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Discussions on the fixed points of Suzuki–Edelstein *E*-contractions

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Abstract
Recently, fixed point results via E -contractions (also called P- contractions in some papers) for self-mappings in metric spaces have been investigated. Some results generalize the well - known Edelstein's theorem. In this paper, we provide some new fixed point theorems for mappings satisfying conditions of Edelstein - Suzuki type involving E -contractions. We also present several illustrative examples to compare our finding with some know results in the literature.

Keywords E-contractions · Contractive mapping · P-contractive mappings · Edelsteins–Suzuki type · Fixed point

Mathematics Subject Classification 54H25 · 47H10

1 Introduction

Fixed point theory plays a fundamental role in nonlinear analysis and its applications. One of the basic and important fixed point results is the Banach contraction principle. It states that if T is a contraction mapping on a complete metric (X, d), i.e., there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X, \tag{1}$$

then T has a unique fixed point. This principle has been a powerful tool for proving the existence of solutions of differential equations, integral equations, nonlinear equations, etc., and for solving various problems in mathematical science and engineering. The Banach

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principle has been extended, generalized and improved in several directions. In 1962, Edelstein [8] proved the following fixed point theorem.

Theorem 1 [8] *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a self-mapping. If*

$$d(Tx, Ty) < d(x, y) \quad \forall x, y \in X, x \neq y, \tag{2}$$

then T *has a unique fixed point.*

Examples show that the compactness of the metric space (X, d) cannot be replaced by its completeness. The following is such an example which may be known.

Example 1 Let $X = [2, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then, (X, d) is a complete metric space but it is not compact. We consider the mapping $T : X \to X$ defined by $Tx = x + \frac{2}{x}$ for all $x \in X$. For all $x, y \in X$ with $x \neq y$, we have

$$d(Tx, Ty) = \left| x + \frac{2}{x} - y - \frac{2}{y} \right| < |x - y| = \left(1 - \frac{2}{xy}\right) d(x, y) = d(x, y).$$

Thus, condition (2) is satisfied. However, T has no fixed point in X.

It is noted that if $T : X \to X$ satisfies condition (2), then T is continuous on X. Thus, the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx)$ is continuous on X. The standard proof of Theorem 1 is based on the fact that the continuous function f on the compact space X attains its minimum on X. Many results which generalize and extend Edelstein's result use this proof technique and require some continuity properties of the considered mappings, see, for examples [6, 9, 13, 14, 20] and references therein.

In 2009, Suzuki [23] obtained a very interesting generalization of Edelstein's theorem which is stated as follows.

Theorem 2 [23] *Let* (X, d) *be a compact metric space and let* $T : X \to X$*. Assume that*

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Longrightarrow \quad d(Tx, Ty) < d(x, y) \tag{3}$$

holds for all $x, y \in X$*. Then* T *has a unique fixed point in* X*.*

Note that a mapping $T : X \to X$ satisfying condition (3) is not necessarily continuous and Theorem 2 is a real generalization of Theorem 1. The idea of the proof of Theorem 2 may be used to prove some generalizations of Edelstein's theorem without assuming that the involving mappings are continuous (see, e.g., [7, 15, 16]). For further fixed point results involving Suzuki or Suzuki–Edelstein type contractions, we refer the reader to [2, 3, 17, 18] and references therein.

Recently, Altun et.al. [4] proved a new extension of Edelstein's theorem.

Theorem 3 [4, *Theorem* 2.11] *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a P-contractive mapping, i.e.,*

$$d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)|, \quad \text{for all} \quad x, y \in X, x \neq y. \tag{4}$$

If $f(x) = d(x, Tx)$ *is lower semi-continuous, then* T *has a unique fixed point.*

In [4], the authors also presented several examples showing that Theorem 3 and Theorem 2 are independent. We refer the reader to [5] for further results concerning P- contractive mappings.

