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Strong convergence of the viscosity approximation method for the split generalized equilibrium problem

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Abstract
In this paper, we consider a common solution of three problems in real Hilbert spaces including the split generalized equilibrium problem, the variational inequality problem and the fixed point problem for nonexpansive multivalued mappings. For finding the solution, we present a modified viscosity approximation method and prove a strong convergence theorem under mild conditions. Moreover, we also provide a numerical example to illustrate the convergence behavior of the proposed iterative method.

Keywords Split generalized equilibrium problems · Variational inequality problems · Viscosity approximation method · Nonexpansive multivalued mappings · Hilbert spaces

Mathematics Subject Classification 46C05 · 47H09 · 47H10

1 Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point x in a Hilbert space by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

The *classical variational inequality problem* is the problem to find $u \in C$ such that

$$\langle Du, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.1}$$

where $D : C \to H_1$ is a bounded linear operator. The solution set of the variational inequality problem (1.1) is denoted by $VI(C, D)$. It is well known that the variational inequality

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problem (1.1) has a unique solution when the operator D is a strongly monotone and Lipschitz continuous mapping on C.

The *equilibrium problem* for a bifunction $F : C \times C \to \mathbb{R}$ is to find a point $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.2}$$

The solution set of the equilibrium problem (1.2) is denoted by $EP(F)$. It is easy to see that $EP(F) = VI(C, D)$ when $F(x, y) = \langle Dx, y - x \rangle$ for all $x, y \in C$. Let $\varphi : C \times C \to \mathbb{R}$ be a nonlinear bifunction, then the *generalized equilibrium problem* is to find $x^* \in C$ such that

$$F(x^*, x) + \varphi(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.3}$$

The solution set of the generalized equilibrium problem (1.3) is denoted by $GEP(F, \varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem (1.2).

In this paper, we are interested to find the solution of the *split generalized equilibrium problem* which is introduced by Kazmi and Rizvi [16] in 2013 as the following problem; find $x^* \in C$ such that

$$F_1(x^*, x) + \varphi_1(x^*, x) \geq 0, \quad \forall x \in C \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \varphi_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{1.5}$$

where $F_1, \varphi_1 : C \times C \to \mathbb{R}$ and $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator.

The solution set of the split generalized equilibrium problem (1.4)–(1.5) is denoted by

$$SGEP(F_1, \varphi_1, F_2, \varphi_2) := \{x^* \in C : x^* \in GEP(F_1, \varphi_1) \text{ and } Ax^* \in GEP(F_2, \varphi_2)\}.$$

If $\varphi_1 = 0$ and $\varphi_2 = 0$, the split generalized equilibrium problem reduces to the split equilibrium problem; see [27]. If $F_2 = 0$ and $\varphi_2 = 0$, the split generalized equilibrium problem reduces to the generalized equilibrium problem considered by Cianciaruso *et al.* [9].

The split generalized equilibrium problem generalizes multiple-sets split feasibility problem. It also includes as special case, the split variational inequality problem [3] which is the generalization of split zero problems and split feasibility problems, see for details [4–6, 8, 11, 12, 15, 19, 20, 25, 27, 30, 32, 33]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real world problems; for instance, in sensor networks in computerized tomography and data compression; see, e.g., [1, 2, 7, 10].

A single-valued mapping $S : C \to C$ is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad x, y \in C.$$

A point $x \in C$ is called a *fixed point* of a mapping S if $Sx = x$ and denote by $F(S)$ the set of all fixed points of S. A single-valued mapping $g : C \to C$ is called *contraction* if there exists a constant $k \in (0, 1)$ such that $\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in C$. There are some algorithms for approximation of fixed points of a nonexpansive single-valued mapping. In 2000, Moudafi [26] introduced the following iterative algorithm, which is known as the *viscosity approximation method*, for finding a fixed point of a nonexpansive single-valued mapping in Hilbert spaces under some suitable conditions:

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)Sx_n, \quad n \in \mathbb{N}, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in [0, 1], g is a contraction and S is a nonexpansive single-valued mapping on C. We note that the *Halpern approximation method* [14],

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, \quad n \in \mathbb{N}, \tag{1.7}$$

where u is a fixed element in C, is a special case of (1.6).

Viscosity approximation methods are very important because they are applied to linear programming, convex optimization and monotone inclusions. In Hilbert spaces, many authors have studied the fixed points problems of the fixed points for the nonexpansive single-valued mappings and monotone mappings by the viscosity approximation methods, and obtained a series of good results (see [12, 22, 23, 26, 29, 34, 38]).

Recently, Kazmi and Rizvi [17] introduced the iterative process combined with Halpern approximation method (1.7) for finding a common solution of the split equilibrium problem, the variational inequality problem and the fixed point problem for nonexpansive single-valued mapping in real Hilbert spaces.

Motivated by the works of Kazmi and Rizvi [16, 17] and Moudafi [26], we introduce and study a modified viscosity approximation method for approximating a common solution of three problems in real Hilbert spaces including the split generalized equilibrium problem, the variational inequality problem for a τ-inverse strongly monotone mapping and the fixed point problem for a nonexpansive multivalued mapping. We prove the strong convergence of the purposed iterative method under mild conditions. Our results extend and improve recent results announced by many others. Moreover, we give a numerical example to illustrate our main result.

2 Preliminaries

In this section, we recall some concepts and results which are needed in sequel. Let C be a nonempty closed convex subset of a real Hilbert space H. We denote by $CB(C)$ and $K(C)$ the collections of all nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The *Hausdorff metric* $\mathcal{H}$ on $CB(C)$ is defined by

$$\mathcal{H}(B_1, B_2) := \max\left\{ \sup_{x\in B_1} \operatorname{dist}(x, B_2), \sup_{y\in B_2} \operatorname{dist}(y, B_1) \right\}, \quad \forall B_1, B_2 \in CB(C),$$

where $\operatorname{dist}(x, B_2) = \inf\{d(x,y) : y \in B_2\}$ is the distance from a point x to a subset B_2. Let $S : C \to CB(C)$ be a multivalued mapping. An element $x \in C$ is called a *fixed point* of a multivalued mapping S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$. Recall that a multivalued mapping $S : C \to CB(C)$ is called *nonexpansive* if

$$\mathcal{H}(Sx, Sy) \le \|x - y\|, \ \forall x, y \in C.$$

If S is a nonexpansive single-valued mapping on a closed convex subset of a Hilbert space, then $F(S)$ is always closed and convex. The closedness of $F(S)$ can be easily extended to the multivalued case. But the convexity of $F(S)$ cannot be extended (see, *e.g.*, [18]). However, if S is a nonexpansive multivalued mapping and $Sp = \{p\}$ for each $p \in F(S)$, then $F(S)$ is always closed and convex.

For every point x in a real Hilbert space H, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the *metric projection* from H onto C. It means that $z = P_C x$ if and only if $\|x - z\| \le \|x - y\|$ for all $y \in C$. Moreover, it is equivalent to

$$\langle x - z, y - z \rangle \leq 0, \ \forall y \in C. \tag{2.1}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following properties:

(i) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$, $\forall x, y \in H$;
(ii) $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2$, $\forall x \in H, y \in C$;
(iii) $\|x - y\|^2 - \|P_C x - P_C y\|^2 \leq \|(x - y) - (P_C x - P_C y)\|^2$, $\forall x, y \in H$.

