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Nonnegative weak solution for a periodic parabolic equation with bounded Radon measure

Abderrahim Charkaoui[1] · **Nour Eddine Alaa**[1]

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Abstract
The purpose of this work is to study a class of periodic parabolic equations having a critical growth nonlinearity with respect to the gradient and bounded Radon measure. By the main of the sub- and super-solution method, we employ some new technics to prove the existence of a nonnegative weak periodic solution to the studied problems.

Keywords Periodic solution · Weak solution · Radon measure · Sub-solution · Super-solution

Mathematics Subject Classification 35K59 · 35K55 · 35K57 · 34C25

1 Introduction

Partial differential equations appears naturally in the mathematical modeling of a wide variety of phenomena, not only in the natural sciences but also in engineering and economics, such as gas dynamics, fusion processes, some biological models, cellular processes, and chemical reactions and others. Mathematical analysis of PDEs has gained considerable attention in the early literature. For instance, the book by Rădulescu et al. [23] presents the most suitable materials for the functional analysis of partial differential equations (PDEs). This book also makes it possible to introduce the readers to the most important topological methods in the case of PDEs described by non-homogeneous differential operators. On the other way, several authors have interested in the famous question about the existence and uniqueness of the solution to the different cases of PDEs such as elliptic or parabolic under either Dirichlet or Neumann boundary conditions, we refer the readers to see the references [1–3, 12, 15, 17, 22, 24, 25]. Meanwhile, the periodic partial differential equations has undergone a great huge amount of interest in a wide range of researchers. The importance of periodic PDEs arises in the

✉ Abderrahim Charkaoui
abderrahim.charkaoui@edu.uca.ma

Nour Eddine Alaa
n.alaa@uca.ac.ma

1 Laboratory LAMAI, Faculty of Science and Technology of Marrakesh, Cadi Ayyad University, Marrakesh, Morocco

modeling of many real-world phenomena, see for examples: distributed biological models and computational physics [4, 8, 18, 19].

In this work, we propose to study a nonlinear periodic equation involving critical growth nonlinearity with respect to the gradient modeled as

$$\begin{cases} \partial_t u - \Delta u + g(t,x,u,\nabla u) = \mu & \text{in } \; Q_T \\ u(0,.) \;=\; u(T,.) & \text{in } \; \Omega \\ u(t,x) = 0 & \text{on } \; \Sigma_T \end{cases} \tag{1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $T>0$ is the period, $Q_T=]0,T[\times\Omega$, $\Sigma_T=]0,T[\times\partial\Omega$, μ is a bounded nonnegative Radon measure in Q_T and $g: Q_T\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty[$ is a Carathéodory function satisfying some growth assumptions. To the best of our knowledge, several authors studied the well-posedness of a periodic solution to PDEs (see e.g [5, 9–11, 13, 14, 16, 21]). Hence, to enhance our result and relates it to recent studies in the early literature, we propose to recall some periodic work that presents particular cases of our problem. Amann [5] studied the existence of a classical periodic solution to (1) when μ is more regular and g has a subquadratic growth on the gradient. Their proof was based on the sub and super-solutions method and involved Schauder's fixed point theorem. When the nonlinearity term g is equal to zero, a major and comprehensive existence results still is the book by Lions [20]. The author assumed that μ belongs to $L^2(Q_T)$ and proved the existence and uniqueness of a weak periodic solution to the considered problem via maximal monotone operator theory. He also established the regularity properties and proved some a priori estimates on the obtained solutions. In [14] Deuel & Hess extended the results of [5, 20] to the case when μ belongs to $L^2(Q_T)$ and g has a sublinear growth with respect to the gradient. The authors assumed the existence of an ordered couple of functions $(\underline{u},\overline{u})$ which are a sub- and super solution to the considered model. They proved the existence of a weak periodic solution belonging to space $L^2(0,T;H_0^1(\Omega))\cap\mathcal{C}\big([0,T];L^2(\Omega)\big)$. Alaa et al. [11] have been interested by Eq. (1) when the source μ is a nonnegative function belonging to $L^1(Q_T)$. Indeed, they used the sub- and super-solution method to establish the existence of a weak periodic solution in the distributional sense. Later, in [9] Charkaoui et al. generalized the work [11] by considering a quasilinear periodic system with quadratic growth with respect to the gradient and L^1 data. The authors presented a simple method to construct a weak periodic super-solution to the considered system where they based on Schauder's fixed point theorem. Thereafter, by employing the obtained super-solution, the authors established the existence of a weak periodic solution in the space $L^1(0,T;W_0^{1,1}(\Omega))\cap\mathcal{C}([0,T];L^1(\Omega))$. Contrary to early mentioned works, here we investigate the existence of a nonnegative weak periodic solution to (1) in the case when μ is a nonnegative bounded Radon measure and the nonlinearity g has a critical growth with respect to the gradient. The mathematical tools used to prove the existence result of our work involves the sub- and super-solution technic combined with the truncation method. Note that the difficulties comeback in the presence of the non regular data μ, but we overcome these difficulties by employing some new techniques. As can be viewed the result of our paper generalize the works [5, 9–11, 14].