It is worth noting that the expression

$$E(x, y) := d(x, y) + |d(x, Tx) - d(y, Ty)|$$

in the right-hand side of (4) was first appeared in the papers [10, 11] by A. Fulga and A. M. Proca where they presented some fixed point theorems for a self-mapping T in a metric space (X, d) satisfying some generalized contractive conditions which involve the expression $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$. For more interesting fixed point results for mappings satisfying conditions involving E-contractions (i.e., conditions which involve the expression $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$), we refer the reader to, e.g., [12, 19] and references therein.

Some questions may arise:

Question 1

(i) Do the conclusion of Theorem 3 still hold true if we do not require any assumption on the function f?
(ii) Can we provide a Theorem which generalize both Theorem 2 and Theorem 3?
(iii) Can we replace the compactness of X in Theorem 2 by other conditions on X?

Our aim is to answer these questions and provide some new fixed point of Edelstein - Suzuki type involving E-contraction mappings. Our results extend, improve and unify some known results in the literature. Several examples are also given to illustrate our results.

2 Fixed point results

Our first result is a generalization of Theorem 3.

Theorem 4 *Let (X, d) be a compact metric space and let T be a self mapping on X. Assume that there exists $C \geq 0$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx) \tag{5}$$

for all $x, y \in X$. If $f(x) = d(x, Tx)$ is lower semi-continuous, then T has a fixed point in X.

Proof Since $f(x) = d(x, Tx)$ is lower semi-continuous, there exists $x_0 \in X$ such that $f(x_0) \leq f(Tx_0)$, i.e., $d(x_0, Tx_0) \leq d(Tx_0, T^2x_0)$ (see, e.g., [1]). Assume that $x_0 \neq Tx_0$. Then

$$\frac{1}{2}d(x_0, Tx_0) < d(x_0, Tx_0) \leq d(Tx_0, T^2x_0).$$

By (5), we have

$$\begin{aligned} d(Tx_0, T^2x_0) <&d(x_0, Tx_0) + |d(x_0, Tx_0) - d(Tx_0, T^2x_0)| + Cd(Tx_0, Tx_0) \\ =&d(x_0, Tx_0) + d(Tx_0, T^2x_0) - d(x_0, Tx_0) \\ =&d(Tx_0, T^2x_0) \end{aligned}$$

which is a contradiction. Thus, $x_0 = Tx_0$ and T has a fixed point. □

The following simple example showing that if $C > 0$, then T may have more than one fixed point.

Example 2 Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $T : X \to X$ be defined by $Tx = x$ for all $x \in X$. For all $x, y \in X$, $x \neq y$, we have $d(Tx, Ty) = |x - y|$ and

$$d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx) = |x - y| + C|x - y| = (1 + C)|x - y|.$$

Since $C > 0$, we have

$$d(Tx, Ty) = |x - y| < (1 + C)|x - y| = d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx).$$

Thus all conditions of Theorem 4 are satisfied. Of course, every $x \in X$ is a fixed point of T.

The following corollary is a generalization of Theorem 2.11 and Theorem 2.14 in [4].

Corollary 1 *Let (X, d) be a compact metric space and let T be a self mapping on X. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \Longrightarrow \quad d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)|$$

for $x, y \in X$. If $f(x) = d(x, Tx)$ is lower semi-continuous, then T has a unique fixed point in X.

Proof According to Theorem 4, T has a fixed point $z \in X$. Assume that T has another fixed point $y \in X$ with $y \neq z$. Then $\frac{1}{2}d(z, Tz) = 0 < d(z, y)$. Hence, by (6), we have

$$d(z, y) = d(Tz, Ty) < d(z, y) + L|d(z, Tz) - d(y, Ty)| = d(z, y),$$

a contradiction. Therefore, T has a unique fixed point. □

Question 2 Does the conclusion of Theorem 4 still hold true if we remove condition "*$f(x) = d(x, Tx)$ is lower semi-continuous*"?

In the next theorem, we give a partial answer to the above question. This is also a partial answer to Question 1(ii).

Theorem 5 *Let (X, d) be a compact metric space and let T be a self mapping on X. Assume that there exist $L \in [0, 1)$ and $C \geq 0$ such that*

$$\frac{1}{2}d(x,Tx) < d(x,y) \\ \implies \quad d(Tx,Ty) < d(x,y) + L|d(x,Tx) - d(y,Ty)| + Cd(y,Tx) \tag{6}$$

for $x, y \in X$. Then T has a fixed point in X.

