 3 Let $X = \{(0,0),(1,0),(0,1),(2,0),(0,2),(2,3),(3,2)\}$ be equipped with the metric d defined as follows

$$d((x^{(1)},x^{(2)}),(y^{(1)},y^{(2)})) = |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|.$$

Define $f : X \to X$ by

$$f(0,0) = (0,0),\ f(1,0) = (0,0),\ f(0,1) = (0,0),\ f(2,0) = (1,0),\ f(0,2) = (0,1),$$
$$f(2,3) = (2,0),\ f(3,2) = (0,2).$$

It can be easily verified that f satisfies (7) for any $M \geq 0.5$ and $\mu \geq 5$. However, for $x = (2,3)$, $y = (3,2)$ and any $k \in [0,1)$, we have

$$\begin{aligned} d(f(x),f(y)) = 4 &> k \max\{2,3,3,3,4,1,4,3\} \\ &= k \max\Big\{ d(x,y), \frac{d(x,f(x)) + d(y,f(y))}{2}, d(x,f(y)), d(y,f(x)), \\ &\quad d(f^2(x),x), d(f^2(x),f(x)), d(f^2(x),y), d(f^2(x),f(y))\Big\}. \end{aligned}$$

Hence f does not satisfy (8).

Now, we present a theorem without continuity assumption of the mapping.

Theorem 7 *Suppose (X, d) is a complete metric space and $f : X \to X$ is an asymptotically regular mapping satisfying condition (7). Then there exists a unique fixed point $p \in X$ for f and for any $x \in X$ we have $\lim_{n\to\infty} f^n(x) = p$.*

Proof Let $x_0 \in X$ and define $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$. For any $m > 0$, by the triangle inequality and from (7), we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},x_{n+m+1}) + d(x_{n+m+1},x_n) \\ &\leq d(x_{n+m},x_{n+m+1}) + d(x_n,f(x_{n+m})) \\ &\leq d(x_{n+m},x_{n+m+1}) + Md(x_n,x_{n+m}) + \mu[d(x_n,f(x_n)) \\ &\quad + d(f^j(x_n),f^{j+1}(x_n))] \\ &\leq d(x_{n+m},x_{n+m+1}) + Md(x_n,x_{n+m}) + \mu[d(x_n,x_{n+1}) + d(x_{n+j},x_{n+j+1})]. \end{aligned}$$

This implies that

$$(1 - M)d(x_{n+m},x_n) \leq d(x_{n+m},x_{n+m+1}) + \mu[d(x_n,x_{n+1}) + d(x_{n+j},x_{n+j+1})].$$

By the asymptotically regularity of f, we obtain $d(x_{n+m},x_n) \to \infty$ as $n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Next we show that p is a fixed point of f, from (7), it follows that

$$d(x_n,f(p)) \leq Md(x_n,p) + \mu[d(x_n,f(x_n)) + d(f^j(x_n),f^{j+1}(x_n))].$$

It implies that $x_n \to f(p)$ as $n \to \infty$ and $f(p) = p$. Suppose q is another fixed point of f. Then

$$0 < d(p,q) = d(p,f(q)) \leq Md(p,q) + \mu[d(p,f(p)) + d(f^j(p),f^{j+1}(p))].$$
$$= Md(p,q) < d(p,q),$$

a contradiction unless $p = q$. This proves the uniqueness of fixed point. Further, for any $x \in X$, we have

$$d(f^n(x),p) = d(f^n(x),f(p)) \leq Md(f^n(x),p) + \mu[d(f^n(x),f^{n+1}(x) + d(f^{j+n}(x),f^{j+n+1}(x))].$$

This implies that

$$(1-M)d(f^n(x),p) \leq \mu[d(f^n(x),f^{n+1}(x)) + d(f^{j+n}(x),f^{j+n+1}(x))] \to 0 \text{ as } n \to \infty.$$

This shows that $\lim_{n\to\infty} f^n(x) = p$ for any $x \in X$. □

The following example shows that the asymptotic regularity condition on the mapping f can not be dropped in Theorem 7.