We have organized this work as follows. In Sect. 2, we state the assumptions involving our problem, we define the adapted notion of weak periodic solution to (1) and we state the main result of our work. Section 3 is devoted to proving the main result. We start initially by introducing an approximate problem to (1). Next, we establish necessaries a priori estimate of the approximate solution. Thereafter, we justify the passage to the limit in all the terms of the approximate problem.

We close this paragraph by giving the used notation throughout this work.

- $\mathcal{C}_c^\infty(Q_T) = \{\varphi : Q_T \to \mathbb{R}$, indefinitely derivable with compact support in $Q_T\}$
- $\mathcal{C}_b(Q_T) = \{\varphi : Q_T \to \mathbb{R}$, continuous and bounded in $Q_T\}$
- $\mathcal{M}_b(Q_T) = \{\mu$ bounded Radon measure in $Q_T\}$
- $\mathcal{M}_b^+(Q_T) = \{\mu$ bounded nonnegative Radon measure in $Q_T\}$.
- $\mathcal{V}_T = L^r(0,T;W_0^{1,r}(\Omega))$, with $1 \le r < \frac{N+2}{N+1}$.
- For every $k > 0$, we define $T_k(s) = \min\{k, \max\{s,-k\}\}$.
- We note $S_k(v) = \int_0^v T_k(s)ds$.

2 Main result

This section is devoted to enunciate the main result of our work. First of all, let us now introduce the hypothesizes which we suppose throughout this paper. We assume that

($\mathbf{A}_1$) $\mu \in \mathcal{M}_b^+(Q_T)$

($\mathbf{A}_2$) $g : Q_T \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty[$ Carathéodory function, which means that

$$(s,\xi) \mapsto g(t,x,s,\xi) \text{ is continuous for a.e } (t,x)$$
$$(t,x) \mapsto g(t,x,s,\xi) \text{ is measurable for all } (s,\xi)$$

($\mathbf{A}_3$) $g(t,x,s,0) = \min\{g(t,x,s,r), r \in \mathbb{R}^N\} = 0.$

($\mathbf{A}_4$) $|g(t,x,s,\xi)| \le H(t,x) + d(|s|^r + |\xi|^r)$, for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ and a.e $(t,x) \in Q_T$

where H is a nonnegative function belonging to $L^1(Q_T)$, $d \ge 0$ and $r \in [1, \frac{N+2}{N+1}[$.

Before giving the main result of our work, we need to clarify in which sense we want to solve problem (1). For this reason, we introduce the notion of weak periodic solution to (1).

Definition 1 A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak periodic solution to the problem (1) if it satisfies

$$\begin{aligned} &u \in \mathcal{C}([0,T], L^1(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega)), \quad g(t,x,u,\nabla u) \in L^1(Q_T) \\ &\partial_t u - \Delta u + g(t,x,u,\nabla u) = \mu \text{ in } \mathcal{D}'(Q_T) \end{aligned} \tag{2}$$

$$u(0,.) = u(T,.) \text{ in } L^1(\Omega) \tag{3}$$

Definition 2 A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak periodic sub-solution (resp. super-solution) to the problem (1) if it satisfies the conditions of the Definition 1 with }} $=''$ replaced by }} $\le''$ (resp. }} $\ge''$) in (2) and (3).