Proof Set

$$m = \inf\{d(x,Tx) : x \in X\}.$$

Then, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = m.$$

Since X is compact, we may assume that

$$\lim_{n\to\infty} x_n = a, \quad \text{and} \quad \lim_{n\to\infty} Tx_n = b$$

for some $a, b \in X$. Hence,

$$m = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, b) = \lim_{n\to\infty} d(a, Tx_n) = d(a,b).$$

We are going to show that $m = 0$. Assume to the contrary that $m > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, b) > \frac{2}{3}m \quad \text{and} \quad d(x_n, Tx_n) < \frac{4}{3}m$$

for all $n \geq n_0$. That is,

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, b) \quad \text{for all} \quad n \geq n_0.$$

By (6), one has, for all $n \geq n_0$, that

$$d(Tx_n, Tb) < d(x_n, b) + L|d(x_n, Tx_n) - d(b, Tb)| + Cd(b, Tx_n).$$

Letting $n \to \infty$, we get

$$d(b, Tb) \leq m + L|m - d(b, Tb)| + C.0.$$

By the definition of m, it follows from the latter inequality that

$$d(b, Tb) \leq m + L(d(b, Tb) - m) = (1 - L)m + Ld(b, Tb)$$

which yields $d(b, Tb) \leq m$ since $L \in [0, 1)$. Again, by the definition of m, we obtain $d(b, Tb) = m$. This implies that $\frac{1}{2}d(b, Tb) < d(b, Tb)$. By (6), we have

$$d(Tb, T^2b) < d(b, Tb) + L|d(b, Tb) - d(Tb, T^2b)| + Cd(Tb, Tb) = m + L|m - d(Tb, T^2b)|.$$

Again, by the definition of m and by the fact that $L \in [0, 1)$, we arrive at $d(Tb, T^2b) < m$ which is a contradiction. Thus, $m = 0$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = a.$$

We shall show that T has a fixed point in X. Assume to the contrary that T has no fixed point. Then $d(x_n, Tx_n) > 0$ for all $n \in \mathbb{N}$. This implies that

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, Tx_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

By (6), we have for all n that

$$\begin{aligned} d(Tx_n, T^2x_n) <& d(x_n, Tx_n) + L|d(x_n, Tx_n) - d(Tx_n, T^2x_n)| + Cd(Tx_n, Tx_n) \\ =& d(x_n, Tx_n) + L|d(x_n, Tx_n) - d(Tx_n, T^2x_n)|. \end{aligned} \tag{7}$$

Suppose that there exists $k \in \mathbb{N}$ such that $d(Tx_k, T^2x_k) \geq d(x_k, Tx_k)$. By (7),

$$\begin{aligned} d(Tx_k, T^2x_k) <& d(x_k, Tx_k) + L(d(Tx_k, T^2x_k) - d(x_k, Tx_k) \\ =& (1 - L)d(x_k, Tx_k) + Ld(Tx_k, T^2x_k) \end{aligned}$$

which yields $d(Tx_k, T^2x_k) < d(x_k, Tx_k)$, a contradiction. Thus,

$$d(Tx_n, T^2x_n) < d(x_n, Tx_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

We have

$$\begin{aligned} \lim_{n\to\infty} d(a, T^2x_n) \leq& \lim_{n\to\infty}[d(a, Tx_n) + d(Tx_n, T^2x_n)] \\ \leq& \lim_{n\to\infty}[d(a, Tx_n) + d(x_n, Tx_n)] = 0. \end{aligned}$$

Hence, $\lim_{n\to\infty} T^2x_n = a$.