For more properties of P_C can be found in [13, 21].

We now give some concepts of the monotonicity of a nonlinear mapping.

Definition 2.1 Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $D : C \to H$ is said to be:

(i) *monotone* if $\langle Dx - Dy, x - y \rangle \geq 0$, $\forall x, y \in C$;
(ii) *τ-inverse strongly monotone* if there exists a constant $\tau > 0$ such that

$$\langle Dx - Dy, x - y \rangle \geq \tau \|Dx - Dy\|^2, \ \forall x, y \in C.$$

It is easy to observe that every τ-inverse strongly monotone mapping D is monotone.

Lemma 2.2 ([36]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and D be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$, $u = P_C(I - \lambda D)u$ if and only if $u \in VI(C, D)$.*

Lemma 2.3 ([28]) *Let $\{x_n\}$ be any sequence in a Hilbert space H. Then, we have $\{x_n\}$ satisfies Opial's condition, that is, if $x_n \rightharpoonup x$, then the inequality*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.4 ([39]) *Let H be a Hilbert space. Let $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$. Then, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.5 *In a real Hilbert space H, the following inequalities hold:*

(i) $\|x - y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$, $\forall x, y \in H$;
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, $\forall x, y \in H$.

Lemma 2.6 ([37]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \ \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 ([35]) *Let* $\{x_n\}$ *and* $\{w_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)w_n + \beta_n x_n$ *for all integer* $n \ge 1$ *and*

$$\limsup_{n\to\infty} \left(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|\right) \le 0.$$

Then $\lim_{n\to\infty} \|w_n - x_n\| = 0$.

For solving the generalized equilibrium problem, we assume that the bifunctions $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ satisfy the following assumption:

Assumption 2.8 Let C be a nonempty closed convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfy the following conditions:

(A1) $F_1(x,x) = 0$ for all $x \in C$;
(A2) F_1 is monotone, i.e., $F_1(x,y) + F_1(y,x) \le 0$, $\forall x, y \in C$;
(A3)
$F_1$is upper hemicontinuous, *i.e.*, for each $x, y, z \in C$, $\lim_{t\downarrow 0} F_1(tz + (1-t)x, y) \le F_1(x,y)$;
(A4) For each $x \in C$, $y \mapsto F_1(x,y)$ is convex and lower semicontinuous;
(A5) $\varphi_1(x,x) \ge 0$ for all $x \in C$;
(A6) For each $y \in C$, $x \mapsto \varphi_1(x,y)$ is upper semicontinuous;
(A7) For each $x \in C$, $y \mapsto \varphi_1(x,y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y,x) + \varphi_1(y,x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C \backslash K.$$

Lemma 2.9 ([24]) *Let C be a nonempty closed convex subset of a Hilbert space* H_1. *Let* $F_1 : C \times C \to \mathbb{R}$ *and* $\varphi_1 : C \times C \to \mathbb{R}$ *be two bifunctions satisfy Assumption* 2.8. *Assume* φ_1 *is monotone. For* $r > 0$ *and* $x \in H_1$. *Define a mapping* $T_r^{(F_1,\varphi_1)} : H_1 \to C$ *as follows*:

$$T_r^{(F_1,\varphi_1)}(x) = \left\{ z \in C : F_1(z,y) + \varphi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C \right\},$$

for all $x \in H_1$. *Then, the following conclusions hold*:

(1) For each $x \in H_1$, $T_r^{(F_1,\varphi_1)} \ne \emptyset$;
(2) $T_r^{(F_1,\varphi_1)}$ is single-valued;
(3) $T_r^{(F_1,\varphi_1)}$ is firmly nonexpansive, *i.e.*, for any $x, y \in H_1$,

$$\left\|T_r^{(F_1,\varphi_1)}x - T_r^{(F_1,\varphi_1)}y\right\|^2 \le \left\langle T_r^{(F_1,\varphi_1)}x - T_r^{(F_1,\varphi_1)}y, x - y\right\rangle;$$

(4) $F\left(T_r^{(F_1,\varphi_1)}\right) = GEP(F_1, \varphi_1)$;
(5) $GEP(F_1, \varphi_1)$ is compact and convex.

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ and $\varphi_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.8, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{(F_2,\varphi_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,\varphi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \varphi_2(w,d) + \frac{1}{r}\langle d - w, w - v\rangle \geq 0, \ \ \forall d \in Q \right\}.$$

Then we have the following:

(6) For each $v \in H_2$, $T_s^{(F_2,\varphi_2)} \neq \emptyset$;
(7) $T_s^{(F_2,\varphi_2)}$ is single-valued;
(8) $T_s^{(F_2,\varphi_2)}$ is firmly nonexpansive;
(9) $F\left(T_s^{(F_2,\varphi_2)}\right) = GEP(F_2, \varphi_2)$;
(10) $GEP(F_2, \varphi_2)$ is closed and convex, where $GEP(F_2, \varphi_2)$ is the solution set of the following generalized equilibrium problem:

$$\text{Find } y^* \in Q \text{ such that } F_2(y^*, y) + \varphi_2(y^*, y) \geq 0 \text{ for all } y \in Q.$$

Further, it is easy to prove that $SGEP(F_1, \varphi_1, F_2, \varphi_2)$ is closed and convex.

Lemma 2.10 ([9]) *Let C be a nonempty closed convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfy Assumption 2.8 and let $T_r^{(F_1,\varphi_1)}$ be defined as in Lemma 2.9 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then,*

$$\left\| T_{r_2}^{(F_1,\varphi_1)}y - T_{r_1}^{(F_1,\varphi_1)}x \right\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| T_{r_2}^{(F_1,\varphi_1)}y - y \right\|.$$

3 Main results

In this section, we prove the strong convergence theorems for finding a common element of the set of solutions of the split generalized equilibrium problem, the variational inequality problem for a τ-inverse strongly monotone mapping and the fixed point problem for a nonexpansive multivalued mapping in real Hilbert spaces.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, $D : C \to H_1$ be a τ-inverse strongly monotone mapping, and $S : C \to K(C)$ be a nonexpansive multivalued mapping. Let $F_1, \varphi_1 : C \times C \to \mathbb{R}$, $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.8. Let φ_1, φ_2 be monotone, φ_1 be upper hemicontinuous, and F_2 and φ_2 be upper semicontinuous in the first argument. Assume that $\Gamma = F(S) \cap SGEP(F_1, \varphi_1, F_2, \varphi_2) \cap \mathrm{VI(C,D)} \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let g be a contraction of C into itself with coefficient $k \in (0, 1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n, \\ y_n = P_C(u_n - \lambda_n D u_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $z_n \in Sy_n$ such that $\|z_{n+1} - z_n\| \leq \mathcal{H}(Sy_{n+1}, Sy_n) + \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$, and $r_n \in (0, 1)$, $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\tau$, and $\xi \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ *satisfy $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions are satisfied:*

- *(C1)* $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- *(C2)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- *(C3)* $\gamma_n \in [c, 1]$ for some $c \in (0, 1)$;
- *(C4)* $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
- *(C5)* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma g(z)$.

Proof We shall divide our proof into six steps.