Remark 1 We say that $u(0,.) = u(T,.)$ in $L^1(\Omega)$, if for all $\varphi \in L^\infty(\Omega)$ we have

$$\lim_{s \mapsto 0} \int_\Omega (u(T-s,x) - u(s,x))\varphi(x)dx = 0.$$

In the following theorem, we state the main result of our work.

Theorem 1 *Under the hypotheses* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, *we assume that the problem* (1) *has a nonnegative weak periodic super-solution* w. *Then* (1) *has a weak periodic solution* u *in the sense of the Definition* 1 *such that:* $0 \leq u \leq w$.

3 Proof of the main result

The purpose of this section is to prove the result of Theorem 1. To do this, we propose to approach the weak periodic solution of (1) by an approximate periodic solution which is more regular. Thereafter, we establish necessaries estimates to pass to the limit on the approximate scheme. Let us introduce the approximate problem associated with Eq. (1).

3.1 Approximate scheme

Let $k > 0$, we can build a truncation function $\tau_k \in \mathcal{C}^2$, such that

$$\begin{cases} \tau_k(r) = r & \text{if} \quad 0 \leq r \leq k \\ \tau_k(r) \leq k+1 & \text{if} \quad r \geq k \\ 0 \leq \tau_k'(r) \leq 1 & \text{if} \quad r \geq 0 \\ \tau_k'(r) = 0 & \text{if} \quad r \geq k+1 \\ 0 \leq -\tau_k''(r) \leq \ C(k) \end{cases}$$

A typical construction of the truncation τ_k can be given as follows

$$\tau_k(s) = \begin{cases} s & \text{in } [0,k] \\ \frac{1}{2}(s-k)^4 - (s-k)^3 + s & \text{in } [k,k+1] \\ \frac{1}{2}(k+1) & \text{for } s > k+1 \end{cases}$$

Let w be the nonnegative weak super-solution of problem (1). For all $n \in \mathbb{N}^*$, we consider $w_n = \tau_n(w)$ and we introduce the Carathédory function g_n as follows

$$g_n(t,x,s,\xi) = \frac{g(t,x,s,\xi)}{1 + \frac{1}{n}|g(t,x,s,\xi)|}.\mathbf{1}_{\{w \leq n\}} \qquad \text{a.e in } Q_T$$

Using the standard convolution arguments, we can construct a nonnegative sequence $f_n \in \mathcal{C}_c^2(Q_T)$, such that (f_n) converges to μ in $\mathcal{M}_b^+(Q_T)$ and bounded in $L^1(Q_T)$. Let us define

$$\mu_n(t,x) = f_n(t,x).\mathbf{1}_{\{w \leq n\}} \qquad \text{a.e in } Q_T$$

It is clear that the sequence (μ_n) verifies

$$\mu_n \to \mu \text{ in } \mathcal{M}_b^+(Q_T) \text{ and } \|\mu_n\|_{L^1(Q_T)} \leq \|\mu\|_{\mathcal{M}_b^+(Q_T)} \tag{4}$$

Now, we define the approximate problem of (1) as follows

$$\begin{cases} \partial_t u_n - \Delta u_n + g_n(t,x,u_n,\nabla u_n) = \mu_n & \text{in } Q_T \\ u_n(0,.) = u_n(T,.) & \text{in } \Omega \\ u_n(t,x) = 0 & \text{on } \Sigma_T \end{cases} \tag{5}$$

As well known, we need to ensure the existence of a weak periodic solution to the approximate problem (5). This will be achieved by the following lemma.

Lemma 1 *For any $n \in \mathbb{N}^*$, problem (5) has a weak periodic solution $u_n \in L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega))$ in the sens that*

$$\begin{gathered} \partial_t u_n \in L^2(0,T;H^{-1}(\Omega)), \quad u_n(0,x) = u_n(T,x) \text{ in } L^2(\Omega) \\ \langle \partial_t u_n, \varphi\rangle + \int_{Q_T} \nabla u_n \nabla\varphi + \int_{Q_T} g_n(t,x,u_n,\nabla u_n)\varphi = \int_{Q_T} \mu_n\varphi \end{gathered} \tag{6}$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$. Where $\langle .,.\rangle$ denote the duality pairing between $L^2(0,T;H^{-1}(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. *Furthermore, we have*