Assume now that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_N, Tx_N) \geq d(x_N, a) \quad \text{and} \quad \frac{1}{2}d(Tx_N, T^2x_N) \geq d(Tx_N, a).$$

Then, by the triangular inequality, one has

$$\begin{aligned} d(x_N, Tx_N) \leq& d(x_N, a) + d(a, Tx_N) \leq \frac{1}{2}d(x_N, Tx_N) + \frac{1}{2}d(Tx_N, T^2x_N \\ <& \frac{1}{2}d(x_N, Tx_N) + \frac{1}{2}d(x_N, Tx_N) = d(x_N, Tx_N), \end{aligned}$$

a contradiction. Thus, for each $n \in \mathbb{N}$,

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, a) \quad \text{or} \quad \frac{1}{2}d(Tx_n, T^2x_n) < d(Tx_n, a)$$

holds true. For the first case, one gets

$$d(Tx_n, Ta) < d(x_n, a) + L|d(x_n, Tx_n) - d(a, Ta)| + Cd(a, Tx_n), \tag{8}$$

and for the second case, one gets

$$d(T^2x_n, Ta) < d(Tx_n, a) + L|d(Tx_n, T^2x_n) - d(a, Ta)| + Cd(a, T^2x_n). \tag{9}$$

If (8) holds true for n in an infinite subset I of $\mathbb{N}$, then passing to the limit when $n \in I$, $n \to \infty$, we obtain

$$d(a, Ta) \leq d(a, a) + L|d(a, a) - d(a, Ta)| + Cd(a, a) = Ld(a, Ta)$$

which implies $d(a, Ta) = 0$, or $Ta = a$ since $L \in [0, 1)$.

If (9) holds for infinitely many n in $\mathbb{N}$, then we can treat in the same way with the first case and arrive at $Ta = a$. Hence, in any case, $Ta = a$. This contradicts to the assumption that T has no fixed point. Therefore, there exists $z \in X$ such that $Tz = z$. □

Corollary 2 *Let (X, d) be a compact metric space and let T be a self mapping on X. Assume that there exist $L \in [0, 1)$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \text{implies} \quad d(Tx, Ty) < d(x, y) + L|d(x, Tx) - d(y, Ty)| \tag{10}$$

for $x, y \in X$. Then T has a unique fixed point in X.

The following corollary is a partial answer to Question 1(i).

Corollary 3 *Let (X, d) be a compact metric space and let T be a self mapping on X. Assume that there exist $L \in [0, 1)$ such that*

$$d(Tx, Ty) < d(x, y) + L|d(x, Tx) - d(y, Ty)| \tag{11}$$

for $x, y \in X$. Then T has a unique fixed point in X.

Remark 1 In Corollary 2, if we let $L = 0$, then we get Theorem 2 - the Suzuki fixed point theorem. We do not know whether the conclusion of Corollary 3 hold or not if we let $L = 1$ in Corollary 3.

Example 3 Let $X = [-1, 10]$ with the usual metric and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } x \in (-1, 10], \\ 5 & \text{if } x = -1. \end{cases}$$

Then

(i) T has a unique fixed point.
(ii) T satisfies condition (10) with $L = \dfrac{39}{40}$.
(iii) T does not satisfy Suzuki's condition (3).
(iv) $x \mapsto d(x, Tx)$ is not lower semi-continuous.

Proof (i) It is obvious that T has a unique fixed point $x = 0$.

(ii) *Case 1:* For $x, y \in (-1, 10]$ and $x \neq y$, we have $d(Tx, Ty) = |x - y|/2 < |x - y| = d(x, y)$. Thus, T satisfies condition (10) for all $x, y \in (-1, 10]$.

Case 2: Let $x = -1$ and $y \in (-1, 10]$. We have

$$\frac{1}{2}d(x, Tx) = 3 < 1 + y = d(x, y)$$

when $y > 2$. In this case,

$$d(x,y) + L|d(x,Tx) - d(y,Ty)| = 1 + y + \frac{39}{40}\left|6 - \frac{y}{2}\right| = \frac{137}{20} + \frac{41y}{80} > 5 - \frac{y}{2} = d(Tx,Ty),$$

that is, condition (10) is satisfied for $x = -1$, $y \in (-1, 10]$.