Example 4 [7]. $X = \{0\} \cup [1,\infty)$ be a metric space endowed with the usual metric d. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

We consider the following two cases:

Case (a) If $x \neq 0$ and $y = 0$ then

$$d(x,f(y)) = x - 1 \leq Mx + \mu(x+1) = Md(x,y) + \mu[d(x,f(x)) + d(f^1(x),f^2(x))].$$

Case (b) If $x = 0$ and $y \neq 0$ then

$$d(x,f(y)) = 0 \leq Md(x,y) + \mu[d(x,f(x)) + d(f^1(x),f^2(x))].$$

Then f satisfies condition (7) for $\mu \geq 1$, $M > 0$ and $j = 2$. But f is not asymptotically regular at any point in X and f is a fixed point free mapping.

The example below shows the validity of our Theorem 7.

Example 5 Let $X = [0,1] \times [0,1]$ be a metric space endowed with the metric d defined as

$$d((x^{(1)},x^{(2)}),(y^{(1)},y^{(2)})) = |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|.$$

Let $f : X \to X$ be a mapping defined by

$$f(x^{(1)},x^{(2)}) = \begin{cases} \left(\frac{1}{4}(x^{(1)} + \frac{1}{5})^2, 1 - \frac{2}{3}x^{(2)}\right), & \text{if } x^{(1)} \in [0,\frac{2}{3}), \\ \left(\frac{x^{(1)}}{7} + \frac{1}{3}, 1 - \frac{2}{3}x^{(2)}\right), & \text{if } x^{(1)} \in [\frac{2}{3},1]. \end{cases}$$

We consider two cases and show that f satisfies condition (7) for $\mu = 5$ and $M = 0.9$:

Case (a) Let $x^{(1)} \in \left[0, \frac{2}{3}\right)$ and $y^{(1)} \in \left[0, \frac{2}{3}\right)$. Then

$$\begin{aligned} d(f(x), f(y)) &= \frac{1}{4}\left|\left(x^{(1)} + \frac{1}{5}\right)^2 - \left(y^{(1)} + \frac{1}{5}\right)^2\right| + \frac{2}{3}|x^{(2)} - y^{(2)}| \\ &\le \frac{1}{4}|(x^{(1)} + y^{(1)})(x^{(1)} - y^{(1)})| + \frac{1}{10}|x^{(1)} - y^{(1)}| + \frac{2}{3}|x^{(2)} - y^{(2)}| \\ &\le \frac{13}{30}|x^{(1)} - y^{(1)}| + \frac{2}{3}|x^{(2)} - y^{(2)}| \le Md(x, y). \end{aligned}$$

By the triangle inequality and above inequality, we get

$$d(x, f(y)) \le d(x, f(x)) + Md(x, y).$$

Now, let $y^{(1)} \in \left[\frac{2}{3}, 1\right]$. Then f satisfies (7) if the following condition holds:

$$\begin{aligned} \left|x^{(1)} - \frac{y^{(1)}}{7} - \frac{1}{3}\right| + \left|x^{(2)} - 1 + \frac{2}{3}y^{(2)}\right| \le 5\left\{\left|x^{(1)} - \frac{1}{4}\left(x^{(1)} + \frac{1}{5}\right)^2\right| + \left|x^{(2)} - 1 + \frac{2}{3}x^{(2)}\right|\right\} \\ + M|x^{(1)} - y^{(1)}| + M|x^{(2)} - y^{(2)}|. \end{aligned}$$

We split the above inequality into two parts. First, we show the following inequality is true:

$$\left|x^{(1)} - \frac{y^{(1)}}{7} - \frac{1}{3}\right| \le 5\left|x^{(1)} - \frac{1}{4}\left(x^{(1)} + \frac{1}{5}\right)^2\right| + M|x^{(1)} - y^{(1)}|. \tag{9}$$