Step 1. We will show that $\{x_n\}$ is bounded. Since D is τ-inverse strongly monotone mapping, we obtain $\langle x - y, Dx - Dy \rangle \geq \tau \|Dx - Dy\|^2$. Then for any $x, y \in C$, we have

$$\begin{aligned} \|(I - \lambda_n D)x - (I - \lambda_n D)y\|^2 &= \|(x - y) - \lambda_n (Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n \langle x - y, Dx - Dy \rangle + \lambda_n^2 \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - 2\tau\lambda_n \|Dx - Dy\|^2 + \lambda_n \|Dx - Dy\|^2 \\ &= \|x - y\|^2 - \lambda_n (2\tau - \lambda_n) \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.2}$$

This shows that the mapping $(I - \lambda_n D)$ is a nonexpansive mapping from C to H_1.

Let $p \in \Gamma$, that is, $p \in SGEP(F_1, \varphi_1, F_2, \varphi_2)$, we have $p = T_{r_n}^{(F_1,\varphi_1)} p$ and $Ap = T_{r_n}^{(F_2,\varphi_2)} Ap$. Thus, we get that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{(F_1,\varphi_1)}\Big(I - \xi A^*\Big(I - T_{r_n}^{(F_2,\varphi_2)}\Big)A\Big)x_n - T_{r_n}^{(F_1,\varphi_1)}p\|^2 \\
&\leq \|(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \xi^2\|A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|^2 + 2\xi\langle p - x_n, A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\rangle \\
&\leq \|x_n - p\|^2 + \xi^2\langle Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n, AA^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\rangle \\
&\quad + 2\xi\langle A(p - x_n), Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\xi^2\langle Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n, Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\rangle \\
&\quad + 2\xi\langle A(p - x_n) + (Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n) \\
&\quad - (Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n), Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\xi^2\|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2 \\
&\quad + 2\xi\Big(\langle Ap - T_{r_n}^{(F_2,\varphi_2)}Ax_n, Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\rangle - \|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2\Big) \\
&\leq \|x_n - p\|^2 + L\xi^2\|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2 \\
&\quad + 2\xi\Big(\frac{1}{2}\|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2 - \|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2\Big) \\
&= \|x_n - p\|^2 + \xi(L\xi - 1)\|Ax_n - T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2.
\end{aligned} \tag{3.3}$$

Since $\xi \in (0, \frac{1}{L})$, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.4}$$

Now, we estimate

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2 \\
&\leq \|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\|^2 \\
&\leq \|u_n - p\|^2 - \lambda_n(2\tau - \lambda_n)\|Du_n - Dp\|^2 \\
&\leq \|u_n - p\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.5}$$

Further, we estimate

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n g(x_n)+\beta_n x_n+\gamma_n z_n-p\| \\
&\le \alpha_n\|g(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|z_n-p\| \\
&\le \alpha_n(\|g(x_n)-g(p)\|+\|g(p)-p\|)+\beta_n\|x_n-p\|+\gamma_n \mathrm{dist}(z_n,Sp) \\
&\le \alpha_n(k\|x_n-p\|+\|g(p)-p\|)+\beta_n\|x_n-p\|+\gamma_n \mathrm{dist}(z_n,Sp) \\
&= (k\alpha_n+\beta_n)\|x_n-p\|+\alpha_n\|g(p)-p\|+\gamma_n \mathrm{dist}(z_n,Sp) \\
&\le (k\alpha_n+\beta_n)\|x_n-p\|+\alpha_n\|g(p)-p\|+\gamma_n \mathcal{H}(Sy_n,Sp) \\
&\le (k\alpha_n+\beta_n)\|x_n-p\|+\alpha_n\|g(p)-p\|+\gamma_n\|y_n-p\| \\
&\le (k\alpha_n+\beta_n)\|x_n-p\|+\alpha_n\|g(p)-p\|+\gamma_n\|x_n-p\| \\
&= (k\alpha_n+\beta_n+\gamma_n)\|x_n-p\|+\alpha_n\|g(p)-p\| \\
&= (1-(\alpha_n(1-k)))\|x_n-p\|+\alpha_n(1-k)\frac{\|g(p)-p\|}{1-k} \\
&\le \max\left\{\|x_n-p\|,\frac{\|g(p)-p\|}{1-k}\right\}.
\end{aligned}$$

For every $n \ge 1$, we can conclude that

$$\|x_n-p\| \le \max\left\{\|x_1-p\|,\frac{\|g(p)-p\|}{1-k}\right\}$$

for a fixed element $x_1 \in C$ by using the mathematical induction. Hence $\{x_n\}$ is bounded; so are $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From the nonexpansivity of the mapping $(I-\lambda_n D)$, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}Du_{n+1})-P_C(u_n-\lambda_n Du_n)\| \\
&\le \|(u_{n+1}-\lambda_{n+1}Du_{n+1})-(u_n-\lambda_n Du_n)\| \\
&= \|(u_{n+1}-u_n)-\lambda_{n+1}(Du_{n+1}-Du_n)+(\lambda_{n+1}-\lambda_n)Du_n\| \\
&\le \|(u_{n+1}-u_n)-\lambda_{n+1}(Du_{n+1}-Du_n)\|+|(\lambda_{n+1}-\lambda_n)|\|Du_n\| \\
&\le \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Du_n\|.
\end{aligned} \tag{3.6}$$

Since $T_{r_n}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)$ is nonexpansive, $u_n = T_{r_n}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)x_n$ and $u_{n+1}=T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_{n+1}$, it follows from Lemmma 2.10 that

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_{n+1}-T_{r_n}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)x_n\| \\
&\le \|T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_{n+1}-T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_n\| \\
&\quad+\|T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_n-T_{r_n}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)x_n\| \\
&\le \|x_{n+1}-x_n\|+\|(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_n-(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)x_n\| \\
&\quad+\left|1-\frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}^{(F_1,\varphi_1)}(I-\xi A^*(I-T_{r_n}^{(F_2,\varphi_2)})A)x_n-(I-\xi A^*(I-T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_n\| \\
&\le \|x_{n+1}-x_n\|+\xi\|A\|\|T_{r_{n+1}}^{(F_2,\varphi_2)}Ax_n-T_{r_n}^{(F_2,\varphi_2)}Ax_n\|+\eta_n \\
&\le \|x_{n+1}-x_n\|+\xi\|A\|\left|1-\frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}^{(F_2,\varphi_2)}Ax_n-Ax_n\|+\eta_n \\
&= \|x_{n+1}-x_n\|+\xi\|A\|\kappa_n+\eta_n,
\end{aligned} \tag{3.7}$$

where

$$\kappa_n := \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}}^{(F_2,\varphi_2)} Ax_n - Ax_n\|$$

and

$$\eta_n := \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}}^{(F_1,\varphi_1)}(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - (I - \xi A^*(I - T_{r_{n+1}}^{(F_2,\varphi_2)})A)x_n\|.$$

By using (3.6) and (3.7), we get

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \xi\|A\|\kappa_n + \eta_n + |\lambda_{n+1} - \lambda_n|\|Du_n\|. \tag{3.8}$$

Setting $x_{n+1} = \beta_n x_n + (1 - \beta_n)w_n$, which implies from (3.1) that

$$w_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n g(x_n) + \gamma_n z_n}{1 - \beta_n}.$$