$$0 \le u_n \le w_n \le w \tag{7}$$

Proof To prove the result of Lemma 1, we propose to apply the result of [14]. By using the assumptions $(\mathbf{A}_1)$ and $(\mathbf{A}_3)$, it comes that 0 is a weak periodic sub-solution to the approximate problem 5. On the other hand, by a simple computation we obtain

$$\begin{aligned} \partial_t w_n &= \partial_t w.\tau_n'(w) = \partial_t w.\mathbf{1}_{\{w\le n\}} \\ \nabla w_n &= \nabla w.\tau_n'(w) = \nabla w.\mathbf{1}_{\{w\le n\}} \\ \Delta w_n &= \Delta w.\mathbf{1}_{\{w\le n\}} + |\nabla w|^2.\tau_n''(w) \end{aligned}$$

We recall that $0 \le -\tau_n''(s) \le C(n)$, then, the fact that w is a weak super solution to (2) permit us to deduce that w_n is weak periodic super-solution to the approximate problem 5. Since the nonlinearity $(g_n(t,x,u_n,\nabla u_n))$ is bounded, we can apply the result of [14] to deduce the existence of a weak periodic solution $u_n \in L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega))$ to (5) such that

$$\begin{gathered} \partial_t u_n \in L^2(0,T;H^{-1}(\Omega)), \quad u_n(0,x) = u_n(T,x) \text{ in } L^2(\Omega) \\ \langle \partial_t u_n, \varphi\rangle + \int_{Q_T} \nabla u_n \nabla\varphi + \int_{Q_T} g_n(t,x,u_n,\nabla u_n)\varphi = \int_{Q_T} \mu_n\varphi \end{gathered}$$

for all test function $\varphi \in L^2(0,T;H_0^1(\Omega))$. As a consequence of [14], we have

$$0 \le u_n \le w_n \le w$$

□

3.2 A priori estimates

In this section, we will state the priori estimates on the approximate solution u_n.

Lemma 2 *Let u_n be the weak periodic solution to the approximate problem (5). Then*

i) *for every $k > 0$*

$$\int_{Q_T} |\nabla T_k(u_n)|^2 + k \int_{Q_T \cap \{k<u_n\}} |g_n(t,x,u_n,\nabla u_n)| \le k\|\mu\|_{\mathcal{M}_b^+(Q_T)}$$

ii) *there exists constants C_1 and C_2 independents on n such that*

$$\int_{Q_T} |g_n(t,x,u_n,\nabla u_n)| \le C_1$$
$$\|u_n\|_{\mathcal{V}_T} \le C_2$$

Proof *i)* By choosing $\varphi = T_k(u_n)$ as a test function in (6), it follows

$$\langle \partial_t u_n, T_k(u_n)\rangle + \int_{Q_T} |\nabla T_k(u_n)|^2 + \int_{Q_T} g_n(t,x,u_n,\nabla u_n) T_k(u_n) = \int_{Q_T} \mu_n T_k(u_n) \tag{8}$$

Since u_n is periodic with respect to time, one obtains

$$\langle \partial_t u_n, T_k(u_n)\rangle = \int_\Omega S_k(u_n(T)) - \int_\Omega S_k(u_n(T)) = 0$$

On the other hand, by employing (4) and (7) the relation (8) becomes

$$\int_{Q_T} |\nabla T_k(u_n)|^2 + k \int_{Q_T \cap \{k<u_n\}} |g_n(t,x,u_n,\nabla u_n)| \le k\|\mu\|_{\mathcal{M}_b^+(Q_T)}$$

ii) Let us remark that

$$\int_{Q_T} |g_n(t,x,u_n,\nabla u_n)| = \int_{Q_T \cap \{1<u_n\}} |g_n(t,x,u_n,\nabla u_n)| + \int_{Q_T \cap \{u_n \le 1\}} |g_n(t,x,u_n,\nabla u_n)|$$