From the considered range of $x^{(1)}$ and $y^{(1)}$, it follows that $\left|x^{(1)} - \frac{y^{(1)}}{7} - \frac{1}{3}\right| \le \frac{10}{21}$. For $x^{(1)} \in \left[0, \frac{26}{189}\right)$, it can be seen that $M|y^{(1)} - x^{(1)}| \ge \frac{10}{21}$ and (9) is true for this case. For $x^{(1)} \in \left[\frac{26}{189}, \frac{2}{3}\right)$, the function $x^{(1)} - \frac{1}{4}\left(x^{(1)} + \frac{1}{5}\right)^2$ is increasing and $\left|x^{(1)} - \frac{1}{4}\left(x^{(1)} + \frac{1}{5}\right)^2\right| \ge \frac{389639}{3572100}$. Thus $5\left|x^{(1)} - \frac{1}{4}\left(x^{(1)} + \frac{1}{5}\right)^2\right| \ge \frac{10}{21}$ and (9) is true for this case too. Moreover, by the triangle inequality

$$\begin{aligned} \left|x^{(2)} - 1 + \frac{2}{3}y^{(2)}\right| &\le \left|x^{(2)} - 1 + \frac{2}{3}x^{(2)}\right| + \frac{2}{3}|x^{(2)} - y^{(2)}| \\ &\le \left|x^{(2)} - 1 + \frac{2}{3}x^{(2)}\right| + M|x^{(2)} - y^{(2)}|. \end{aligned} \tag{10}$$

Combining (9) and (10), it follows that f satisfies the condition (7) for the case considered.

Case (b) Let $x^{(1)} \in \left[\frac{2}{3}, 1\right]$ and $y^{(1)} \in \left[\frac{2}{3}, 1\right]$. Then it is evident that f satisfies condition (7).

Let $y^{(1)} \in \left[0, \frac{2}{3}\right)$. Then f satisfies condition (7) if the following condition holds:

$$\begin{aligned} \left|x^{(1)} - \frac{1}{4}\left(y^{(1)} + \frac{1}{5}\right)^2\right| + \left|x^{(2)} - 1 + \frac{2}{3}y^{(2)}\right| \le 5\left\{\left|\frac{6}{7}x^{(1)} - \frac{1}{3}\right| + \left|\frac{5}{3}x^{(2)} - 1\right|\right\} \\ + M|x^{(1)} - y^{(1)}| + M|x^{(2)} - y^{(2)}|. \end{aligned}$$

We split the above inequality into two parts. First, we prove the following inequality is true:

$$\left|x^{(1)} - \frac{1}{4}\left(y^{(1)} + \frac{1}{5}\right)^2\right| \le 5\left\{\left|\frac{6}{7}x^{(1)} - \frac{1}{3}\right|\right\} + M|x^{(1)} - y^{(1)}|. \tag{11}$$

From the considered range of $x^{(1)}$ and $y^{(1)}$, it can be seen that $\left|x^{(1)} - \frac{1}{4}\left(y^{(1)} + \frac{1}{5}\right)^2\right| \le \frac{99}{100}$ and $\left|\frac{6}{7}x^{(1)} - \frac{1}{3}\right| \ge \frac{5}{21}$. Therefore, $5\left|\frac{6}{7}x^{(1)} - \frac{1}{3}\right| \ge \frac{99}{100}$. Further, by the triangle inequality

$$\left|x^{(2)} - 1 + \frac{2}{3}y^{(2)}\right| \le \left|\frac{5}{3}x^{(2)} - 1\right| + \frac{2}{3}|x^{(2)} - y^{(2)}|. \tag{12}$$

Combining (11) and (12), it follows that f satisfies the condition (7). Therefore f satisfies the hypotheses of Theorem 7. We note that f is not continuous on X.

3 Applications to nonlinear integral equations

In this section, we present an application of our results to integral equations.