Therefore, by using (3.8), we obtain that

$$
\begin{aligned}
\|w_{n+1}-w_n\| &= \left\| \frac{\alpha_{n+1}g(x_{n+1})+\gamma_{n+1}z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n g(x_n)+\gamma_n z_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}(g(x_{n+1})-g(x_n)) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) g(x_n) \right. \\
&\quad \left. + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(z_{n+1}-z_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) z_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|g(x_{n+1})-g(x_n)\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \|g(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|z_{n+1}-z_n\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|z_n\| \\
&\le \frac{k\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|z_{n+1}-z_n\| \\
&\le \frac{k\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\mathcal{H}(Sy_{n+1}, Sy_n) + \varepsilon_n) \\
&\le \frac{k\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\|y_{n+1}-y_n\| + \varepsilon_n) \\
&\le \frac{k\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\|x_{n+1}-x_n\| + \xi\|A\|\kappa_n + \eta_n + |\lambda_{n+1}-\lambda_n| \|Du_n\| + \varepsilon_n) \\
&= \left(1 - \frac{(1-k)\alpha_{n+1}}{1-\beta_{n+1}} \right) \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\xi\|A\|\kappa_n + \eta_n + |\lambda_{n+1}-\lambda_n| \|Du_n\| + \varepsilon_n) \\
&\le \|x_{n+1}-x_n\| + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\xi\|A\|\kappa_n + \eta_n + |\lambda_{n+1}-\lambda_n| \|Du_n\| + \varepsilon_n).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\| \le{}& \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| (\|g(x_n)\| + \|z_n\|) \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\xi\|A\|\kappa_n + \eta_n + |\lambda_{n+1}-\lambda_n| \|Du_n\| + \varepsilon_n).
\end{aligned}
$$

By the conditions ($C1$), ($C2$), ($C4$) and ($C5$), we have

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

This implies by Lemma 2.7 that $\lim_{n\to\infty}\|w_n-x_n\|=0$ and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.9}$$

Step 3. We will show that $\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|z_n-x_n\|=0$.

It follows from (3.3), (3.4), (3.5), and Lemma 2.4 that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|z_n-p\|^2\\ &= \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\operatorname{dist}(z_n,Sp)^2\\ &\leq \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\mathcal{H}(Sy_n,Sp)^2\\ &\leq \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\leq \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|u_n-p\|^2\\ &\leq \alpha_n\|g(x_n)-p\|^2+\beta_n\|x_n-p\|^2\\ &\qquad+\gamma_n(\|x_n-p\|^2+\xi(L\xi-1)\|Ax_n-T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2)\\ &\leq \alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\ &\qquad-\xi(1-L\xi)\gamma_n\|Ax_n-T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2.\end{aligned} \tag{3.10}$$

Then we have

$$\begin{aligned}\xi(1-L\xi)\gamma_n\|Ax_n-T_{r_n}^{(F_2,\varphi_2)}Ax_n\|^2) &\leq \alpha_n\|g(x_n)-p\|^2+(\|x_n-p\|^2-\|x_{n+1}-p\|^2)\\ &\leq \alpha_n\|g(x_n)-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|.\end{aligned}$$

By the conditions ($C1$), ($C3$), $\xi(1-L\xi)>0$, $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|Ax_n-T_{r_n}^{(F_2,\varphi_2)}Ax_n\|=0. \tag{3.11}$$

For $p\in\Gamma$, $p=T_{r_n}^{(F_1,\varphi_1)}p$, $T_{r_n}^{(F_1,\varphi_1)}$ is firmly nonexpansive, and $I-\gamma A^*(I-T_{r_n}^{(F_2,\varphi_2)})A$ is nonexpansive, we obtain that

$$
\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{(F_1,\varphi_1)}(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - T_{r_n}^{(F_1,\varphi_1)}p\|^2 \\
&\le \langle T_{r_n}^{(F_1,\varphi_1)}(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - T_{r_n}^{(F_1,\varphi_1)}p, (I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - p\rangle \\
&\le \langle u_n - p, (I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - p\rangle \\
&= \frac{1}{2}\Big(\|u_n - p\|^2 + \|(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - p\|^2 \\
&\quad - \|u_n - x_n - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n - p\|^2 + \|(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n - p\|^2 \\
&\quad - \|(u_n - p) - (x_n + \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n - p)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \xi^2\|A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|^2 \\
&\quad - 2\xi\langle u_n - x_n, A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\rangle)\Big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\xi\langle u_n - x_n, A^*(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\rangle \\
&\le \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\xi\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|.
\end{aligned} \tag{3.12}
$$

It follows from (3.10) and (3.12) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|g(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 \\
&\le \alpha_n\|g(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - \|u_n - x_n\|^2 \\
&\quad + 2\xi\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|) \\
&\le \alpha_n\|g(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n\|u_n - x_n\|^2 \\
&\quad + 2\xi\gamma_n\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\| \\
&\le \alpha_n\|g(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|u_n - x_n\|^2 \\
&\quad + 2\xi\gamma_n\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|.
\end{aligned}
$$

Therefore, we get that

$$
\begin{aligned}
\gamma_n\|u_n - x_n\|^2 &\le \alpha_n\|g(x_n) - p\|^2 + (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\
&\quad + 2\xi\gamma_n\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\| \\
&\le \alpha_n\|g(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\xi\gamma_n\|A(u_n - x_n)\|\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\|.
\end{aligned}
$$

By the conditions $(C1)$, $(C3)$, $\|x_n - x_{n+1}\| \to 0$, and $\|(I - T_{r_n}^{(F_2,\varphi_2)})Ax_n\| \to 0$ as $n \to \infty$, we have

$$
\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.13}
$$

Consider

$$\begin{aligned}\|x_n - z_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n - z_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n \|g(x_n) - z_n\| + \beta_n \|z_n - x_n\|\end{aligned}$$

and then,

$$\|z_n - x_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|g(x_n) - z_n\|.$$

By the conditions $(C1)$, $(C2)$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.14}$$

Step 4. We will show that $\lim_{n\to\infty} \|z_n - y_n\| = 0$.

For each $p \in \Gamma$, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|z_n - p\|^2 \\ &= \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \operatorname{dist}(z_n, Sp)^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \mathcal{H}(Sy_n, Sp)^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|) \\ &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|u_n - p\|^2 + \lambda_n(\lambda_n - 2\tau)\|Du_n - Dp\|^2) \\ &\le \alpha_n \|g(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\tau)\|Du_n - Dp\|^2) \\ &\le \alpha_n \|g(x_n) - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 + \gamma_n \lambda_n(\lambda_n - 2\tau)\|Du_n - Dp\|^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + \|x_n - p\|^2 + \gamma_n \lambda_n(\lambda_n - 2\tau)\|Du_n - Dp\|^2 \\ &= \alpha_n \|g(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n \lambda_n(2\tau - \lambda_n)\|Du_n - Dp\|^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n a(2\tau - b)\|Du_n - Dp\|^2,\end{aligned}$$

which yields

$$\begin{aligned}-\gamma_n a(2\tau - b)\|Du_n - Dp\|^2 &\le \alpha_n \|g(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n \|g(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.\end{aligned}$$

By the conditions $(C1)$, $(C3)$, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|Du_n - Dp\| = 0. \tag{3.15}$$