Case 3: Let $x \in (-1, 10]$ and $y = -1$. We have

$$\frac{1}{2}d(x,Tx) = \frac{|x|}{4} < 1 + x = d(x,y)$$

when $x > -\frac{4}{5}$. In this case,

$$d(Tx,Ty) < d(x,y) + \frac{39}{40}|d(x,Tx) - d(y,Ty)|$$

if and only if

$$5 - \frac{x}{2} < 1 + x + \frac{39}{40}\left|\frac{|x|}{2} - 6\right|.$$

Similar to *Case 2*, the latter inequality holds for $x \geq 0$. When $-\frac{4}{5} < x < 0$, the latter inequality is equivalent to

$$4 < \frac{3}{2}x + \frac{117}{20} + \frac{39}{80}x$$

or

$$x > -\frac{148}{159}.$$

Thus, condition (10) is satisfied for $x \in (-1, 10]$ and $y = -1$.

Therefore, T satisfies condition (10) with $L = \frac{39}{40}$.

(iii) For $x = 0$ and $y = -1$, we have

$$\frac{1}{2}d(x,Tx) = 0 < 1 = d(x,y) \quad \text{and} \quad d(Tx,Ty) = 5 > 1 = d(x,y)$$

and then (3) does not hold.

(iv) One has

$$d(x,Tx) = \begin{cases} \frac{|x|}{2} & \text{if } x \in (-1, 10], \\ 6 & \text{if } \quad x = -1. \end{cases}$$

It is easy to see that $x \mapsto d(x,Tx)$ is not lower semi-continuous at $x = -1$. Thus, we cannot apply Theorem 4 and Theorem 3 to this example. □

Example 4 Let $X = [0, 1]$ with the usual metric d. Then, (X, d) is compact. We define a mapping $T : X \to X$ as

$$Tx = \begin{cases} \frac{x}{2} & \text{if} \quad x \in [0,1), \\ 1 & \text{if} \quad x = 1. \end{cases}$$

We first check that condition (10) is not satisfied. Assume to the contrary that (10) is satisfied. Since

$$\frac{1}{2}d(1, T1) = 0 < 1 - y = d(1, y) \forall y \in [0, 1),$$

we have for all $y \in [0, 1)$ that

$$1 - \frac{y}{2} = d(T1, Ty) < d(1, y) + L|d(1, T1) - d(y, Ty)| + Cd(y, T1) = 1 - y + L\frac{y}{2} + C(1 - y)$$

or, equivalently,

$$(1 - L)\frac{y}{2} < C(1 - y).$$

Letting $y \to 1^-$ in both sides of the latter inequality, one gets $1 - L \leq 0$ which contradicts to $L < 1$. Thus, condition (10) is not satisfied.

For $x, y \in X$, set

$$R = d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx).$$

If $x, y \in [0, 1)$ and $x \neq y$, then $d(Tx, Ty) = \frac{1}{2}|x - y| < |x - y| = d(x, y)$. That is, (5) is satisfied in this case.

If $x = 1$ and $y \in [0, 1)$, then

$$R = 1 - \frac{y}{2} + C(1 - y) > 1 - \frac{y}{2} = d(T1, Ty).$$

Thus, (5) is satisfied in this case.

If $x \in [0, 1)$ and $y = 1$, then

$$R = 1 - \frac{x}{2} + C\left(1 - \frac{x}{2}\right) > 1 - \frac{x}{2} = d(Tx, T1).$$

Thus, (5) is satisfied in this case.

We have checked that condition (5) is satisfied. Moreover, since

$$d(x, Tx) = \begin{cases} \frac{x}{2} & \text{if} \quad x \in [0,1), \\ 0 & \text{if} \quad x = 1 \end{cases}$$

the function $f(x) = d(x, Tx)$ is lower semi-continuous on X. Thus, all conditions of Theorem 4 are satisfied and Theorem 4 is applicable to this example.

Remark 2 From Example 3 and Example 4, one can see that the results in Theorem 4 and Theorem 5 are independent.