By employing the result of *i*), one gets

$$\int_{Q_T} |g_n(t,x,u_n,\nabla u_n)| \le \|\mu\|_{\mathcal{M}_b^+(Q_T)} + \int_{Q_T \cap \{u_n \le 1\}} |g_n(t,x,u_n,\nabla u_n)| \tag{9}$$

On the other hand, thanks to the growth assumption ($\mathbf{A}_4$) and Hölder's inequality, we have

$$\begin{aligned}\int_{Q_T \cap \{u_n \le 1\}} |g_n(t,x,u_n,\nabla u_n)| &\le \int_{Q_T} H(t,x) + d\left(\int_{Q_T} |T_1(u_n)|^r + \int_{Q_T} |\nabla T_1(u_n)|^r \right) \\ &\le \|H\|_{L^1(Q_T)} + d\left(T|\Omega| + (T|\Omega|)^{\frac{2-r}{2}} \|T_1(u_n)\|^{\frac{r}{2}}_{L^2(0,T;H_0^1(\Omega))} \right)\end{aligned}$$

Hence, the result of *i*) permit us to deduce that

$$\int_{Q_T\cap\{u_n\le 1\}} |g_n(t,x,u_n,\nabla u_n)| \le \|H\|_{L^1(Q_T)} + d\left(T|\Omega| + (T|\Omega|)^{\frac{2-r}{2}}\|\mu\|^{\frac{r}{2}}_{\mathcal{M}_b^+(Q_T)}\right) \quad (10)$$

Consequently, one may conclude that $(g_n(t,x,u_n,\nabla u_n))$ is bounded in $L^1(Q_T)$. To prove that (u_n) is bounded in $\mathcal{V}_T$, one may use the result of [6], we have

$$\|u_n\|_{L^s\left(0,T;W_0^{1,r}(\Omega)\right)} \le C(\Omega,s,r)\left[\|g_n\|_{L^1(Q_T)} + \|\mu_n\|_{L^1(Q_T)} + \|u_n(0)\|_{L^1(\Omega)}\right]$$

such that $s,r\ge 1$ and $\frac{2}{s}+\frac{N}{r} > N+1$. Hence, choosing $s=r$ in the last inequality and employing (4), (7) and the boundness of (g_n) in $L^1(Q_T)$, one obtains

$$\|u_n\|_{\mathcal{V}_T} \le C(\Omega,r)\left[C_1 + \|\mu\|_{\mathcal{M}_b^+(Q_T)} + \|w(0)\|_{L^1(\Omega)}\right]$$

where $C(\Omega,r)$ is a constant depending only on Ω and r. □

Lemma 3 *Let u_n be the sequence defined as above. Then, there exists a sub-sequence of u_n still denoted by u_n for simplicity such that*

$$\begin{aligned} u_n &\longrightarrow u \text{ strongly in } L^1(0,T;W_0^{1,1}(\Omega)) \\ (\nabla u_n,u_n) &\longrightarrow (\nabla u,u) \text{ a.e in } Q_T \\ u_n &\longrightarrow u \text{ strongly in } \mathcal{V}_T \end{aligned}$$

Proof We set $\gamma_n(t,x) := \mu_n(t,x) - g_n(t,x,u_n,\nabla u_n)$ for a.e (t,x) in Q_T. Therefore, from the result *ii*) of the Lemma 2 and the relation (4) it is clear that (γ_n) is bounded in $L^1(Q_T)$. Furthermore, thanks to the compactness result of [7], we obtain that the application $(u_n(0),\gamma_n)\longmapsto u_n$ is compact from $L^1(\Omega)\times L^1(Q_T)$ into $L^1(0,T;W_0^{1,1}(\Omega))$. Consequently, we can extract a sub-sequence, still denoted by u_n for simplicity, such that

$$\begin{aligned} u_n &\longrightarrow u \text{ strongly in } L^1(0,T;W_0^{1,1}(\Omega)) \\ (\nabla u_n,u_n) &\longrightarrow (\nabla u,u) \text{ a.e in } Q_T \end{aligned}$$