Now, we consider the following nonlinear integral equation

$$x(t) = \varpi(t) + \lambda \int_a^b F(t,s)\varrho(s, x(s))ds, \quad t \in [a,b], \quad \lambda \ge 0. \tag{13}$$

Theorem 8 *Let $X = C[a,b]$ be the space of continuous functions on $[a, b]$ with metric defined by $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$. Suppose that the following assumptions are true:*

(i) $\varpi : [a,b] \to \mathbb{R}$ *is a continuous function;*

(ii) $\varrho : [a,b] \times X \to X$ *is continuous,* $\varrho(t,x) \ge 0$ *and there is a constant* $L \ge 0$ *such that for all* $x, y \in X$,

$$|\varrho(t,x) - \varrho(t,y)| \le L|x(t) - y(t)|;$$

(iii) $F : [a,b] \times [a,b] \to \mathbb{R}$ *is continuous for all* $(t,x) \in [a,b] \times [a,b]$ *such that* $F(t,x) \ge 0$ *and* $\int_0^1 F(t,s)ds \le K$;

(iv) $M = \lambda KL < 1$;

(v) $f : X \to X$ *is a mapping defined by*

$$f(x(t)) = \varpi(t) + \lambda \int_a^b F(t,s)\varrho(s,x(s))ds, \quad t \in [a,b], \quad \lambda \ge 0$$

and f is asymptotically regular. Then, the nonlinear integral equation (13) has a unique solution in X. Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ defined as

$$(x_n)(t) = \varpi(t) + \lambda \int_a^b F(t,s)\varrho(s, x_{n-1}(s))ds \quad \text{for all } n \in \mathbb{N}$$

converges to the unique solution of (13).

Proof For $x, y \in X$, we have

$$\begin{aligned}|x(t)-f(y(t))| &= \left|\left(x(t)-\varpi(t)-\lambda\int\limits_a^b F(t,s)\varrho(s,x(s))ds\right)+(\varpi(t)\right.\\ &\quad \left.+\ \lambda\int\limits_a^b F(t,s)\varrho(s,x(s))ds-\varpi(t)-\lambda\int\limits_a^b F(t,s)\varrho(s,y(s))ds\right)\Bigg|\\ &\le |x(t)-f(x(t))|+\lambda\left|\int\limits_a^b F(t,s)\varrho(s,x(s))ds-\int\limits_a^b F(t,s)\varrho(s,y(s))ds\right|\\ &\le |x(t)-f(x(t))|+\lambda\int\limits_a^b F(t,s)|\varrho(s,x(s))-\varrho(s,y(s))|ds\\ &\le |x(t)-f(x(t))|+\lambda\int\limits_a^b F(t,s)L|x(s)-y(s)|ds.\end{aligned}$$

Taking supremum over $[a, b]$ on both sides, we get

$$\begin{aligned}d(x,f(y)) &\le d(x,f(x))+\lambda KLd(x,y)\\ &= d(x,f(x))+Md(x,y).\end{aligned}$$

Thus, the mapping f satisfying condition (7) and all the hypothesis of Theorem 7 hold. Therefore, (13) has a unique solution in X. □

Example 6 Let us consider the following Fredholm integral equation:

$$x(t)=\left[\cos\left(\frac{\pi}{4}t\right)-\frac{620}{719}t\right]+\frac{1}{2}\int_0^1(ts^2+t^2s)x(s)ds,\qquad t\in[0,1] \tag{14}$$

It can be seen that the Fredholm integral equation (14) is a particular case of (13) with

$$\varpi(t)=\cos\left(\frac{\pi}{4}t\right)-\frac{6200}{719}t;\quad F(t,s)=ts^2+t^2s\quad\text{and}\quad\varrho(t,x)=x(s).$$

For any $x, y \in \mathbb{R}$ and for $t \in [0, 1]$, we have

$$|\varrho(t,x)-\varrho(t,y)|=|x-y|.$$

It can be easily seen that ϖ is a continuous function and $t \in [0, 1]$

$$\int_0^1 F(t,s)ds=\int_0^1(ts^2+t^2s)ds=\frac{t}{3}+\frac{t^2}{2}\le\frac{5}{6}.$$

Further, $L = 1$, $K = \frac{5}{6}$, $\lambda = \frac{1}{2}$ with $M = LK\lambda = \frac{5}{12} < 1$. Therefore, all the assumptions of Theorem 8 are satisfied. Hence, there exists a solution of the Fredholm integral equation (14). It can be seen that $x(t) = \cos\left(\frac{\pi}{4}t\right) + \frac{220}{719}t + \frac{160}{719}t^2$ is a solution of nonlinear integral equation (14).