Existence of fixed points of a mapping $T : X \to X$ satisfying condition (5) is still guaranteed if we replace both the lower semi-continuity of $x \mapsto d(x, Tx)$ and the compactness of X by another condition on T and the completeness of X. We recall here two results ([22, Theorem 5] and [24, Theorem 5]) given by Suzuki in which compactness are replaced by other conditions; see [21] and references therein for more interesting results.

Theorem 6 [22] *Let* (X, d) *be a complete metric space and* T *be a mapping on* X*. Assume that for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

(c1) $\frac{1}{2}d(x, Tx) < d(x, y)$ and $d(x, y) < \varepsilon + \delta$ imply $d(Tx, Ty) \leq \varepsilon$ *and*
(c2) $\frac{1}{2}d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$

for all $x, y \in X$*. Then, there exists a unique fixed point* z *of* T*. Moreover,* $\{T^n x\}$ *converges to* z *for any* $x \in X$*.*

Theorem 7 [24] *Let* T *be a mapping on a complete metric space* (X, d)*. Assume that*

(d1) $\frac{1}{2}d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ *for all* $x, y \in X$ and
(d2) *for any* $x \in X$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $d(T^i x, T^j x) < \varepsilon + \delta$ *implies* $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ *for any* $i, j \in \mathbb{N}$*.*

Then T *has a unique fixed point* z *and* $\{T^n x\}$ *converges to* z *for any* $x \in X$*.*

Similarly, we next replace the compactness of X in Theorem 5 and provide a generalization of both Theorem 6 and Theorem 7. For our aim, recalling that if $T : X \to X$ and $x \in X$, then the orbit of T at x is the set $O(T, x) = \{x, Tx, T^2x, \cdots, T^n x, \cdots\}$.

Theorem 8 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Assume that*

(a) *there exists* $C \geq 0$ *such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx) \tag{12}$$

for $x, y \in X$*,*

and

(b) *for any* $x \in X$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\frac{1}{2}d(u, Tu) < d(u, v) \quad \text{and} \quad d(u, v) < \varepsilon + \delta \quad \text{imply} \quad d(Tu, Tv) \leq \varepsilon$$

for all $u, v \in O(T, x)$*.*

Then, T *has a fixed point in* X *and for any* $x \in X$*, the sequence of iterates* $\{T^n x\}$ *converges to a fixed point of* T*.*

Proof Let $x_0 \in X$ be arbitrary but fixed and define the sequence $\{x_n\}$ in X by $x_n = T^n x$ for each $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and the sequence $\{x_n\}$ converges to x_{n_0}. Assume now that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$\frac{1}{2}d(x_n, Tx_n) = d(x_n, x_{n+1}) < d(x_n, x_{n+1}).$$

By (a), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) =& d(Tx_n, Tx_{n+1}) \\ <& d(x_n, x_{n+1}) + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+2})| + Cd(x_{n+1}, Tx_n) \\ =& d(x_n, x_{n+1}) + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})| \end{aligned}$$

This implies that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, i.e., $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers. Thus there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \alpha$. We shall show that $\alpha = 0$. Assume to the contrary that $\alpha > 0$. Then $d(x_n, x_{n+1}) > \alpha$ for each n. Moreover, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$d(T^n x, T^{n+1} x) = d(x_n, x_{n+1}) < \alpha + \delta$$

for all $n \geq N$. By condition (b), the latter inequality implies $d(x_{n+1}, x_{n+2}) = d(T^{n+1}x, T^{n+2}x) \leq \alpha$ for $n \geq N$. This is a contradiction. Hence, $\alpha = 0$. We now prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary but fixed and let $\delta > 0$ be fixed. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists $p \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \delta$ for all $n \geq p$. We shall show by induction that

$$d(x_{p+1}, x_{p+m}) \leq \varepsilon \tag{13}$$

for all $m \in \mathbb{N}$. It is evident that (13) holds for $m = 0$ and $m = 1$. Assume that (13) holds for some $m \in \mathbb{N}$, $m \geq 1$. We have

$$d(x_p, x_{p+m}) \leq d(x_p, x_{p+1}) + d(x_{p+1}, x_{p+m}) < \varepsilon + \delta.$$