Furthermore, we observe that

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2 \\ &\le \langle y_n - p, (u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\rangle \\ &\le \frac{1}{2}(\|y_n - p\|^2 + \|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\|^2 - \|(y_n - u_n) + \lambda_n(Du_n - Dp)\|^2) \\ &\le \frac{1}{2}(\|y_n - p\|^2 + \|u_n - p\|^2 - \|(y_n - u_n) + \lambda_n(Du_n - Dp)\|^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}\|y_n - p\|^2 &\le \|u_n - p\|^2 - \|y_n - u_n\|^2 - \lambda_n^2\|Du_n - Dp\|^2 + 2\lambda_n\langle y_n - u_n, Du_n - Dp\rangle \\ &\le \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|y_n - u_n\|\|Du_n - Dp\| \\ &\le \|x_n - p\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|y_n - u_n\|\|Du_n - Dp\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|g(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\ &\le \alpha_n\|g(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - \|y_n - u_n\|^2 \\ &\quad + 2\lambda_n\|y_n - u_n\|\|Du_n - Dp\|) \\ &= \alpha_n\|g(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n\|y_n - u_n\|^2 \\ &\quad + 2\gamma_n\lambda_n\|y_n - u_n\|\|Du_n - Dp\| \\ &\le \alpha_n\|g(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|y_n - u_n\|^2 + 2\gamma_n\lambda_n\|y_n - u_n\|\|Du_n - Dp\|.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\gamma_n\|y_n - u_n\|^2 &\le \alpha_n\|g(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\lambda_n\|y_n - u_n\|\|Du_n - Dp\| \\ &\le \alpha_n\|g(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad + 2\gamma_n\lambda_n\|y_n - u_n\|\|Du_n - Dp\|.\end{aligned}$$

By the conditions $(C1)$, $(C3)$, $\|x_n - x_{n+1}\| \to 0$, $\|Du_n - Dp\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.16}$$

Observe that

$$\|z_n - y_n\| \le \|z_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\|,$$

from (3.13), (3.14) and (3.16), we get

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.17}$$

Step 5. We will show that $\limsup_{n\to\infty}\langle g(z) - z, x_n - z\rangle \le 0$ where $z = P_\Gamma g(z)$.

To show this, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle g(z) - z, z_n - z\rangle = \lim_{i\to\infty}\langle g(z) - z, z_{n_i} - z\rangle. \tag{3.18}$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $z_{n_i} \rightharpoonup w$. Since $\|z_n - y_n\| \to 0$, we obtain $y_{n_i} \rightharpoonup w$ as $i \to \infty$.

Next, we show that $w \in \Gamma$, that is, $w \in F(S) \cap SGEP(F_1, \varphi_1, F_2, \varphi_2) \cap VI(C, D)$.

Step 5.1. We will show that $w \in F(S)$. Since Sw is compact, we can choose $q'_n \in Sw$ such that $\|z_n - q'_n\| = \mathrm{dist}(z_n, Sw)$ and the sequence $\{q'_n\}$ has a convergent subsequence $\{q'_{n_i}\}$ with $\lim_{i\to\infty} q'_{n_i} = q' \in Sw$. By nonexpansiveness of S, we obtain that

$$\begin{aligned}\|y_{n_i} - q'\| &\le \|y_{n_i} - z_{n_i}\| + \|z_{n_i} - q'_{n_i}\| + \|q'_{n_i} - q'\| \\ &\le \|y_{n_i} - z_{n_i}\| + \operatorname{dist}(z_{n_i}, Sw) + \|q'_{n_i} - q'\| \\ &\le \|y_{n_i} - z_{n_i}\| + \mathcal{H}(Sy_{n_i}, Sw) + \|q'_{n_i} - q'\| \\ &\le \|y_{n_i} - z_{n_i}\| + \|y_{n_i} - w\| + \|q'_{n_i} - q'\|.\end{aligned}$$

This implies by (3.17) and $\lim_{i\to\infty} q'_{n_i} = q'$ that

$$\limsup_{i\to\infty} \|y_{n_i} - q'\| \le \limsup_{i\to\infty} \|y_{n_i} - w\|.$$

By Opial's condition, we get $w = q' \in Tw$. Hence, $w \in F(S)$.

Step 5.2. We will show that $w \in SGEP(F_1, \varphi_1, F_2, \varphi_2)$. First, we will show that $w \in GEP(F_1, \varphi_1)$. Since $u_n = T_{r_n}^{(F_1,\varphi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n$, we have

$$F_1(u_n, y) + \varphi_1(u_n, y) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n - \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)Ax_n\right\rangle \ge 0,$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \varphi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\left\langle y - u_n, \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)Ax_n\right\rangle \ge 0,$$

for all $y \in C$. It follows from the monotonicity of F_1 and φ_1 that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\left\langle y - u_n, \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)Ax_n\right\rangle \ge F_1(y, u_n) + \varphi_1(y, u_n),$$

for all $y \in C$. Since $\|u_n - x_n\| \to 0$, $\|z_n - x_n\| \to 0$, $\|z_n - y_n\| \to 0$, and $y_{n_i} \rightharpoonup w$, we have $u_{n_i} \rightharpoonup w$ and $u_{n_i} - x_{n_i} \to 0$ as $i \to \infty$. It follows by the condition ($C4$), (3.11), (3.13), Assumption 2.8 (A4) and (A7) that $0 \ge F_1(y, w) + \varphi_1(y, w)$ for all $y \in C$. Put $y_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get $y_t \in C$ and hence $F_1(y_t, w) + \varphi_1(y_t, w) \le 0$. So, by Assumption 2.8 (A1)-(A7), we have

$$\begin{aligned}0 &\le F_1(y_t, y_t) + \varphi_1(y_t, y_t) \\ &\le t(F_1(y_t, y) + \varphi_1(y_t, y)) + (1-t)(F_1(y_t, w) + \varphi_1(y_t, w)) \\ &\le t(F_1(y_t, y) + \varphi_1(y_t, y)) + (1-t)(F_1(w, y_t) + \varphi_1(w, y_t)) \\ &\le F_1(y_t, y) + \varphi_1(y_t, y).\end{aligned}$$

Hence, we have $F_1(y_t, y) + \varphi_1(y_t, y) \ge 0$ for all $y \in C$. Letting $t \to 0$, by Assumption 2.8 (A3) and upper hemicontinuity of φ_1, we have $F_1(q, y) + \varphi_1(q, y) \ge 0$ for all $y \in C$. This implies that $w \in GEP(F_1, \varphi_1)$.

Next, we show that $Aw \in GEP(F_2, \varphi_2)$. Since $\|z_n - x_n\| \to 0$, $\|z_n - y_n\| \to 0$, and $y_{n_i} \rightharpoonup w$, we have $x_{n_i} \rightharpoonup w$. Since A is a bounded linear operator, we get $Ax_{n_i} \to Aw$.