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References

1. Browder, F.E.: On the convergence of successive approximations for nonlinear functional equations. Nederl. Akad. Wetensch. Proc. Ser. A 30, 27-35 (1968)
2. Browder, F.E., Petryshyn, W.V.: The solution by iteration of nonlinear functional equations in Banach spaces. Bull. Amer. Math. Soc. 72, 571–575 (1966)
3. Boyd, D.W., Wong, J.S.W.: On nonlinear contractions. Proc. Amer. Math. Soc. 20, 458–464 (1969)
4. Ćirić, L.B.: A generalization of Banach's contraction principle. Proc. Amer. Math. Soc. 45, 267–273 (1974)
5. Fulga, A., Proca, A.: A new Generalization of Wardowski Fixed Point Theorem in Complete Metric Spaces. Adv. Theory Nonlinear Anal. Appl. 1, 57–63 (2017)
6. Fulga, A.: Fixed point theorems in rational form via Suzuki approaches. Results in Nonlinear Analysis. 1, 19–29 (2018)
7. Guay, M.D., Singh, K.L.: Fixed points of asymptotically regular mappings. Mat. Vesnik. 35, 101–106 (1983)
8. Hardy, G.E., Rogers, T.G.: Generalization of a fixed point theorem of Reich. Can. Math. Bull. 16, 201–206 (1973)
9. Kannan, R.: Some results on fixed points. Bull. Cal. Math. Soc. 60, 71–76 (1968)
10. Kannan, R.: Some results on fixed points-II. Amer. Math. Monthly 76, 405–408 (1969)
11. Karapınar, E., De La Sen, M., Fulga, A.: A note on the Górnicki-Proinov type contraction. J. Funct. Spaces 2021, Art. ID 6686644, 1–8 (2021)
12. Karapınar, E., Fulga, A.: A fixed point theorem for Proinov mappings with a contractive iterate. Appl. Math. J. Chin. Univ. **(in press)**
13. Matkowski, J.: Fixed point theorems for contractive mappings in metric spaces. Cas. Pest. Mat. 105, 341–344 (1980)
14. Meir, A., Keeler, E.: A theorem on contraction mappings. J. Math. Anal. Appl. 28, 326–329 (1969)
15. Nadler, S.B., Jr.: Multi-valued contraction mappings. Pacific J. Math. 30, 475–488 (1969)
16. Pant, R., Shukla, R., Nashine, H.K., Panicker, R.: Some new fixed point theorems in partial metric spaces with applications. J. Funct. Spaces 2017, Art. ID 1072750, 1–13 (2017)
17. Proinov, P.D.: Fixed point theorems in metric spaces. Nonlinear Anal. 64, 546–557 (2006)
18. Rakotch, E.: A note on contractive mappings. Proc. Amer. Math. Soc. 13, 459–465 (1962)
19. Reich, S.: Some remarks concerning contraction mappings. Canad. Math. Bull. 14, 121–124 (1971)
20. Rhoades, B.E.: A comparison of various definitions of contractive mappings. Tran. Amer. Math. Soc. 226, 257–290 (1977)
21. Singh, S.L., Mishra, S.N., Pant, R.: New fixed point theorems for asymptotically regular multi-valued maps. Nonlinear Anal. 71, 3299–3304 (2009)
22. Singh, S.L., Mishra, S.N., Chugh, R., Kamal, R.: General common fixed point theorems and applications. J. Appl. Math. 2012, Art. ID 902312, 1–14 (2012)
23. Shukla, R., Pant, R.: Fixed point results for nonlinear contractions with application to integral equations. Asian-Eur. J. Math. 12(2050007), 1–17 (2019)
24. Suzuki, T.: A generalized Banach contraction principle that characterizes metric completeness. Proc. Amer. Math. Soc. 136, 1861–1869 (2008)