Then, by condition (b), one has $d(x_{p+1}, x_{p+m+1}) = d(Tx_p, Tx_{p+m}) \leq \varepsilon$. Thus, by induction, (13) holds for all $m \in \mathbb{N}$. This implies that the sequence $\{x_n\}$ is Cauchy. Since X is complete, the sequence $\{x_n\}$ converges to some $z \in X$. Our attempt is to show that z is a fixed point of T. Assume now that there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_k, x_{k+1}) \geq d(x_n, z) \quad \text{and} \quad \frac{1}{2}d(x_{n+1}, x_{n+2}) \geq d(x_{n+1}, z).$$

Then,

$$\begin{aligned} d(x_n, x_{n+1}) \leq& d(x_n, z) + d(x_{n+1}, z) \\ \leq& \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ <& d(x_n, x_{n+1}) \end{aligned}$$

which is a contradiction. Thus, it holds either

(i) $\frac{1}{2}d(x_n, x_{n+1}) < d(x_n, z)$ for all n in some infinite subset I of $\mathbb{N}$, or

(ii) $\frac{1}{2}d(x_{n+1}, x_{n+2}) < d(x_{n+1}, z)$ for all n in some infinite subset J of $\mathbb{N}$.

If (i) holds, then, by Condition (a), we have for all $n \in I$ that

$$\begin{aligned} d(x_{n+1}, Tz) =& d(Tx_n, Tz) < d(x_n, z) + |d(x_n, Tx_{n+1}) - d(z, Tz)| + Cd(z, Tx_n) \\ =& d(x_n, z) + d(x_n, x_{n+1}) + Cd(z, x_{n+1}). \end{aligned}$$

This yields

$$\lim_{n\in I, n\to\infty} x_{n+1} = Tz,$$

that is, $\{x_n\}$ has a subsequence converging to Tz. Similarly, if (ii) holds, then we also get that $\{x_n\}$ has a subsequence converging to Tz. Since $\{x_n\}$ converges to z and has a subsequence converging to Tz, we obtain $Tz = z$. Therefore, we have shown that T has a fixed point z and the sequence $\{T^n x\}$ converges to this fixed point. □

We next provide a simple example for which we can apply Theorem 8 but cannot apply Theorem 6 and Theorem 7.

Example 5 Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then, (X, d) is complete but it is not compact. We define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 2 & \text{if } \ x \geq 2, \\ 0 & \text{if } \ x < 2. \end{cases}$$

First, we can see that T has two fixed points $z = 0$ and $z = 2$. Thus, we cannot apply Theorem 6 and Theorem 7. to this example. We now check that all conditions of Theorem 8 are satisfied.

We first check Condition (b). If $x \in [0, 2)$, then $O(T, x) = \{x, 0, 0, \cdots, \}$. Hence, if $u, v \in O(T, x)$ and $u \neq v$, then $d(u, v) = x$ and $d(Tu, Tv) = 0$. Similarly, if $x \in [2, \infty)$, $u, v \in O(T, x)$ with $u \neq v$, then $d(u, v) = x$ and $d(Tu, Tv) = 0$. Thus, Condition (b) is satisfied.

We now check that Condition (a) is satisfied with $C = 2$. If $x \neq y$ and $x, y \in [2, \infty)$ or $x, y \in [0, 2)$, then $d(Tx, Ty) = 0 < |x - y| = d(x, y)$. Thus, (12) holds for this case.

Consider $x \in [2, \infty)$ and $y \in [0, 2)$. We have

$$\begin{aligned} RHS :=& d(x, y) + |d(x, Tx) - d(y, Ty)| + Cd(y, Tx) \\ =& x - y + |x - 2 - y| + 2(2 - y) \\ =& x + 2 - 2y + |x - y - 2| + 2 > 2 = d(Tx, Ty). \end{aligned}$$

Thus, (12) holds.