Now, setting $v_{n_i} = Ax_{n_i} - T_{r_{n_i}}^{(F_2,\varphi_2)}Ax_{n_i}$. It follows from (3.11) that

$$\lim_{i\to\infty} v_{n_i} = 0 \text{ and } Ax_{n_i} - v_{n_i} = T_{r_{n_i}}^{(F_2,\varphi_2)}Ax_{n_i}.$$

Therefore, from Lemma 2.9, we have

$$0 \le F_2(Ax_{n_i} - v_{n_i}, z) + \varphi_2(Ax_{n_i} - v_{n_i}, z) + \frac{1}{r_{n_i}}\langle z - (Ax_{n_i} - v_{n_i}), (Ax_{n_i} - v_{n_i}) - Ax_{n_i}\rangle,$$

for all $z \in Q$. Since F_2 and φ_2 are upper semicontinuous in the first argument, it follows that

$$F_2(Aw, z) + \varphi_2(Aw, z) \ge 0 \text{ for all } z \in Q.$$

This means that $Aw \in GEP(F_2, \varphi_2)$ and hence $w \in SGEP(F_1, \varphi_1, F_2, \varphi_2)$.

Step 5.3. We will show that $w \in VI(C, D)$. Let $U : H_1 \to 2^{H_1}$ be a multivalued mapping defined by

$$Uw = \begin{cases} Dw + N_C w, & w \in C \\ \emptyset, & w \notin C \end{cases}$$

where $N_C w$ is the normal cone to C at $w \in C$. Then U is maximal monotone, and $0 \in Uw$ if and only if $w \in VI(C, D)$. Let $G(U)$ be the graph of U and let $(w, y) \in G(U)$. Then we have $y \in Uw = Dw + N_C w$ and hence $y - Dw \in N_C w$. Since $y_n \in C$ for all $n \in \mathbb{N}$, we have

$$\langle w - y_n, y - Dw\rangle \ge 0. \tag{3.19}$$

On the other hand, from $y_n = P_C(u_n - \lambda_n Du_n)$, we have

$$\langle w - y_n, y_n - (u_n - \lambda_n Du_n)\rangle \ge 0,$$

that is,

$$\left\langle w - y_n, \frac{y_n - u_n}{\lambda_n} + Du_n \right\rangle \ge 0.$$

Therefore, we have

$$\begin{aligned}
\langle w - y_{n_i}, y\rangle &\ge \langle w - y_{n_i}, Dw\rangle \\
&\ge \langle w - y_{n_i}, Dw\rangle - \left\langle w - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Du_{n_i} \right\rangle \\
&= \left\langle w - y_{n_i}, Dw - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} - Du_{n_i} \right\rangle \\
&= \langle w - y_{n_i}, Dw - Dy_{n_i}\rangle + \langle w - y_{n_i}, Dy_{n_i} - Du_{n_i}\rangle \\
&\quad - \left\langle w - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\ge \langle w - y_{n_i}, Dy_{n_i}\rangle - \left\langle w - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Du_{n_i} \right\rangle \\
&\ge \|w - y_{n_i}\| \|Dy_{n_i} - Du_{n_i}\| - \|w - y_{n_i}\| \left\| \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\|.
\end{aligned} \tag{3.20}$$

Noting that $\|y_{n_i} - u_{n_i}\| \to 0$ as $i \to \infty$ and D is τ-inverse strongly monotone, hence from the inequality (3.20), we have $\langle w - z, y \rangle \geq 0$ as $i \to \infty$. Since U is maximal monotone, we have $w \in U^{-1}0$, and hence $w \in VI(C, D)$. By Steps 5.1, 5.2 and 5.3, we can conclude that $w \in F(S) \cap SGEP(F_1, \varphi_1, F_2, \varphi_2) \cap VI(C, D)$, that is, $w \in \Gamma$.

Since $z = P_\Gamma g(z)$ and $z_{n_i} \rightharpoonup w$ as $i \to \infty$, it implies by (2.1) that

$$\begin{aligned}\limsup_{n\to\infty}\langle g(z) - z, x_n - z\rangle &= \limsup_{n\to\infty}\langle g(z) - z, z_n - z\rangle \\ &= \limsup_{i\to\infty}\langle g(z) - z, z_{n_i} - z\rangle \\ &= \langle g(z) - z, w - z\rangle \\ &\leq 0.\end{aligned} \tag{3.21}$$

Step 6. Finally, we will show that $\{x_n\}$ converges strongly to z.

Consider

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle x_{n+1} - z, x_{n+1} - z\rangle \\ &= \langle \alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n - (\alpha_n + \beta_n + \gamma_n)z, x_{n+1} - z\rangle \\ &= \alpha_n\langle g(x_n) - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle + \gamma_n\langle z_n - z, x_{n+1} - z\rangle \\ &\leq \alpha_n\|g(x_n) - g(z)\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\|z_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\leq k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\mathrm{dist}(z_n, Sz)^2 + \|x_{n+1} - z\|^2) \\ &\leq k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\mathcal{H}(Sy_n, Sz)^2 + \|x_{n+1} - z\|^2) \\ &\leq k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{\beta_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{\gamma_n}{2}(\|y_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\leq k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{\beta_n + \gamma_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &= k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{(1 - \alpha_n)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\leq k\alpha_n\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{(1 - \alpha_n)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n)\|x_n-z\|^2 + 2\alpha_n\langle g(z)-z, x_{n+1}-z\rangle \\ &\quad + 2k\alpha_n\|x_n-z\|\,\|x_{n+1}-z\| \\ &\leq (1-\alpha_n)\|x_n-z\|^2 + \delta_n,\end{aligned}$$

where

$$\delta_n = 2\alpha_n\langle g(z)-z, x_{n+1}-z\rangle + 2k\alpha_n\|x_n-z\|\,\|x_{n+1}-z\|.$$

By the inequality (3.21) and condition ($C1$), we get $\delta_n \to 0$ as $n \to \infty$. By using Lemma 2.6, it implies that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 3.2 Theorem 3.1 extends the corresponding one of Kazmi and Rizvi [16, 17] and Moudafi [26] to a nonexpansive multivalued mapping and to a split generalized equilibrium problem. In fact, we present a new viscosity approximation method for finding a common solution of three problems including the split generalized equilibrium problem, the variational inequality problem for a τ-inverse strongly monotone mapping and the fixed point problem for a nonexpansive multivalued mapping.

If $\varphi_1 = \varphi_2 = 0$, then the split generalized equilibrium problem reduces to split equilibrium problem. So, the following result can be obtained from Theorem 3.1 immediately.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, $D : C \to H_1$ be τ-inverse strongly monotone mapping, and $S : C \to K(C)$ be a nonexpansive multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.8. Let F_2 be upper semicontinuous in the first argument. Assume that $\Gamma = F(S) \cap SEP(F_1, F_2) \cap \mathrm{VI(C,D)} \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let g be a contraction of C into itself with coefficient $k \in (0,1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \xi A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = P_C(u_n - \lambda_n D u_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n, \quad n \in \mathbb{N}, \end{cases}$$

where $z_n \in Sy_n$ such that $\|z_{n+1} - z_n\| \leq \mathcal{H}(Sy_{n+1}, Sy_n) + \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$, and $r_n \in (0,1)$, $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\tau$, and $\xi \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions are satisfied:

- *(C1)* $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- *(C2)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- *(C3)* $\gamma_n \in [c, 1]$ for some $c \in (0,1)$;
- *(C4)* $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
- *(C5)* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma g(z)$.