It remains to prove that (u_n) converges strongly in $\mathcal{V}_T$. To do this, we will show that (∇u_n) is a Cauchy sequence in $L^r(Q_T)^N$. For $m,n\ge 1$ and $0<\alpha<1$, we can employ Hölder's inequality to obtain

$$\int_{Q_T} |\nabla u_n - \nabla u_m|^r \le \left(\int_{Q_T} |\nabla u_n - \nabla u_m|\right)^{\alpha}\left(\int_{Q_T} |\nabla u_n - \nabla u_m|^{\frac{r-\alpha}{1-\alpha}}\right)^{1-\alpha} \quad (11)$$

Choosing $\alpha\in]0,1[$ such that $\frac{r-\alpha}{1-\alpha}\in[1,\frac{N+2}{N+1}[$ and employing the result *ii*) of the Lemma 2, one has

$$\|\nabla u_n - \nabla u_m\|^r_{L^r(Q_T)} \le C\|\nabla u_n - \nabla u_m\|^{\alpha}_{L^1(Q_T)}$$

Since (∇u_n) is strongly convergent in $L^1(Q_T)^N$, one may deduce that (∇u_n) is a Cauchy sequence in $L^r(Q_T)^N$. Which is equivalent to say that (u_n) strongly convergent in $\mathcal{V}_T$. □

3.3 Passing to the limit

In this section, we are concerned with the passage to the limit in the approximate scheme (5). By employing the growth condition ($\mathbf{A}_4$) and the result of Lemma 3, it comes that

$$g_n(t,x,u_n,\nabla u_n) \longrightarrow g(t,x,,u,\nabla u) \text{ in } L^1(Q_T)$$

To prove that the limit u of u_n is a weak periodic solution to (2), we take $\varphi \in \mathcal{C}_c^\infty(Q_T)$ and by employing the convergence results of the Lemma 3, we arrive at

$$\langle \partial_t u_n, \varphi\rangle \longrightarrow \langle \partial_t u, \varphi\rangle$$
$$\int_{Q_T} \nabla u_n \nabla\varphi \longrightarrow \int_{Q_T} \nabla u \nabla \varphi$$
$$\int_{Q_T} g_n(t,x,u_n,\nabla u_n)\varphi \longrightarrow \int_{Q_T} g(t,x,u,\nabla u)\varphi$$
$$\int_{Q_T} \mu_n \varphi \longrightarrow \int_{Q_T} \varphi d\mu$$

Which implies that u satisfies

$$u \in \mathcal{C}\big([0,T], L^1(\Omega)\big) \cap L^1(0,T;W_0^{1,1}(\Omega)), \quad g(t,x,u,\nabla u) \in L^1(Q_T)$$
$$\partial_t u - \Delta u + g(t,x,u,\nabla u) = \mu \text{ in } \mathcal{D}'\big(Q_T\big)$$

It remains to verify the periodicity condition (3). To do this, we propose to apply the semi-group theory, we have

$$u_n(T,.) = \mathcal{S}(T)u_n(0,.) + \int_0^T \mathcal{S}(T-s)\gamma_n(s,.)ds$$

where $\mathcal{S}(t)$ present the semigroup of contractions in $L^1(\Omega)$ generated by the Laplacian $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$. Employing the fact that $u_n(0,.) = u_n(T,.)$ in $L^1(\Omega)$, we obtain for all $\varphi \in L^\infty(\Omega)$

$$\int_\Omega u_n(0,x)\varphi(x) = \int_\Omega \mathcal{S}(T)u_n(0,x)\varphi(x) + \int_\Omega \int_0^T \mathcal{S}(T-s)\gamma_n(s,x)\varphi(x) \tag{12}$$

We recall that

$$\gamma_n \to \gamma \text{ strongly in } L^1(Q_T)$$

where $\gamma(t,x) := \mu(t,x) - g(t,x,u,\nabla u)$. On the other hand, by employing the continuity of $\mathcal{S}(t)$ in $L^1(Q_T)$, we can pass to the limit in (12) when n tends to infinity, we obtain

$$\int_\Omega u(0,x)\varphi(x) = \int_\Omega \mathcal{S}(T)u(0,x)\varphi(x) + \int_\Omega \int_0^T \mathcal{S}(T-s)\gamma(s,x)\varphi(x) = \int_\Omega u(T,x)\varphi(x)$$

Hence, we deduce that $u(0,.) = u(T,.)$ in $L^1(\Omega)$. Which ends the proof of Theorem 1.

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