If $x \in [0, 2)$ and $y \in [2, \infty)$, then

$$RHS = y - x + |x - y + 2| + 2y > 2y > 2 = d(Tx, Ty).$$

Thus, (12) holds. Therefore, all conditions of Theorem 8 are satisfied. It is evident that if $x \in [2, \infty)$ then $\{T^n x\}$ converges to 2 and if $x \in [0, 2)$ then $\{T^n x\}$ converges to 0.

The following corollary is an answer to Question 1(iii).

Corollary 4 *In Theorem 8, if $C = 0$, then T has a unique fixed point in X and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to the fixed point of T.*

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References

1. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian-type Mappings and Applications. Springer, New York (2009)
2. Alsulami, H.H., Karapinar, E., Kutbi, M.A., Roldán, A.: An illusion: a Suzuki type coupled fixed point theorem. Abstr. Appl. Anal. 2014, Art. ID 235731, 8 pp
3. Alqahtani, O., Himabindu, V.M., Karapinar, E.: On Pata-Suzuki-type contractions. Mathematics **7**, 720 (2019)
4. I. Altun, G. Durmaz, M. Olgun, P- contractive mappings on metric spaces. J. Nonlinear Funct. Anal. 2018 (2018) Article ID 43
5. Altun, I., Hancer, H.A.: Almost Picard operators. AIP Conf. Proc. **2183**, 060003 (2019)
6. Chen, H.Y., Yeh, C.C.: A note on fixed point in compact metric spaces. Indian J. Pure Appl. Math. **11**, 297–298 (1980)
7. Dorić, D., Kadelburg, Z., Radenović, S.: Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces. Nonlinear Anal. **75**(4), 1927–1932 (2012)
8. Edelstein, M.: On fixed and periodic points under contractive mappings. J. London Math. Soc. **37**, 74–79 (1962)
9. Fisher, B.: Fixed point and constant mappings on metric spaces. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. MatNat. **61**(8), 329–332 (1976)
10. Fulga, A., Proca, A.M.: Fixed points for φ_E-Geraghty contractions. J. Nonlinear Sci. Appl. **10**, 5125–5131 (2017)
11. Fulga, A., Proca, A.M.: A new generalization of Wardowski fixed point theorem in complete metric spaces. Adv. Theory Nonlinear Anal. Appl. **1**, 57–63 (2017)
12. Fulga, A., Karapinar, E.: Revisiting of some outstanding metric fixed point theorems via E-contraction. An. St. Univ. Ovidius Constanta Ser. Mat. **26**, 73–97 (2017)
13. Górnicki, J.: Fixed point theorems for Kannan type mappings. J. Fixed Point Theory Appl. **19**, 2145–2152 (2017)
14. Garai, H., Dey, L.K., Senapati, T.: On Kannan-type contractive mappings. Numer. Funct. Anal. Optim. **39**, 1466–1476 (2018)
15. Karapinar, E.: Edelstein type fixed point theorems. Fixed Point Theory Appl. **107**, 12 (2012)
16. Karapinar, E.: Edelstein type fixed point theorems. Ann. Funct. Anal. **2**, 5158 (2011)
17. Karapinar, E., Salimi, P.: Suzuki-Edelstein type contractions via auxiliary functions, Math. Probl. Eng. 2013, Article ID 648528
18. Karapinar, E., Hima Bindu, V.M.L.: On Pata-Suzuki-type contractions. Mathematics **8**, 389 (2020)
19. Karapinar, E., Fulga, A., Aydi, H.: Study on Pata E-contractions. Adv. Differ. Eq. **2020**, 539 (2020)
20. Khan, M.S.: On fixed point theorems. Math. Japonica. **23**, 201–204 (1978)
21. Pasicki, L.: Four to one. Fixed Point Theory **21**, 715–726 (2020)
22. Suzuki, T.: A generalized Banach contraction principle that characterizes metric completeness. Proc. Amer. Math. Soc. **136**, 1861–1869 (2008)
23. Suzuki, T.: A new type of fixed point theorem in metric spaces. Nonlinear Anal. **71**, 5313–5317 (2009)
24. Suzuki, T.: The weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces. J. Fixed Point Theory Appl. **19**, 2361–2368 (2017)