Recall that a multivalued mapping $S : C \subseteq H_1 \to CB(C)$ is said to satisfy *Condition* (*) if $\|x - p\| = \text{dist}(x, Sp)$ for all $x \in H_1$ and $p \in F(S)$; see [31]. We see that S satisfies Condition (*) if and only if $Sp = \{p\}$ for all $p \in F(S)$. Then the following results can be obtained from Theorem 3.1 and Corollary 3.3 immediately.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, $D : C \to H_1$ be a τ-inverse strongly monotone mapping, and $S : C \to K(C)$ be a nonexpansive multivalued mapping. Let $F_1, \varphi_1 : C \times C \to \mathbb{R}$, $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.8. Let φ_1, φ_2 be monotone, φ_1 be upper hemicontinuous, and F_2 and φ_2 be upper semicontinuous in the first argument. Assume that $\Gamma = F(S) \cap SGEP(F_1, \varphi_1, F_2, \varphi_2) \cap \mathrm{VI(C,D)} \neq \emptyset$ and S satisfies Condition (*). Let g be a contraction of C into itself with coefficient $k \in (0, 1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(I - \xi A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n, \\ y_n = P_C(u_n - \lambda_n Du_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n, \quad n \in \mathbb{N}, \end{cases}$$

where $z_n \in Sy_n$ such that $\|z_{n+1} - z_n\| \leq \mathcal{H}(Sy_{n+1}, Sy_n) + \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$, and $r_n \in (0, 1)$, $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\tau$, and $\xi \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions are satisfied:

- *(C1)* $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- *(C2)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- *(C3)* $\gamma_n \in [c, 1]$ for some $c \in (0, 1)$;
- *(C4)* $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
- *(C5)* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma g(z)$.

Corollary 3.5 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, $D : C \to H_1$ be a τ-inverse strongly monotone mapping, and $S : C \to K(C)$ be a nonexpansive multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.8. Let F_2 be upper semicontinuous in the first argument. Assume that $\Gamma = F(S) \cap SEP(F_1, F_2) \cap \mathrm{VI(C,D)} \neq \emptyset$ and S satisfies Condition (*). Let g be a contraction of C into itself with coefficient $k \in (0, 1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \xi A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = P_C(u_n - \lambda_n Du_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n z_n, \quad n \in \mathbb{N}, \end{cases}$$

where $z_n \in Sy_n$ *such that* $\|z_{n+1} - z_n\| \leq \mathcal{H}(Sy_{n+1}, Sy_n) + \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$, *and* $r_n \in (0, 1)$, $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < 2\tau$, *and* $\xi \in (0, \frac{1}{L})$ *with* L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $(0, 1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C3) $\gamma_n \in [c, 1]$ for some $c \in (0, 1)$;
(C4) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C5) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma g(z)$.

We now present a numerical example to demonstrate the performance and convergence of our theoretical results. All codes were written in Scilab.

Example 3.6 Let $H_1 = H_2 = \mathbb{R}$, $C = Q = [0, 15]$. Let $A : H_1 \to H_2$ be defined by $Ax = x$ for each $x \in H_1$. Then $A^*y = y$ for each $y \in H_2$. Let $D : C \to H_2$ defined by $Dx = \frac{x}{5}$ for each $x \in C$. For each $x \in C$, we define a multivalued mapping S on C as follows:

$$Sx = \left[0, \frac{7x}{10}\right].$$

For each $x, y \in C$, define bifunctions $F_1, \varphi_1 : C \times C \to \mathbb{R}$ by

$$F_1(x, y) = 3y^2 + 6xy - 9x^2 \text{ and } \varphi_1(x, y) = y^2 - x^2.$$

For each $w, v \in Q$, define $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ by

$$F_2(w, v) = 4v^2 + 2wv - 6w^2 \text{ and } \varphi_2(w, v) = w - v.$$

Choose $r_n = \frac{n}{n+1}$, $\gamma = \frac{1}{4}$. It is easy to check that S, A, D, F_1, F_2, φ_1, φ_2, and $\{r_n\}$ satisfy all conditions in Theorem 3.1 with $\Gamma = \{0\}$.

For each $x \in C$ and each $n \in \mathbb{N}$, we compute $T_r^{(F_2,\varphi_2)}Ax$. Find w such that

$$\begin{aligned}
0 &\leq F_2(w, v) + \varphi_2(w, v) + \frac{1}{r}\langle v - w, w - Ax\rangle \\
&= 4v^2 + 2wv - 6w^2 + w - v + \frac{1}{r}(v - w)(w - x) \\
&\Leftrightarrow \\
0 &\leq 4rv^2 + 2rwv - 6rw^2 + rw - rv + (v - w)(w - x) \\
&= 4rv^2 + 2rwv - 6rw^2 + rw - rv + wv - vx - w^2 + wx \\
&= 4rv^2 + (2rw - r + w - x)v + (-6rw^2 + rw - w^2 + wx)
\end{aligned}$$

for all $v \in Q$. Let $J_2(v) = 4rv^2 + (2rw - r + w - x)v + (-6rw^2 + rw - w^2 + wx)$. $J_2(v)$ is s a quadratic function of v with coefficient $a = 4r$, $b = 2rw - r - x - w$, and $c = -6rw^2 + rw - w^2 + wx$. Determine the discriminant Δ of J_2 as follows:

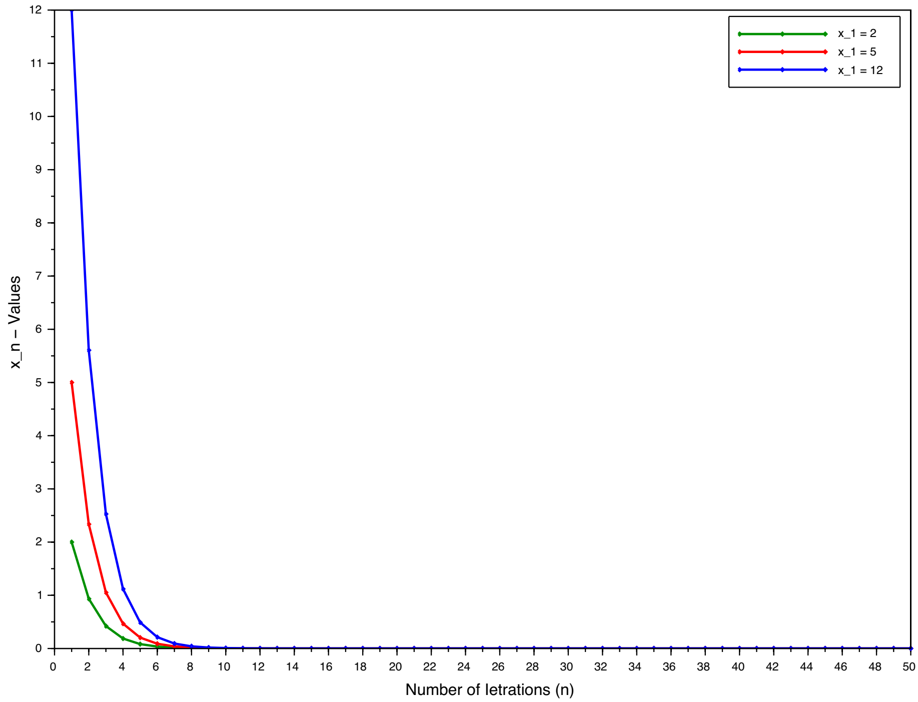

Fig. 1 Behaviours of x_n with three random initial points x_1

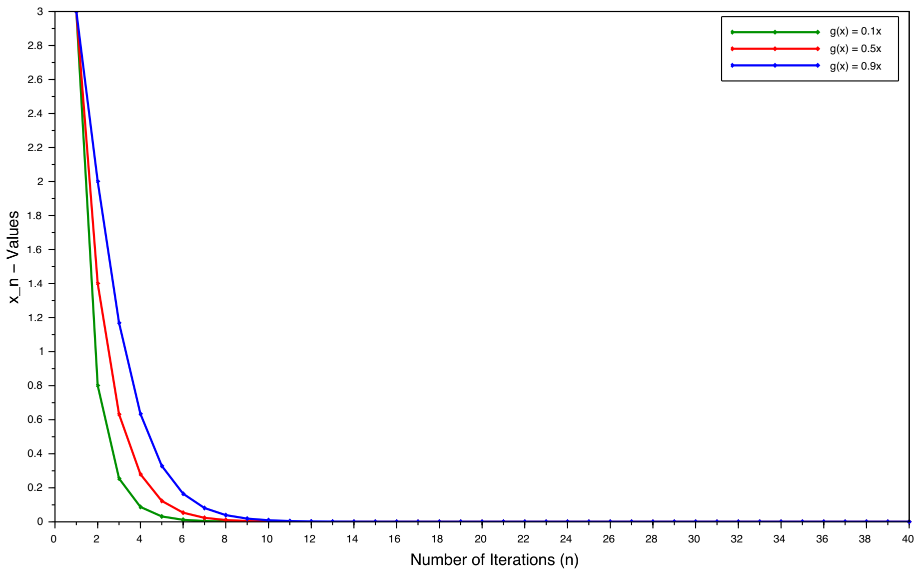

Fig. 2 Behaviours of x_n with three different contraction mappings g

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (2rw - r + w - x)^2 - 4(4r)(-6rw^2 + rw - w^2 + wx)\\ &= 100r^2w^2 - 20r^2w + 20rw^2 - 20rwx + r^2 - 2rw + 2rx + w^2 - 2wx + x^2\\ &= (100r^2 + 20r + 1)w^2 + (-20r^2 - 20rx - 2r - 2x)w + (2rx + x^2 + r^2)\\ &= (10r + 1)^2w^2 - 2w(10r + 1)(x + r) + (x + r)^2\\ &= ((10r + 1)w - (x + r))^2.\end{aligned}$$

We know that $J_2(v) \geq 0$ for all $v \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we have $w = \frac{x+r}{10r+1}$. This implies that

$$T_r^{(F_2,\varphi_2)}Ax = \frac{x + r}{10r + 1}.$$

Furthermore, we can get

$$\begin{aligned}\left(I - \gamma A^*\left(I - T_r^{(F_2,\varphi_2)}\right)A\right)x &= x - \gamma A^*(Ax - T_r^{(F_2,\varphi_2)}Ax)\\ &= x - \frac{1}{4}A^*\left(x - \frac{x + r}{10r + 1}\right)\\ &= x - \frac{1}{4}\left(\frac{10rx - r}{10r + 1}\right)\\ &= \frac{30xr + 4x + r}{40r + 4}.\end{aligned}$$

Next, we find $u \in C$ such that $F_1(u, z) + \varphi_1(u, z) + \frac{1}{r}\langle z - u, u - s\rangle \geq 0$ for all $z \in C$, where $s = \left(I - \gamma A^*\left(I - T_r^{(F_2,\varphi_2)}\right)A\right)x$. Note that

$$\begin{aligned}0 &\leq F_1(u, z) + \varphi_1(u, z) + \frac{1}{r}\langle z - u, u - s\rangle\\ &= 4z^2 + 6uz - 10u^2 + \frac{1}{r}\langle v - u, u - s\rangle\\ &\Leftrightarrow\\ 0 &\leq 4rz^2 + 6ruz - 10ru^2 + (z - u)(u - s)\\ &= 4rz^2 + 6ruz - 10ru^2 + uz - sz - u^2 + us\\ &= 4rz^2 + (6ru + u - s)z + (-10ru^2 - u^2 + us)\end{aligned}$$

for all $z \in C$. Let $J_1(z) = 4rz^2 + (6ru + u - s)z + (-10ru^2 - u^2 + us)$. $J_1(z)$ is s a quadratic function of z with coefficient $a = 4r$, $b = 6ru + u - s$, and $c = -10ru^2 - u^2 + us$. Determine the discriminant Δ of J_1 as follows:

$$\begin{aligned}\Delta &= (6ru + u - s)^2 - 4(4r)(-10ru^2 - u^2 + us)\\ &= 196r^2u^2 + 28ru^2 - 28rus + u^2 - 2us + s^2\\ &= ((14r + 1)u - s)^2.\end{aligned}$$

We know that $J_1(z) \geq 0$ for all $z \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we have $u = \frac{s}{14r+1}$. This implies that

$$u_n = T_{r_n}^{(F_1,\varphi_1)}\Big(I - \gamma A^*\Big(I - T_{r_n}^{(F_2,\varphi_2)}\Big)A\Big)x_n,$$
$$= \frac{30x_n r_n + 4x_n + r_n}{(40r_n + 4)(14r_n + 1)}$$
$$= \frac{30x_n r_n + 4x_n + r_n}{560r_n^2 + 96r_n + 4}.$$

We put $z_n = \frac{7y_n}{10}$ for all $n \in \mathbb{N}$. Then the algorithm (3.1) becomes:

$$\begin{cases} r_n = \dfrac{n}{n+1}, \\ u_n = \dfrac{30x_n r_n + 4x_n + r_n}{560r_n^2 + 96r_n + 4}, \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \dfrac{7\gamma_n P_C\Big(u_n - \frac{\lambda_n u_n}{5}\Big)}{10}, \quad n \in \mathbb{N}. \end{cases} \tag{3.22}$$

In this example, we set the parameter on algorithm (3.22) by $\lambda_n = \frac{1}{20}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{4n}{10n+10}$ and $\gamma_n = \frac{5n}{10n+10}$ for all $n \in \mathbb{N}$.

Figure 1 indicates the behavior of x_n for algorithm (3.22) with $g(x) = 0.5x$ that converges to the same solution, that is, $0 \in \Gamma$ as a solution of this example.

Moreover, we test the effect of the different contraction mappings g on the convergence of the algorithm (3.22). In this test, Figure 2 presents the behaviour of x_n by choosing three different contraction mappings $g(x) = 0.1x$, $g(x) = 0.5x$ and $g(x) = 0.9x$. We see that the sequence $\{x_n\}$ by choosing the contraction $g(x) = 0.1x$ converges to the solution $0 \in \Gamma$ faster than the others.

4 Conclusion

The results presented in this paper modify, extend, and improve the corresponding results of Kazmi and Rizvi [16, 17] and Moudafi [26], and others. The main aim of this paper is to propose an iterative algorithm based on the modified viscosity approximation method to find an element for solving a class of split generalized equilibrium problems, the variational inequality problems, and fixed point problems for nonexpansive multivalued mappings in real Hilbert spaces.

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